GRAVITATIONAL RADIATION
and the
MOTION OF TWO POINT MASSES

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1964

(Submitted February 14, 1964)

ACKNOWLEDGEMENTS

The author wishes to acknowledge the help and guidance of Dr. Jon Mathews throughout the research leading to this work. The financial support of the National Science Foundation in the form of graduate fellowships from 1960 to 1964 is gratefully acknowledged.

The author wishes to thank his wife, Linda, for encouragement and understanding during this project.

ABSTRACT

The expansion of the field equations of general relativity in powers of the gravitational coupling constant yield conservation laws of energy, momentum, and angular momentum. From these laws, the loss of energy, momentum and angular momentum of a system due to the radiation of gravitational waves is found. Two techniques, radiation reaction and flux across a large sphere, are used in this calculation and are shown to be in agreement over a time average. These results are then applied to the system of two point masses moving in elliptical orbits around each other. The secular decays of the semi-major axis and eccentricity are found as functions of time and are integrated to specify the decay by gravitational radiation of such systems as functions of their initial conditions. For completeness, the secular change in the perihelion of the orbit for two arbitrary masses is found by a method which is in the spirit of the above calculations. The case of gravitational radiation when the bodies are relativistic is then considered, and an equation for the radiation similar to that of electromagnetic radiation is found. Also a proof is given that, regardless of coordinate

systems or conditions, the energy of a system must decrease as a result of the radiation of gravitational waves, providing the potentials are inversely proportional to the distance from the source for large distances.

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I. INTRODUCTION

The existence of gravitational radiation was predicted by Einstein[1,2] shortly after he formulated the general theory of gravitation. It was found that physical systems emit gravitational waves much in analogy with the emission of electromagnetic waves by a system of moving charges. Early attempts to calculate the energy in these waves were based on using a pseudo-stress-energy tensor for the evaluation of the energy flux. One disadvantage of this was that one could always choose a coordinate system where the energy flux would vanish.[3] This led many people to be skeptical about the reality of gravitational radiation. Another disadvantage of the calculation was that it was valid only for systems which were not gravitationally bound. Thus the problem in which one had the most chance of finding effects of the radiation, the case of double stars, had no solution at that time.

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[1] A. Einstein, Sb. Preuss. Akad. Wiss. (1916), 688.

[2] A. Einstein, Sb. Preuss. Akad. Wiss. (1918), 154.

[3] For a detailed discussion of the status of the theories of gravitational radiation and their objections, the reader is referred to the review article by F. A. E. Pirani in <u>Gravitation: An Introduction to Current Research</u>, L. Witten, ed.,(John Wiley and Sons Inc., New York, 1962), Chap. 6.

Later Eddington found the radiation from a system by calculating the radiation reaction of the system on itself.[4] However, like Einstein's method, this was not valid for gravitationally bound systems. For problems in which the radiation is constant, the two methods are in agreement; for problems where the radiation is time dependent, the answers differ. One can show that over a time average of the motion, the two answers agree. This is in analogy with similar results from the theory of electromagnetic radiation and radiation reaction.

For systems in which the velocities of the masses are small compared to the velocity of light, the calculation of Einstein has been extended to include gravitationally bound systems.[5] The problem concerning the choice of the stress-energy of the gravitational field is still debated. Also the selection of certain preferred coordinate systems and conditions is subject to much criticism. One can find references in the current literature which describe the radiation from the system as carrying away energy,[6] bring-

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[4]A. S. Eddington, Proc. Roy. Soc. (London) 102A, 268 (1922).

[5]See, for example, L. Landau and E. Lifshitz, The Classical Theory of Fields (Addison-Wesley Publishing Co., Inc., Reading, Mass.), Chap. 11.

[6]Ibid.

ing in energy,[7] carrying no energy,[8] or having an energy dependent on the coordinate system used.[9] Clearly a consistent picture of gravitational radiation is needed.

One approach to gravitational radiation is to consider only exact solutions of the non-linear field equations of general relativity. Although some solutions have been found, they correspond to un-physical systems.[10] Therefore one usually employs some approximation procedure in solving the field equations. The field equations are sometimes expanded in powers of the gravitational coupling constant because of the weakness of the gravitational interaction. In addition, one encounters expansions in powers of the ratio of the velocities of the masses of the system to the velocity of light and also expansions in inverse powers of the distance from the system under consideration. Each approximation method is not independent of the others. Throughout this paper, we will be concerned only with

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[7] P. Havas and J. N. Goldberg, Phys. Rev. <u>128</u>, 398 (1962).

[8] L. Infeld and J. Plebanski, <u>Motion and Relativity</u> (Pergamon Press, Inc., New York, 1960), Chap. VI.

[9] Ibid.

[10] For an example, see J. Weber, <u>General Relativity and Gravitational Waves</u> (Interscience Publishers, 1961), pp. 99-105.

solutions obtained through the use of these approximation methods and not with any exact solutions of the field equations.

It will be assumed in the following that the reader has a knowledge of the fundamentals of general relativity theory. However, because of different notations used, a brief summary will be given so that there is no confusion about the symbols used.

II. BASIC CONCEPTS OF GENERAL RELATIVITY

We denote the metric tensor by $g_{\mu\nu}$, where in flat space $g_{\mu\nu} = \delta_{\mu\nu}$ and

$$\delta_{\mu\nu} = \left\{1, \mu=\nu=4;\ -1, \mu=\nu\neq 4;\ 0, \mu\neq\nu\right\}. \quad (2.1)$$

The index 4 denotes a time component and 1, 2, 3 denote spatial components. Greek indices can take the values 1, 2, 3, and 4, whereas Latin indices can take the values 1, 2, 3 of the spatial components. We may have components of a vector or a tensor with lower or upper indices. The former are called covariant components and the latter contravariant components. We raise or lower indices through the use of the metric tensor $g^{\mu\nu}$ or $g_{\mu\nu}$, where $g^{\mu\nu}$ is defined by $\sum_\alpha g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$ and

$$\delta^\nu_\mu = \left\{1,\ \mu=\nu\ ;\ 0,\ \mu\neq\nu\right\} . \quad (2.2)$$

As a shorthand we employ the Einstein summation convention for summing over repeated indices: $A_\alpha A^\alpha \equiv \sum_\alpha A_\alpha A^\alpha = A_4A^4 + A_1A^1 + A_2A^2 + A_3A^3$. This is valid only if one index is an upper index and one is a lower index. In later work we will be dealing with quantities such as $\delta^{\mu\nu}A_\mu A_\nu$, where $\delta^{\mu\nu} = \delta_{\mu\nu}$; we then define (Feynman summation convention) $A_\mu A_\mu = \delta^{\mu\nu} A_\mu A_\nu = A_4A_4 - A_1A_1 - A_2A_2 - A_3A_3$. Thus if we have two repeated lower indices, we sum them a la Feynman; if we have one up and one down, we sum them a la Einstein.

The components of x_α are (x_1 , x_2 , x_3 , ct). We have a shorthand for the ordinary derivative with respect to x_α : $A_{,\alpha} = \frac{\partial A}{\partial x_\alpha}$. For the covariant derivative of a quantity with respect to x_α we use the shorthand $A_{;\alpha}$. The covariant derivative of a tensor involves the metric tensor as well as the ordinary derivatives of the tensor and the tensor itself. The metric tensor enters through combinations called "Christoffel symbols of the second kind." Christoffel symbols of the first kind are denoted by $[\alpha\beta,\gamma]$ and are defined by

$$[\alpha\beta,\gamma] = \tfrac{1}{2}\left[g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma} \right] . \qquad (2.3)$$

Christoffel symbols of the second kind are denoted by $\left\{ {\sigma \atop \alpha\beta} \right\}$ and are defined by

$$\left\{ {\sigma \atop \alpha\beta} \right\} = g^{\sigma\gamma}[\alpha\beta,\gamma] \quad . \qquad (2.4)$$

Two successive covariant differentiations do not, in general, commute. In particular, if we differentiate covariantly a vector A_μ or A^μ with respect to x_α and then with respect to x_β , we get that

$$A^\mu{}_{;\alpha;\beta} = A^\mu{}_{;\beta;\alpha} - A^\lambda R^\mu{}_{\lambda\alpha\beta}$$

or

$$A_{\mu;\alpha;\beta} = A_{\mu;\beta;\alpha} + A_\lambda R^\lambda{}_{\mu\alpha\beta} \quad .$$

$R^\lambda{}_{\mu\alpha\beta}$ is an expression involving the metric tensor and its derivatives. The quantity $R_{\nu\mu\alpha\beta} \equiv g_{\nu\lambda} R^\lambda{}_{\mu\alpha\beta}$ is called the curvature tensor. The quantity $R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}$ is called the Ricci tensor. The quantity $R \equiv g^{\mu\nu}R_{\mu\nu}$ is called the curvature scalar. The $R^\lambda{}_{\mu\alpha\beta}$ satisfy the Bianchi identities

$$R^\lambda{}_{\mu\alpha\beta;\gamma} + R^\lambda{}_{\mu\gamma\alpha;\beta} + R^\lambda{}_{\mu\beta\gamma;\alpha} = 0 \quad . \qquad (2.5)$$

From this it follows that

$$(R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu} R)_{;\nu} = 0 \quad . \qquad (2.6)$$

so that the tensor $R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R$ has zero covariant divergence. In order to relate the curvature of space to matter, we would like to put a second rank symmetric tensor on the right side of equation 2.6. We <u>require</u> it to have zero covariant divergence for otherwise we would not have a consistent equation. In flat space the tensor would then have to be an ordinary divergenceless tensor of rank two, of which the most obvious choice is the stress-energy-momentum tensor $T^{\mu\nu}$. Thus we can write the field equations of general relativity as

$$(R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu} R) = \frac{8\pi G}{c^4}\, T^{\mu\nu}, \qquad (2.7)$$

where the constant has been chosen appropriately for later work and G is the ordinary gravitational constant. $T^{\mu\nu}$ is the stress-energy tensor of all matter and fields <u>except</u> gravity. For a classical particle of mass m, we may write $T^{\mu\nu}$ as

$$T^{\mu\nu} = \frac{m}{\sqrt{-g}}\int \delta^4(x - z(s))\frac{dz^\mu}{ds}\frac{dz^\nu}{ds}\,ds \qquad (2.8)$$

where $g = \det g_{\alpha\beta}$ and s is the proper time of the particle with s satisfying

$$g_{\mu\nu}\frac{dz^\mu}{ds}\frac{dz^\nu}{ds} = 1 \quad . \qquad (2.9)$$

From equation 2.8 and the fact that the covariant divergence of $T^{\mu\nu}$ vanishes, we can find the equations of motion (equation of the geodesic path) for a particle of mass m in a gravitational field

$$m\frac{d^2x^\mu}{ds^2} = -m\left\{ {\mu \atop \alpha\beta} \right\}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \quad . \qquad (2.10)$$

We will need to have the expression for $R_{\mu\nu}$ explicitly. For this we use

$$\begin{aligned} R_{\mu\nu} = {} & \left\{ {\alpha \atop \mu\nu} \right\}_{,\alpha} - \left\{ {\alpha \atop \mu\alpha} \right\}_{,\nu} + \\ & + \left\{ {\alpha \atop \mu\nu} \right\}\left\{ {\beta \atop \alpha\beta} \right\} - \left\{ {\alpha \atop \mu\beta} \right\}\left\{ {\beta \atop \nu\alpha} \right\} \quad . \end{aligned} \qquad (2.11)$$

One can easily see that this contains both first and second derivatives of the metric tensor as well as the metric tensor itself. The field equations are therefore non-linear equations and one can not find a general exact solution. It is for this reason that we consider an expansion in terms of the deviation of the metric from the flat space metric.

III. CONSERVATION LAWS and the GRAVITATIONAL FIELD STRESSES

A. Expansion of the Field Equations

Let $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ and expand quantities in powers of $h_{\mu\nu}$. In practice we see that this is a reasonable expansion. At a distance r from a mass M, $h_{\mu\nu}$ is of the order $GM/(rc^2)$. At the surface of the earth this becomes $\sim 10^{-9}$, at the surface of the sun $\sim 10^{-6}$, and at the surface of a white dwarf $\sim 5 \times 10^{-5}$. Thus where one term of the expansion contains one more factor of h than another, one would expect the first to be much smaller than the second.

The metric $g^{\mu\nu}$ can be found as an infinite series in h and is determined from the equation $g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}$. This gives

$$g^{\mu\nu} = \delta_{\mu\nu} - h_{\mu\nu} + h_{\mu\sigma} h_{\sigma\nu} + \cdots \tag{3.1}$$

Expanding the field equations, equation 2.7, we get

$$-\tfrac{1}{2}\left[\bar{h}_{\mu\nu},_{\lambda\lambda} - \bar{h}_{\mu\lambda},_{\lambda\nu} - \bar{h}_{\nu\lambda},_{\lambda\mu} + \delta_{\mu\nu}\bar{h}_{\lambda\sigma},_{\sigma\lambda}\right] + \\ + F^{(2)}(h,h) + F^{(3)}(h,h,h) + \cdots = \frac{8\pi G}{c^4} T^{\mu\nu}, \tag{3.2}$$

where $F^{(k)}$ is of order h^k and $\bar{h}_{\mu\nu}$ is defined by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu} h_{\sigma\sigma} \quad . \tag{3.3}$$

The only terms linear in h are within the brackets.

If we take the ordinary divergence of the linear terms, we get

$$\left[\bar{h}_{\mu\nu,\lambda\lambda} - \bar{h}_{\mu\lambda,\lambda\nu} - \bar{h}_{\nu\lambda,\lambda\mu} + \bar{h}_{\lambda\sigma,\lambda\sigma}\delta_{\mu\nu}\right]_{,\nu} = \\ = \bar{h}_{\mu\nu,\lambda\lambda\nu} - \bar{h}_{\mu\sigma,\sigma\nu\nu} - \bar{h}_{\nu\sigma,\sigma\mu\nu} + \bar{h}_{\lambda\sigma,\lambda\sigma\mu} = 0 \;. \tag{3.4}$$

If in equation 3.2 we take the terms non-linear in h to the right hand side, we have (setting c = 1)

$$\bar{h}_{\mu\nu,\lambda\lambda} - \bar{h}_{\mu\sigma,\sigma\nu} - \bar{h}_{\nu\sigma,\sigma\mu} + \delta_{\mu\nu}\bar{h}_{\lambda\sigma,\lambda\sigma} = \\ = -16\pi G\left[T^{\mu\nu} + X(\mu,\nu)\right] , \tag{3.5}$$

where $X(\mu,\nu)$ has the expansion $X(\mu,\nu) = \sum_{k=2}^{\infty} X^{(k)}(\mu,\nu)$. The upper index again indicates the power of h present. $X(\mu,\nu)$ is not a tensor in the sense of general relativity; it is however a tensor with respect to Lorentz transformations of special relativity. Because of equation 3.4, the right side of equation 3.5 has zero ordinary divergence, i. e.

$$\left[T^{\mu\nu} + X(\mu,\nu)\right]_{,\nu} = 0 \;. \tag{3.6}$$

We can therefore write integral conservation laws for the quantity $T^{\mu\nu} + X(\mu,\nu) \equiv S(\mu,\nu)$:

$$\frac{d}{dt}\int_V S(4,\mu)\, dV \quad - \quad \int_S S(i,\mu)\, dS_i \;= 0 \;. \tag{3.7}$$

Because of these conservation laws, we can interpret $S(4,4) = T^{44} + X(4,4)$ as the total energy density of the system and $S(4,i) = T^{4i} + X(4,i)$ as the total momentum density of the system including gravity. Also because $S(\mu,\nu) = S(\nu,\mu)$, we can write the law of conservation of total angular momentum as

$$\frac{d}{dt}\int\left[x_i\ S(4,j) - x_j\ S(4,i)\right] dV - \\ - \int\left[x_i\ S(k,j) - x_j\ S(k,i)\right] dS_k = 0\ . \tag{3.8}$$

The question arises as to whether $S(\mu,\nu)$ is uniquely defined; we could just as well have expanded different forms of the field equations, say,

$$R_\mu{}^\nu - \tfrac{1}{2}\, g_\mu{}^\nu R = 8\pi G\, T_\mu{}^\nu \tag{3.9}$$

or

$$(R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu}R)\sqrt{-g} = 8\pi G\, T^{\mu\nu}\sqrt{-g} \tag{3.10}$$

However, it is obvious that one will obtain the same $S(\mu,\nu)$ from these, since equations 3.9 and 3.10 are equivalent to equation 2.7. We shall find it advantageous to work with the quantity $T^{\mu\nu}\sqrt{-g}$, which we shall hereafter call $\tilde{T}^{\mu\nu}$. The $X(\mu,\nu)$ obtained from the expansion of equation 3.10 and defined by $S(\mu,\nu) = \tilde{T}^{\mu\nu} + X(\mu,\nu)$, we shall call $\tilde{X}_{\mu\nu}$, the gravitational field stresses corresponding to the stress-energy tensor $\tilde{T}^{\mu\nu}$. $S(\mu,\nu)$ we shall hereafter write as $S_{\mu\nu}$, the position of the indices being unimportant since it is an expression which is independent of the T used to find it. It is a uniquely defined quantity, but it is not a tensor with respect to arbitrary transformations of coordinates. It is, however, a conserved quantity, and, as we saw above, can be called the total stress-energy-momentum of the system including gravity.

B. <u>Field Theory Approach</u>

The quantity $S_{\mu\nu}$ could have also been obtained by another method, namely from Feynman's field theory of a spin-two meson coupled to energy.[1] The classical limit of this quantum theory is precisely the expanded version of Einstein's field equations of general relativity. In the derivation of this theory, there is some arbitrariness in that one must require as an additional postulate that the resulting field equations come from the variation of an action. As first pointed out by Feynman and Huggins[2], this requirement seems to be the only way to uniquely define the theory and still give one the feeling that the equations have not been pulled out of a hat. Since this question has bearing on the problem of gravitational radiation and the motion of two point masses, a discussion of the arbitrariness of the stress-energy of the gravitational fields will be given in the appendix.

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[1] R. P. Feynman, lecture notes, California Institute of Technology (unpublished).

[2] E. Huggins, Ph. D. thesis, California Institute of Technology (1962).

The first order field equations are

$$\bar{h}_{\mu\nu,\lambda\lambda} - \bar{h}_{\mu\lambda,\lambda\nu} - \bar{h}_{\nu\lambda,\lambda\mu} + \delta_{\mu\nu}\bar{h}_{\sigma\lambda,\sigma\lambda} = -16\pi G\, \tilde{T}^{\mu\nu}. \quad (3.11)$$

The divergence of the left side is zero; however the divergence of the right side is not. We find that $\tilde{T}^{\mu\nu}{}_{,\nu} \sim O(h^2)$. Physically we can say that gravity is coupled to all energy including the energy of gravity itself, and we have not included the energy of gravity in our equations. Thus we must add to the right side an expression which gives the stress-energy of the gravitational field. To lowest order, we expect this to be bilinear in $h_{\alpha\beta}$ and to have just two derivatives. If we assume that this must come from the variation of an action, where in the action Lagrangian we take all possible independent combinations of three $h_{\alpha\beta}$'s and two derivatives, then we have specified the Lagrangian and the second order stresses when we impose the requirement that $(\tilde{T}^{\mu\nu} + \tilde{X}^{(2)}_{\mu\nu})_{,\nu} \sim O(h^3)$. In a like manner we could solve for $\tilde{X}^{(3)}_{\mu\nu}$ in which case $(\tilde{T}^{\mu\nu} + \tilde{X}^{(2)}_{\mu\nu} + \tilde{X}^{(3)}_{\mu\nu})_{,\nu} \sim O(h^4)$ and so on for higher orders. This reproduces the expanded version of Einstein's field equations.

From this point of view, therefore, the expanded equations appear to be the fundamental equations. It is the expanded equations which one uses in calculating processes in the quantized gravity theory. Of course one can sum the series in the equations by introducing the metric tensor and get back to Einstein's equations and all of their geometrical beauty. However, one can imagine a world where only the expanded equations were set down, and the geometrical significance of the theory was never explored.

Because of the analogy of the expanded field equations with other field theories, we shall call $h_{\alpha\beta}$ the gravitational potential. To be sure, the rate at which clocks run and the lengths of rulers depend on this potential, but usually we will be considering cases where $h_{\alpha\beta}$ is small and the effects of it are also small.

We shall assume the expanded field equations to be valid:

$$\begin{aligned}&\bar{h}_{\mu\nu,\lambda\lambda} - \bar{h}_{\mu\lambda,\lambda\nu} - \bar{h}_{\nu\lambda,\lambda\mu} + \delta_{\mu\nu}\bar{h}_{\sigma\lambda,\sigma\lambda} = \\ &= -16\pi G\, S_{\mu\nu} = -16\pi G\left[\tilde{T}^{\mu\nu} + \sum_{k=2}^{\infty}\tilde{X}^{(k)}_{\mu\nu}\right].\end{aligned} \tag{3.12}$$

In the following we will make extensive use of $\tilde{X}^{(2)}_{\mu\nu}$. It is given by

$$\begin{aligned}\tilde{X}^{(2)}_{\mu\nu} = -\frac{1}{32\pi G}\Big\{ & 4[\mu\beta,\alpha][\nu\beta,\alpha] + \\ & - 2\,\delta_{\mu\nu}[\gamma\beta,\alpha][\gamma\beta,\alpha] + \\ +2h_{\alpha\beta}&\left[h_{\mu\nu,\alpha\beta} - h_{\mu\alpha,\beta\nu} - h_{\nu\alpha,\beta\mu} + h_{\alpha\beta,\mu\nu}\right] + \\ +2h_{\mu\alpha}&\left[h_{\nu\alpha,\lambda\lambda} - h_{\alpha\lambda,\lambda\nu} - h_{\nu\lambda,\lambda\alpha} + h_{\lambda\lambda,\alpha\nu}\right] + \\ +2h_{\nu\alpha}&\left[h_{\mu\alpha,\lambda\lambda} - h_{\alpha\lambda,\lambda\mu} - h_{\mu\lambda,\lambda\alpha} + h_{\lambda\lambda,\alpha\mu}\right] + \\ - h_{\sigma\sigma}&\left[h_{\mu\nu,\alpha\alpha} - h_{\mu\alpha,\alpha\nu} - h_{\nu\alpha,\alpha\mu} + h_{\alpha\alpha,\mu\nu}\right] +\end{aligned}$$

$$+ 2h_{\mu\nu}\left[h_{\gamma\sigma,\sigma\gamma} - h_{\gamma\gamma,\sigma\sigma}\right] +$$
$$- 2\delta_{\mu\nu}h_{\alpha\beta}\left[h_{\alpha\beta,\lambda\lambda} - 2h_{\alpha\lambda,\lambda\beta} + h_{\lambda\lambda,\alpha\beta}\right] +$$
$$+ \delta_{\mu\nu}h_{\sigma\sigma}\left[h_{\gamma\gamma,\alpha\alpha} - h_{\gamma\alpha,\gamma\alpha}\right]\Big\}. \qquad (3.13)$$

C. Coordinate Conditions

One would like to put equation 3.12 in the form of an integral equation in order to solve for $\bar{h}_{\alpha\beta}$ by an approximation method. This would be possible if we had a wave equation, i. e. a $\Box \equiv \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}$ operating on the $\bar{h}_{\mu\nu}$ on the left side of equation 3.12. We have the corresponding problem in electromagnetic theory. The equation for the electromagnetic potentials is given by

$$A_{\mu,\lambda\lambda} - A_{\lambda,\mu\lambda} = -J_\mu \quad . \qquad (3.14)$$

We note that equation 3.14 is invariant under the transformation $A_\mu \rightarrow A'_\mu + \chi_{,\mu}$. Under this transformation, $A_{\mu,\mu}$ transforms like $A_{\mu,\mu} \rightarrow A'_{\mu,\mu} + \chi_{,\mu\mu}$, so that we can always find a χ such that $A_{\mu,\mu}$ can be made to vanish; we just choose

$$\chi = -\frac{1}{4\pi}\int\left[\frac{A_{\mu,\mu}}{r}\right]_{ret.} dV \quad . \qquad (3.15)$$

A corresponding choice can be made in equation 3.12. If we look at the linear terms only, we see that they are invariant under the transformation

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu} \quad , \qquad (3.16)$$

where η_μ is arbitrary. If we form the quantity $\bar{h}_{\mu\nu,\nu}$, we see that $\bar{h}_{\mu\nu,\nu} \longrightarrow \bar{h}'_{\mu\nu,\nu} + \eta_{\mu,\nu\nu}$, and as before we can find an η_μ such that $\bar{h}_{\mu\nu,\nu} = 0$. This type of constraint on the $h_{\mu\nu}$ is called a coordinate condition. This has, however, neglected the higher order terms on the right side of equation 3.12. They are not necessarily invariant under the transformation given by equation 3.16. But it turns out that we can find a transformation such that the entire field equations are left invariant. We might also wish to impose a different coordinate condition than $\bar{h}_{\mu\nu,\nu} = 0$. We would thus like to examine to what extent any coordinate condition can be chosen in a given problem. The technique consists in showing that one can always make a coordinate transformation so that in the new system the specified condition is satisfied.

The field equations of general relativity are invariant under arbitrary coordinate transformations. Under such a transformation, the $g_{\mu\nu}$ transforms like

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = g_{\alpha\beta}(x')\,\frac{dx'^{\alpha}}{dx^{\mu}}\frac{dx'^{\beta}}{dx^{\nu}} \;. \qquad (3.17)$$

If we let $x'^{\alpha} = x^{\alpha} + \eta^{\alpha}(x)$, where $\eta^{\alpha}(x)$ is arbitrary (not necessarily infinitesimal), then we can write equation 3.17 as

$$g_{\mu\nu}(x) \longrightarrow g_{\mu\nu}(x) + g_{\nu\beta}\eta^{\beta}_{,\mu} + g_{\mu\alpha}\eta^{\alpha}_{,\nu} + \qquad (3.18)$$
$$+ g_{\alpha\beta}\eta^{\alpha}_{,\mu}\eta^{\beta}_{,\nu} + g_{\mu\nu,\lambda}\eta^{\lambda} + \tfrac{1}{2} g_{\mu\nu,\lambda\gamma}\eta^{\lambda}\eta^{\gamma} + \cdots$$

Rewriting this symbolically,

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + [\eta_{\nu,\mu} + \eta_{\mu,\nu}] + [\eta^2 + \eta h] + [\eta^3 + \eta^2 h] + \cdots$$
$$\rightarrow h_{\mu\nu} + [\eta_{\nu,\mu} + \eta_{\mu,\nu}] + F_{2\mu\nu} + F_{3\mu\nu} + \cdots$$

and forming the quantity $\bar{h}_{\mu\nu,\nu}$ yields

$$\bar{h}_{\mu\nu,\nu} \longrightarrow \bar{h}_{\mu\nu,\nu} + \eta_{\mu,\nu\nu} + F'_{2\mu} + F'_{3\mu} + \cdots$$

If in the new system we want $\bar{h}_{\mu\nu,\nu} = 0$, we need to solve

$$\bar{h}_{\mu\nu,\nu} = -\eta_{\mu,\nu\nu} + F'_{2\mu} + F'_{3\mu} + \cdots$$

To do this we let $\eta_\mu = \eta^{(1)}_\mu + \eta^{(2)}_\mu + \cdots$. Then $\eta^{(1)}_\mu$, the lowest order η_μ, is given by

$$\eta^{(1)}_\mu = \frac{1}{4\pi}\int\left[\frac{\bar{h}_{\mu\nu,\nu}}{r}\right]_{\text{ret.}} dV ,$$

and thus in the transformed system we have that $\bar{h}_{\mu\nu,\nu} \sim F''_{2\mu}$, a quantity of second order in h and η. We then solve $F''_{2\mu} = \eta^{(2)}_{\mu,\lambda\lambda}$, where $F''_{2\mu}$ contains terms like h^2, $h\eta^{(1)}$, and $[\eta^{(1)}]^2$, which are known. Thus

$$\eta^{(2)}_\mu = -\frac{1}{4\pi}\int\left[\frac{F''_{2\mu}}{r}\right] dV .$$

Similarly we get

$$\eta^{(n)}_\mu = -\frac{1}{4\pi}\int\left[\frac{F''_{n\mu}}{r}\right] dV ,$$

so that we can choose $\bar{h}_{\mu\nu,\nu} = 0 + O(h^n)$, where n is as large as we wish. If this series is convergent, we can choose $\bar{h}_{\mu\nu,\nu} = 0$. Even if the series were not convergent, we could effectively choose $\bar{h}_{\mu\nu,\nu} = 0$ since we are never concerned with higher than third order terms anyway.

The condition $\bar{h}_{\mu\nu,\nu} = 0$ is not the only coordinate condition used in the literature. One can divide the conditions used into two classes - those which distinguish time and those which do not. Another example of the latter is that one used by Fock, $(g^{\mu\nu}\sqrt{-g})_{,\nu} = 0$. The relation of this condition to the condition $\bar{h}_{\mu\nu,\nu} = 0$ is

$$(g^{\mu\nu}\sqrt{-g})_{,} = - \bar{h}_{\mu\nu,\nu} + \tfrac{1}{2} h_{\sigma\sigma} \bar{h}_{\mu\nu,\nu} + + h_{\mu\sigma} \bar{h}_{\sigma\nu,\nu} + h_{\mu\sigma,\nu} h_{\sigma\nu} - \tfrac{1}{2} h_{\alpha\beta} h_{\alpha\beta,\mu} + O(h^3).$$

Since the extra terms can be incorporated into the F_n, the previous arguments apply. Therefore one could always choose a coordinate system in which $(g^{\mu\nu}\sqrt{-g})_{,\nu} = = 0$ rather than $\bar{h}_{\mu\nu,\nu} = 0$.

An example of a coordinate condition which distinguishes time is that used by Infeld and his coworkers. This condition is equivalent to the conditions

$$\bar{h}_{4\lambda,\lambda} = 0 \ ; \ \bar{h}_{ik,k} = 0 \ . \tag{3.19}$$

We can show that this condition is valid in a similar manner: first we solve $\eta_{4,\lambda\lambda}^{(1)} = \bar{h}_{4\lambda,\lambda}$, and having obtained $\eta_4^{(1)}$, we then find the transformation of the quantity $\bar{h}_{il,l}$.

$$\bar{h}_{il,l} \longrightarrow \bar{h}_{il,l} + \eta_{i,ll} - \eta_{4,4i} + O(h^2) .$$

Then we must solve for

$$\eta_{i,ll}^{(1)} = \bar{h}_{il,l} + \eta_{4,4i}^{(1)}$$

which can be done since we already know $\eta_4^{(1)}$. We note that we now have $\nabla^2 \eta_i^{(1)}$ instead of $\Box\, \eta_i^{(1)}$, so that the solution for $\eta_i^{(1)}$ will not be a retarded one. The process is carried to further orders as before.

In most of the following discussion the coordinate condition $\bar{h}_{\mu\nu,\nu} = 0$ will be employed to reduce expressions to a simpler form. In section VII we will explore the effect, if any, of the different coordinate conditions on the results obtained in the previous sections. We will find that with reasonable assumptions concerning the asymptotic behaviour of $h_{\mu\nu}$, there will be no effect on the radiation if we choose different coordinate conditions.

D. Reduced Equations

With the coordinate condition $\bar{h}_{\mu\nu,\nu} = 0$, we can reduce the field equations to the form of an inhomogeneous wave equation.

$$\bar{h}_{\mu\nu,\lambda\lambda} = - 16\pi G \, S_{\mu\nu} \quad . \tag{3.20}$$

We can write an integral solution of these equations

$$\bar{h}_{\mu\nu} = - 4G \int \left[\frac{S_{\mu\nu}}{r} \right] dV \quad , \tag{3.21}$$

where the bracket indicates that the quantity within it is to be evaluated at the retarded time t - r/c. The quantity $S_{\mu\nu}$ is given by $\tilde{T}^{\mu\nu} + \sum_{k=2}^{\infty} \tilde{X}^{(k)}_{\mu\nu}$ as before, where $\tilde{X}^{(2)}_{\mu\nu}$ is now given by

$$\begin{aligned} \tilde{X}^{(2)}_{\mu\nu} = - \frac{1}{32\pi G} \Big\{ & h_{\alpha\beta,\mu} h_{\alpha\beta,\nu} - \tfrac{3}{2} \delta_{\mu\nu} h_{\alpha\beta,\gamma} h_{\alpha\beta,\gamma} + \\ & + 2 h_{\gamma\mu,\delta} h_{\gamma\nu,\delta} - 2 h_{\mu\gamma,\delta} h_{\nu\delta,\gamma} + \delta_{\mu\nu} h_{\alpha\gamma,\delta} h_{\alpha\delta,\gamma} + \\ & + 2 h_{\mu\nu,\alpha\beta} h_{\alpha\beta} + 2 h_{\mu\alpha} h_{\nu\alpha,\lambda\lambda} + 2 h_{\nu\alpha} h_{\mu\alpha,\lambda\lambda} + \\ & + 2 h_{\alpha\beta} h_{\alpha\beta,\mu\nu} - 2 h_{\alpha\beta} h_{\mu\alpha,\beta\nu} - 2 h_{\alpha\beta} h_{\nu\alpha,\beta\mu} + \\ & - 2 \delta_{\mu\nu} h_{\alpha\beta} h_{\alpha\beta,\gamma\gamma} - h_{\mu\nu,\lambda\lambda} h_{\sigma\sigma} - h_{\mu\nu} h_{\alpha\alpha,\lambda\lambda} + \\ & + \tfrac{1}{2} \delta_{\mu\nu} h_{\alpha\alpha,\beta\beta} h_{\gamma\gamma} \Big\} \quad . \end{aligned} \tag{3.22}$$

Equations 3.20 and 3.21 are non-linear equations and we must solve them by some approximation method.

The divergence of $\tilde{X}^{(2)}_{\mu\nu}$ is given by

$$\begin{aligned} \tilde{X}^{(2)}_{\mu\nu,\nu} &= - \frac{1}{16\pi G} (h_{\alpha\mu,\beta} - \tfrac{1}{2} h_{\alpha\beta,\mu}) \bar{h}_{\alpha\beta,\lambda\lambda} = \\ &= (h_{\alpha\mu,\beta} - \tfrac{1}{2} h_{\alpha\beta,\mu}) S_{\alpha\beta} \quad . \end{aligned} \tag{3.23}$$

From the covariant divergence of $\widetilde{T}^{\mu\nu}$, we have that

$$\delta_{\mu\lambda}\widetilde{T}^{\mu\nu}{}_{,\nu} = -\,(h_{\alpha\lambda,\beta} - \tfrac{1}{2}\,h_{\alpha\beta,\lambda})\,\widetilde{T}^{\alpha\beta} - h_{\lambda\alpha}\widetilde{T}^{\alpha\nu}{}_{,\nu}. \quad (3.24)$$

Therefore, since $(\widetilde{T}^{\mu\nu} + \widetilde{X}^{(2)}_{\mu\nu} + \widetilde{X}^{(3)}_{\mu\nu})_{,\nu} \sim O(h^4)$, $\widetilde{X}^{(3)}_{\mu\nu,\nu}$ must be given by

$$\widetilde{X}^{(3)}_{\mu\nu,\nu} = -\,(h_{\alpha\mu,\beta} - \tfrac{1}{2}\,h_{\alpha\beta,\mu})\,\widetilde{X}^{(2)}_{\alpha\beta} + $$
$$+(16\pi G)^{-1}h_{\mu\lambda}\,(h_{\alpha\lambda,\beta} - \tfrac{1}{2}\,h_{\alpha\beta,\lambda})\,\overline{h}_{\alpha\beta,\lambda\lambda}\;. \quad (3.25)$$

The divergence of higher order $\widetilde{X}^{(n)}_{\mu\nu}$ can be obtained in a like manner, without knowing explicitly the $\widetilde{X}^{(n)}_{\mu\nu}$ themselves. Thus we shall see that $\widetilde{X}^{(3)}_{\mu\nu}$ plays a role in the radiation problem, but we never need to calculate the $\widetilde{X}^{(3)}_{\mu\nu}$ explicitly.

There is another expression for $S_{\mu\nu}$ which we will find useful in that our equations will have fewer terms, and the structure will be more apparent. Let

$$S_{\mu\nu} = \left[\widetilde{T}_{\mu}{}^{\nu} + \overline{X}^{(2)}_{\mu\nu} + \overline{X}^{(3)}_{\mu\nu} + \cdots\right] =$$
$$= \left[g_{\mu\alpha}\,\widetilde{T}^{\alpha\nu} + \overline{X}^{(2)}_{\mu\nu} + \overline{X}^{(3)}_{\mu\nu} + \cdots\right] .$$

Since $\widetilde{T}_{\mu}{}^{\nu} \neq \widetilde{T}_{\nu}{}^{\mu}$, we see that the $\overline{X}^{(k)}_{\mu\nu}$ defined here are not symmetric with respect to an interchange of indices. But the quantity $S_{\mu\nu}$ is symmetric since it is independent of the way the stress-energy is broken up into the matter terms and the field stress terms. $\overline{X}^{(2)}_{\mu\nu}$ is related to $\widetilde{X}^{(2)}_{\mu\nu}$ through the following (in the gauge $\bar{h}_{\mu\nu,\nu} = 0$)

$$\bar{X}^{(2)}_{\mu\nu} = \tilde{X}^{(2)}_{\mu\nu} + (32\pi G)^{-1}\Big[2\, h_{\mu\alpha} h_{\alpha\nu,\lambda\lambda} + - h_{\mu\nu} h_{\sigma\sigma,\lambda\lambda}\Big]. \qquad (3.26)$$

The divergence of $\tilde{T}_{\mu}{}^{\nu}$ and of the $\bar{X}^{(k)}_{\mu\nu}$ is especially simple:

$$(\tilde{T}_{\mu}{}^{\nu})_{,\nu} = \tfrac{1}{2}\, h_{\alpha\beta,\mu}\, \tilde{T}^{\alpha\beta} \qquad (3.27)$$

$$\bar{X}^{(2)}_{\mu\nu,\nu} = (32\pi G)^{-1} h_{\alpha\beta,\mu}\, \bar{h}_{\alpha\beta,\lambda\lambda} = -\tfrac{1}{2}\, h_{\alpha\beta,\mu}\, S_{\alpha\beta} \qquad (3.28)$$

$$\bar{X}^{(3)}_{\mu\nu,\nu} = \tfrac{1}{2}\, h_{\alpha\beta,\mu}\, \tilde{X}^{(2)}_{\alpha\beta} \qquad (3.29)$$

$$\bar{X}^{(k)}_{\mu\nu,\nu} = \tfrac{1}{2}\, h_{\alpha\beta,\mu}\, \tilde{X}^{(k-1)}_{\alpha\beta}$$

where $\bar{X}^{(k)}_{\mu\nu}$ is related to $\tilde{X}^{(k)}_{\mu\nu}$ through

$$\bar{X}^{(k)}_{\mu\nu} = \tilde{X}^{(k)}_{\mu\nu} + h_{\mu\alpha}\, \tilde{X}^{(k-1)}_{\alpha\nu} \; ; \; k > 2 \; .$$

We will refer to both of these forms from time to time.

E. Equations of Motion and Conservation Laws

The equations of motion of a particle follow from the assumption that $T^{\mu\nu}$ has zero covariant divergence. Thus

$$(g_{\mu\alpha}\tilde{T}^{\alpha\nu})_{,\nu} = \tfrac{1}{2}\, h_{\alpha\beta,\mu}\, \tilde{T}^{\alpha\beta} \qquad (3.30)$$

implies

$$\frac{d}{ds_i}\, g_{\mu\alpha} \frac{dx_i^{\alpha}}{ds_i} = \tfrac{1}{2}\, h_{\alpha\beta,\mu} \frac{dx_i^{\alpha}}{ds_i} \frac{dx_i^{\beta}}{ds_i} \; , \qquad (3.31)$$

where

$$g_{\alpha\beta}\frac{dx_i^\alpha}{ds_i}\;\frac{dx_i^\beta}{ds_i} = 1\;. \tag{3.32}$$

If we change from s to t as the parameter of the path, then since $\frac{dt}{ds} = (1 - v^2 + h_{\alpha\beta}\,v^\alpha v^\beta)^{-\frac{1}{2}}$, equation 3.31 with $\mu = 4$ and summed over all the particles becomes exactly the same as

$$\frac{d}{dt}\int g_{4\alpha}\,\widetilde{T}^{\alpha 4}\;dV = \tfrac{1}{2}\int h_{\alpha\beta,4}\;\widetilde{T}^{\alpha\beta}\,dV \tag{3.33}$$

where $\widetilde{T}^{\mu\nu}$ is given by equation 2.8. For example, for one mass, $\int\widetilde{T}^{44}\;dV$ is given by

$$\begin{aligned}\int\widetilde{T}^{44}\;dV &= m\int\delta^4(x_\alpha - z_\alpha(s))\;\frac{dz^4}{ds}\;\frac{dz^4}{ds}\;dx\;dy\;dz\;ds = \\ &= m\int\delta(t - z_4(s))\;\frac{dz^4}{ds}\;\frac{dz^4}{ds}\;ds\;= \\ &= m\int\delta(t - z_4)\;\frac{dz^4}{ds}\,dz^4 = m\;\frac{dt}{ds}\;= \\ &= m\;(1 - v^2 + h_{\alpha\beta}\,v^\alpha\,v^\beta\,)^{-\frac{1}{2}}\;.\end{aligned} \tag{3.34}$$

Equation 3.30 can be written in integral form

$$\begin{aligned}\frac{d}{dt}\int g_{4\alpha}\,\widetilde{T}^{\alpha 4}\,dV + \int\frac{\partial}{\partial x_i}(g_{4\alpha}\,\widetilde{T}^{\alpha i})\;dV &= \\ = \tfrac{1}{2}\int h_{\alpha\beta,4}\;\widetilde{T}^{\alpha\beta}\,dV&\;.\end{aligned} \tag{3.35}$$

The second term on the left side of equation 3.35 can be converted to a surface integral which then vanishes if there are no particles entering or leaving the system. Thus equation 3.35 is equivalent to equation 3.31, and we may therefore denote both of them as the gravitational equations of motion.

We can also write conservation laws for the system.

$$\frac{d}{dt}\int_V S_{4\mu}\, dV - \int_S S_{4i}\, ds^i = 0 \quad . \qquad (3.36)$$

From equations 3.27 to 3.29, it is obvious that the conservation laws together with the expressions for the gravitational field stresses to all orders imply equations 3.31 or 3.35, the equations of motion. The converse is not true. We can not, for example, determine the energy of the system to all orders only from the equations of motion. If in the approximation in which we are working, $\int \breve{X}_{\mu i}\, ds^i$ can be neglected, then the conservation laws can be derived from the equations of motion. All we must do is convert $\frac{1}{2}\int h_{\alpha\beta,\mu}\, \tilde{T}^{\alpha\beta}\, dV$ into $\frac{\partial}{\partial t}[\quad]$. That this is possible is seen from an examination of the conservation laws. That this is unique is seen since two resulting expressions whose time derivatives vanish can differ by only a constant. This type of calculation is much easier to perform than the explicit integrations over the $\tilde{X}^{(n)}_{\mu\nu}$. If $\int X_{\mu i}\, ds^i$ can not be neglected in the approximation in which we are working, we are in trouble if we attempt to find $\frac{dE}{dt}$ from the equations of motion. Although we could form the quantity $\sum_a \underset{\sim}{f}^a \cdot \underset{\sim}{v}^a$, where $\underset{\sim}{f}^a$ is the total gravitational force acting on particle a and $\underset{\sim}{v}^a$ is the velocity of this particle, there is no assurance that this expression would give the correct energy loss of the system. In fact, it will be seen that this method ignores the contributions to the radiation coming from the third order field stresses, $\tilde{X}^{(3)}_{\mu\nu}$. Although $\tilde{X}^{(3)}_{\mu\nu}$ does not contribute to the energy of the system to order $(v/c)^0$, it does give a contribution to the radiation in order $(v/c)^5$, the order in which we are going to be calculating the radiation. Explicitly, from equation 3.29,

$$\frac{d}{dt}\int \overline{X}^{(3)}_{44}\, dV = \frac{1}{2}\int h_{\alpha\beta,4}\, \tilde{X}^{(2)}_{\alpha\beta}\, dV \quad .$$

This term represents the contributions of the reactions of the masses on the fields and the fields on themselves. Thus we must go to the conservation laws to find the correct expression for $\frac{dE}{dt}$.

The conservation laws are therefore more fundamental than the equations of motion since the latter are derivable from the former, and there are problems (e.g. gravitational radiation) where the equations of motion are inadequate to give a correct description. There are some objections to the conservation laws because of the possible ambiguity in the definition of the field stresses. However, in the appendix it is shown that this choice is unique when we assume that the stresses come from the variation of an action or from the expansion of the field equations themselves.

There is, of course, no analogy of this in electromagnetism. There are only bilinear combinations of A_μ in the stress-energy expression. The equations of motion and conservation laws carry the same information, even in the case of electromagnetic radiation. It is in the non-linearity of the gravity theory that this distinction comes in.

F. Newtonian Approximation

For a non-relativistic system S_{44} is much larger than any other components. It is approximately given by

$$S_{44} \approx \widetilde{T}_{44} \approx \sum_i m_i c^2 \delta^3(\underset{\sim}{r} - \underset{\sim}{r}_i) \quad . \tag{3.37}$$

Thus to lowest order only $\bar{h}_{44}$ is large, and from equation 3.21, it is given by

$$\bar{h}_{44} = -4 G \sum_i \frac{m_i}{r_i} \equiv 2\phi \quad . \tag{3.38}$$

Since $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\bar{h}_{\sigma\sigma}$, we have that

$$h_{44} \approx h_{11} \approx h_{22} \approx h_{33} \approx \phi \quad . \tag{3.39}$$

The equations of motion, equation 2.10, then take the following form

$$m_1\, \delta_{\mu i}\, \ddot{x}^i = \tfrac{1}{2}\, h_{44,\mu}$$

or

$$m_1\, \ddot{x}^i = -\, G\, m_1\, m_2\, x^i/r^3$$

which is the force law of the Newtonian theory of gravitation.

We can find $\int S_{44}\, dV$, the total energy of the system, to order c^0. If we consider the case of two particles, then using equations 2.8 and 3.22, and the expression for the lowest order $h_{\mu\nu}$, equation 3.38, we have that

$$E = \int S_{44}\, dV = (m_1 + m_2)c^2 + \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 + -(32\pi G)^{-1}\int\left[3\, \phi_{,i}\, \phi_{,i} + 2\phi\phi_{,ii}\right]dV \quad . \qquad (3.40)$$

The last term is easily integrated to give $-Gm_1 m_2/r_{12}$ as we would have obtained directly from the Newtonian theory.

The form $(3\,\phi_{,i}\,\phi_{,i} + 2\,\phi\,\phi_{,ii})$ for the energy density of the gravitational field is of some interest. From a Newtonian theory one would have expected an energy density like $\phi_{,i}\ \phi_{,i}$. The additional term $2\ \phi_{,i}\ \phi_{,i} + 2\phi\ \phi_{,ii} = 2(\phi\ \phi_{,i})_{,i}$ would have made no difference in any problem, since we could convert the integral $\int(\phi\phi_{,i})_{,i}\, dV$ to a surface integral $\int \phi\ \phi_{,i}\, ds_i$, which would vanish. In the general theory of gravitation there is a fundamental difference; physical effects, i.e. the perihelion shift, depend on the distribution of energy in the field, and one can not add an arbitrary function, even if it is the divergence of something. There is a physical meaning to the form $(3\,\phi_{,i}\,\phi_{,i} + 2\phi\,\phi_{,ii})$. We may say that there is a field energy $2\phi\,\phi_{,ii}$ concentrated at the masses and a field energy $3\,\phi_{,i}\,\phi_{,i}$ spread throughout space. In the non-relativistic approximation, these balance to give the total gravitational energy, but in higher orders of (v/c), they may give rise to observable effects where we have gravity interacting with the energy density of gravity itself.

We also see that $\int \tilde{X}^{(2)}_{44}\, dV$ is of the same order of magnitude as $\int \tilde{T}_{ij}\, dV$. In a non-relativistic system, all of the $\tilde{X}^{(2)}_{ij}$ will be of the same order of magnitude as the spatial components of the mass energy tensor $\tilde{T}_{ij}$. However, since $\tilde{X}^{(2)}_{\mu\nu}$ to lowest order depends only on $\bar{h}_{44}$, and $\bar{h}_{44}$ depends only on $\tilde{T}_{44}$, we have a consistent procedure for solving the field equations term by term.

The total momentum and angular momentum integrals are, of course, also the same as one would expect from a Newtonian theory since the field's momentum and stresses are completely negligible in the 0^{th} order approximation.

IV. RADIATION from a NON-RELATIVISTIC SYSTEM

A. Electromagnetic Radiation

We are now in a position to apply the formalism developed to the problem of gravitational radiation. If, in the following, one finds analogies with the theory of electromagnetic radiation, he should not be too surprised. There are many parallels between the two theories, and notations which are used to describe electromagnetic radiation can be carried directly over to the case of gravitational radiation.

Let us briefly summarize the results of electromagnetic radiation theory. We have, in the gauge $A_{\mu,\mu} = 0$, the wave equation

$$A_{\mu,\lambda\lambda} = -J_{\mu} \quad , \tag{4.1}$$

where J_{μ} satisfies the conservation equation

$$J_{\mu,\mu} = 0 \quad . \tag{4.2}$$

If we define the fields to be $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$, we see that the fields are invariant under the gauge transformation $A_{\mu} \rightarrow A_{\mu} + \chi_{,\mu}$. (In gravity, we cannot form fields as above which are invariant under arbitrary coordinate transformations - e.g. see the Appendix). The force density equation or equation of motion is given by

$$f_\mu = F_{\mu\nu} J_\nu \quad , \qquad (4.3)$$

where the force density is also invariant under the gauge transformation. (In gravity, we cannot introduce such a gauge invariant force.) f_μ is, by definition, the divergence of the matter tensor

$$f_\mu = T^m_{\mu\nu;\nu} \quad ,$$

and since energy and momentum conservation laws can be written, we have that

$$f_\mu = - T^s_{\mu\nu;\nu} \quad ,$$

where $T^s_{\mu\nu}$ is the stress-energy tensor of the fields and is given by

$$T^s_{\mu\nu} = - F_{\mu\alpha} F_{\nu\alpha} + \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} \quad . \qquad (4.4)$$

$\int f_i v_i \, dV = \int f_4 \, dV$ is the power radiated as calculated by the radiation reaction method. If we consider the law of conservation of energy, we get

$$\frac{d}{dt} \int_V (T^m_{44} + T^s_{44}) \, dV - \int_V \frac{\partial}{\partial x_i} (T^m_{4i} + T^s_{4i}) \, dV = 0$$

or

$$\frac{d}{dt} \int_V (T^m_{44} + T^s_{44}) \, dV = \int_S T^s_{4i} \, ds_i \quad . \qquad (4.5)$$

so that the rate of change of energy of the system can also be found by calculating the flux of energy across a large sphere. Over a time average these two methods are seen to agree since $\int dt \int_S T^s_{4i}\, ds_i =$ $= \int dt \int_V T^s_{4i,i}\, dV = -\int dt \int_V T^s_{4\nu,\nu}\, dV = \int dt \int_V f_4\, dV$. We may therefore either calculate the force on a charge due to the combined fields of all of the other charges and the charge itself or find the flux of energy, where we only need the wave fields ($\propto 1/r$) in the expression. One could also decompose the radiation into multipoles and relate their amplitude to the multipole expansion of the source. In the non-relativistic approximation, one of these, the electric dipole, will usually dominate the radiation. This discussion has implicitly assumed that the solution of the wave equation is to be the retarded one. This is the standard approach and is based on the idea that the effect of a field should take place at a later time than the time of the motion of the charges which produced the field. Wheeler and Feynman[(1)] have formulated the radiation problem in terms of the interaction of an absorber at large distances and the half-retarded, half-advanced potentials of the sources.

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[1] J. Wheeler and R. P. Feynman, Rev. Mod. Phys. 17, 157 (1945).

This gives an effective retarded potential, but it eliminates the annoyances of the infinite self-energy of the interaction of a charge with itself.

We can carry over the ideas of the radiation reaction calculation and the flux of energy leaving the system to the case of gravitational radiation. To generalize the absorber theory of radiation to the case of gravitation appears at present to be quite difficult. We wish to assume that space becomes flat at infinity; however, if there are systems at indefinitely large distances which can absorb gravitational radiation, then space cannot become flat. Hence the asymptotic boundary condition that $h_{\mu\nu} \to 0$ as $r \to \infty$ does not hold.

There is also some difficulty in gravitational theory concerning the choice of half-retarded, half-advanced potentials. This is due to the non-linearity of the field equations. In gravity the principle of superposition does not hold. Suppose we wished to find the gravitational potential in a given problem to all orders. If we took our lowest order potentials to be retarded, $h_{\mu\nu}^{ret}$, and use this to find the higher order corrections to $h_{\mu\nu}^{ret}$, we would arrive at a complete retarded solution to the problem, which will be denoted by $H_{\mu\nu}^{ret}$. We could also have started with $h_{\mu\nu}^{adv}$

and used this in the iteration, in which case we would get a solution given by $H^{adv}_{\mu\nu}$. Similarly, we could get $H^{sym}_{\mu\nu}$ starting with $h^{sym}_{\mu\nu} = \frac{1}{2}\,(h^{ret}_{\mu\nu} + h^{adv}_{\mu\nu})$ and $H^{anti}_{\mu\nu}$ starting with $h^{anti}_{\mu\nu} = \frac{1}{2}\,(h^{ret}_{\mu\nu} - h^{adv}_{\mu\nu})$. An important fact to note is that $\frac{1}{2}\,(H^{adv}_{\mu\nu} + H^{ret}_{\mu\nu}) \neq H^{sym}_{\mu\nu}$ and $\frac{1}{2}\,(H^{ret}_{\mu\nu} - H^{adv}_{\mu\nu}) \neq H^{anti}_{\mu\nu}$ in general. In fact the linear combinations of the H's are not necessarily even solutions of the problem.

The analogue of the absorber in gravity is probably connected with a similar problem involving the action of distant masses on a system, namely Mach's principle. This principle states that the effects of acceleration or rotation with respect to the distant stars are caused by the action of these stars on the system. There are semi-quantitative arguments which lead one to believe that this is true, but no completely quantitative explanation has yet been given. One of the primary reasons for this failure seems to be that the metric is always assumed to become flat at infinity, i.e. after one has gone past the distant stars. In the actual universe, as far as one can see, this does not happen. Thus if one had an explanation of how the distant masses affect the choice of inertial frames of reference, then one might have a much better view of what the role of the distant absorbers is in the theory of gravitational radiation.

We will thus assume in the following that the retarded solutions are the ones which are to be found and used in the calculation of gravitational radiation.

B. Energy radiation

Let us assume that the gravitational energy and stresses are small compared to the energy and stresses binding the system under consideration. Then the total energy of the system is approximately given by $\int T^{44}\, dV$. Of course it is also given by $\int \tilde{T}^{44}\, dV$ or $\int \tilde{T}_4{}^4\, dV$ since the $h_{\mu\nu}$ are assumed to be small. Let us then calculate $\frac{dE}{dt}$ by finding $\frac{d}{dt}\int \tilde{T}_4{}^4\, dV$. From equation 3.27 we then get

$$\frac{dE}{dt} = \frac{d}{dt}\int \tilde{T}_4{}^4\, dV = \tfrac{1}{2}\int h_{\alpha\beta,4}\, \tilde{T}^{\alpha\beta}\, dV\ , \qquad (4.6)$$

where in the last term $\tilde{T}^{\alpha\beta}$ can be approximated by $T^{\alpha\beta}$. This result, which we shall call the radiation reaction, was first derived by Eddington[2]. We could also have calculated $\frac{dE}{dt}$ using a slightly different energy, say $\int \tilde{T}^{44}\, dV$. From equation 3.24 we then get that

$$\frac{dE}{dt} = \frac{d}{dt}\int \tilde{T}^{44}\, dV = -\int h_{\alpha 4,\beta}\, \tilde{T}^{\alpha\beta}\, dV + \tfrac{1}{2}\int h_{\alpha\beta,4}\tilde{T}^{\alpha\beta} dV\ . \quad (4.7)$$

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[2] A. S. Eddington, Proc. Roy. Soc. (London) 102A, 268 (1922).

The two expressions differ by the term $\int h_{\alpha 4,\beta}\, \tilde{T}^{\alpha\beta} dV$, and one would be tempted to say that these two expressions predict different energy losses. However, by integrating by parts with respect to the spatial derivative of $h_{\alpha 4}$ and using the fact that $\tilde{T}^{\alpha\beta}{}_{,\beta} = 0$ for this system, we can reduce the term to a pure time derivative term:

$$\int h_{\alpha 4,\beta}\, \tilde{T}^{\alpha\beta}\, dV = \int h_{\alpha 4,4}\, \tilde{T}^{\alpha 4}\, dV - \int h_{\alpha 4,i}\, \tilde{T}^{\alpha i}\, dV =$$

$$= \int h_{\alpha 4,4}\, \tilde{T}^{\alpha 4}\, dV + \int h_{\alpha 4}\, \tilde{T}^{\alpha i}{}_{,i}\, dV =$$

$$= \int h_{\alpha 4,4}\, \tilde{T}^{\alpha 4}\, dV + \int h_{\alpha 4}\, \tilde{T}^{\alpha 4}{}_{,4}\, dV =$$

$$= \frac{d}{dt}\int h_{\alpha 4}\, \tilde{T}^{\alpha 4}\, dV. \qquad (4.8)$$

Usually we will be dealing with periodic systems in which the effect of the radiation on the motion can be neglected when the motion of the system is used to calculate the energy emission. In other words, we consider the parameters of the system to return to the same value after every multiple of the period τ. If we want the secular change in energy due to the radiation, we write

$$\Delta E = \int_0^{\tau} \frac{dE}{dt}\, dt \quad .$$

We then assume that the radiation is small so that in the formula for $\frac{dE}{dt}$ we can use the motion of the system as if there were no radiation. Then any term in the radiation formula which can be reduced to a pure time derivative does not contribute anything to the secular change in energy ΔE and thus can be neglected. Then because of equation 4.8, equation 4.7 can be reduced to equation 4.6 over a time average, and we are led back to Eddington's original equation.

In electromagnetism we were introduced to the idea that different methods of calculation of the energy emission need only agree over a time average. We essentially had only two methods of calculation available - the radiation reaction and energy emission across a large sphere. In gravity, we have, even within a given technique, e.g. radiation reaction, a number of different formulae for $\frac{dE}{dt}$ which agree only over a time average. Thus we shall make extensive use of this concept of time averages in deriving the various radiation formulae and in reducing these formulae to show that they predict the same energy emission.

The energy flux in the wave zone can be reduced to the form

$$X_{44} = (32\pi G)^{-1} \left[h_{\alpha\beta,4} \, \bar{h}_{\alpha\beta,4} \right] \quad . \qquad (4.9)$$

This has been obtained by Landau and Lifshitz[3] from

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[3]L. Landau and E. Lifshitz, The Classical Theory of Fields (Addison-Wesley, 1959), Chap. 11.

a consideration of a pseudo-stress-energy tensor and also has been obtained by Feynman[4] by decomposing the radiation fields into outgoing waves of frequency ω , finding the energy carried in each wave, and summing over all frequencies. We will find later that this form can be directly found from the expression for $S_{\mu\nu}$. Following Feynman's method for the moment, the $\bar{h}_{\mu\nu}$ are decomposed into components with time dependence $e^{-i\omega t}$; then for large r,

$$\bar{h}_{\mu\nu}(r,\omega) = -\frac{4G\, e^{i\omega r}}{r} \int S_{\mu\nu}(r',\omega)\, e^{i\frac{r\cdot r'}{r}\omega} dV'. \quad (4.10)$$

If the bodies in the system are moving non-relativistically and the system is small enough so that retardation effects can be ignored, we can approximate the exponential in the integral by 1 since $\omega r'/c$ will be small. Then equation 4.10 becomes

$$\bar{h}_{\mu\nu}(r,\omega) = -\frac{4G\, e^{i\omega r}}{r} \int S_{\mu\nu}(r',\omega)\, dV', \qquad (4.11)$$

or hence, putting the time dependence back,

$$\bar{h}_{\mu\nu}(r,t) = -\frac{4G}{r} \int S_{\mu\nu}(r',t')\, dV' , \qquad (4.12)$$

where the quantity on the right is to be evaluated at the retarded time $t' = t - r/c$, where r is the radius

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[4] R. P. Feynman, lecture notes, California Institute of Technology (unpublished).

vector to the system. For a non-relativistic system $\int S_{ij}\, dV$ can be written

$$\int S_{ij}\, dV = \tfrac{1}{2} \frac{d^3}{dt^3} Q_{ij} \equiv \tfrac{1}{2} \dddot{Q}_{ij} \ , \qquad (4.13)$$

where Q_{ij} is the moment of inertia tensor

$$Q_{ij} = \sum_a m_a\, x_i^a\, x_j^a \quad . \qquad (4.14)$$

If we look at the radiation in a certain direction, we find that it can be broken up into contributions from two different polarizations, much in analogy with the corresponding result from electromagnetic theory. These polarizations can be chosen to be $e_1 = \frac{\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}}{2}$ and $e_2 = \frac{\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}}{2}$, which are, of course, transverse to the direction of propagation of the radiation. $\hat{\theta}$ and $\hat{\phi}$ are unit vectors in the θ and ϕ directions, and the polarization tensor $e_{\mu\nu}$ has only two independent components. If we look at the radiation in the 3-direction, then we get (summing over polarizations)

$$\frac{d^2E}{dt\, d\Omega} = - \frac{G}{8\pi} \left\{ \tfrac{1}{2} (\dddot{Q}_{11} - \dddot{Q}_{22})^2 + 2(\dddot{Q}_{12})^2 \right\} , \qquad (4.15)$$

where each squared term represents the energy radiated into each polarization. We can get the radiation into an arbitrary direction by taking all possible combinations of two tensors, $\dddot{Q}_{ij}$, and unit vectors n_i , and requiring that the resulting expression reduce to equation 4.15 when we set $n_1 = n_2 = 0$ and $n_3 = 1$. This yields

$$\frac{d^2E}{dt\,d\Omega} = -\frac{G}{8\pi}\left\{\frac{1}{2}(n_i n_j \dddot{Q}_{ij})^2 - 2\, n_j n_k \dddot{Q}_{ij}\dddot{Q}_{ik} + \right.$$

$$\left. + \dddot{Q}_{ij}\dddot{Q}_{ij} + n_i n_j \dddot{Q}_{ij}\dddot{Q}_{kk} - \frac{1}{2}(\dddot{Q}_{kk})^2\right\} . \qquad (4.16)$$

The only angular dependence is in the unit vectors n_i. Thus the integrals over solid angles are trivial to perform. This gives

$$\frac{dE}{dt} = -\frac{G}{5}\left\{\dddot{Q}_{ij}\dddot{Q}_{ij} - \frac{1}{3}\dddot{Q}_{kk}\dddot{Q}_{ll}\right\} , \qquad (4.17)$$

which is then the total power radiated from a non-relativistic system. This result does not depend on whether or not the systems are gravitationally bound.

Mathews(5) has decomposed the radiation fields into multipoles in analogy with the similar decomposition in the case of electromagnetic radiation. His results agree with equation 4.17 in the quadrupole limit (equivalent to the non-relativistic approximation), where the terms denoted by magnetic quadrupole or M2 dominate. This is in analogy with electromagnetism where the electric dipole or E1 terms dominate.

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[5] J. Mathews, J. Soc. Ind. Appl. Math. 10, 768 (1962).

It seems desirable to extend equation 4.6 to include the case of gravitationally bound systems and show that, at least non-relativistically, the results are in agreement over a time average with the results of the energy flux calculation given by equation 4.17. It is also desirable, since we have defined a unique $S_{\mu\nu}$ through the use of the conservation equations, to find the energy flux using this $S_{\mu\nu}$ and find the total radiation leaving the system.

The quantity $\int S_{44}\, dV$ is the total energy of the system. Therefore $\frac{d}{dt}\int S_{44}\, dV$ is the time rate of change of energy of the system and can be found by using the conservation laws

$$\frac{dE}{dt} = \frac{d}{dt}\int_V S_{44}\, dV = \int_V \frac{\partial}{\partial x_i} S_{4i}\, dV = \int_S S_{4i}\, ds_i \quad . \qquad (4.18)$$

S_{4i} is composed of the matter terms $\tilde{T}^{4i}$ and the stress terms $\tilde{X}_{4i}$. The mass tensor contributes nothing to the surface integral when we consider the surface to be a large sphere of radius r, and let $r \rightarrow \infty$. Thus we have assumed a system in which no particles enter or leave the system, and in which there are no other radiation processes, e.g. electromagnetic, which are acting on the system. If we did include the other radiation processes, we would have that

$$\frac{dE}{dt} = \sum_{i \neq \text{gravity}} (\text{RAD})_i + (\text{RAD})_{\text{gravity}}$$

so that our discussion will be simpler if we consider only gravitational waves being emitted. Thus we have

$$\frac{dE}{dt} = \int \tilde{X}_{4i}\, ds_i \quad . \tag{4.19}$$

From equation 3.21 we expect that in the far zone (large r) $h_{\alpha\beta} \propto 1/r$ and $h_{\alpha\beta,\lambda} \propto 1/r$. The analogous breakdown of the derivatives of the potentials into induction zone and wave zone parts is well known in electromagnetic theory. Then $\frac{dE}{dt}$ will be

$$\begin{aligned} \frac{dE}{dt} &= \int_S \tilde{X}_{4i}\, ds_i = \int_S \tilde{X}^{(2)}_{4i} ds_i + \int_S \tilde{X}^{(3)}_{4i} ds_i + \cdots \\ &= \int_S \tilde{X}^{(2)}_{4i} ds_i + 0(1/r) \quad . \end{aligned}$$

Since $\tilde{X}^{(k)}_{4i}$ for $k \geq 3$ contains terms with products of at least three $h_{\alpha\beta}$'s or their derivatives, they will yield a surface integral with a radius dependence of 1/r or smaller. Thus when we let $r \to \infty$, so that the volume V can be thought of as encompassing the system, only the second order stresses will contribute to the surface integral. Therefore we have that equation 4.19 becomes

$$\frac{dE}{dt} = \int_S \tilde{X}^{(2)}_{4i}\, ds_i \quad , \tag{4.20}$$

where $\tilde{X}^{(2)}_{4i}$ is given by equation 3.22.

In order to calculate the radiation we need the terms in $\tilde{X}^{(2)}_{4i}$ which are proportional to $1/r^2$. Some terms in $\widetilde{X}^{(2)}_{4i}$ can therefore be neglected at the start. Since $\widetilde{T}^{\mu\nu}$ is zero for large r, we have that

$$\bar{h}_{\alpha\beta,\lambda\lambda} \propto \widetilde{X}^{(2)}_{\alpha\beta} \propto 1/r^2 \qquad (4.21)$$

so that we can neglect all terms in $\widetilde{X}^{(2)}_{4i}$ which are of that form. Evaluating $\widetilde{X}^{(2)}_{4i}$ without these terms gives

$$\widetilde{X}^{(2)}_{4i} = -(32\pi G)^{-1}\Big\{ h_{\alpha\beta,4}\, h_{\alpha\beta,i} + 2\, h_{\alpha 4,\beta}\, h_{\alpha i,\beta} + \\ - 2\, h_{4\alpha,\beta}\, h_{i\beta,\alpha} + 2\, h_{4i,\alpha\beta}\, h_{\alpha\beta} + 2\, h_{\alpha\beta}\, h_{\alpha\beta,4i} + \\ - 2\, h_{\alpha\beta}\, h_{4\alpha,\beta i} - 2\, h_{\alpha\beta}\, h_{i\alpha,\beta 4}\Big\} \quad .$$

Also because of the retarded solution to the wave equation, we can write

$$h_{\alpha\beta,i} = -(x_i/r)\, h_{\alpha\beta,4} = -n_i\, h_{\alpha\beta,4} \qquad (4.22)$$

so that in the wave zone all spatial derivatives can be reduced to time derivatives multiplied by the unit direction vectors. If we consider now the time average of the radiation (letting the limits of integration from 0 to τ be implicit and the division by τ also implicit), we get

$$\int \frac{dE}{dt}\, dt = \iint \widetilde{X}^{(2)}_{4i}\, ds_i\, dt$$

so that we can integrate any derivatives by parts we choose through the use of equation 4.22. This then gives

$$\int \widetilde{X}_{4i}^{(2)} dt = (32\pi G)^{-1} \int \Big\{ h_{\alpha\beta,4} h_{\alpha\beta,i} + 2\, h_{\alpha 4}\, h_{\alpha i,\beta\beta} +$$

$$+ 2\, h_{4\alpha,\alpha}\, h_{i\beta,\beta} - 2\, h_{4i}\, h_{\alpha\beta,\alpha\beta} + \qquad (4.23)$$

$$- 2\, h_{\alpha\beta,\beta}\, h_{4\alpha,i} - 2\, h_{\alpha\beta,\beta}\, h_{i\alpha,4} \Big\} dt \quad .$$

The we can apply the coordinate condition $\bar{h}_{\mu\nu,\nu} = h_{\mu\nu,\nu} - \frac{1}{2} h_{\sigma\sigma,\mu} = 0$ and also equation 4.21 again to reduce the average $\widetilde{X}_{4i}^{(2)}$ to the desired form

$$\iint \widetilde{X}_{4i}^{(2)} ds_i\, dt = (32\pi G)^{-1} \iint \left[h_{\alpha\beta,4} h_{\alpha\beta,i} - \tfrac{1}{2}\, h_{\alpha\alpha,4} h_{\beta\beta,4} \right] ds_i dt$$

$$= (32\pi G)^{-1} \iint h_{\alpha\beta,4}\, \bar{h}_{\alpha\beta,i}\, dt\, ds_i$$

$$= - (32\pi G)^{-1} \iint_{\text{sphere}} h_{\alpha\beta,4}\, \bar{h}_{\alpha\beta,4}\, dt\, ds \;, \quad (4.24)$$

which is the same as equation 4.9.

We can now derive the radiation reaction loss if we convert the surface integral back to a volume integral. Then

$$\iint_S \widetilde{X}_{4i}^{(2)} ds_i dt = (32\pi G)^{-1} \iint_S h_{\alpha\beta,4}\, \bar{h}_{\alpha\beta,i}\, ds_i dt$$

$$= (32\pi G)^{-1} \int dt \int_V dV \frac{\partial}{\partial x_i} h_{\alpha\beta,4} \bar{h}_{\alpha\beta,i} = (32\pi G)^{-1} \int dt \int_V dV\, h_{\alpha\beta,4} \bar{h}_{\alpha\beta,ii}$$

$$+ (32\pi G)^{-1} \int dt \int dV\, h_{\alpha\beta,4i}\, \bar{h}_{\alpha\beta,i} \quad .$$

The integrand of the second integral is $\frac{\partial}{\partial t} h_{\alpha\beta,i} \bar{h}_{\alpha\beta,i}$ and thus the integral over time gives a zero contribution. The first integral can be written as

$$(32\pi G)^{-1}\int dt \int dV \left[h_{\alpha\beta,4}(-\bar{h}_{\alpha\beta,\lambda\lambda} + \bar{h}_{\alpha\beta,44})\right] =$$

$$= (32\pi G)^{-1}\int dt \int dV \left[h_{\alpha\beta,4}(16\pi G\ S_{\alpha\beta} + \bar{h}_{\alpha\beta,44})\right] .$$

Since $h_{\alpha\beta,4}\bar{h}_{\alpha\beta,44} = \frac{\partial}{\partial t} h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4}$, that term goes out and we get

$$\int dt \int ds_i\ \tilde{X}^{(2)}_{4i} = \tfrac{1}{2}\int dt \int dV\ h_{\alpha\beta,4}\ S_{\alpha\beta}$$

as the flux of energy out of the system and thus we can write

$$\int \frac{dE}{dt}\, dt = \tfrac{1}{2}\int dt \int dV\ h_{\alpha\beta,4}\ S_{\alpha\beta} \tag{4.25}$$

which is the generalized version of equation 4.6, valid for all kinds of energies and stresses present.

We can show that equation 4.25 agrees with equation 4.17. The time average in the following is implicitly assumed. Equations 4.25 and 3.21 imply that

$$\frac{dE}{dt} = -2G\int_V\int_{V'}\left\{S_{\alpha\beta}\frac{\partial}{\partial t}\left[\frac{S'_{\alpha\beta}}{r}\right] - \tfrac{1}{2}\ S_{\alpha\alpha}\frac{\partial}{\partial t}\left[\frac{S'_{\beta\beta}}{r}\right]\right\}dV\ dV' , \tag{4.26}$$

where the brackets indicate that the quantities are to be evaluated at the retarded time. When the velocities and accelerations are small, it is useful to expand quantities evaluated at retarded times in a Taylor series so that all the quantities can be taken at the present time. Thus in equation 4.26 we can expand $\left[\frac{S_{\alpha\beta}}{r}\right]$

in the well known series[6]

$$\left[\frac{S_{\alpha\beta}}{r}\right] = \frac{S_{\alpha\beta}}{r} - \frac{d}{dt} S_{\alpha\beta} + \sum_{n=2}^{\infty} \frac{(-)^n}{n!} \frac{d^n}{dt^n}(r^{n-1} S_{\alpha\beta}) \quad . \qquad (4.27)$$

In equation 4.26 the only time dependence is with the $S_{\alpha\beta}$ so that the time derivatives can be commuted with the r's. We see immediately that the integral of terms with an odd power of r, say r^n, vanishes since when we integrate by parts n times with respect to time, we will get what we had to begin with, but with a minus sign. Also we want to keep terms that will contribute to $\frac{dE}{dt}$ only in order $1/c^5$ or lower. Thus we get contributions only from the following terms:

$$\int \frac{dE}{dt} dt = 2G \iiint dV\, dV' dt \Big\{ S_{\alpha\beta} \frac{d^2}{dt^2} S'_{\alpha\beta} + \\ + \frac{1}{6} S_{\alpha\beta} |\underset{\sim}{r} - \underset{\sim}{r}'|^2 \frac{d^4}{dt^4} S'_{\alpha\beta} + \frac{1}{120} S_{\alpha\beta} |\underset{\sim}{r} - \underset{\sim}{r}'|^4 \frac{d^6}{dt^6} S'_{\alpha\beta} + \\ - \frac{1}{2} S_{\alpha\alpha} \frac{d^2}{dt^2} S'_{\beta\beta} - \frac{1}{12} S_{\alpha\alpha} |\underset{\sim}{r} - \underset{\sim}{r}'|^2 \frac{d^4}{dt^4} S'_{\beta\beta} \\ - \frac{1}{240} S_{\alpha\alpha} |\underset{\sim}{r} - \underset{\sim}{r}'|^4 \frac{d^6}{dt^6} S'_{\beta\beta} \Big\} \quad .$$

Consider, for example, the first term of this equation. Neither α nor β can be 4 since if they were, we could use the condition $S_{\alpha\beta,\beta} = 0$ to reduce the term to a surface integral which would vanish to the order in which we are calculating the energy loss. Therefore

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[6] A. S. Eddington, The Mathematical Theory of Relativity (Cambridge, 1959), p. 253.

we get a contribution only from $-2G\int \dot{S}_{ij}\, dV \int \dot{S}'_{ij}\, dV'$. In a term which contains r^2, for instance, the condition $S_{\alpha\beta,\beta} = 0$ is used to reduce the expression to a product of two integrals over the S_{ij} or S_{ii}. For instance, in the term

$$\iiint S_{\alpha\beta}\, r^2\, \overset{....}{S}{}'_{\alpha\beta}\, dV\, dV'dt = \iiint \ddot{S}_{44} r^2 \ddot{S}'_{44}\, dV\, dV'dt -$$

$$- 2\iiint \ddot{S}_{4i}\, r^2\, \ddot{S}'_{4i}\, dV\, dV'dt + \iiint \ddot{S}_{ij} r^2 \ddot{S}'_{ij}\, dV\, dV'dt\ ,$$

the last term on the right is of order $1/c^7$ so we can neglect it. In the first term, if we use the fact that $\ddot{S}_{44} = S_{ij,ij}$, then integrating by parts twice with respect to the spatial derivatives takes away the r^2, and a further integration by parts gives a surface integral which again vanishes. In the second term, $\ddot{S}_{4i} = \dot{S}_{ik,k}$ so that

$$\iiint \ddot{S}_{4i}\, r^2\, \ddot{S}'_{4i}\, dV\, dV'dt = \iiint \dot{S}_{ik,k}\, r^2\, \dot{S}'_{im,m}\, dV\, dV'dt =$$

$$= -2\iiint \dot{S}_{ik}(x_k - x'_k)\dot{S}'_{im,m} dV\, dV'dt = -2\iiint \dot{S}_{ik}\,\delta_{km}\dot{S}'_{im} dV\, dV'dt =$$

$$= -2\iiint \dot{S}_{im}\, \dot{S}'_{im}\, dV\, dV'dt = -2\int dt \left[\int \dot{S}_{im}\, dV\right]\left[\int \dot{S}'_{im}\, dV'\right]\ .$$

In a like manner we find that the terms are either 0, of higher order in $1/c$ than $1/c^5$ or are separable into the product of two integrals rather than the double integral which we started out with. In this manner we can reduce equation 4.26 to

$$\int\frac{dE}{dt}\,dt = -\,2\,G\int dt\Big\{(1-\tfrac{2}{3}+\tfrac{1}{10}-\tfrac{1}{30})\int\dot{S}_{ij}dV\int\dot{S}'_{ij}dV' + \\ + (\tfrac{1}{20}-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{60})\int\dot{S}_{kk}dV\int\dot{S}'_{mm}dV'\Big\} \;.$$

These terms can be simplified using the fact that in the non-relativistic approximation $\int S_{ij}\,dV = \frac{1}{2}\,\ddot{Q}_{ij}$. This yields, remembering that the time average is implicitly assumed,

$$\frac{dE}{dt} = -\,\frac{G}{5}\left[\dddot{Q}_{ij}\,\dddot{Q}_{ij} - \frac{1}{3}\,\dddot{Q}_{kk}\,\dddot{Q}_{mm}\right] . \qquad (4.28)$$

This agrees with the previous calculation.

We can also get to equation 4.28 by way of the energy flux calculation without mention of the polarizations and arguments about the energy in waves of different frequencies. $\int\frac{dE}{dt}\,dt$ is given by

$$\int\frac{dE}{dt}dt = -(32\pi G)^{-1}\int dt\int ds\Big[\bar{h}_{\alpha\beta,4}\bar{h}_{\alpha\beta,4} - \tfrac{1}{2}\,\bar{h}_{\alpha\alpha,4}\bar{h}_{\beta\beta,4}\Big] =$$

$$= -(32\pi G)^{-1}\int dt\int ds\Big\{\tfrac{1}{2}\,\bar{h}_{44,4}\bar{h}_{44,4} - 2\bar{h}_{4i,4}\bar{h}_{4i,4} + \\ + \bar{h}_{ij,4}\bar{h}_{ij,4} + \bar{h}_{ii,4}\bar{h}_{44,4} - \tfrac{1}{2}\,\bar{h}_{ii,4}\bar{h}_{jj,4}\Big\} \;.$$

In this expression the only restriction is that the time average is allowed and the coordinate condition $\bar{h}_{\mu\nu,\nu} = 0 = \bar{h}_{4\mu,4} + n_i\,\bar{h}_{\mu i,4}$ is employed. Then we get

$$\int \frac{dE}{dt}\, dt = -\,(32\pi G)^{-1} \int dt \int ds \Big[\tfrac{1}{2}\, (n_i n_j \bar{h}_{ij},_4)^2 + $$
$$- 2n_j n_k \bar{h}_{ij},_4 \bar{h}_{ik},_4 + \bar{h}_{ij},_4 \bar{h}_{ij},_4 + \qquad (4.29)$$
$$+ n_i n_j \bar{h}_{ij},_4 \bar{h}_{kk},_4 - \tfrac{1}{2}\, (\bar{h}_{kk},_4)^2 \Big] \ ,$$

which is the same as equation 4.16 if the solution for $\bar{h}_{\alpha\beta}$, equations 4.13 and 4.14, are used. The quantity in the brackets is a scalar and we can evaluate it in any coordinate system we choose. Let us choose $n_1 = n_2 = 0$ and $n_3 = 1$. Then we get, after summing on the indices as indicated

$$\int \frac{d^2E}{dt\,ds}\, dt = -(32\pi G)^{-1} \int dt \Big[\tfrac{1}{2}(\bar{h}_{11},_4 - \bar{h}_{22},_4)^2 + 2\bar{h}^2_{12},_4 \Big] . \qquad (4.30)$$

This gives equation 4.15 if the solutions for $\bar{h}_{ij}$ in terms of the $\ddot{Q}_{ij}$ are used. We see an additional feature in this. The assumption that we are dealing only with a non-relativistic system has not been made in deriving equation 4.30. It is only when we use the restricted solution for $\bar{h}_{ij}$ that this assumption enters in. Also note that the quantity $\frac{1}{2}(\bar{h}_{11},_4 - \bar{h}_{22},_4)^2 + 2\,(\bar{h}_{12},_4)^2$ is always positive. Thus when we integrate over all angles, we are integrating a positive function and the answer must also be positive. Because of the minus sign in equation 4.30, we can conclude that the gravitational radiation must always correspond to a decrease of energy of the system with time. This is opposed to the results of Havas and Goldberg[7] who have reported an apparent increase in energy of the system in time. Their result was obtained in an approximation method where only the matter tensor, $T^{\mu\nu}$, was considered as the source of the potentials, $h_{\mu\nu}$, and the stresses were neglected. The potentials obtained in this way obviously do not satisfy $\bar{h}_{\mu\nu,\nu} = 0$, the gauge we have chosen here. In addition, the potentials chosen in this manner contradict the asymptotic field equations and are therefore not valid. A more thorough discussion of the decrease of energy by radiation when the coordinate condition $\bar{h}_{\mu\nu,\nu} = 0$ is relaxed and the role of the asymptotic field equations is given in section VII.

We cannot explicity integrate equation 4.29 over angles since the $\bar{h}_{\mu\nu}$ may be functions of the angles. It is only in the non-relativistic case where retardation effects within the source are neglected that we can reduce equation 4.29 to an expression where the direction vectors contain the only angular dependence. In that case the integration can be carried out. We get, as before,

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[7] P. Havas and J. N. Goldberg, Phys. Rev. 128, 398 (1962).

$$\frac{dE}{dt} = -\frac{G}{5}\left[\dddot{Q}_{ij}\,\dddot{Q}_{ij} - \frac{1}{3}\,\dddot{Q}_{ii}\,\dddot{Q}_{jj}\right] . \qquad (4.31)$$

Let us examine some features of these derivations and results. Equation 4.31 is valid only for a closed system. Usually calculations are carried out in the system of coordinates where the origin is taken to be the center of mass of the system. However, Galilean relativity states that we can transform our equations to another system moving with a constant velocity with respect to the first and still get the same results. Let $x^i \to x^{i'} = x^i + a^i + b^i t$. The moment of inertia tensor Q_{ij} then is obviously not invariant under such a transformation. However our formulae contain only $\dddot{Q}_{ij}$. If we evaluate $\dddot{Q}_{ij}$, we get

$$\dddot{Q}_{ij} = \sum_a m_a\left[3\, v_a^i\, \dot{v}_a^j + x_a^i\, \ddot{v}_a^j + (i \to j)\right] ,$$

so that $\dddot{Q}'_{ij}$ is given by

$$\dddot{Q}'_{ij} = \sum_a m_a\, \dot{v}_a^j(3\, v_a^i + 3b^i) + \sum_a m_a \ddot{v}_a^j(x_a^i + a^i + b^i t) + $$
$$+ (i \to j)$$

$$= \dddot{Q}_{ij} + \left\{3b^i \sum_a m_a \dot{v}_a^j + (a^i + b^i t)\sum_a m_a \ddot{v}_a^j + \right.$$
$$\left. + (i \to j)\right\} .$$

Thus the results will be consistent only if $\sum_a m_a \dot{v}_a^i = 0$. But this is just the requirement that $\sum_a f_a^i = 0$, where

f_a^i is the total force on mass a; thus the system must have no external forces acting on it. Therefore, in applying this formalism, we must consider only systems in which the bodies are moving only under their own mutual forces.

If we wish to find the energy in a gravitational wave $h_{\mu\nu} = \mathrm{Re}\ e_{\mu\nu} \exp(\underset{\approx}{k} \cdot \underset{\approx}{r} - \omega t)$, where $e_{\mu\nu} e_{\mu\nu} = 1$, we can apply equation 4.9 to get the result that the wave carries an energy $\frac{1}{2}\omega^2/32\pi G$, the factor $\frac{1}{2}$ coming from the average over the phases of the waves. The argument can be used in reverse; if $\frac{1}{2}\omega^2/32\pi G$ is the energy in a wave of frequency ω, then the energy in the field including all waves is $(32\pi G)^{-1} h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4}$. This has, however, neglected the transient energies in the interference between waves of different frequency. We must therefore use the argument about only the time average quantity being important, since the interference terms then drop out. We will find, in discussing the radiation from a relativistic system, that these transient energies are to some degree important when we try to relate the radiation to the parameters of the motion.

$\frac{dE}{dt}$, given by equation 4.25, represents the rate of change of energy of the system averaged over one cycle. The expression for S_{44} contains the mass terms,

$\tilde{T}^{44}$, and the stress terms, $\tilde{X}^{(2)}_{44} + \tilde{X}^{(3)}_{44} + \cdots$. Let us instead break up S_{44} into $\tilde{T}_4{}^4$ and the $\bar{X}^{(k)}_{44}$, and find the change in $\bar{X}^{(2)}_{44}$ over one cycle of the motion. From equation 3.29, we have

$$\frac{d}{dt}\int \bar{X}^{(2)}_{44}\, dV = \int \bar{X}^{(2)}_{4i}\, ds_i - \tfrac{1}{2}\int h_{\alpha\beta,4}\, S_{\alpha\beta}\, dV \quad .$$

But over a time average we have found that

$$\int \bar{X}^{(2)}_{4i}\, ds_i = \tfrac{1}{2}\int h_{\alpha\beta,4}\, S_{\alpha\beta}\, dV \quad ,$$

so that

$$\int dt \int \bar{X}^{(2)}_{44}\, dV = 0 \quad . \tag{4.32}$$

In other words, the second order field stresses do not contribute to the gravitational radiation over a period of the motion. This would also have been found if we had considered, say, the change in $\tilde{X}^{(2)}_{44}$. At first, this seems to be a paradox. If we consider a system which is gravitationally bound, we have from the virial theorem applied to an inverse square law that the kinetic energy T and potential energy V are always related over a time average by

$$2\,\bar{T} = -\,\bar{V}$$

where the total energy is $E = T + V$. This would seem to show that a change in E implies a change in T and a change in V. Since the $X^{(2)}_{\mu\nu}$ are the only terms in the gravitational field stresses which contribute to the non-relativistic potential energy, equation 4.32 implies that the change in V is zero, and thus the change in E must be zero.

The answer to this problem is that we are working with a system in which we have assumed that the radiation has no effect on the motion of the system over one cycle. The radiation, depending on parameters such as v, a, m, etc., should not depend on whether we use the actual motion of the system, say a spiral type decay as would be observed over a long time, or a slightly perturbed motion, where the system is assumed to move as if there were no radiation.

In electromagnetic radiation, one has the same problem. Suppose we calculate the radiation from a bound system of two charges. Assume that one has a heavy mass so that the radiation comes from the smaller mass. One method to find the energy loss due to electromagnetic radiation is to find the radiation reaction force on the charge, which one always finds to be in the direction such that the particle's velocity has a negative component in the direction of this force. One then says that the system is losing energy and at the end of one period it will have lost an energy given by $\Delta E = \int \underset{\sim}{f} \cdot \underset{\sim}{v}\, dt$. One never asks what the change in V is because it is trivially zero. This is because all of the parameters are assumed for simplicity to return to the same value at the end of one period, and the potential energy depends only on these parameters, e.g. the separation of the charges. When one wants to apply this calculation to find the actual decay of a system, he first writes, knowing ΔE and $E = T + V$, $\Delta E = \Delta T + \Delta V$, and since $2T = -V$ on the average, he gets

$$\Delta T = - \Delta E \quad ; \qquad \Delta V = 2\Delta E$$

which is the actual secular decay of these quantities.

The same thing works in the case of gravity. Once one has found ΔE, one can then proceed to find the actual secular change in the quantities he wanted, even though in the calculation of ΔE, these quantities were assumed to have no secular decay. For example, if one applies the method which was used on $X^{(2)}_{\mu\nu}$ to find the secular change in $X^{(3)}_{\mu\nu}$, one finds that it is not zero. Although there is no meaning in separating out the different parts of the total energy, we will see in section IV D. that we will not be able to get the energy loss from any force calculation because there is energy lost directly from the energy in the fields themselves, and it is this energy loss which is represented by the third order field stress change.

C. Angular Momentum Radiation

In addition to the conservation law of energy, we have the conservation laws of total momentum and angular momentum. The change in these quantities due to the radiation will give us additional information about the decay of the system.

The change in momentum of the system is found as in the energy radiation case

$$\frac{dP_i}{dt} = - \frac{d}{dt}\int S_{4i}dV = - \int S_{ij}ds_j \quad . \qquad (4.33)$$

The introduction of the minus sign in equation 4.33 is so that P_i will represent the total momentum of the system. In a non-relativistic system $S_{4i} \approx T_{4i} \approx \delta_{i\alpha}\delta_{4\beta}T^{\alpha\beta} \approx - T^{4i} = - mv^i$ so that the equationsfor momentum and angular momentum in terms of the S_{4i} have the opposite sign from what one might expect. Thus since P_i is approximately $\int T^{4i}dV$ and in the absence of radiation it is conserved, P_i is therefore equal to $- \int S_{4i}dV$. Since in the case of angular momentum we will also be dealing with the quantity S_{4i}, we will have to put a minus sign in also so that the angular momentum is what we usually mean and not its negative. As in the energy radiation case, the only contribution to S_{ij} comes from the second order stresses. Over a time average we can reduce $\tilde{X}^{(2)}_{ij}$ for large r to the form

$$\int \tilde{X}_{ij}^{(2)} dt = (32\pi G)^{-1} \int h_{\alpha\beta,i}\, \bar{h}_{\alpha\beta,j}\, dt \quad , \qquad (4.34)$$

so that

$$\begin{aligned}\int \frac{dP_i}{dt}\, dt &= -(32\pi G)^{-1} \int dt \int ds_j\, \bar{h}_{\alpha\beta,j}\, h_{\alpha\beta,i} \\ &= +(32\pi G)^{-1} \int dt \int_{\text{sphere} \equiv \odot} ds\, \bar{h}_{\alpha\beta,4}\, h_{\alpha\beta,i} \\ &= -(32\pi G)^{-1} \int dt \int_{\odot} ds\, n_i\, h_{\alpha\beta,4}\, \bar{h}_{\alpha\beta,4} \quad .\end{aligned}$$

We can reduce $h_{\alpha\beta,4}\, \bar{h}_{\alpha\beta,4}$ as we did in equation 4.29, so that

$$\begin{aligned}\int \frac{dP_i}{dt}\, dt = -(32\pi G)^{-1} \int dt \int_{\odot} ds\, n_i \Big[&\tfrac{1}{2} (n_k n_m \bar{h}_{km,4})^2 - \\ &- 2\, n_j n_k\, \bar{h}_{mk,4}\, \bar{h}_{mj,4} + \bar{h}_{mk,4}\, \bar{h}_{mk,4} + \\ &+ n_j n_k\, \bar{h}_{jk,4}\, \bar{h}_{mm,4} - \tfrac{1}{2} (\bar{h}_{jj,4})^2 \Big] \quad .\end{aligned}$$

In the approximation in which the masses are moving slowly and retardation effects within the source are ignored, $\bar{h}_{ij,4}$ is independent of the angles. Thus the only angular dependence is in the n_k, and we see that each term contains an odd number of the direction vectors n_k and thus the change in total momentum vanishes when the integration over angles is carried out.

$$\int \frac{dP_i}{dt}\, dt = 0 \quad . \qquad (4.35)$$

This result is valid only if the above approximation holds. One can imagine systems which radiate preferentially in one direction, but they require a phase lag between different components of the system. In this case equation 4.35 would not be true. In our analysis we have assumed that the phase lag between different parts of the system is negligible. This will be valid so long as the dimensions of the system are small compared with the characteristic wavelength of the radiation.

The change in the total angular momentum of the system is given by

$$\frac{dL_i}{dt} = - \epsilon_{ijk} \frac{d}{dt} \int x_j \, S_{4k} \, dV \qquad (4.36)$$

where $\epsilon_{ijk} = 1$ if i, j, k are 1,2,3 or cyclic permutations therof; - 1 if i,j,k are 1,3,2 or cyclic permutations thereof; 0 if any two indices are the same. This L_i is seen to be the angular momentum since in the limit of no radiation it is a conserved quantity. Also in the non-relativistic limit, L_i corresponds to the usual definitions of the i^{th} component of the angular momentum of a system. From the conservation law of angular momentum, equation 4.36 becomes

$$\frac{dL_i}{dt} = - \epsilon_{ijk} \int x_j \frac{\partial}{\partial x_m} S_{mk} \, dV$$

or

$$\frac{dL_i}{dt} = -\epsilon_{ijk}\int \frac{\partial}{\partial x_m} x_j S_{mk}\, dV + \epsilon_{ijk}\int S_{jk}\, dV \quad ,$$

and since S_{jk} is symmertic and ϵ_{ijk} is anti-symmetric, the last term becomes zero. The first term can be converted to a surface integral to give

$$\frac{dL_i}{dt} = -\epsilon_{ijk}\int x_j S_{km} ds_m = -\epsilon_{ijk}\int x_j \widetilde{X}_{km} ds_m \qquad (4.37)$$

since $\widetilde{T}^{mk} = 0$ for large r. We must be careful in our argument that $\widetilde{X}_{mk}$ can be replaced by $\widetilde{X}^{(2)}_{mk}$. Since $\widetilde{X}^{(2)}_{mk}$ is proportional to $1/r^2$, and $x_j \propto r$, we see that the integral over the surface is proportional to r for large r and therefore appears to diverge for large distances. However, if we look at the part of the $\widetilde{X}^{(2)}_{mk}$ which is proportional to $1/r^2$, we see that it is given by (over a time average)

$$\int X^{(2)}_{mk}\, dt = (32\pi G)^{-1} \frac{x_m\, x_k}{r^2} \int h_{\alpha\beta,4}\, \bar{h}_{\alpha\beta,4}\, dt \quad .$$

Thus the lowest order contribution to equation 4.37 will come from

$$-\epsilon_{ijk}\, (32\pi G)^{-1} \int dt \int ds_m \frac{x_j x_m x_k}{r^2}\, h_{\alpha\beta,4}\, \bar{h}_{\alpha\beta,4}$$

which vanishes because ϵ_{ijk} is antisymmetric in any two indices. Thus we can have a contribution only from the $1/r^3$ part of $\widetilde{X}^{(2)}_{mk}$ and the $1/r^3$ part of $\widetilde{X}^{(3)}_{mk}$.

We can show that $\widetilde{X}^{(3)}_{mk}$ does not count in finding the angular momentum radiated in the approximation in which we are working. Assume that $\widetilde{X}^{(3)}_{mk}$ yields an angular momentum loss

$$\int \frac{dL_i'}{dt}\, dt = -\,\epsilon_{ijk}\int dt \int ds_m\, x_j\, \widetilde{X}^{(3)}_{mk} \quad .$$

$\widetilde{X}^{(3)}_{mk}$ contains three $h_{\alpha\beta}$ and two derivatives. Furthermore, only the lowest order $h_{\alpha\beta}$ ($\propto 1/r$) is needed in this expression. Remembering that $h_{\alpha\beta,k} = -\,n_k\, h_{\alpha\beta,4}$ to this order, we can integrate all components by parts by first converting them to time derivatives and then integrating by parts with respect to time. Thus the contribution from any one term of $\widetilde{X}^{(3)}_{mk}$ can be reduced to the form

$$\epsilon_{ijk}\int dt \int ds_m\, x_j\, \bar{h}_{\alpha\beta,\gamma}\bar{h}_{\delta\epsilon,\lambda}\bar{h}_{\sigma\tau} \quad .$$

Through the use of the coordinate conditions, we can reduce the components α, β, δ, and ϵ to spatial components multiplied by the direction vectors n_i, and the derivatives with respect to x_γ and x_λ to time derivatives multiplied by $-n_\gamma$ and $-n_\lambda$, where $n_\gamma = (-1,\, n_i)$. This reduction gives

$$\epsilon_{ijk}\int dt \int ds\; x_j n_m \begin{Bmatrix}\text{direction}\\ \text{vectors}\end{Bmatrix} \bar{h}_{np,4}\, \bar{h}_{qr,4}\, \bar{h}_{\sigma\tau} \; .$$

If we write $x_i = n_i\, r$, we see that we have an angular momentum radiation distribution which is of the order of $r\,\bar{h}_{\sigma\tau}$ times the angular distribution of the energy radiation to lowest order. Since $\bar{h}_{44} \approx -4GM/r$ and all of the other components $\bar{h}_{4m}$ and $\bar{h}_{mn}$ are smaller than this, then at most $\frac{dL_i}{dt} \approx GM \frac{dE}{dt}$, whereas we are expecting terms such that $\frac{dE}{dt} \approx \omega \frac{dL_i}{dt}$ for a system which is periodic with frequency ω . Since $GM\omega \approx GMv/r \approx (GM/rc^2)(v/c) \approx (v/c)^3$, the contribution from $\tilde{X}^{(3)}_{mk}$ will be of order $(v/c)^3$ smaller than the contributions we expect from $\tilde{X}^{(2)}_{mk}$. With these simplifications, equation 4.37 can be written

$$\int \frac{dL_i}{dt} dt = -\epsilon_{ijk} \int dt \int ds_m \tilde{X}^{(2)}_{mk} x_j \qquad (4.38)$$

where only the part of $\tilde{X}^{(2)}_{mk}$ which is proportional to $1/r^3$ counts.

The formula for $\tilde{X}^{(2)}_{mk}$ given by equation 4.34 does not apply in this case. We must therefore go back to the original expression for $\tilde{X}^{(2)}_{mk}$. Any term in this which is proportional to $h_{\alpha\beta,\lambda\lambda}$ will not contribute to this radiation. Since $h_{\alpha\beta,\lambda\lambda} \propto h^2$, this type of term will be of order h^3. By the same argument as used in eliminating the contribution of the third order stresses, one can show that this term will not give anything in the order in which we are finding the radiation. Also any terms in the second order stresses $\tilde{X}^{(2)}_{mk}$ which are proportional to δ_{mk} can be neglected for then we would have the integrand in equation 4.38

be symmetrical in j and k and the indicated sum would then vanish. Writing the remaining terms which can contribute gives (letting $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\bar{h}$)

$$\begin{aligned}\tilde{X}^{(2)}_{mk} = (32\pi G)^{-1} \Big\{ &- \bar{h}_{\alpha\beta,k}\bar{h}_{\alpha\beta,m} - 2\,\bar{h}_{\gamma k,\gamma}\bar{h}_{\delta m,\delta} + \\ &+ 2\,\bar{h}_{mk,\delta}\bar{h}_{\sigma\sigma,\delta} + 2\,\bar{h}_{k\gamma,\delta}\bar{h}_{m\delta,\gamma} - \bar{h}_{k\gamma,m}\bar{h}_{\sigma\sigma,\gamma} - \\ &- \bar{h}_{\sigma\sigma,\delta}\bar{h}_{m\delta,k} - \tfrac{1}{2}\,\bar{h}_{\sigma\sigma,m}\,\bar{h}_{\gamma\gamma,k} - 2\,\bar{h}_{km,\alpha\beta}\bar{h}_{\alpha\beta} - \\ &- 2\,\bar{h}_{\alpha\beta}\bar{h}_{\alpha\beta,mk} - \bar{h}_{k\beta}\,\bar{h}_{\sigma\sigma,\beta m} - \bar{h}_{m\beta}\,\bar{h}_{\sigma\sigma,\beta k} + \\ &+ 2\,\bar{h}_{\alpha\beta}\bar{h}_{k\alpha,m\beta} + 2\,\bar{h}_{\alpha\beta}\bar{h}_{m\alpha,k\beta} + \bar{h}_{\sigma\sigma}\bar{h}_{\delta\delta,km}\Big\} . \qquad (4.39)\end{aligned}$$

In each term we have the product of two h's. In order that the product be proportional to $1/r^3$, one of the h's must be proportional to $1/r^2$ and the other proportional to $1/r$. In order to reduce equation 4.39, one property of the $1/r^2$ solution is needed. If we have only the $1/r^2$ part of $\bar{h}_{\mu\nu,\lambda}$ and differentiate this again with respect to x_σ, we must have that

$$(\bar{h}_{\mu\nu,\lambda})_{,\sigma} = - n_\sigma\,\bar{h}_{\mu\nu,\lambda 4} \quad .$$

We can see this from an examination of the solution for $\bar{h}_{\mu\nu,\lambda}$.

$$\bar{h}_{\mu\nu,\lambda}(\underset{\sim}{r},t) = -4G\,\frac{\partial}{\partial x_\lambda}\int \frac{S_{\mu\nu}(\underset{\sim}{r}',t')\quad(t'-t+|\underset{\sim}{r}-\underset{\sim}{r}'|)dt'\,dV'}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \quad .$$

Assume that the expansion to order $1/r^2$ has been carried out and consider only the $1/r^2$ part of $\bar{h}_{\mu\nu,\lambda}$, denoted by $\bar{h}^{(2)}_{\mu\nu,\lambda}$. The only way a further spatial derivative can act on $\bar{h}^{(2)}_{\mu\nu,\lambda}$ and still have a term proportional to $1/r^2$ is to act on the retarded time δ-function, which gives $(\bar{h}^{(2)}_{\mu\nu,\lambda})_{,i} = -n_i\, \bar{h}^{(2)}_{\mu\nu,\lambda 4}$. Suppose there was a product of two h's, say $\bar{h}_{\alpha\beta,\lambda}\bar{h}_{\sigma\epsilon,\delta}$, and only the $1/r^3$ part is considered. Then one of the terms must be $\propto 1/r$ and the other $\propto 1/r^2$. If we consider the quantity $\frac{\partial}{\partial x_i}(\bar{h}_{\alpha\beta,\lambda}\, \bar{h}_{\sigma\epsilon,\delta})^{(3)}$, we see that if the resulting expression is still to be of order $1/r^3$, we must again have that $\frac{\partial}{\partial x_i}(\quad) = -n_i\frac{\partial}{\partial t}(\quad)$, because only derivatives on the retarded time delta-function give a result which does not change the order of the expression.

This, together with the fact that we can neglect the terms mentioned before, allows us to eliminate many terms in $\tilde{X}^{(2)}_{mk}$ which do not contribute to the radiation of angular momentum and also simplify others which do remain. The following terms of equation 4.39 yield no contribution in equation 4.38 in the order in which we are calculating the angular momentum loss:

$$\text{a) } \int dt\int ds_m x_j \bar{h}_{\gamma k,\delta}\bar{h}_{\gamma m,\delta} = -\tfrac{1}{2}\int dt\int ds_m x_j \bar{h}_{\gamma k,\delta\delta}\bar{h}_{\gamma m} -$$

$$-\tfrac{1}{2}\int dt\int ds_m x_j \bar{h}_{\gamma k}\bar{h}_{\gamma m,\delta\delta} - \tfrac{1}{2}\int dt\int ds_m x_j \frac{\partial}{\partial x_p}(\bar{h}_{\gamma k,p}\bar{h}_{\gamma m} + \bar{h}_{\gamma k}\bar{h}_{\gamma m,p}).$$

The first two terms are neglected because of the $,\delta\delta$.

In the last term, if the part of the expression within the parentheses is proportional to $1/r^3$, then $\frac{\partial}{\partial x_p} =$ $= - n_p \frac{\partial}{\partial t}$, and the integral over time vanishes. Therefore the only part of the quantity which counts is the part proportional to $1/r^2$ or thus where each term is proportional to $1/r$. But then

$$\bar{h}_{\gamma k,p}\bar{h}_{\gamma m} + \bar{h}_{\gamma k}\bar{h}_{\gamma m,p} = - n_p(\bar{h}_{\gamma k,4}\bar{h}_{\gamma m} + \bar{h}_{\gamma k}\bar{h}_{\gamma m,4}) =$$

$$= - n_p \frac{\partial}{\partial t} (\bar{h}_{k\gamma}\, \bar{h}_{m\gamma}) \quad ,$$

and the time integral eliminates this. A similar argument applies to the term $\bar{h}_{mk,\delta}\bar{h}_{\sigma\sigma,\delta}$.

$$\text{b)} \quad \int dt \int ds_m x_j \left[\bar{h}_{\sigma\sigma,\delta}\bar{h}_{m\delta,k} + \bar{h}_{\sigma\sigma,\beta k}\bar{h}_{m\beta} \right] =$$

$$= \int dt \int ds_m x_j \frac{\partial}{\partial x_k}(\bar{h}_{\sigma\sigma,\delta}\bar{h}_{m\delta}) \quad .$$

As before the part in the parentheses has to be proportional to $1/r^2$, so that we can write $\bar{h}_{\sigma\sigma,\delta}\bar{h}_{m\delta} =$ $= - n_\delta \bar{h}_{\sigma\sigma,4}\bar{h}_{m\delta}$, and a time integration gives $n_\delta \bar{h}_{\sigma\sigma}\, \bar{h}_{m\delta,4} = - \bar{h}_{\sigma\sigma}\, \bar{h}_{m\delta,\delta} = 0$. Thus this term also vanishes.

$$\text{c)} \quad \int dt \int ds_m x_j \bar{h}_{\alpha\beta}\bar{h}_{km,\alpha\beta} = -\int dt \int ds_m x_j \frac{\partial}{\partial x_p}(\bar{h}_{\alpha p}\bar{h}_{km,\alpha}).$$

As in b), $\bar{h}_{\alpha p}\, \bar{h}_{km,\alpha} = - \bar{h}_{\alpha p,\alpha}\, \bar{h}_{km} = 0$.

$$\text{d)} \quad \int dt \int ds_m x_j\, \bar{h}_{k\gamma,\delta}\, \bar{h}_{m\delta,\gamma} \; =$$

$$= -\int dt \int ds_m x_j \frac{\partial}{\partial x_p}(\bar{h}_{kp,\delta}\bar{h}_{m\delta}) = 0.$$

e) $$\int dt \int ds_m x_j \;(\bar{h}_{k\beta,m}\;\bar{h}_{\sigma\sigma,\beta} + \bar{h}_{k\beta}\;\bar{h}_{\sigma\sigma,\beta m}) =$$

$$= \int dt \int ds_m x_j \;\frac{\partial}{\partial x_m}(\bar{h}_{k\beta}\;\bar{h}_{\sigma\sigma,\beta}) = 0.$$

The following terms simplify:

f) $$-\;2\int dt \int ds_m \;x_j\;\bar{h}_{\alpha\beta}\;\bar{h}_{\alpha\beta,km} =$$

$$= 2\int dt \int ds_m x_j \bar{h}_{\alpha\beta,m}\bar{h}_{\alpha\beta,k} - 2\int dt \int ds_m x_j \;\frac{\partial}{\partial x_m}(\bar{h}_{\alpha\beta}\bar{h}_{\alpha\beta,k}) \quad .$$

The last term becomes

$$\int dt \int ds_m x_j \;\frac{\partial}{\partial x_m}\left[\frac{x_k}{r}\;\frac{\partial}{\partial t}\;(\bar{h}_{\alpha\beta}\bar{h}_{\alpha\beta})\right] = 0.$$

The term $\bar{h}_{\sigma\sigma}\;\bar{h}_{\lambda\lambda,km}$ can similarly be reduced.

g) $$2\int dt \int ds_m \;x_j\;\bar{h}_{\alpha\beta}\;\bar{h}_{k\alpha,\beta m} =$$

$$= -2\int dt \int ds_m x_j \bar{h}_{\alpha\beta,m}\bar{h}_{k\alpha,\beta} + 2\int dt \int ds_m x_j \frac{\partial}{\partial x_m}(\bar{h}_{\alpha\beta}\bar{h}_{k\alpha,\beta}) \quad ,$$

where the last term is zero. A similar transformation applies to the term $2\;\bar{h}_{\alpha\beta}\;\bar{h}_{m\alpha,\beta k}$.

With the above results, equation 4.38 can be written

$$\int \frac{dL_i}{dt}\;dt = -\;\epsilon_{ijk}(32\pi G)^{-1}\int dt \int ds_m \;x_j\Big[h_{\alpha\beta,k}\;\bar{h}_{\alpha\beta,m} -$$

$$-\;2\;\bar{h}_{\alpha\beta,m}\;\bar{h}_{k\alpha,\beta} - 2\;\bar{h}_{\alpha\beta,k}\;\bar{h}_{m\alpha,\beta}\Big] \;. \qquad (4.40)$$

This is the angular momentum analogue of equation 4.24.

The angular momentum change in the quadrupole approximation can be found by two methods analogous to those used in the energy radiation case. Let us first consider the analogue of the radiation reaction calculation. The surface integral in equation 4.40 is converted back to a volume integral. This gives, after some simplification

$$\int \frac{dL_i}{dt} dt = \epsilon_{ijk}(32\pi G)^{-1}\int dt \int dV \Big\{ x_j (h_{\alpha\beta,k} - 2\bar{h}_{k\alpha,\beta})\bar{h}_{\alpha\beta,\lambda\lambda}$$

$$- \tfrac{1}{2} x_j \frac{\partial}{\partial x_k}(h_{\alpha\beta,\lambda}\bar{h}_{\alpha\beta,\lambda}) + 2\, x_j \bar{h}_{\alpha\beta,\lambda}\bar{h}_{k\alpha,\beta\lambda} -$$

$$- 2\, x_j \frac{\partial}{\partial x_m}(\bar{h}_{\alpha\beta,k}\bar{h}_{m\alpha,\beta}) \quad = (a) + (b) + (c) + (d).$$

Term (b) is proportional to

$$\int ds_k\, x_j(\quad) - \int dV\, \delta_{jk}(\quad)$$

which vanishes. Term (c) becomes

$$-2\,\epsilon_{ijk}\int dt \int ds_m x_j \bar{h}_{\alpha m,\lambda}\bar{h}_{\alpha k,\lambda} + 2\,\epsilon_{ijk}\int dt \int dV\, \bar{h}_{\alpha j,\lambda}\bar{h}_{\alpha k,\lambda} =$$

$$= \epsilon_{ijk}\int dt \int ds_m x_j \frac{\partial}{\partial x_p}(\bar{h}_{\alpha m,p}\bar{h}_{\alpha k} + \bar{h}_{\alpha m}\bar{h}_{\alpha k,p}) = 0.$$

Term (d) becomes

$$-2\,\epsilon_{ijk}\int dt \int ds_m x_j \bar{h}_{\alpha\beta,k}\bar{h}_{m\alpha,\beta} + 2\,\epsilon_{ijk}\int dt \int dV\, \bar{h}_{\alpha\beta,k}\bar{h}_{j\alpha,\beta} =$$

$$= 2\,\epsilon_{ijk}\int dt \int ds_m x_j \frac{\partial}{\partial x_p}(\bar{h}_{\alpha p,k}\bar{h}_{m\alpha}) -$$

$$- 2\,\epsilon_{ijk}\int dt \int ds_m\, \bar{h}_{\alpha m,k}\bar{h}_{j\alpha} \quad . \; (*)$$

The first term of this is

$$2\,\epsilon_{ijk}\int dt\int ds_m\, x_j\,\frac{\partial}{\partial x_p}\left(-\frac{x_k}{r}\,\bar{h}_{\alpha p,4}\,\bar{h}_{m\alpha}\right) =$$

$$= -\,2\,\epsilon_{ijk}\int dt\int ds_m\,\frac{x_j}{r}\,\bar{h}_{\alpha k,4}\,\bar{h}_{m\alpha} =$$

$$= +\,2\,\epsilon_{ijk}\int dt\int ds_m\,\bar{h}_{\alpha k}\,\bar{h}_{m\alpha,4}\,n_j =$$

$$= -\,2\,\epsilon_{ijk}\int dt\int ds_m\,\bar{h}_{\alpha m,j}\,\bar{h}_{\alpha k}\quad, \qquad (**)$$

so that the sum of (*) and (**) is symmetric in j and k and thus vanishes. We then get that

$$\int\frac{dL_i}{dt}\,dt = \epsilon_{ijk}(32\pi G)^{-1}\int dt\int dV\,x_j(h_{\alpha\beta,k} - 2\bar{h}_{k\alpha,\beta})\bar{h}_{\alpha\beta,\lambda\lambda} =$$

$$= -\tfrac{1}{2}\,\epsilon_{ijk}\int dt\int dV\,x_j(h_{\alpha\beta,k} - 2\,\bar{h}_{k\alpha,\beta})\,S_{\alpha\beta}\quad, \qquad (4.41)$$

which is the equation analogous to the radiation reaction equation for the energy loss.

Equation 4.41 can be simplified by a further integration by parts. This yields

$$\int\frac{dL_i}{dt}\,dt = -\,\tfrac{1}{2}\,\epsilon_{ijk}\int dt\int dV(x_j h_{\alpha\beta,k}S_{\alpha\beta} - 2\,h_{\alpha k}\,S_{\alpha j})\;. \qquad (4.42)$$

Since $\bar{h}_{\alpha\beta} = -\,4G\int\left[\frac{S_{\alpha\beta}}{r} - \dot{S}_{\alpha\beta} + \frac{r}{2}\,\ddot{S}_{\alpha\beta} - \frac{r^2}{6}\dddot{S}_{\alpha\beta} + \cdots\right]$, only the terms with even powers of r survive time integrations in equation 4.42. Thus we get the following terms:

$$\int \frac{dL_i}{dt} dt = -2G\,\epsilon_{ijk}\int dt\int dV\int dV'\Big\{ x_j\Big[S_{\alpha\beta}\frac{\partial}{\partial x_k}\dot{S}'_{\alpha\beta} +$$

$$+\frac{1}{6}S_{\alpha\beta}\frac{\partial}{\partial x_k}|\underset{\sim}{r}-\underset{\sim}{r}'|^2\,\dddot{S}'_{\alpha\beta} + \frac{1}{120}S_{\alpha\beta}\frac{\partial}{\partial x_k}|\underset{\sim}{r}-\underset{\sim}{r}'|^4\,\overset{.....}{S}'_{\alpha\beta} +$$

$$-\frac{1}{2}S_{\alpha\alpha}\frac{\partial}{\partial x_k}\dot{S}'_{\beta\beta} - \frac{1}{12}S_{\alpha\alpha}\frac{\partial}{\partial x_k}|\underset{\sim}{r}-\underset{\sim}{r}'|^2\,\dddot{S}'_{\beta\beta} +$$

$$-\frac{1}{240}S_{\alpha\alpha}\frac{\partial}{\partial x_k}|\underset{\sim}{r}-\underset{\sim}{r}'|^4\,\overset{.....}{S}'_{\beta\beta}\Big] - 2S_{\alpha j}\dot{S}'_{\alpha k} +$$

$$-\frac{1}{3}S_{\alpha j}|\underset{\sim}{r}-\underset{\sim}{r}'|^2\,\dddot{S}'_{\alpha k}\Big\} \quad .$$

Explicitly reducing the expression using $S_{\alpha\beta,\beta} = 0$ and $\int S_{ij}\,dV = \frac{1}{2}\ddot{Q}_{ij}$ as was done in the energy radiation case, we get the contributions

$$\int \frac{dL_i}{dt} dt = -2G\,\epsilon_{ijk}\int dt\,\ddot{Q}_{jm}\,\dddot{Q}_{km}\left(-\frac{2}{3}+\frac{4}{15}-\frac{2}{15}+2-\frac{2}{3}\right)$$

which is then summed to give

$$\int \frac{dL_i}{dt} dt = -\frac{2}{5}G\,\epsilon_{ijk}\int dt\,\ddot{Q}_{jm}\,\dddot{Q}_{km} \quad . \qquad (4.43)$$

This is then the expression for the total i-component of angular momentum radiated by the system.

We may also get to equation 4.43 by way of the calculation of the angular momentum flux crossing a large sphere. This allows us to get the angular momentum radiated as a function of the angles as well as the total radiation. For this we need $\bar{h}_{\mu\nu,\lambda}$ to order $1/r^2$. $\bar{h}_{\mu\nu,4}$ is, of course,

$$\bar{h}_{\mu\nu,4}(\underset{\sim}{r},t) = -4G\int \frac{\dot{S}_{\mu\nu}(\underset{\sim}{r}',t')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \delta(t'-t+|\underset{\sim}{r}-\underset{\sim}{r}'|)dt'dV' \quad .$$

We need $\bar{h}_{\mu\nu,k}$ more explicitly than the above:

$$\bar{h}_{\mu\nu,k}(\underset{\sim}{r},t) = 4G\int \dot{S}_{\mu\nu}(\underset{\sim}{r}',t') \quad \frac{x_k - x_k'}{|\underset{\sim}{r} - \underset{\sim}{r}'|^2} \delta(\ \ )dt'dV' +$$

$$+ 4G\int S_{\mu\nu}(\underset{\sim}{r}',t') \quad \frac{x_k - x_k'}{|\underset{\sim}{r} - \underset{\sim}{r}'|^3} \delta(\ \ )dt'dV' \quad .$$

Thus

$$\bar{h}_{\mu\nu,k} = -\frac{x_k}{r}\bar{h}_{\mu\nu,4} - \frac{x_k}{r^2}\bar{h}_{\mu\nu} - \frac{4G}{r^2}\int \dot{S}_{\mu\nu}x_k' \,\delta(\ \ )dt'dV' +$$

$$+ \frac{4G}{r^3} x_k \int \dot{S}_{\mu\nu}\, \underset{\sim}{r}\cdot\underset{\sim}{r}' \;\delta(\ \ )dt'dV' \quad .$$

Then

$$\bar{h}_{ij,k} = -\frac{x_k}{r}\bar{h}_{ij,4} - \frac{x_k}{r^2}\bar{h}_{ij} - \frac{4G}{r^2}\int \dot{S}_{ij}x_k' \,\delta(\ \ )dt'dV' +$$

$$+ \frac{4G}{r^3} x_k \int \dot{S}_{ij}\, \underset{\sim}{r}\cdot\underset{\sim}{r}' \;\delta(\ \ )dt'dV' \quad .$$

The last two terms will be much smaller than the first in the order of the quadrupole approximation. To see this we let everything have a time dependence $e^{-i\omega t}$. Then

$$\bar{h}_{ij}(\underset{\sim}{r},\omega) \;\propto\; \int S_{ij}(\underset{\sim}{r}',\omega)\; e^{-i\omega(\frac{\underset{\sim}{r}\cdot\underset{\sim}{r}'}{r})}\; dV'$$

whereas the last two terms have the form

$$\int S_{ij}(\underset{\sim}{r}',\omega)\;(-i\omega x_k')\; e^{-i\omega(\frac{\underset{\sim}{r}\cdot\underset{\sim}{r}'}{r})}\; dV' \quad .$$

In the quadrupole approximation, we let $\omega x_k' \ll 1$, so that the exponential $\exp(-i\omega\frac{\underset{\sim}{r}\cdot\underset{\sim}{r}'}{r})$ can be set equal to one. The latter terms have the form of the second term in the expansion of the exponential and thus, in this approximation, can be neglected. Therefore, to order $1/r^2$, we can write

$$\bar{h}_{ij,k} = -\frac{n_k}{r}\bar{h}_{ij} - n_k\,\bar{h}_{ij,4} \qquad (4.44)$$

where $\bar{h}_{ij}$ is then given by $\bar{h}_{ij} = -\frac{4G}{r}\int S_{ij}\,dV$. Let us consider $\bar{h}_{4k,m}$. $\bar{h}_{4k,m} = -n_m r^{-1}\bar{h}_{4k} - n_m\bar{h}_{4k,4} - - 4G\,r^{-2}\int\dot{S}_{4k}\,x'_m\,\delta\,dt'dV' + 4G\,x_k\,r^{-3}\int\dot{S}_{4m}\,\underset{\sim}{r}\cdot\underset{\sim}{r}'\,\delta\,dt'dV'$. In the last two terms, we let $\dot{S}_{4k} = S_{kj,j}$, in which case we can integrate by parts to get

$$\bar{h}_{4k,m} = -\frac{n_m}{r}\bar{h}_{4k} - n_m\bar{h}_{44,4} - \frac{1}{r}\bar{h}_{mk} + \frac{n_k n_j}{r}\bar{h}_{mj}\,. \qquad (4.45)$$

Similarly,

$$\bar{h}_{44,k} = -\frac{n_k}{r}\bar{h}_{44} - n_k\,\bar{h}_{44,4} - \frac{n_k n_m n_s}{r}\bar{h}_{ms} + + \frac{n_k n_m}{r}\bar{h}_{4m} - \frac{1}{r}\bar{h}_{4k} + \frac{n_m}{r}\bar{h}_{mk}\,. \qquad (4.46)$$

The coordinate conditions $\bar{h}_{44,4} = \bar{h}_{4m,m}$ and $\bar{h}_{k4,4} = \bar{h}_{km,m}$ are used to eliminate the time derivative terms. With these expressions, we can find the solution to equation 4.40 directly.

$$\int\frac{dL_i}{dt}\,dt = -\epsilon_{ijk}(32\pi G)^{-1}\int dt\oint ds\; n_m\Big[h_{\alpha\beta,k}\,\bar{h}_{\alpha\beta,m} - - 2\,\bar{h}_{\alpha\beta,k}\,\bar{h}_{m\alpha,\beta} - 2\,\bar{h}_{\alpha\beta,m}\,\bar{h}_{k\alpha,\beta}\Big]\,. \qquad (4.47)$$

In the first term, $h_{\alpha\beta,k}\,\bar{h}_{\alpha\beta,m}$, $h_{\alpha\beta,k}$ must be of order $1/r^2$ since any term proportional to x_k will yield zero upon summation with $\epsilon_{ijk}x_j$. Then $\bar{h}_{\alpha\beta,m} = -\,n_m\,\bar{h}_{\alpha\beta,4}$. From equations 4.44-6 we have that

$$\epsilon_{ijk}x_j\,\bar{h}_{44,k4} = \epsilon_{ijk}x_j\,\frac{2n_m}{r}\,\bar{h}_{mk,4}$$

$$\epsilon_{ijk}x_j\,\bar{h}_{4m,k} = -\,\epsilon_{ijk}x_j\,\bar{h}_{mk}$$

$$\epsilon_{ijk}x_j\,\bar{h}_{pq,k} = 0$$

so that for the first term of equation 4.47 we get

$$+\ \epsilon_{ijk}(32\pi G)^{-1}\int dt\oint ds\left[n_q\,n_j\,n_m\,n_p\,\bar{h}_{qk}\,\bar{h}_{mp,4}\,-\right.$$
$$\left.-\,2\,n_j\,n_m\,\bar{h}_{qk}\,\bar{h}_{qm,4} + 2\,n_j\,n_q\,\bar{h}_{qk}\,\bar{h}_{pp,4}\right] .$$

For the second term of equation 4.47 we get in an analogous manner

$$-\,2\,\epsilon_{ijk}(32\pi G)^{-1}\int dt\oint ds\left[n_q\,n_j\,n_m\,n_p\,\bar{h}_{pk}\,\bar{h}_{qm,4}\,-\right.$$
$$\left.-\,n_q\,n_j\,\bar{h}_{mk}\,\bar{h}_{mq,4}\right] .$$

The third term can be written

$$2\,\epsilon_{ijk}\int dt\oint ds\;n_m\,n_j\,r\,\bar{h}_{\alpha\beta,m}\,\bar{h}_{k\alpha,\beta} =$$
$$= -\,2\,\epsilon_{ijk}\int dt\oint ds\;n_p\,n_j\,r\,\frac{\partial}{\partial x_m}(\bar{h}_{\alpha m,p}\,\bar{h}_{k\alpha}) ,$$

and since $\int dt\,\bar{h}_{\alpha m,p}\,\bar{h}_{k\alpha}$ to order $1/r^2$ is equal to

$$\int dt\left[-\,n_p\,n_r\,n_q\,\bar{h}_{rm,4}\,\bar{h}_{kq} + n_p\,\bar{h}_{rm,4}\,\bar{h}_{kr}\right] ,$$

the term becomes

$$- 2\,\epsilon_{ijk}\int dt\oint ds\Big[n_p\, n_j\, r\,\frac{\partial}{\partial x_m}\big(- n_p\, n_r\, n_q\, \bar{h}_{rm,4}\, \bar{h}_{kq} + $$

$$+ n_p\, \bar{h}_{rm,4}\, \bar{h}_{kr}\big)\Big] \quad .$$

Carrying out the differentiation with respect to x_m then yields an expression in which we can use the solutions given 4.44-6 to get

$$- 2\,\epsilon_{ijk}\int dt\oint ds\,\big(4\, n_j\, n_p\, n_q\, n_m\, \bar{h}_{pm,4}\, \bar{h}_{kq} -$$

$$- n_j\, n_q\, \bar{h}_{mm,4}\, \bar{h}_{kq} - 3\, n_j\, n_p\, \bar{h}_{pm,4}\, \bar{h}_{km}\big) \quad .$$

Thus the total angular momentum radiation distribution is

$$\int\frac{d^2L_i}{dt\,d\Omega}\,dt = \frac{\epsilon_{ijk}}{8\pi}\int dt\Big[\,6\, n_j\, n_p\, \ddot{Q}_{mk}\, \dddot{Q}_{mp} -$$

$$- 9\, n_j\, n_m\, n_p\, n_q\, \ddot{Q}_{mk}\, \dddot{Q}_{pq} + \qquad (4.48)$$

$$+ 4\, n_j\, n_m\, \ddot{Q}_{mk}\, \dddot{Q}_{pp}\Big] ,$$

where the solutions for $\bar{h}_{ji}$ in terms of the Q_{ji} have been used. The integral over angles is trivially performed. This gives

$$\int\frac{dL_i}{dt}\,dt = \frac{2}{5}\,\epsilon_{ijk}\int dt\, \dddot{Q}_{mj}\, \ddot{Q}_{mk}$$

which agrees with that found in equation 4.43.

There remains yet another way of finding the angular momentum radiated. We may use the multipole analysis developed by Mathews[8]. A wave of frequency ω with quantum numbers J and M will have a z-component of angular momentum flux of M/ω times the energy flux. Thus where the total energy radiated is given by

$$\frac{dE}{dt} = -\tfrac{1}{2} \sum_{J,M} \left[|e_{JM}|^2 + |m_{JM}|^2 \right] ,$$

the total z-component of angular momentum radiated will be given by

$$\frac{dL_z}{dt} = -\frac{1}{2\omega} \sum_{J,M} \left[M\,|e_{JM}|^2 + M\,|m_{JM}|^2 \right] . \qquad (4.49)$$

e_{JM} and m_{JM} are the amplitudes of the various multipoles present. In the quadrupole approximation we have that m_{2M} dominates the radiation, so that

$$\frac{dL_z}{dt} = -\frac{1}{2\omega} \sum_{M=-2}^{2} M\,|m_{2M}|^2 , \qquad (4.50)$$

where m_{2M} is given by Mathews as

$$m_{2M} = -i\omega\sqrt{\frac{4G}{5}} \int dV \begin{cases} \tfrac{1}{2}(S_{xx} - S_{yy}) \mp i\, S_{xy} & M = \pm 2 \\ \mp S_{xz} + i\, S_{yz} & M = \pm 1 \\ \tfrac{1}{6}(2\, S_{zz} - S_{xx} - S_{yy}) & M = 0 . \end{cases}$$

Also in the quadrupole approximation we can let $\int S_{ij} dV = \tfrac{1}{2}\, \ddot{Q}_{ij}$.

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[8] J. Mathews, J. Soc. Ind. Appl. Math. 10, 768 (1962).

In performing the sum indicated in equation 4.50, we must remember that Q_{ij} here are really phasors, with real and imaginary parts. We then let $Q_{ij} = \underline{Q}_{ij} + i\,\underset{\sim}{Q}_{ij}$, where $\underline{Q}_{ij} = \operatorname{Re} Q_{ij}$ and $\underset{\sim}{Q}_{ij} = \operatorname{Im} Q_{ij}$. Summing then gives

$$\frac{dL_z}{dt} = -\frac{G}{10}\,\omega^5 \Big\{ 4\,\underset{\sim}{Q}_{xy}\,(\underline{Q}_{xx} - \underline{Q}_{yy}) - 4\,\underline{Q}_{xy}\,(\underset{\sim}{Q}_{xx} - \underset{\sim}{Q}_{yy}) + (\underline{Q}_{xz} + \underset{\sim}{Q}_{yz})^2 + (\underline{Q}_{yz} - \underset{\sim}{Q}_{xz})^2 - (\underline{Q}_{xz} - \underset{\sim}{Q}_{yz})^2 - (\underline{Q}_{yz} + \underset{\sim}{Q}_{xz})^2 \Big\},$$

which reduces to

$$\frac{dL_z}{dt} = -\frac{2G}{5}\,\omega^5 \operatorname{Re}\left[\frac{1}{-i}\left(Q_{xx}\,Q^*_{xy} + Q_{xy}\,Q^*_{yy} + Q_{xz}\,Q^*_{yz}\right)\right].$$

Putting this back into the form with time dependence gives

$$\frac{dL_z}{dt} = -\frac{4G}{5}\left[\ddot{Q}_{xx}\,\dddot{Q}_{xy} + \ddot{Q}_{xy}\,\dddot{Q}_{yy} + \ddot{Q}_{xz}\,\dddot{Q}_{yz}\right],$$

which is the z-component of the following equation

$$\frac{dL_z}{dt} = -\frac{2G}{5}\,\epsilon_{ijk}\,\ddot{Q}_{jm}\,\dddot{Q}_{km},$$

which was found before.

For a system which is circularly rotating, say a spinning rod, there is only one parameter, ω, which specifies the state of the system. We have, however, two equations, $\frac{dE}{dt}$ and $\frac{dL_i}{dt}$ to specify the secular change in ω over one period. It is necessary, therefore, to show that the formulae for $\frac{dE}{dt}$ and $\frac{dL_i}{dt}$ give consistent results for circular motion.

For circular motion in the x-y plane, we have

$$Q_{xx} = I\cos^2\omega t$$
$$Q_{xy} = I\cos\omega t\,\sin\omega t$$
$$Q_{yy} = I\sin^2\omega t \quad ,$$

where I is the moment of inertia of the masses or rod. Therefore

$$\ddot{Q}_{xx} = -2I\omega^2\cos(2\omega t)\ ;\quad \dddot{Q}_{xx} = 4I\omega^3\sin(2\omega t)$$
$$\ddot{Q}_{xy} = -2I\omega^2\sin(2\omega t)\ ;\quad \dddot{Q}_{xy} = -4I\omega^3\cos(2\omega t)$$
$$\ddot{Q}_{yy} = +2I\omega^2\cos(2\omega t)\ ;\quad \dddot{Q}_{yy} = -4I\omega^3\sin(2\omega t).$$

L_z and E are related by $L_z = I\omega$ and $E = \frac{1}{2}I\omega^2$; therefore $\frac{dE}{dt}$ should be equal to $\omega\frac{dL_z}{dt}$. From equation 4.28 we get

$$\frac{dE}{dt} = -\frac{G}{5}\left[\dddot{Q}^2_{xx} + 2\,\dddot{Q}^2_{xy} + \dddot{Q}^2_{yy}\right]$$
$$= -\frac{16G\,I^2\omega^6}{5}\left[2\sin^2 2\omega t + 2\cos^2 2\omega t\right]$$
$$= -\frac{32G\,I^2\omega^6}{5} \quad ,$$

and from equation 4.40 we get

$$\frac{dL_z}{dt} = -\frac{2G}{5}(\ddot{Q}_{xx}\dddot{Q}_{xy} + \ddot{Q}_{yx}\dddot{Q}_{yy} - \ddot{Q}_{xy}\dddot{Q}_{xx} - \ddot{Q}_{yy}\dddot{Q}_{yx})$$
$$= -\frac{16\,G\,I^2\omega^5}{5}(2\cos^2 2\omega t + \sin^2 2\omega t)$$
$$= -\frac{32\,G\,I^2\omega^5}{5} \quad .$$

Thus one could use either formalism to find the radiation from a circularly rotating system. We will find, in the case of two point masses moving in elliptical orbits, that the two equations give different information and allow one to find the secular change in the eccentricity as well as in the energy.

D. Radiation from the Equations of Motion

One would like to find an expression for the gravitational radiation reaction force analogous to the electromagnetic radiation reaction force. From this, one could predict both the energy radiation and the angular momentum radiation of a system of masses. As we have see, in the case of a gravitationally bound system, the third order field stresses contribute to the energy radiation, whereas we will see that the equations of motion do not give this. Even in the case of non-gravitationally bound systems we must have some stress-energy tensor as a result of the binding forces. If the mass tensor is $T^{\mu\nu}$ and this stress tensor is $W^{\mu\nu}$, then we have that

$$(T^{\mu\nu} + W^{\mu\nu})_{;\nu} = 0 ,$$

or that

$$(\widetilde{T}^{\mu\nu} + \widetilde{W}^{\mu\nu})_{,\nu} + \left\{ {\mu \atop \alpha\beta} \right\} (\widetilde{T}^{\alpha\beta} + \widetilde{W}^{\alpha\beta}) = 0 .$$

If $-\widetilde{W}^{\mu\nu}{}_{,\nu} = f^{\mu}$ is interpreted as the force density due to the binding fields and $\tilde{T}^{\mu\nu}{}_{,\nu}$ the effect of this force on matter, e.g. $\rho\dot{v}^{\mu}$ in the N.R. case, then the terms of the right are interpreted as the force density due to the gravitational fields. However, there is a flaw in this: we would like to have the force density be non-zero only where there is matter on which the force acts. This is true for $\tilde{T}^{\mu\nu}{}_{,\nu}$ and $\left\{{\mu \atop \alpha\beta}\right\}\tilde{T}^{\alpha\beta}$. However $\left\{{\mu \atop \alpha\beta}\right\}\tilde{W}^{\alpha\beta}$ is not localized at the masses, and thus there is a gravitational force density even where there is no matter. For example, this implies that there might be a contribution to the radiation from sources other than the masses themselves. In general this will be true for both gravitationally and non-gravitationally bound systems. This is not to imply, however, that information cannot be obtained from the equations of motion. The equations of motion can show explicitly why we do not get a dipole gravitational radiation. Also we can get the energy loss for a circularly rotating system correctly, although for general motion the method fails.

In electromagnetism, the radiation can be thought of as being caused by the reaction of a given charge to the motion of all of the charges present including itself. In gravity, the radiation is caused by four types of processes:

a) reaction of a given mass to the motion of all of the masses present

b) reaction of the masses to the changes in the stress-energy of the fields produced by all of the masses

c) reaction of the fields to the motion of all of the masses

d) reaction of the fields to the changes in the stress-energy of the fields themselves.

Clearly a consideration of the equations of motion of a given mass can yield only the radiation due to processes a) and b). With these limitations in mind, we can now proceed to describe the radiation from the point of view of the equations of motion.

Our starting point will be the equations for the geodesic path, which describe the motion of a mass in a gravitational field. In the following we will assume that gravity is the only field affecting the motion. If the position in space-time of the mass is denoted by x^σ, then x^σ is given by

$$g_{\mu\sigma} x''^{\sigma} = - [\alpha\beta,\mu] x'^{\alpha} x'^{\beta} \quad , \qquad (4.51)$$

where $' = \frac{d}{ds}$, and $g_{\mu\nu} x'^{\mu} x'^{\nu} = 1$. If we change from s to u as the independent variable, then we get

$$g_{\mu\sigma}\ddot{x}^{\sigma} = -[\alpha\beta,\mu]\,\dot{x}^{\alpha}\,\dot{x}^{\beta} - g_{\mu\sigma}\,\dot{x}^{\sigma}\frac{d}{ds}\ln\frac{du}{ds}\,, \qquad (4.52)$$

where $\cdot = \frac{d}{du}$. The same equation with $\mu \rightarrow \nu$ is

$$g_{\nu\sigma}\ddot{x}^{\sigma} = -[\alpha\beta,\nu]\,\dot{x}^{\alpha}\,\dot{x}^{\beta} - g_{\nu\sigma}\,\dot{x}^{\sigma}\frac{d}{ds}\ln\frac{du}{ds}\;.$$

If we multiply the first equation by $g_{\nu\sigma}\,\dot{x}^{\sigma}$ and the second equation by $-\,g_{\mu\sigma}\dot{x}^{\sigma}$ and add, we get

$$\begin{aligned} g_{\mu\sigma}\ddot{x}^{\sigma}\,g_{\nu\beta}\dot{x}^{\beta} - g_{\mu\sigma}\dot{x}^{\sigma}\,g_{\nu\beta}\ddot{x}^{\beta} &= \\ &= -\,g_{\nu\sigma}[\alpha\beta,\mu]\,\dot{x}^{\alpha}\,\dot{x}^{\beta}\,\dot{x}^{\sigma} + g_{\mu\sigma}[\alpha\beta,\nu]\,\dot{x}^{\alpha}\,\dot{x}^{\beta}\,\dot{x}^{\sigma}. \qquad (4.53)\end{aligned}$$

Since u can be chosen to be anything we wish, we take it to be the time t. Then denoting differentiation with respect to t by $\cdot$, we have that $\dot{t} = 1$ and $\ddot{t} = 0$. In equation 4.53 let $\mu = i$ and $\nu = 4$. Then

$$\begin{aligned} g_{i\sigma}\ddot{x}^{\sigma}\,g_{4\beta}\dot{x}^{\beta} - g_{4\sigma}\ddot{x}^{\sigma}\,g_{i\beta}\dot{x}^{\beta} &= \\ &= -\,g_{4\sigma}[\alpha\beta,i]\,\dot{x}^{\alpha}\dot{x}^{\beta}\dot{x}^{\sigma} + g_{i\sigma}[\alpha\beta,4]\,\dot{x}^{\alpha}\dot{x}^{\beta}\dot{x}^{\sigma}\;.\end{aligned}$$

To lowest order in $h_{\alpha\beta}$, we thus get that

$$\delta_{i\sigma}\ddot{x}^{\sigma} = -\,[\alpha\beta,i]\,\dot{x}^{\alpha}\,\dot{x}^{\beta} + \delta_{i\sigma}[\alpha\beta,4]\,\dot{x}^{\alpha}\,\dot{x}^{\beta}\,\dot{x}^{\sigma}\;.$$

An examination of the last term shows that it is of order $(v/c)^2$ smaller than the first, and we will neglect it. Therefore

$$\ddot{x}^{i} = [\alpha\beta,i]\;\dot{x}^{\alpha}\,\dot{x}^{\beta} = \left[h_{\alpha i,\beta} - \tfrac{1}{2}\,h_{\alpha\beta,i}\right]\dot{x}^{\alpha}\,\dot{x}^{\beta}\,,$$

so that we may define the "force" due to gravitational interaction to lowest order in $h_{\alpha\beta}$ in the N.R. limit as

$$f_i = m \left[h_{\alpha i,\beta} - \tfrac{1}{2} h_{\alpha\beta,i} \right] v^\alpha v^\beta \quad , \qquad (4.54)$$

where $v^\alpha = \dot{x}^\alpha$. This is the analogue of the Lorentz force on a charge.

The calculation of the radiation reaction force of a charge on itself in the limit of slowly varying parameters is well-known in electromagnetism. The result is that

$$f_i = -\frac{4}{3} W_o \dot{v}_i + \frac{2}{3} e^2 \ddot{v}_i + O(a) \quad , \qquad (4.55)$$

where W_o is the self energy of the charge, and a is the approximate radius of the charge distribution. The result is most easily found by calculating the force in a system in which the charge is momentarily at rest, noting that the result is independent of the coordinate system (for small velocities). The term O(a) will be small for small radii and can be neglected. The term $-\frac{4}{3} W_o v_i$ is usually taken to the left side of the equation and included in the $m \dot{v}_i$. The power loss is then $f_i v_i$ and its time integral is

$$\int f_i v_i \, dt = \frac{2}{3} e^2 \int \ddot{v}_i v_i \, dt = -\frac{2}{3} e^2 \int \dot{v}_i \dot{v}_i \, dt \quad ,$$

so that we get the familiar equation for the power radiated by a charge.

The analogous calculation can be done in gravity using equation 4.54. In a system in which the mass is at rest, the force is

$$f_i = m\left[h_{4i,4} - \tfrac{1}{2} h_{44,i}\right]$$
$$= m\left[\bar{h}_{4i,4} - \tfrac{1}{2} \bar{h}_{44,i} + \tfrac{1}{4} \bar{h}_{\sigma\sigma,i}\right] .$$

During this calculation the effect of the stresses will be ignored in the computation of the $h_{\alpha\beta}$ and only the terms arising from the masses themselves will be considered. Thus instead of the charge density ρ_c and $\rho_c v_i$ as sources in electromagnetism, we now have the stress-energy density as the sources. The sources are therefore taken to be

$$T_{44} = \frac{\rho}{\sqrt{1 - v^2}} \ ; \quad T_{4i} = - \frac{\rho v_i}{\sqrt{1 - v^2}} \ ; \ T_{ij} = \frac{\rho v_i v_j}{\sqrt{1-v^2}} ; \quad (4.56)$$

From these we obtain the potentials (analogous to the Lienard-Wiechert potentials of electromagnetism)

$$\bar{h}_{44} = - \frac{4Gm}{\sqrt{1 - v^2}\ s}$$
$$\bar{h}_{4i} = + \frac{4Gm\ v_i}{\sqrt{1 - v^2}\ s}$$
$$\bar{h}_{ij} = - \frac{4Gm\ v_i v_j}{\sqrt{1 - v^2}\ s}$$
$$\bar{h}_{\sigma\sigma} = - \frac{4Gm\sqrt{1 - v^2}}{s} ,$$

where

$$s = |\underset{\sim}{r} - \underset{\sim}{r}_p| - (\underset{\sim}{r} - \underset{\sim}{r}_p)\cdot\underset{\sim}{v}(t')$$

and

$$\underset{\sim}{r}_p = \underset{\sim}{r}_p(t') \quad .$$

The self force of a spherically symmetric mass distribution is then found in the same manner as for the corresponding problem in electromagnetism[9]. The details are more lengthy, but the steps are the same. This calculation gives the following terms:

$$\bar{h}_{4i,4} \longrightarrow 8\, W_o\, \dot{v}_i - 4\, G\, m^2\, \ddot{v}_i + O(a)$$

$$-\tfrac{1}{2}\, \bar{h}_{44,i} \longrightarrow -\frac{4}{3}\, W_o\, \dot{v}_i + \frac{2}{3}\, G\, m^2\, \ddot{v}_i + O(a)$$

$$\tfrac{1}{4}\, \bar{h}_{\sigma\sigma,i} \longrightarrow \frac{2}{3}\, W_o\, \dot{v}_i - \frac{1}{3}\, G\, m^2\, \ddot{v}_i + O(a) \quad ,$$

where $W_o = G \iint \frac{dm\ dm'}{r}$ is the "self energy of the mass" and a is the "radius" of the mass distribution. The total reaction force is then

$$f_i = \frac{22}{3}\, W_o\, \dot{v}_i - \frac{11}{3}\, G\, m^2\, \ddot{v}_i + O(a). \quad (4.57)$$

The self energy term can be treated as before and the $O(a)$ term neglected. We would be tempted to say that $f_i\, v_i$ is the power radiated by the mass. Over a time

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[9] For a discussion of the electromagnetic case, see W. K. H. Panofsky and M. Phillips, Classical Electricity and Magnetism (Addison-Wesley Pub. Co., Inc., Reading, Mass., 1956), Chap. 20.

average this gives

$$\int f_i\ v_i dt = -\frac{11}{3}\ Gm^2\int \ddot{v}_i\ v_i dt = +\frac{11}{3}\ Gm^2\int \dot{v}_i\ \dot{v}_i dt \quad ,$$

which implies that the mass is gaining or absorbing energy, not losing it as we would expect. In addition, this term is of order $(v/c)^3$, which is $(c/v)^2$ greater than the contribution we had found from the previous analysis. The answer to these difficulties lies in the fact that the effects of the other masses of the system have not been included in the calculation of the force. If we find the force on mass a due to all of the other masses to this same order, we get

$$f_i(\text{on a}) \ = -\frac{11}{3}\ G \sum_{k\neq a} m_a\ m_k\ \ddot{v}_{ki} \quad ,$$

so that the total radiation reaction force, to order $(v/c)^3$, is

$$f_{ia} \ = \sum_k\ m_a\ m_k\ \ddot{v}_{ki} = m_a\ \frac{d}{dt} \sum_k\ m_k\ \dot{v}_{ki} \ = 0 \quad ,$$

where k runs over all of the masses, and where the system is assumed to have no external force acting on it. A similar cancellation occurs sometimes in electromagnetism. If a localized non-relativistic closed system contains only particles which have the same charge to mass ratio, then the radiation in the dipole approximation vanishes (order $(v/c)^3$), and we must go to the quadrupole approximation (order

$(v/c)^5$) in order to find an effect. In gravitation, since all masses have the same mass to mass ratio, the lowest possible order of the radiation is the quadrupole approximation. This can be seen explicitly also in the case where the contributions of the stresses are included in the solution for $h_{\alpha\beta}$.

Still working with equation 4.54 with the mass at rest, we write the solution for $h_{\alpha\beta}$,

$$h_{\alpha\beta} = -4G\int\left[\frac{S'_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta}\,S'_{\sigma\sigma}}{r}\right] dV' \quad .$$

The expansion of the retarded brackets can be carried out and the solution used in equation 4.54. This gives

$$\begin{aligned} f_i = & -4Gm\left\{ \frac{\partial}{\partial t}\int\frac{S'_{4i}}{r}\,dV' - \frac{\partial^2}{\partial t^2}\int S'_{4i}\,dV' + \right. \\ & \left. + \frac{1}{2}\frac{\partial^3}{\partial t^3}\int rS'_{4i}\,dV' - \frac{1}{6}\frac{\partial^4}{\partial t^4}\int r^2\,S'_{4i}\,dV'\right\} + \\ & + 2Gm\left\{\frac{\partial}{\partial x_i}\int\frac{S'_{44} - \frac{1}{2}S'_{\sigma\sigma}}{r}\,dV' - \frac{\partial}{\partial x_i}\frac{\partial}{\partial t}\int\frac{S'_{44} - \frac{1}{2}S'_{\sigma\sigma}}{r}\,dV' + \right. \\ & + \frac{1}{2}\frac{\partial}{\partial x_i}\frac{\partial^2}{\partial t^2}\int r(S'_{44} - \tfrac{1}{2}S'_{\sigma\sigma})\,dV' - \frac{1}{6}\frac{\partial}{\partial x_i}\frac{\partial^3}{\partial t^3}\int r^2(S'_{44} - \tfrac{1}{2}S'_{\sigma\sigma})\,dV' + \\ & \left. + \frac{1}{24}\frac{\partial}{\partial x_i}\frac{\partial^4}{\partial t^4}\int r^3(S'_{44} - \tfrac{1}{2}S'_{\sigma\sigma})\,dV' - \frac{1}{120}\frac{\partial}{\partial x_i}\frac{\partial^5}{\partial t^5}\int r^4(S'_{44} - \tfrac{1}{2}S'_{\sigma\sigma})\,dV'\right\}. \end{aligned}$$

Terms with an odd power of r can be eliminated by a trick. If we had used the advanced solution for the $h_{\alpha\beta}$ in this expression, we would have changed the sign of each term with an even power of r. One would expect a radiation with the opposite sign since, to this order, the $S_{\alpha\beta}$ does not depend on whether the retarded or advanced solutions are taken. Thus to get the radiation terms, we can take the quantity

$$\tfrac{1}{2}(h^{ret}_{\alpha\beta} - h^{adv}_{\alpha\beta})$$

as the potential.

The terms of order (v/c) and $(v/c)^3$ vanish. For the term of order (v/c), we have zero because the quantity does not depend on r, and we are taking $\frac{\partial}{\partial x_i}$. For the terms of order $(v/c)^3$, we have

$$\begin{aligned}\ddot{x}_i &= 4\,G\int \ddot{S}'_{i4}\,dV' - \frac{G}{3}\int \dddot{S}'_{44}(x_i - x_i')dV' \\ &= 4\,G\int \dot{S}'_{ij,j}dV' - \frac{G}{3}\int \dot{S}'_{km,km}(x_i - x_i')dV' = 0\end{aligned}$$

by integrating by parts. If only one mass is considered, then since $\int S'_{4i}\,dV' = -mv_i$ to this order, $m\ddot{x}_i$ is given by

$$m\,\ddot{x}_i = -\,4Gm^2\,\ddot{v}_i + \frac{1}{3}\,Gm^2\,\ddot{v}_i = -\,\frac{11}{3}\,Gm^2\,\ddot{v}_i$$

which was found before. Then the cancellation is seen to occur because $\frac{\partial}{\partial t}\int S'_{4i}\,dV' = 0$, or because the total momentum of the system remains constant.

The fifth order terms will, in general, give a non-vanishing reaction force. These terms may arise from the expansion of the third order terms to order $(v/c)^2$ or from the lowest order part of the fifth order terms. In evaluating the integrals, we use the fact that for mass k,

$$T_{44} \cong m_k \, \delta(r - r_k(t))$$
$$T_{4i} \cong - m_k \, v_{ki} \, \delta(r - r_k(t)) \quad .$$

In a system of two masses, the force on mass 1 will then be

$$f_{1i} = - \frac{2}{3} Gm_1 m_2 \, \overset{....}{(r^2 v_i)} - \frac{Gm_1 m_2}{30} \, \overset{.....}{(r^2 x_i)} - - \frac{Gm_1 m_2}{3} \, \overset{...}{(x_i v^2)} - \frac{Gm_1}{3} \overset{...}{\left[\int x_i \, X_{kk} \, dV \right]} , \qquad (4.58)$$

where the first two terms come from the fifth order terms and the last two are from the expansion of the third order terms. x_i denotes $(x_{pi} - x_{2i}(t))$, where x_{pi} denotes the observation point which becomes x_{1i} in the final evaluation of the term. Since we are in a system of coordinates in which mass 1 is momentarily at rest, we get that $v_i = v_{1i} - v_{2i}(t)$, where v_{1i} is assumed to be constant throughout the indicated time differentiations. The terms with m_1 acting on m_1 vanish in this special coordinate system.

In the case where the two masses are moving circularly around each other, the last term vanishes. Then we get the following contributions to f_{1i}:

$$-\frac{2}{3}\,\overset{....}{(r^2v_i)} = \left\{\frac{2}{3}\,v_i\left[-8\underset{\sim}{v}\cdot\ddot{\underset{\sim}{v}}_2 + 6\dot{\underset{\sim}{v}}_2\cdot\dot{\underset{\sim}{v}}_2 - 2\underset{\sim}{r}\cdot\dddot{\underset{\sim}{v}}_2\right] - \right.$$

$$\left. - 4\,\ddot{v}_{2i}\left[2v^2 - 2\underset{\sim}{r}\cdot\dot{\underset{\sim}{v}}_2\right] - \frac{2}{3}\,\ddddot{v}_{2i}r^2\right\}$$

$$-\frac{1}{3}\,\overset{...}{(x_iv^2)} = \left\{\frac{1}{3}\,\ddot{v}_{2i}v^2 - 2v_i\left[\dot{\underset{\sim}{v}}_2\cdot\dot{\underset{\sim}{v}}_2 - \underset{\sim}{v}\cdot\ddot{\underset{\sim}{v}}_2\right]\right\}$$

$$-\frac{1}{30}\overset{.....}{(r^2x_i)} = \left\{-\frac{1}{6}\,v_i\left[-8\underset{\sim}{v}\cdot\ddot{\underset{\sim}{v}}_2 + 6\dot{\underset{\sim}{v}}_2\cdot\dot{\underset{\sim}{v}}_2 - 2\underset{\sim}{r}\cdot\dddot{\underset{\sim}{v}}_2\right] + \right.$$

$$\left. + \frac{1}{3}\,\ddot{v}_{2i}\left[2v^2 - 2\underset{\sim}{r}\cdot\dot{\underset{\sim}{v}}_2\right] + \frac{1}{30}\,\ddddot{v}_{2i}r^2\right\}$$

$$\overset{...}{\left[\int x_i\;X_{kk}\;dV\right]} = 0 \quad ,$$

where $v_i = v_{1i} - v_{2i}$. Thus

$$f_{1i} = Gm_1m_2\left\{v_i\left[\dot{\underset{\sim}{v}}_2\cdot\dot{\underset{\sim}{v}}_2 - 2\underset{\sim}{v}\cdot\ddot{\underset{\sim}{v}}_2 - \underset{\sim}{r}\cdot\dddot{\underset{\sim}{v}}_2\right] + \right.$$

$$\left. + \ddot{v}_{2i}\left[-7v^2 + \frac{22}{3}\,\underset{\sim}{r}\cdot\dot{\underset{\sim}{v}}_2\right] - \frac{19}{30}\,\ddddot{v}_{2i}r^2\right\}.$$

If the frequency of revolution is ω, then $\ddot{v}_i = -\omega^2 v_i$, etc. Evaluating $f_1\cdot v_1$ then gives

$$f_1\cdot v_1 = -\left[\frac{221}{15} - \frac{50}{3}\,\frac{m_1}{m_1+m_2}\right]\,\frac{G\;I^2\,\omega^6}{2}$$

Then interchanging 1 and 2 and adding to find the total radiation of the system gives

$$\frac{dE}{dt} = -\left[\frac{442}{15} - \frac{50}{3}\right]\tfrac{1}{2}G\;I^2\,\omega^6 = -\,\frac{32}{5}\,G\;I^2\,\omega^6 \qquad (4.59)$$

which is the same as is found from the analysis given in section IV.C. for a rotating system.

The same calculation can be made in the more general case of elliptical motion, although the work is considerably longer. Both the angular momentum radiation and the energy radiation can be found from the force, but they do not agree with that found by the previous methods. One finds an expression for the energy radiated of

$$E = E_{circle} \frac{\left[1 + \frac{1}{192}(684\ e^2 + 99\ e^4)\right]}{(1 - e^2)^{7/2}}$$

whereas the correct energy radiation should be (see section V.A.)

$$E = E_{circle} \frac{\left[1 + \frac{1}{192}(584\ e^2 + 74\ e^4)\right]}{(1 - e^2)^{7/2}} \quad .$$

The difference is, of course, only in the terms proportional to the eccentricity.

We can now show explicitly why this method works for circular motion and fails otherwise. The force equation is

$$m\, \ddot{x}_i = \left[h_{\alpha i, \beta} - \tfrac{1}{2}\, h_{\alpha\beta, i}\right] m\, v^{\alpha} v^{\beta}$$

so that

$$m\, \ddot{x}_i\, \dot{x}_i = \left[h_{\alpha i, \beta} - \tfrac{1}{2}\, h_{\alpha\beta, i}\right] m\, v^{\alpha} v^{\beta} v^i =$$

$$= m\, v^i \left[h_{4i,4} - \tfrac{1}{2}\, h_{44,i}\right] + m\, v^i v^j \left[h_{4i,j} + h_{ij,4} - h_{4j,i}\right] + m\, v^i v^j v^k \left[h_{ij,k} - \tfrac{1}{2}\, h_{jk,i}\right] \quad .$$

The last term is of higher order in v/c and can be neglected. We can then let

$$m\ v^i \longrightarrow \int - T_{4i}\ dV \cong - \int S_{4i}\ dV$$
$$m\ v^i v^j \longrightarrow \int T_{ij}\ dV \cong \int S_{ij} - X_{ij}\ dV \quad .$$

Taking time integrals for the secular change in energy yields

$$\int \frac{dE}{dt}\,dt = m\int \ddot{x}_i \dot{x}_i dt = -\iint \left[h_{4i},_4 - \tfrac{1}{2}\, h_{44},_i\right] S_{4i}\ dV\ dt +$$
$$+ \iint h_{ij},_4\ (S_{ij} - X_{ij})\ dV\ dt \ ,$$

which reduces to

$$\int \frac{dE}{dt}\,dt = \int \left[\tfrac{1}{2}\int h_{44},_4\ S_{44}\ dV - \int h_{4i},_4\ S_{4i}\ dV + \right.$$
$$\left. + \int h_{ij},_4\ (S_{ij} - X_{ij})\ dV\right] dt \quad .$$

Comparison with equation 4.25 shows that we now have an additional term

$$\int h_{ij},_4\ (\tfrac{1}{2} S_{ij} - X_{ij})\ dV$$

It is easy to show that for circular motion

$$\frac{d}{dt}\int X_{ij}\ dV = \tfrac{1}{2}\,\frac{d}{dt}\int S_{ij}\ dV$$

so, in that case, the extra term vanishes. In general it does not cancel out and thus the radiation cannot be found correctly by this method.

The force we have defined has a fourth component given by

$$f_4 = \left[h_{\alpha 4,\beta} - \tfrac{1}{2} h_{\alpha\beta,4}\right] m\, v^{\alpha} v^{\beta} \quad .$$

If the force f_μ is a proper four-vector, then we would expect that f_4 also gives the negative of the power radiated. It can be shown that, over a time average,

$$f_4 = -\tfrac{1}{2}\int h_{\alpha\beta,4}\, S_{\alpha\beta}\, dV - \int (h_{i4,j} - \tfrac{1}{2} h_{ij,4}) X_{ij}\, dV \quad ,$$

so that $f_4 \neq \underset{\sim}{f}\cdot\underset{\sim}{v}$. Neither expression is equal to the correct power radiated in general.

The results of this section are essentially of a negative nature. No matter how one defines a force, it will have terms proportional only to T_{44}, T_{4i}, and T_{ij}. For the 44 and 4i components, we can let $T_{44} = S_{44}$ and $T_{4i} = S_{4i}$, but for the ij components, the stress contribution X_{ij} is of the same order of magnitude as the matter contribution T_{ij}. Since X_{ij} is not localized at the masses, it will never appear explicitly in the force law, and therefore its contribution to the energy radiation will not be included in this type of calculation.

V. THE MOTION OF TWO POINT MASSES

A. Energy Radiation

Making use of the formalism developed in section IV., we can now find the energy and angular momentum radiation of a system of two point masses moving in Keplerian orbits. This has been carried out, and the energy radiation has been reported in a paper published in the July 1, 1963 issue of the Physical Review. Since essentially all of the features of the energy radiation of two bound point masses are included in the paper, this section will consist solely of a reprint of the article. Equations and figures in this chapter will begin with those given in the paper and continue consecutively throughout the remaining part of the chapter.

Reprinted from THE PHYSICAL REVIEW, Vol. 131, No. 1, 435–440, 1 July 1963
Printed in U. S. A.

Gravitational Radiation from Point Masses in a Keplerian Orbit

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(Received 18 January 1963)

The gravitational radiation from two point masses going around each other under their mutual gravitational influence is calculated. Two different methods are outlined; one involves a multipole expansion of the radiation field, while the other uses the inertia tensor of the source. The calculations apply for arbitrary eccentricity of the relative orbit, but assume orbital velocities are small. The total rate, angular distribution, and polarization of the radiated energy are discussed.

I. INTRODUCTION

THE linearized version of Einstein's general theory of relativity is strikingly similar to classical electromagnetism. In particular, one might expect masses in arbitrary motion to radiate gravitational energy. The question has been raised,[1] however, whether the energy so calculated has any physical meaning. We shall not concern ourselves with this question here; we shall take the point of view that the analogy with electromagnetic theory is a correct one, and energy is actually radiated.

In Sec. II we outline briefly two methods which can be used to calculate rates of emission of gravitational energy from a system of masses on which no net external force acts. Only enough details are presented to enable them to be applied to other problems; derivations and proofs are omitted. In Sec. III these methods are applied to obtain the total rate of radiation by two point masses going around each other in the familiar Kepler ellipse. In Sec. IV we discuss the angular distribution and polarization of the radiation.

II. GENERAL METHODS

A. Inertia Tensor

If one linearizes the equations of general relativity, setting[2–4]

$$g_{\mu\nu}=\delta_{\mu\nu}+\kappa h_{\mu\nu}, \quad (|\kappa h_{\mu\nu}|\ll 1),$$

with $\kappa^2=32\pi G$, one obtains

$$\Box \bar{h}_{\mu\nu}=-\tfrac{1}{2}\kappa T_{\mu\nu}, \tag{1}$$

where

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}h_{\lambda\lambda},$$

and $T_{\mu\nu}$ is the total stress-momentum-energy tensor of the source, including the gravitational field stresses. The energy density in a plane wave

$$\bar{h}_{\mu\nu}=h_{\mu\nu}=ae_{\mu\nu}\cos(\omega t-\mathbf{k}\cdot\mathbf{x})$$

is

$$U=\tfrac{1}{2}c^2\omega^2a^2 \tag{2}$$

provided $e_{\mu\nu}$ is a unit polarization tensor, obeying the conditions

$$e_{\mu\nu}=e_{\nu\mu}, \quad e_{\mu\mu}=0, \quad k_\mu e_{\mu\nu}=0, \quad e_{\mu\nu}e_{\mu\nu}=1.$$

Just as in electromagnetic theory, we can work in a gauge in which $e_{\mu\nu}$ is spacelike and transverse; thus, a wave traveling in the z direction has two independent polarizations possible:

$$e_1=\frac{1}{\sqrt{2}}(\hat{x}\hat{x}-\hat{y}\hat{y}) \quad e_2=\frac{1}{\sqrt{2}}(\hat{x}\hat{y}+\hat{y}\hat{x}).$$

One can now solve (1) for the radiation from a system of masses undergoing arbitrary motions, and use (2) to obtain the power radiated. The result,[2] assuming source dimensions are small compared with the wavelength ("quadrupole approximation"), is that the power $dP/d\Omega$ radiated into solid angle Ω with polarization e_{ij} is

$$\frac{dP}{d\Omega}=\frac{G}{8\pi c^5}\left(\frac{d^3Q_{ij}}{dt^3}e_{ij}\right)^2, \tag{3}$$

where Q_{ij} is the tensor

$$Q_{ij}=\sum_\alpha m_\alpha x_{\alpha i}x_{\alpha j}, \tag{4}$$

the sum running over all masses m_α in our system. It is to be noted that the result is independent of the kind of stresses present.

If one sums (3) over the two allowed polarizations, one obtains

$$\sum_{\text{pol}}\frac{dP}{d\Omega}=\frac{G}{8\pi c^5}\left[\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}-2n_i\frac{d^3Q_{ij}}{dt^3}n_k\frac{d^3Q_{kj}}{dt^3}-\frac{1}{2}\left(\frac{d^3Q_{ii}}{dt^3}\right)^2+\frac{1}{2}\left(n_in_j\frac{d^3Q_{ij}}{dt^3}\right)^2+\frac{d^3Q_{ii}}{dt^3}n_jn_k\frac{d^3Q_{jk}}{dt^3}\right], \tag{5}$$

where $\hat{n}$ is the unit vector in the direction of radiation. The total rate of radiation is obtained by integrating

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[1] See, for example, L. Infeld and J. Plebanski, *Motion and Relativity* (Pergamon Press Inc., New York, 1960).
[2] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Chap. 11.
[3] R. P. Feynman, lectures, California Institute of Technology (unpublished).
[4] Greek letters run from 1 to 4; $a_\mu b_\mu=a_4b_4-\mathbf{a}\cdot\mathbf{b}$. Roman letters run from 1 to 3; $a_ib_i=\mathbf{a}\cdot\mathbf{b}$. The Kronecker delta $\delta_{\mu\nu}$ is +1 for $\mu=\nu=4$, −1 for $\mu=\nu=1,2,3$. The d'Alembertian operator is $\Box=\nabla_\mu\nabla_\mu=\partial^2/\partial t^2-\nabla^2$. The phase of a plane wave is $k_\mu x_\mu=\omega t-\mathbf{k}\cdot\mathbf{x}$. G is the usual gravitational constant $\approx 6.67\times10^{-8}$ cgs units.

(5) over all directions of emission; the result is

$$P=\frac{G}{5c^5}\left(\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}-\frac{1}{3}\frac{d^3Q_{ii}}{dt^3}\frac{d^3Q_{jj}}{dt^3}\right). \tag{6}$$

B. Multipole Expansion

The radiation $h_{\mu\nu}(\mathbf{x})$ can be decomposed into multipoles,[5] each with a definite total angular momentum (J) and z component of angular momentum (M). For a given J and M, there are two independent types of radiation, distinguished by their parity; we call them "electric" and "magnetic" to emphasize the analogy with electromagnetic theory.

We analyze the source and field into Fourier components, and treat each separately. If the source is

$$T_{\mu\nu}=\mathrm{Re}\,\hat{T}_{\mu\nu}e^{-i\omega t},$$

then the amplitudes of the electric and magnetic multipole radiation are

$$e_{JM}=-\frac{i\kappa\omega}{2}\int d^3x f_{JM}{}^{e}(\mathbf{x}):\hat{\mathrm{T}}(\mathbf{x}), \tag{7}$$

$$m_{JM}=-\frac{i\kappa\omega}{2}\int d^3x f_{JM}{}^{m}(\mathbf{x}):\hat{\mathrm{T}}(\mathbf{x}), \tag{8}$$

where $A:B$ means $A_{ij}B_{ij}$, and the $f_{JM}{}^{e,m}$ are given in reference 5. In the quadrupole approximation, the dominant type of radiation is "magnetic quadrupole"; in this limit, (8) with $J=2$ becomes

$$m_{2M}=\frac{i\kappa\omega^3}{10\sqrt{3}}\int d^3x\, r^2Y_{2M}(\Omega)\hat{\rho}(\mathbf{x}), \tag{9}$$

where

$$\rho=\mathrm{Re}\,\hat{\rho}e^{-i\omega t}$$

is the mass density in the source.

The total power radiated is given in terms of the multipole amplitudes (7), (8) by

$$P=\tfrac{1}{2}\textstyle\sum_{JM}[|e_{JM}|^2+|m_{JM}|^2]. \tag{10}$$

III. TOTAL RADIATION

Let the masses m_1 and m_2 have coordinates ($d_1\cos\psi, d_1\sin\psi$) and ($-d_2\cos\psi$, $-d_2\sin\psi$) in the xy plane, as in Fig. 1. The origin will be taken to be the center of mass, so that

$$d_1=\left(\frac{m_2}{m_1+m_2}\right)d,\quad d_2=\left(\frac{m_1}{m_1+m_2}\right)d.$$

The simplest way to compute the power radiated is to use the method of Sec. II A, above. The nonvanishing

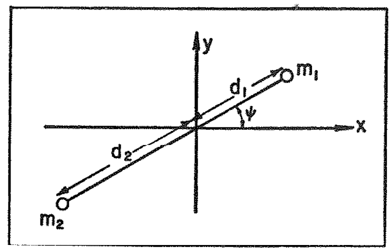

FIG. 1. Coordinate system used in calculation.

Q_{ij} are

$$Q_{xx}=\mu d^2\cos^2\psi,$$
$$Q_{yy}=\mu d^2\sin^2\psi,$$
$$Q_{xy}=Q_{yx}=\mu d^2\sin\psi\cos\psi,$$

where μ is the reduced mass $m_1m_2/(m_1+m_2)$.

For Kepler motion, the orbit equation is[6]

$$d=\frac{a(1-e^2)}{1+e\cos\psi}, \tag{12}$$

while the angular velocity is given by

$$\dot{\psi}=\frac{[G(m_1+m_2)a(1-e^2)]^{1/2}}{d^2}. \tag{13}$$

Using (12) and (13), it is straightforward to calculate the d^3Q_{ij}/dt^3; the results are

$$\begin{aligned}\frac{d^3Q_{xx}}{dt^3}&=\beta(1+e\cos\psi)^2(2\sin 2\psi+3e\sin\psi\cos^2\psi),\\ \frac{d^3Q_{yy}}{dt^3}&=-\beta(1+e\cos\psi)^2\\ &\quad\times[2\sin 2\psi+e\sin\psi(1+3\cos^2\psi)],\\ \frac{d^3Q_{xy}}{dt^3}=\frac{d^3Q_{yx}}{dt^3}&=-\beta(1+e\cos\psi)^2\\ &\quad\times[2\cos 2\psi-e\cos\psi(1-3\cos^2\psi)],\end{aligned} \tag{14}$$

where β is defined by

$$\beta^2=\frac{4G^3m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^5}.$$

The total power radiated is now given by (6);

$$P=\frac{8}{15}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^5}(1+e\cos\psi)^4\times[12(1+e\cos\psi)^2+e^2\sin^2\psi]. \tag{15}$$

[5] J. Mathews, J. Soc. Ind. Appl. Math. 10, 768 (1962). This expansion into multipoles is not to be confused with general multipole expansions usually given. See, for example, *Gravitation, an Introduction to Current Research*, edited by Louis Witten (John Wiley & Sons, Inc., New York, 1962), Chaps. 5 and 6.

[6] a is the semimajor axis and e the eccentricity of our ellipse. Note that we have chosen the x axis to be the direction of m_1 at its closest approach to m_2 (periastron).

In (15), ψ is, of course, the *retarded* position of the system. The *average* rate at which the system radiates energy is obtained by averaging (15) over one period of the elliptical motion; one obtains in this way

$$\langle P\rangle=\frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \quad (16)$$

Thus, the average power equals the power radiated from a circular orbit of equal semimajor axis (or total energy) times an enhancement factor

$$f(e)=\frac{1+(73/24)e^2+(37/96)e^4}{(1-e^2)^{7/2}}. \quad (17)$$

Figure 2 shows $f(e)$ plotted against e. Note that $f(0.6)\sim10$, $f(0.8)\sim10^2$, $f(0.9)\sim10^3$. The power radiated is a steeply rising function of the eccentricity e.

The same result follows from the method of Sec. II B, but the formalism is rather different. We must evaluate the m_{2M} of Eq. (9). In terms of the Q_{ij} defined by (4),

$$m_{2\pm2}=\frac{i\kappa\omega^3}{10\sqrt{3}}\left(\frac{15}{32\pi}\right)^{1/2}(Q_{xx}-Q_{yy}\pm2iQ_{xy}),$$

$$m_{2\pm1}=0,$$

$$m_{20}=\frac{-i\kappa\omega^3}{10\sqrt{3}}\left(\frac{5}{16\pi}\right)^{1/2}(Q_{xx}+Q_{yy}).$$

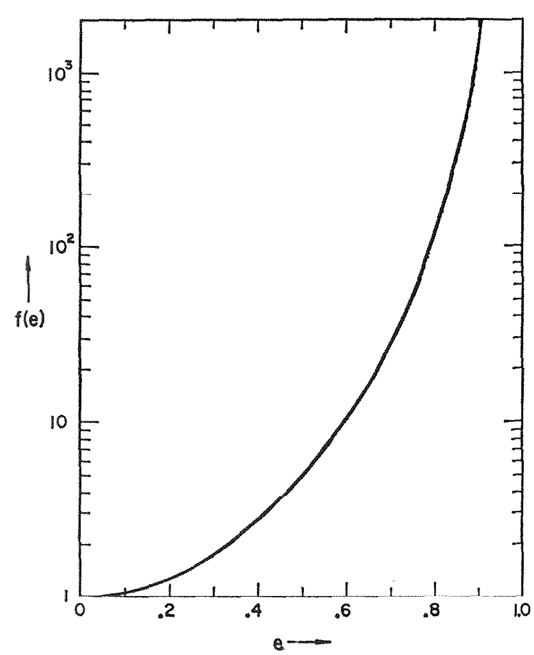

FIG. 2. "Enhancement factor" $f(e)$ plotted against e.

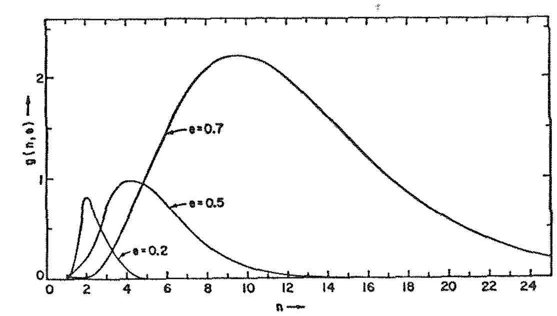

FIG. 3. $g(n,e)$, the relative power radiated into the nth harmonic for $e=0.2$, 0.5, and 0.7.

The Fourier analysis of Kepler motion is well known (to astronomers at least!), so we simply give the results. The components of frequency $n\omega_0$, where $\omega_0=[G(m_1+m_2)/a^3]^{1/2}$ is the average angular velocity, are

$$\begin{aligned}m_{2\pm2}(n)=&\frac{i\kappa\omega^3}{10\sqrt{3}}\left(\frac{15}{32\pi}\right)^{1/2}\mu a^2\frac{2}{n}\\&\times\Big\{J_{n-2}(ne)-2eJ_{n-1}(ne)+\frac{2}{n}J_n(ne)\\&+2eJ_{n+1}(ne)-J_{n+2}(ne)\\&\mp(1-e^2)^{1/2}[J_{n-2}(ne)-2J_n(ne)+J_{n+2}(ne)]\Big\},\\m_{20}(n)=&\frac{i\kappa\omega^3}{10\sqrt{3}}\left(\frac{5}{16\pi}\right)^{1/2}\mu a^2\frac{4}{n^2}J_n(ne).\end{aligned} \quad (18)$$

The power radiated in the nth harmonic is, from (10) and (18),

$$P(n)=\frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}g(n,e), \quad (19)$$

where

$$\begin{aligned}g(n,e)=\frac{n^4}{32}\Big\{&\Big[J_{n-2}(ne)-2eJ_{n-1}(ne)\\&+\frac{2}{n}J_n(ne)+2eJ_{n+1}(ne)-J_{n+2}(ne)\Big]^2\\&+(1-e^2)[J_{n-2}(ne)-2J_n(ne)+J_{n+2}(ne)]^2\\&+\frac{4}{3n^2}[J_n(ne)]^2\Big\}.\end{aligned} \quad (20)$$

In Fig. 3, we plot $g(n,e)$ against n for $e=0.2$, 0.5, and 0.7.

If (16) and (19) are to agree, we must have

$$\sum_{n=1}^{\infty}g(n,e)=f(e)=\frac{1+(73/24)e^2+(37/96)e^4}{(1-e^2)^{7/2}}.$$

This is verified in the Appendix.

That the radiation should depend so strongly on the eccentricity is not surprising. As with electrogmagnetic radiation, the power radiated increases for increasing accelerations. Thus, the bodies will radiate most at their closest approach, and for fixed energy the higher the eccentricity, the higher the power radiated will be. This also explains why the higher harmonics dominate the radiation for e near 1; Fourier components of large n must be present to give such a peaking of the radiation at one part of the path.

IV. ANGULAR DISTRIBUTIONS AND POLARIZATIONS

In this section we only use the method of Sec. II A, as it gives the answers directly without the need of summing over all harmonics.

Let us label the two polarizations

$$e_1=\frac{1}{\sqrt{2}}(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}),\quad e_2=\frac{1}{\sqrt{2}}(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}),\tag{21}$$

where θ and ϕ are conventional polar coordinates. We shall abbreviate the d^3Q_{ij}/dt^3 of (14) by A, B, C:

$$\frac{d^3Q_{xx}}{dt^3}=A,\quad \frac{d^3Q_{yy}}{dt^3}=B,\quad \frac{d^3Q_{xy}}{dt^3}=\frac{d^3Q_{yx}}{dt^3}=C.\tag{22}$$

The power radiated into polarization 1 is obtained by substituting (21) and (22) into (3); we omit the algebra and quote the result:

$$\begin{aligned}\frac{dP_1}{d\Omega}=\frac{G}{8\pi c^5}\{&\tfrac{1}{16}(3A^2+2AB+3B^2+4C^2)(1+\cos^4\theta)\\&-\tfrac{1}{8}(A^2+6AB+B^2-4C^2)\cos^2\theta\\&-\tfrac{1}{4}(A^2-B^2)(1-\cos^4\theta)\cos2\phi\\&-\tfrac{1}{2}C(A+B)(1-\cos^4\theta)\sin2\phi\\&+\tfrac{1}{16}[(A-B)^2-4C^2](1+\cos^2\theta)^2\cos4\phi\\&+\tfrac{1}{4}C(A-B)(1+\cos^2\theta)^2\sin4\phi\}.\end{aligned}\tag{23}$$

The result of averaging (23) over one period of the motion is

$$\begin{aligned}\left\langle\frac{dP_1}{d\Omega}\right\rangle=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\Bigg[&\left(\frac{1}{2}+\frac{99}{64}e^2+\frac{51}{256}e^4\right)\\&\times(1+\cos^4\theta)+\left(1+\frac{95}{32}e^2+\frac{47}{128}e^4\right)\cos^2\theta\\&+\left(\frac{13}{32}e^2+\frac{1}{16}e^4\right)(1-\cos^4\theta)\cos2\phi\\&-\frac{25}{512}e^4(1+\cos^2\theta)^2\cos4\phi\Bigg].\end{aligned}$$

The corresponding results for polarization 2 of (21) are

$$\begin{aligned}\frac{dP_2}{d\Omega}=\frac{G}{8\pi c^5}\Bigg\{&\frac{1}{4}[4C^2+(A-B)^2]\cos^2\theta\\&+\frac{1}{4}[4C^2-(A-B)^2]\cos^2\theta\cos4\phi\\&+C(B-A)\cos^2\theta\sin4\phi\Bigg\},\end{aligned}\tag{24}$$

$$\begin{aligned}\left\langle\frac{dP_2}{d\Omega}\right\rangle=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}&\\\times\Bigg[\left(2+\frac{97}{16}e^2+\frac{49}{64}e^4\right)\cos^2\theta&\\+\frac{25}{128}e^4\cos^2\theta\cos4\phi\Bigg].&\end{aligned}$$

The total power radiated into both polarizations may be obtained either by adding (23) and (24), or by using (5) directly. The result is

$$\begin{aligned}\frac{dP}{d\Omega}=\frac{G}{8\pi c^5}\{&(\tfrac{1}{16})(3A^2+2AB+3B^2+4C^2)(1+\cos^4\theta)\\&+\tfrac{1}{8}(A^2-10AB+B^2+12C^2)\cos^2\theta\\&+\tfrac{1}{4}(B^2-A^2)(1-\cos^4\theta)\cos2\phi\\&-\tfrac{1}{2}C(A+B)(1-\cos^4\theta)\sin2\phi\\&+\tfrac{1}{16}[(A-B)^2-4C^2]\sin^4\theta\cos4\phi\\&+\tfrac{1}{4}C(A-B)\sin^4\theta\sin4\phi\}.\end{aligned}\tag{25}$$

The average of (25) over the orbit is

$$\begin{aligned}\left\langle\frac{dP}{d\Omega}\right\rangle=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}&\\\times\{[\tfrac{1}{2}+(99/64)e^2+(51/256)e^4](1+\cos^4\theta)&\\+[3+(289/32)e^2+(145/128)e^4]\cos^2\theta&\\+(13/32)e^2+(1/16)e^4](1-\cos^4\theta)\cos2\phi&\\-(25/512)e^4\sin^4\theta\cos4\phi\}.&\end{aligned}$$

The basic results of this section, Eqs. (23), (24), and (25), are quite complicated. The quantities A, B, and C are given by (22) and (14) as functions of ψ, the retarded orientation of the line joining the mass points. We may extract some rather simple results from our formulas, however.

For example, in the case of circular motion ($e=0$), the formulas become

$$\frac{dP_1}{d\Omega}=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}(1+\cos^2\theta)^2\sin^2 2(\phi-\psi),$$

$$\frac{dP_2}{d\Omega}=\frac{4}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}\cos^2\theta\cos^2 2(\phi-\psi),$$

$$\frac{dP}{d\Omega}=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}[4\cos^2\theta+\sin^4\theta\sin^2 2(\phi-\psi)].$$

The averages over the orbit are now quite trivially done:

$$\left\langle\frac{dP_1}{d\Omega}\right\rangle=\frac{1}{2\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}(1+\cos^2\theta)^2,$$

$$\left\langle\frac{dP_2}{d\Omega}\right\rangle=\frac{2}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}\cos^2\theta,$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle=\frac{1}{2\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}(1+6\cos^2\theta+\cos^4\theta).$$

Another aspect of Eqs. (23) (25) is that the total power may be obtained by integrating over solid angle, and the result for the total power should agree with (15). Carrying out the integration over all directions, we obtain

$$\begin{aligned}P_1&=(G/120c^5)(11A^2-6AB+11B^2+28C^2),\\ P_2&=(G/120c^5)(5A^2-10AB+5B^2+20C^2),\\ P&=(2G/15c^5)(A^2-AB+B^2+3C^2).\end{aligned}\tag{26}$$

The corresponding averages over the elliptical orbit are

$$\langle P_1\rangle=\frac{32}{5}\frac{G}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(\frac{7}{12}+\frac{683}{384}e^2+\frac{347}{1536}e^4\right),\tag{27}$$

$$\langle P_2\rangle=\frac{32}{5}\frac{G}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(\frac{5}{12}+\frac{485}{384}e^2+\frac{245}{1536}e^4\right).\tag{28}$$

It is straightforward to verify that (26), with A, B, C given by (22) and (14), agrees with our previous result (15), and that the sum of (27) and (28) is just the value (16) for $\langle P\rangle$ given earlier.

ACKNOWLEDGMENTS

One of the authors (JM) would like to acknowledge the support of the Radio Corporation of America during this work.

APPENDIX

We now show that the sum over all harmonics n of $g(n,e)$ is the same as $f(e)$, where $g(n,e)$ is defined by (20) and $f(e)$ is given by (17).

We first reduce the right-hand side of Eq. (20) to terms containing only $[J_n(ne)]^2$, $J_n'(ne)J_n(ne)$, and $[J_n'(ne)]^2$, by use of the recurrence relations and Bessel's equation. Prime denotes differentiation with respect to the argument. This gives

$$\begin{aligned}g(n,e)=\frac{n^4}{32}\Bigg\{&\frac{J_n^2}{n^2}\left(2-\frac{4}{e^2}\right)^2+J_n'^2\left(\frac{4}{e}-4e\right)^2+\frac{2J_nJ_n'}{n}\\ &\times\left(2-\frac{4}{e^2}\right)\left(\frac{4}{e}-4e\right)+(1-e^2)J_n^2\left(\frac{4}{e^2}-4\right)^2\\ &+(1-e^2)\frac{J_n'^2}{n^2}\left(\frac{4}{e^2}\right)^2-\frac{2J_nJ_n'}{n}(1-e^2)\left(\frac{4}{e}\right)\\ &\times\left(\frac{4}{e^2}-4\right)+\frac{4}{3n^2}J_n^2\Bigg\}.\end{aligned}\tag{A1}$$

A solution of the equation $M=E-e\sin E$ for $E(M,e)$ is given by the Fourier expansion

$$E(M,e)=M+2\sum_{n=1}^{\infty}\frac{\sin(nM)}{n}J_n(ne).\tag{A2}$$

If we differentiate (A2) successively with respect to e, we can form series with terms such as $\sin(nM)J_n'$, $\sin(nM)nJ_n$, $\sin(nM)n^2J_n'$, and $\sin(nM)n^3J_n$. We have made use of Bessel's equation to eliminate terms with a higher than first derivative of J_n. If we multiply two such series together, say,

$$\left[\frac{\partial^2E}{\partial e^2}+\frac{1}{e}\frac{\partial E}{\partial e}\right]^2=\frac{4(1-e^2)^2}{e^4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(nM)\times\sin(mM)nmJ_n(ne)J_m(me),$$

and integrate both sides with respect to M from 0 to 2π, we get on the right-hand side

$$\frac{4(1-e^2)^2\pi}{e^4}\sum_{n=1}^{\infty}n^2J_n^2(ne),$$

which is one of the expressions needed to sum (A1). The integral on the left-hand side is straightforward. The formulas obtained in this manner which are necessary to sum (A1) are

$$\sum_{n=0}^{\infty}n^2J_n^2(ne)=\frac{e^2}{4(1-e^2)^{7/2}}\left(1+\frac{e^2}{4}\right),$$

$$\sum_{n=0}^{\infty}n^3J_n'(ne)J_n(ne)=\frac{e}{4(1-e^2)^{9/2}}\left(1+3e^2+\frac{3}{8}e^4\right),$$

$$\sum_{n=0}^{\infty}n^4[J_n'(ne)]^2=\frac{1}{4(1-e^2)^{11/2}}\times\left(1+\frac{39}{4}e^2+\frac{79}{8}e^4+\frac{45}{64}e^6\right),\tag{A3}$$

$$\sum_{n=0}^{\infty} n^2[J_n'(ne)]^2=\frac{1}{4(1-e^2)^{5/2}}\left(1+\frac{3e^2}{4}\right),$$

$$\sum_{n=0}^{\infty} n^4 J_n^2(ne)=\frac{e^2}{4(1-e^2)^{13/2}}\left(1+\frac{37}{4}e^2+\frac{59}{8}e^4+\frac{27}{64}e^6\right).$$

Substitution of (A3) into the sum of the reduced series (Al) yields

$$\sum_{n=1}^{\infty} g(n,e)=\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}},$$

which is the same as $f(e)$ as calculated in (17).

B. Angular Momentum Radiation

The angular momentum radiation of the system of two point masses may be found in the same manner as the energy radiation. The angular distribution of the i-component of angular momentum radiated has been found to be

$$\frac{d^2L_i}{dt\,d\Omega} = \frac{G}{8\pi}\,\epsilon_{ijk}\Big[6\,n_j\,n_p\,\ddot{Q}_{mk}\,\dddot{Q}_{mp} - 9\,n_j\,n_m\,n_p\,n_q\,\ddot{Q}_{mk}\,\dddot{Q}_{pq} + 4\,n_j\,n_m\,\ddot{Q}_{mk}\,\dddot{Q}_{pp}\Big] \quad . \tag{5.29}$$

The total i-component of angular momentum radiated is the integral over angles of equation 5.29:

$$\frac{dL_i}{dt} = -\frac{2G}{5}\,\epsilon_{ijk}\,\ddot{Q}_{mj}\,\dddot{Q}_{mk} \quad . \tag{5.30}$$

Keeping the same notation and conventions as in the paper, we have that

$$\begin{aligned}
\ddot{Q}_{xx} &= -\alpha\left[\cos(2\psi) + e\cos^3(\psi)\right]\\
\ddot{Q}_{yy} &= +\alpha\left[\cos(2\psi) + e\cos(\psi)\left[1+\cos^2(\psi)\right] + e^2\right]\\
\ddot{Q}_{yx} = \ddot{Q}_{xy} &= -\alpha\left[\sin(2\psi) + e\sin(\psi)\left[1+\cos^2(\psi)\right]\right]
\end{aligned} \tag{5.31}$$

where α is defined by

$$\alpha = \frac{2Gm_1m_2}{a(1-e^2)} \quad .$$

The formulae for $\dddot{Q}_{ij}$ are given in equation 5.14.

We can find the total x, y, and z components of the angular momentum radiated from equations 5.30, 5.14, and 5.31. This gives

$$\frac{dL_x}{dt} = \frac{dL_y}{dt} = 0$$

$$\frac{dL_z}{dt} = -\frac{8}{5}\frac{G^{7/2}m_1{}^2m_2{}^2(m_1+m_2)^{\frac{1}{2}}}{a^{7/2}(1-e^2)^{7/2}}\left\{(1+e\cos\Psi)^3 \times \right.$$
$$\left. \times\left[3(1+e\cos\Psi)^2+(1-e^2)\right]\right\}$$

The average angular momentum radiated over one period of the motion is then found to be

$$\frac{dL_z}{dt} = -\frac{32}{5}\frac{G^{7/2}m_1{}^2m_2{}^2(m_1+m_2)^{\frac{1}{2}}}{a^{7/2}(1-e^2)^2}(1+\tfrac{7}{8}e^2) \ . \qquad (5.32)$$

Thus the average angular momentum radiated over one period can be expressed as the product of the angular momentum radiated by a circular orbit of the same semi-major axis (or energy) times an enhancement factor k(e), where k(e) is

$$k(e) = \frac{(1+\frac{7}{8}e^2)}{(1-e^2)^2} \ . \qquad (5.33)$$

k(e) is plotted against e in Figure 4. The enhancement factor for the angular momentum radiation is not as steeply rising as the enhancement factor for the energy radiation.

The angular distributions can also be found from

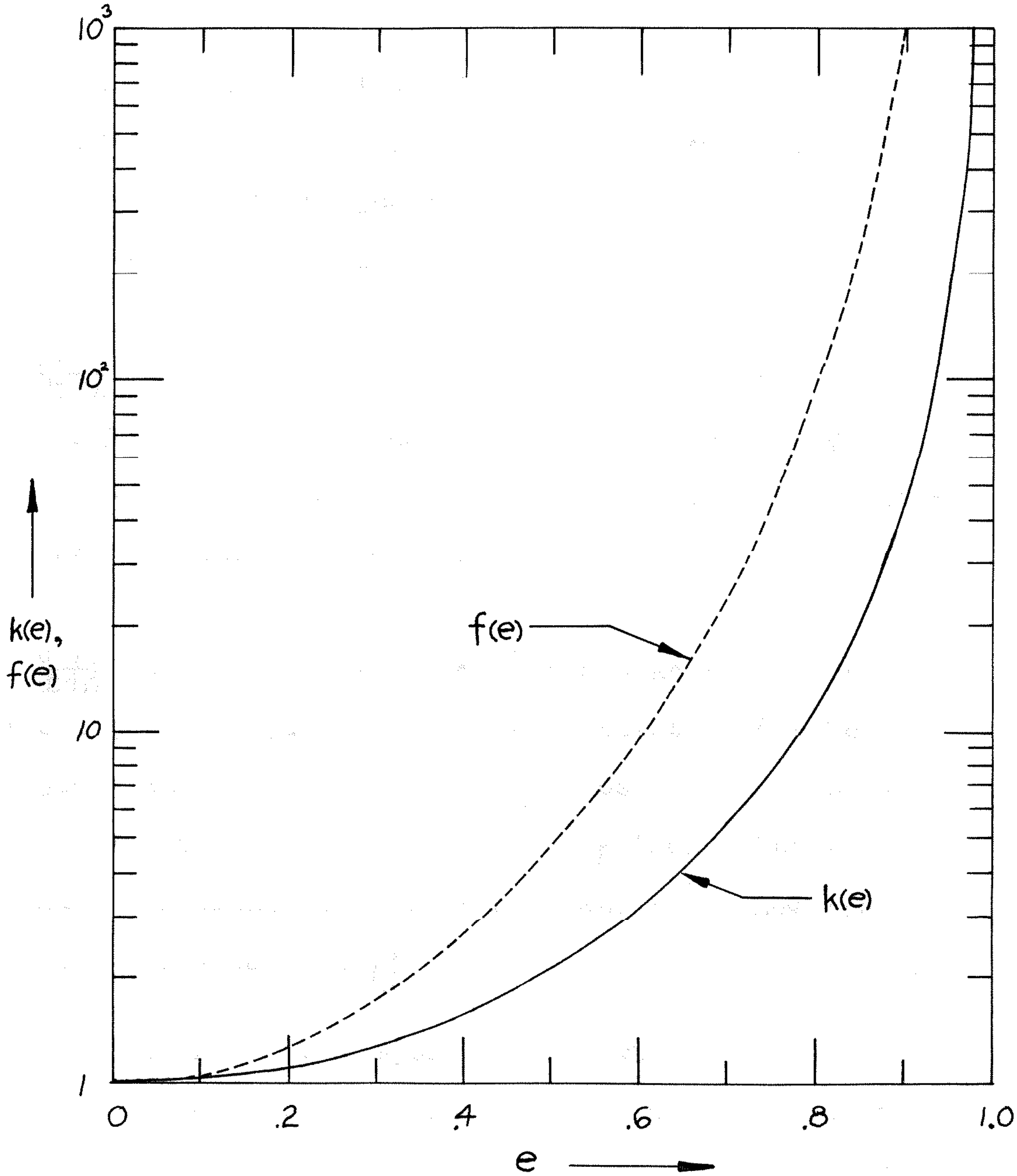

Figure 4. Enhancement factors, k(e) and f(e), plotted against e.

equation 5.29. Let $\ddot{Q}_{xx} = a$, $\ddot{Q}_{yy} = b$, and $\ddot{Q}_{xy} = c$. Then $\dddot{Q}_{xx} = \dot{a}$, $\dddot{Q}_{yy} = \dot{b}$, and $\dddot{Q}_{xy} = \dot{c}$. The angular momentum radiation distribution is given by

$$\frac{d^2L_x}{dt\,d\Omega} = -\frac{G}{8\pi}\Big\{\cos\theta\sin\theta\big[\cos\phi\,(6b\dot{c} + 10c\dot{a} + 4c\dot{b}) + \sin\phi\,(6c\dot{c} + 10b\dot{b} + 4b\dot{a})\big] - \frac{9}{4}\cos\theta\sin^3\theta\big[\cos3\phi\,(c\dot{a} - 2b\dot{c} - c\dot{b}) + \cos\phi\,(3c\dot{a} + 2b\dot{c} + c\dot{b}) + \sin\phi\,(3b\dot{b} + b\dot{a} + 2c\dot{c}) - \sin3\phi(b\dot{b} - b\dot{a} - 2c\dot{c})\big]\Big\}$$

$$\frac{d^2L_y}{dt\,d\Omega} = +\frac{G}{8\pi}\Big\{\cos\theta\sin\theta\big[\cos\phi\,(6c\dot{c} + 10a\dot{a} + 4a\dot{b}) + \sin\phi\,(6a\dot{c} + 10c\dot{b} + 4c\dot{a})\big] - \frac{9}{4}\cos\theta\sin^3\theta\big[\cos3\phi\,(a\dot{a} - 2c\dot{c} - a\dot{b}) + \cos\phi\,(3a\dot{a} + 2c\dot{c} + a\dot{b}) + \sin\phi\,(3c\dot{b} + c\dot{a} + 2a\dot{c}) - \sin3\phi\,(c\dot{b} - c\dot{a} - 2a\dot{c})\big]\Big\}$$

$$\frac{d^2L_z}{dt\,d\Omega} = +\frac{G}{8\pi}\Big\{(1 - \cos^2\theta)\big[3(c\dot{a} - a\dot{c} + b\dot{c} - c\dot{b}) + \sin2\phi\,(5b\dot{b} - 5a\dot{a} + 2b\dot{a} - 2a\dot{b}) + \cos2\phi\,(7c\dot{a} + 7c\dot{b} + 3b\dot{c} + 3a\dot{c})\big] - 9(1 - \cos^2\ )^2\big[\frac{1}{8}\cos4\phi\,(2c\dot{a} - 2b\dot{c} - 2c\dot{b} + 2a\dot{c}) + \tfrac{1}{2}\cos2\phi\,(c\dot{a} + c\dot{b}) + \frac{1}{8}(2c\dot{a} + 2b\dot{c} - 2c\dot{b} - 2a\dot{c}) + \tfrac{1}{4}\sin2\phi\,(b\dot{b} - a\dot{a} + b\dot{a} - a\dot{b}) + \frac{1}{8}\sin4\phi\,(b\dot{a} + a\dot{b} + 4c\dot{c} - a\dot{a} - b\dot{b})\big]\Big\} \quad . \tag{5.34}$$

This of course gives equation 5.30 when integrated over solid angle. The time averages of the products of a, b, c, and $\dot{a}$, $\dot{b}$, $\dot{c}$, are

$$\langle c\dot{b}\rangle = -\langle b\dot{c}\rangle = \gamma\left[1 + \frac{19}{16}e^2\right]$$

$$\langle a\dot{c} \rangle = -\langle c\dot{a} \rangle = \gamma\left[1 + \frac{9}{16}e^2\right]$$

$$\langle a\dot{b} \rangle = \langle b\dot{a} \rangle = \langle a\dot{a} \rangle = \langle b\dot{b} \rangle = \langle c\dot{c} \rangle = 0 \quad ,$$

where γ is

$$\gamma = \frac{4G^{5/2}m_1{}^2m_2{}^2(m_1 + m_2)^{\frac{1}{2}}}{a^{7/2}(1 - e^2)^2} \quad .$$

Thus the time average of the angular distribution of the angular momentum radiation is

$$\left\langle \frac{d^2L_x}{dt\,d\Omega} \right\rangle = \frac{G\gamma}{8\pi}\Big\{12\cos\theta\sin\theta\cos\phi\,(1 + \tfrac{2}{3}e^2) +$$

$$+ \tfrac{9}{4}\cos\theta\sin^3\theta\Big[\tfrac{5}{8}e^2\cos3\phi -$$

$$- 4\cos\phi\,(1 + \tfrac{23}{32}e^2)\Big]\Big\}$$

$$\left\langle \frac{d^2L_y}{dt\,d\Omega} \right\rangle = \frac{G\gamma}{8\pi}\Big\{12\cos\theta\sin\theta\sin\phi\,(1 + \tfrac{13}{12}e^2) +$$

$$+ \tfrac{9}{4}\cos\theta\sin^3\theta\Big[\tfrac{5}{8}e^2\sin3\phi -$$

$$- 4\sin\phi\,(1 + \tfrac{23}{32}e^2)\Big]\Big\}$$

$$\left\langle \frac{d^2L_z}{dt\,d\Omega} \right\rangle = -\frac{G\gamma}{8\pi}\Big\{3(1 + 2\cos^2\theta - 3\cos^4\theta)(1 + \tfrac{7}{8}e^2) +$$

$$+ \tfrac{5}{16}e^2\cos2\phi - \tfrac{25}{8}e^2\cos\theta\cos2\phi +$$

$$+ \tfrac{45}{16}e^2\cos^4\theta\cos2\phi\Big\} \quad . \tag{5.35}$$

The integral over angles reduces to equation 5.32.

These equations appear to be similar to those of the energy radiation case. However, as we shall see in the next section, they give different information about the decay of the system, and they allow us to get both the decay of the semi-major axis and the decay of the eccentricity.

C. Changes in the Elements of the Orbit

The results of the previous two sections can be applied to find the secular change in the elements of the relative orbit due to gravitational radiation. The equation of the relative orbit of the motion is

$$r = \frac{a\,(1 - e^2)}{1 + e\,\cos(\Psi - \Psi_0)} \quad . \tag{5.36}$$

If the plane of the motion is specified, then there are three parameters necessary to describe the orbit: a, e, and Ψ_0 . In the Newtonian theory they are constants of the motion. In the complete gravity theory they will be functions of time which will be slowly varying in the non-relativistic limit. The variation of Ψ_0 occurs in order $(v/c)^2$ and is the well known perihelion precession. Although this has been calculated for the case of two arbitrary masses from the equations of motion, we will show in the next section how it can also be derived from a consideration of the energy and angular momentum conservation laws.

The parameters a and e are related to E and L through the following equations:

$$a = - \frac{Gm_1 m_2}{2E} \qquad (5.37)$$

$$L^2 = \frac{Gm_1{}^2 m_2{}^2}{m_1 + m_2} a (1 - e^2) \quad . \qquad (5.38)$$

We have found that the change in E and L is of the order $(v/c)^5$ and that this change is a result of the radiation of energy and angular momentum by gravitational waves. Equations 5.37 and 5.38 therefore imply that the secular change in a and e will also be of order $(v/c)^5$. Starting with the basic time averages for $\left\langle \frac{dE}{dt} \right\rangle$ and $\left\langle \frac{dL}{dt} \right\rangle$,

$$\left\langle \frac{dE}{dt} \right\rangle = - \frac{32}{5} \frac{G^4 m_1{}^2 m_2{}^2 (m_1 + m_2)}{a^5 (1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \quad (5.39)$$

$$\left\langle \frac{dL}{dt} \right\rangle = - \frac{32}{5} \frac{G^{7/2} m_1{}^2 m_2{}^2 (m_1 + m_2)^{\frac{1}{2}}}{a^{7/2} (1 - e^2)^2} \left(1 + \frac{7}{8} e^2\right) \ , \qquad (5.40)$$

we can find $\left\langle \frac{da}{dt} \right\rangle$ and $\left\langle \frac{de}{dt} \right\rangle$.

$$\left\langle \frac{da}{dt} \right\rangle = - \frac{64}{5} \frac{G^3 m_1 m_2 (m_1 + m_2)}{a^3 (1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \quad (5.41)$$

$$\left\langle \frac{de}{dt} \right\rangle = - \frac{304}{15} e \frac{G^3 m_1 m_2 (m_1 + m_2)}{a^4 (1 - e^2)^{5/2}} \left(1 + \frac{121}{304} e^2\right) \ . \qquad (5.42)$$

From equations 5.41 and 5.42, we can then get $\left\langle \frac{da}{de} \right\rangle$, which relates the changes in a to the changes in e:

$$\left\langle \frac{da}{de} \right\rangle = \frac{12}{19}\,\frac{a}{e}\left[\frac{\left(1 + \frac{73}{24}\,e^2 + \frac{37}{96}\,e^4\right)}{(1 - e^2)\left(1 + \frac{121}{304}\,e^2\right)}\right] . \tag{5.43}$$

If we change from a to the period τ , we get

$$\left\langle \frac{d\tau}{de} \right\rangle = \frac{3\tau}{2a}\left\langle \frac{da}{de} \right\rangle . \tag{5.44}$$

Consider now the decay of a system of two masses moving under their own mutual gravitational attraction in elliptical orbits. If the elements of the orbit at time $t = 0$ are a_o and e_o, then the above equations specify a and e at a later time t, assuming that the only decay process acting is gravitational radiation. In particular, the case of circular motion can be integrated directly to give a(t). Let

$$\beta = \frac{64}{5}\, G^3 m_1 m_2 (m_1 + m_2) .$$

Then if $e = 0$,

$$\frac{da}{dt} = -\frac{\beta}{a^3} ,$$

which gives

$$a = \sqrt[4]{a_o^{\,4} - 4\beta t} . \tag{5.45}$$

This predicts that the system will collapse in a finite time T given by

$$T = \frac{a_o^4}{4\beta} . \tag{5.46}$$

Thus if there are two systems, one with $a_o = A_o$ and the other with $a_o = (10)^{-1}A_o$, then the second will collapse in a time 10^{-4} times that of the first. Most of the time of the collapse of the orbit is therefore spent near $a \cong a_o$. If we consider now an elliptical orbit, then the only effective modification of the time will be for $e \cong e_o$. Thus we can write the time of collapse for $e \neq 0$ as

$$T \cong \frac{a_o^4}{4\beta\, f(e_o)} \tag{5.47}$$

where $f(e_o)$ is the enhancement factor which is plotted in Figure 4.

Equation 5.43 can be integrated to get a(e) during the collapse of a system. The integration is tedious but straight forward. a(e) is then found to be

$$a = \frac{c_o\, e^{12/19}}{(1 - e^2)} \left[1 + \frac{121}{304} e^2\right]^{\frac{870}{2299}} , \tag{5.48}$$

where c_o is determined by the initial conditions $a = a_o$ when $e = e_o$. a is plotted against e in Figure 5.

For small e, this reduces to

$$a = c_o\, e^{12/19} \quad , \quad e^2 \ll 1,$$

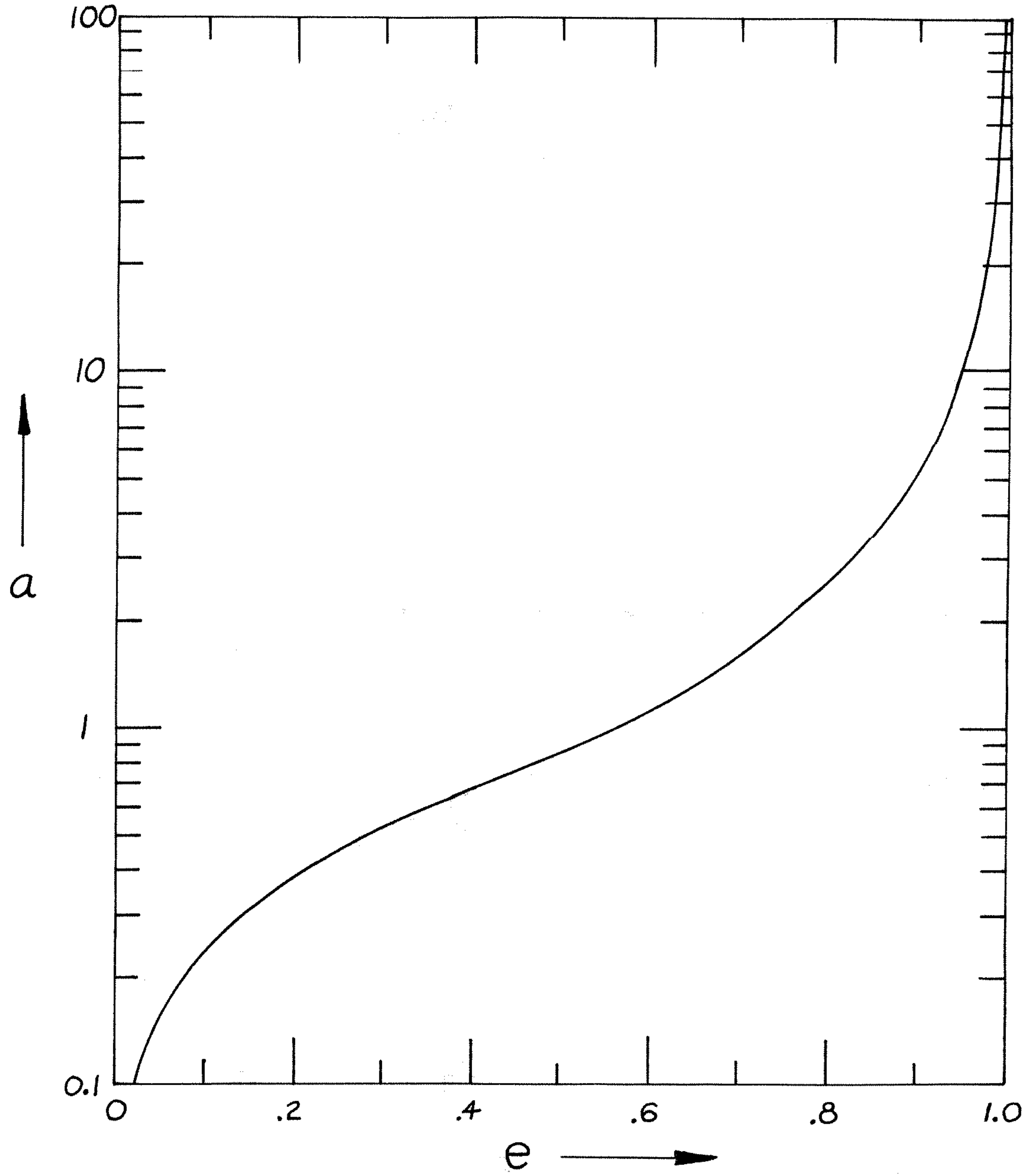

Figure 5. Semi-major axis a plotted against the eccentricity e.

and for e near 1 this becomes

$$a = \frac{c_1}{1 - e^2} \quad , \quad (1 - e^2) \ll 1 ,$$

where

$$c_1 = c_o \left(\frac{425}{304}\right)^{\frac{870}{2299}} \cong 1.135\ c_o \quad .$$

Thus for all practical purposes we can neglect the hideous factor and just consider that a(c) is given by

$$a(e) = \frac{c_o\, e^{12/19}}{(1 - e^2)} \quad . \tag{5.49}$$

With equations 5.42 and 5.48 we can write the equation giving the time decay of an eccentric system exactly. Since as $a \to 0$, $e \to 0$, $\frac{de}{dt}$ may be considered rather than $\frac{da}{dt}$.

$$\frac{de}{dt} = -\frac{19}{12}\frac{\beta}{c_o^4}\frac{e^{-67/19}\,(1 - e^2)^{3/2}}{\left[1 + \frac{121}{304}\, e^2\right]^{\frac{1181}{2299}}} \tag{5.50}$$

This could, in principle, be integrated to give T, the time of collapse. If e is near 1, then

$$\frac{de}{dt} \propto \frac{(1 - e^2)^{3/2}}{c_o^4} \quad ,$$

so that the time T is proportional to

$$T \propto \frac{c_o^4}{(1 - e^2)^{\frac{1}{2}}} \cong a_o^4\, (1 - e_o^2)^{7/2} \quad ,$$

which we can also obtain from equation 5.47 for e' near 1.

When these results are applied to a physical system, it becomes apparent that gravitational radiation is indeed a small effect. The formalism developed here lends itself to finding the radiation from one physical system which may be the system in which effects of the radiation are most easily observed. This is the case of close binary stars. Of course, binary stars are not point masses. They have finite extension and exert tidal forces on each other. In general they will be deformed and mass flow might even occur in them. Other radiation processes are also certain to be acting. With all of these reservations in mind, let us see what contribution gravitational radiation might give in the decay of double stars.

One result we could predict is that a plot of $\ln \tau$ vs. $\ln e$ for the decay of a single binary star should have a slope of 18/19 for small e. Unfortunately, secular changes in the orbit of binary stars have not been observed because they are so small. We would have to rely on a statistical average over binary stars. Aitken[1] has compiled such an average and gives average τ vs. average e for spectroscopic binaries. He gets

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[1] R. Aitken, <u>The Binary Stars</u> (U. of Calif., 1918)

τ (days)	2.7	7.6	14.1	30.6	102.5
e	.05	.16	.22	.35	.31

The two points with the lowest eccentricity give a slope of $\ln \tau$ vs. $\ln e$ of 1.03. No real meaning for the effect of gravitational radiation can be obtained from this, however, since the lifetime of decay of even a 2.7 day binary star due to gravitational radiation is longer than the estimated age of the universe. We can see that the stars will have essentially zero eccentricity by the time, if ever, that gravitational radiation becomes an important energy loss mechanism. Thus we will concern ourselves only with the zero eccentricity case.

For purposes of computation, we can rewrite equation 5.46 for the lifetime of a double star as

$$T = \frac{6.35 \times 10^4 \, a^4}{m_1 m_2 (m_1 + m_2)} \qquad (5.51)$$

$$T = 4.69 \times 10^{10} \tau^{8/3} \frac{(m_1 + m_2)^{\frac{1}{3}}}{m_1 m_2} \qquad (5.52)$$

where m_1 and m_2 are given in solar masses, τ in days, a in units of 10^{10} cm., and T in years. a is related to τ through the numerical conversion

$$\tau^2 = \frac{3.96 \times 10^{-5} \, a^3}{m_1 + m_2} \quad .$$

In figure 6, T is plotted against τ for the case $m_1 = m_2 = 1$. The radius of the sun is $\sim 10^{11}$ cm. This means that two suns revolving around each other separated by 10 solar radii would have a period of 4.5 days and hence a lifetime by gravitational radiation of 3×10^{12} years. We can get a shorter lifetime if we consider the case of two white dwarfs rotating around each other. Since white dwarfs have a radius $\sim 10^{9}$ cm., then the period corresponding to 10 radii would be .0045 days, and the lifetime would be 3×10^{4} years. The most extreme case we could have would be the case in which they are just touching, in which case the lifetime becomes just 50 years. Experimentally, the closest one has come to finding binary stars for which the effects of gravitational radiation might be verified is in a $\tau = 0.057$ day binary reported by Kraft, Mathews, and Greenstein(2). Even here the lifetime is at least 10^{6} years.

One can show that the radiation and thus the lifetime does not depend on whether the stars are point masses or are spherically symmetric bodies providing there is no overlap. Close binaries are certainly deformed. If the deformation is symmetric

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[2] R. Kraft, J. Mathews, and J. L. Greenstein, Astrophys. J. 136, 312 (1962).

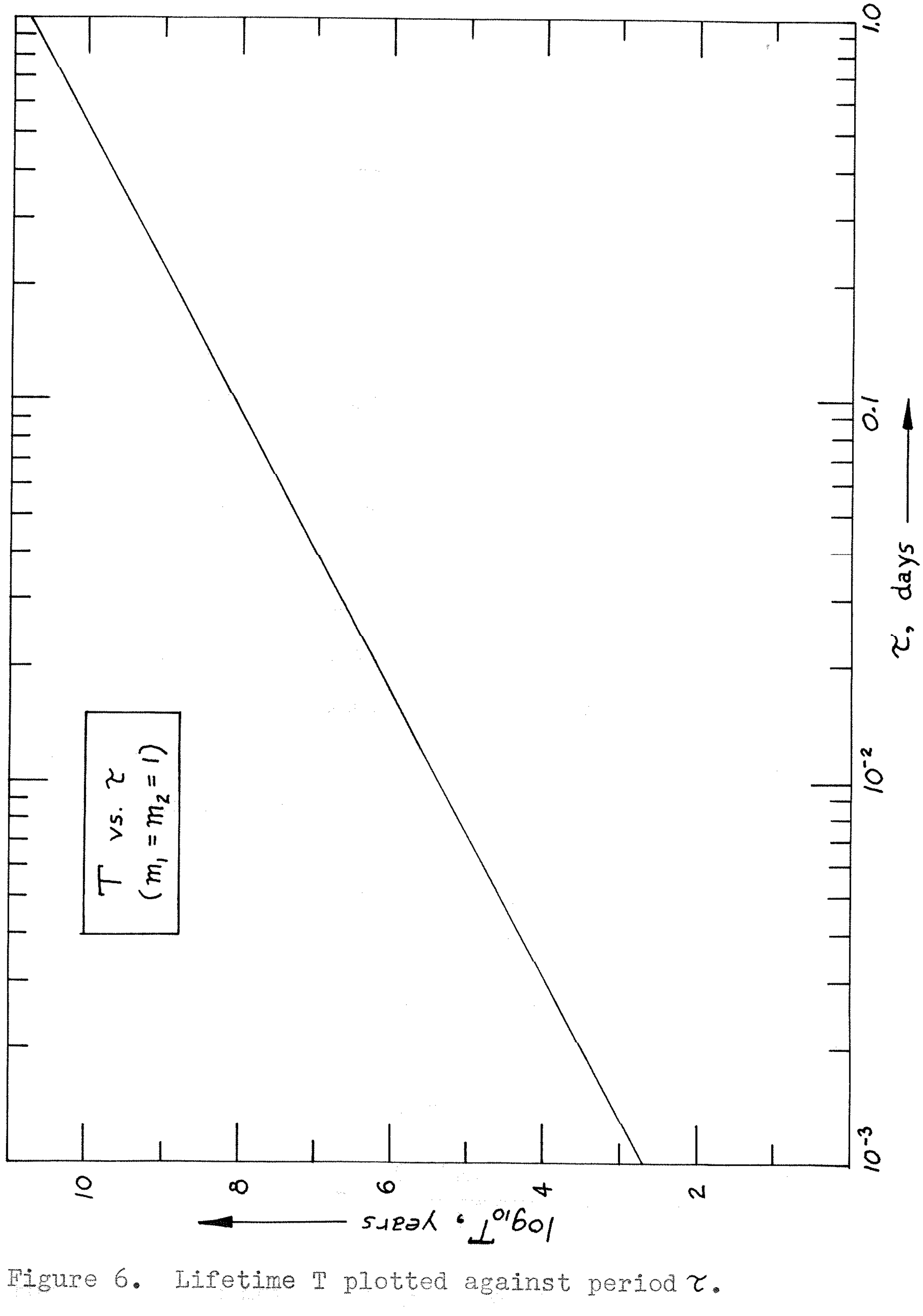

Figure 6. Lifetime T plotted against period τ.

about the line joining the centers of the two stars, then

$$\frac{dE}{dt} = - \frac{32}{5} G \omega^6 (I_1 - I_2)^2 \quad ,$$

where $I_1 = I_{xx}$ and $I_2 = I_{yy}$ when one axis is taken along the line joining the masses. Even if the deformation were great, it is doubtful whether it would affect the order of magnitude of the radiation calculated here by very much.

The energy and angular momentum radiated can also be found for unbound systems providing that the velocities are much smaller than the velocity of light. For $e \geqslant 1$, it is inconvenient to use the formulae developed as they stand. Since the orbits will not now be periodic, let us calculate the energy loss ΔE for one pass. For parabolic motion, $e = 1$, we get

$$\Delta E = - \frac{170}{3} \pi \frac{G^7 m_1{}^9 m_2{}^9}{(m_1 + m_2)^3 L^7} \quad , \qquad (5.53)$$

where L is the relative angular momentum of the masses. If we express this in terms of the closest distance of approach of the masses, b, we get

$$\Delta E = - \frac{85}{12} \pi \frac{G^{7/2} m_1{}^2 m_2{}^2}{b^{7/2}} \left(\frac{m_1 + m_2}{2}\right)^{\frac{1}{2}} . \qquad (5.54)$$

For hyperbolic motion, $e > 1$, we get

$$\Delta E = -\frac{8}{15}\frac{G^7 m_1{}^9 m_2{}^9}{(m_1+m_2)^3 L^7}\left\{\int_{-\alpha}^{\alpha}(1+e\cos\psi)^2 \times\right.$$
$$\times\left[12(1+e\cos\psi)^2+e^2\sin^2\psi\right]d\psi \quad ,(5.55)$$

where α is determined from the equation $\cos\alpha = -1/e$. These results are valid only for small velocities; however, we can formally take the limit as $e \to \infty$ and get the energy loss from a mass which is not appreciably deflected in its transit. This gives

$$\Delta E = -\frac{37\pi}{15}\frac{G^7 m_1{}^9 m_2{}^9 e^4}{(m_1+m_2)^3 L^7} = -\frac{148\pi}{15}\frac{G^3 m_1{}^3 m_2{}^3 E^2}{(m_1+m_2)\, L^3} \quad .(5.56)$$

Let us assume that $m_2 \gg m_1$. Then we can express this in simple form, using the distance of closest approach, b,

$$\Delta E = -\frac{37\pi}{15}\ \frac{G^3 m_1{}^{3/2} m_2{}^2\ (2E)^{\frac{1}{2}}}{b^3}\ . \qquad (5.57)$$

D. Perihelion Advance for Arbitrary Masses

The advance of the perihelion of the elliptical path of a small mass going around a large mass is a standard problem in the general theory of relativity. The solution is usually found from a consideration of the equation for the geodesic path. When the masses are arbitrary and neither one can be considered fixed, this method breaks down. Indeed, it becomes conceptually very difficult to visualize the geometric meaning of the geodesic path in this case.

In 1938, Einstein, Infeld, and Hoffman[(3)] solved the two body problem in general relativity and Robertson[(4)] applied the results to find the perihelion shift for the case of arbitrary masses. A discussion of these methods is found in the book by Infeld and Plebanski[(5)]. Independently Fock also solved this problem; the results can be seen most easily in his book[(6)]. Both of these methods are successive approximation methods, where quantities are expanded in powers

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[3] A. Einstein, L. Infeld, and B. Hoffman, Ann. Math. 39, 66 (1938).

[4] H. P. Robertson, Ann. Math. 39, 101 (1938).

[5] L. Infeld and J. Plebanski, Motion and Relativity (Pergamon Press, 1960), Chap. V.

[6] V. Fock, The Theory of Space, Time and Gravitation (Pergamon Press, 1959), Chap. VI.

of $1/c^2$. Starting from the field equations and the covariant divergence condition on the stress-energy tensor, they find a Lagrangian from which the energy momentum and angular momentum integrals can be found to order $1/c^2$. The derivation is lengthy in both cases, and the physics is obscured (for the author, at least) long before practical results are obtained.

Since the perturbations on the equation of the path of the two bodies can be found from a consideration of the energy and angular momentum integrals (see below), it would be desirable to find the answer in a more direct way than through the intermediary Lagrangian. The method given below is based merely on the conservation laws of energy and angular momentum. In addition, the coordinate conditions used here are different from those used by Infeld or Fock, and there will be some check as to the consistency of choosing different coordinate conditions.

Non-relativistically, if we know the energy and angular momentum integrals of the two masses moving around each other, we can find the equation of the path. Since the energy is given by

$$E = \tfrac{1}{2}\, m_1 v_1^2 + \tfrac{1}{2}\, m_1 v_1^2 - Gm_1 m_2/r = \tfrac{1}{2}\mu v^2 - Gm_1 m_2/r, \quad (5.57)$$

and the angular momentum and angular velocity are given by

$$L = \mu r^2 \dot{\theta} \quad ; \quad h = r^2 \dot{\theta} \quad , \tag{5.58}$$

where

$$\underset{\sim}{r} = \underset{\sim}{r}_1 - \underset{\sim}{r}_2 \ , \quad \underset{\sim}{v} = \underset{\sim}{v}_1 - \underset{\sim}{v}_2 \ , \mu = m_1 m_2/M \ ,$$
$$M = m_1 + m_2 \ , \tag{5.59}$$

we get

$$E = \tfrac{1}{2}\mu\left[\dot{r}^2 + r^2\theta^2\right] - Gm_1m_2/r \quad .$$

Letting $u = 1/r$, E becomes

$$E = \tfrac{1}{2}\mu\, h^2 \left[u'^2 + u^2\right] - Gm_1m_2u \quad .$$

Then we take $\frac{d}{d\theta}$ and divide by $\mu h^2 u'$, where $' = \frac{d}{d\theta}$, to get

$$u'' + u = GM/h^2 \quad . \tag{5.60}$$

This is the differential equation for the elliptical path. Its solution is, of course,

$$u = \frac{GM}{h^2}\,(1 + e\cos(\theta - \theta_o)) \quad , \tag{5.61}$$

where GM/h^2 can be replaced by $a^{-1}(1 - e^2)^{-1}$.

If now instead of the simple expressions for E and L we had expressions with terms of higher order, we would get for equation 5.60 by this method an equation of the form

$$u'' + u = GM/h^2 + \text{pert.}$$

If all of the perturbations are of the same order, then we can just substitute the unperturbed solution, equation 5.61, into the perturbations and get an equation of the form

$$u'' + u = GM/h^2 + f(\theta).$$

The effect of the $f(\theta)$ on the equation of the path can be found and the perihelion advance per revolution can be evaluated.

What remains then is the determination of the next order terms in the energy and angular momentum integrals. We know that energy and angular momentum are radiated starting with order $1/c^5$. Since we will be concerned only with terms of order $1/c^2$, the energy and angular momentum integrals can be considered to be constants of the motion.

Conservation of energy can be stated in the form

$$\frac{d}{dt}\int\left[\tilde{T}^{44} + \tilde{X}^{44}\right]dV = 0 \quad , \qquad (5.62)$$

where $\frac{d}{dt}\int\tilde{X}^{44}dV$ is found to be given by

$$\frac{d}{dt}\int\tilde{X}^{44}dV = \int\left[h_{\alpha 4,\beta}\,\tilde{T}^{\alpha\beta} - \tfrac{1}{2}\,h_{\alpha\beta,4}\,\tilde{T}^{\alpha\beta}\right]dV \; , \qquad (5.63)$$

so that equation 5.62 can be written

$$\frac{d}{dt}\int\left[g_{4\alpha}\,\tilde{T}^{\alpha 4}\right]dV = \tfrac{1}{2}\int h_{\alpha\beta,4}\,\tilde{T}^{\alpha\beta}\,dV \quad . \qquad (5.64)$$

This form could also have been obtained from the condition that the covariant divergence of $T^{\alpha\beta}$ must vanish. We work with equation 5.64 rather than equation 5.62 since equation 5.62 involves $\tilde{X}^{(3)}_{\mu\nu}$ as well as $\tilde{X}^{(2)}_{\mu\nu}$, and $\tilde{X}^{(2)}_{\mu\nu}$ would have to be found to second order in 1/c. The object is then to convert the right side of equation 5.64 into a time derivative of some quantity (to second order), and then this quantity can only differ from $\bar{X}^{44}$ by either a constant or some quantity of higher than second order. The whole quantity whose time derivative vanishes will then be the energy of the system and will be used instead of equation 5.57.

E will have terms of order c^2, c^0, and c^{-2}. That of order c^2 will be just the sum of the masses, which will be constant; that of order c^0 will be the terms given in equation 5.57. In evaluating the terms of order c^{-2}, we see that we can use the solutions of the classical problem, since the errors made will only be of order c^{-4}, and we are neglecting these.

Writing the right side of equation 5.64 out explicitly yields

$$\frac{d}{dt}\int\left[g_{4\alpha}\,\tilde{T}^{\alpha 4}\right]dV = \frac{1}{2}\int h_{44,4}\tilde{T}^{44}dV + \int h_{4i,4}\tilde{T}^{4i}dV + \\ + \int h_{ij,4}\tilde{T}^{ij}dV \quad . \qquad (5.65)$$

The second term of the right side of equation 5.65 can be immediately converted to a time derivative which can then be brought over to the left hand side.

$$\int h_{4i,4}\tilde{T}^{4i}dV = \iint dV\, dV' \frac{\dot{\tilde{T}}^{4i'}\tilde{T}^{4i}}{|\underset{\sim}{r} - \underset{\sim}{r}'|} =$$

$$= \iint dV\, dV' \frac{\tilde{T}^{4i'}\dot{\tilde{T}}^{4i}}{|\underset{\sim}{r} - \underset{\sim}{r}'|} = \frac{1}{2}\frac{d}{dt}\int h_{4i}\, \tilde{T}^{4i}\, dV \quad .$$

Let $\phi \equiv \overset{o}{h}_{44} = \overset{o}{h}_{11} = \overset{o}{h}_{22} = \overset{o}{h}_{33} = -2Gm/r$, where ^{o} means the lowest order part. Then the last term on the right side of equation 5.65 gives

$$\frac{1}{2}\int \phi_{,4}\tilde{T}^{ii}dV = \tfrac{1}{2}m_1 v_1^2 \frac{\partial}{\partial t}\phi(r)\Big|_1 + \tfrac{1}{2}m_2 v_2^2 \frac{\partial}{\partial t}\phi(r)\Big|_2 =$$

$$= - Gm_1 m_2 v_1^2\, \underset{\sim}{r}\cdot\underset{\sim}{v}_2\, r^{-3} + Gm_1 m_2 v_2^2\, \underset{\sim}{r}\cdot\underset{\sim}{v}_1\, r^{-3} =$$

$$= G\mu^2 v^2 \frac{\underset{\sim}{r}\cdot\underset{\sim}{v}}{r^3} = - \frac{d}{dt}(G\mu^2 v^2/r) + 2G\mu^2 \underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}/r.$$

Since $\dot{\underset{\sim}{v}} = -(GM/r^3)\,\underset{\sim}{r}$ to lowest order, the last term gives

$$- 2G^2 m_1 m_2 \mu\, \underset{\sim}{r}\cdot\underset{\sim}{v}/r^4 = \frac{d}{dt}\,(G^2 m_1 m_2 \mu/r^2) \quad .$$

These terms can likewise be taken to the left hand side.

The only remaining term, $\frac{1}{2}\int h_{44,4}\,\tilde{T}^{44}\,dV$, is difficult to work with as it stands because of the time derivative appearing on the h_{44}. It turns out to be more convenient to work with spatial derivatives. To do this we subtract from both sides of equation 5.65 the quantity $\frac{d}{dt}\frac{1}{2}\int h_{44}\,\tilde{T}^{44}\,dV$, and then taking the previously determined time derivatives all to the left side, we get

$$\frac{d}{dt}\left\{\int\left[g_{4\alpha}\tilde{T}^{\alpha 4} - \tfrac{1}{2}h_{4i}\tilde{T}^{4i} - \tfrac{1}{2}h_{44}\tilde{T}^{44}\right]dV + G\mu^2 v^2/r - \right.$$

$$\left. - G^2 m_1 m_2 \mu/r^2\right\} = -\tfrac{1}{2}\int h_{44}\tilde{T}^{44}{}_{,4}dV = +\tfrac{1}{2}\int h_{44}(h_{\alpha 4,\beta}\tilde{T}^{\alpha\beta} -$$

$$- \tfrac{1}{2}h_{\alpha\beta,4}\tilde{T}^{\alpha\beta})dV + \tfrac{1}{2}\int h_{44}\tilde{T}^{4i}{}_{,i}dV = \tfrac{1}{4}\int h_{44}h_{44,4}\tilde{T}^{44}dV +$$

$$+ \tfrac{1}{2}\int h_{44,i}h_{44}\tilde{T}^{4i}dV - \tfrac{1}{2}\int h_{44,i}\tilde{T}^{4i}dV \quad . \qquad (5.66)$$

Here we have used conservation of energy (covariant divergence) to reduce $\tilde{T}^{44}{}_{,4}$ to spatial derivatives and terms of order $1/c^2$. Evaluating the first two terms on the right side of equation 5.66 gives

$$\tfrac{1}{4}\int\phi\phi_{,4}\tilde{T}^{44}\,dV = \tfrac{1}{4}\frac{4G^2m_1m_2^2}{r}\frac{\underset{\sim}{r}\cdot\underset{\sim}{v}_2}{r^3} - \tfrac{1}{4}\frac{4G^2m_1^2m_2}{r}\frac{\underset{\sim}{r}\cdot\underset{\sim}{v}_1}{r^3} =$$

$$= -2G^2m_1m_2\mu\,\underset{\sim}{r}\cdot\underset{\sim}{v}/r^4 = \frac{d}{dt}(G^2m_1m_2\mu/r^2) \quad .$$

$$\tfrac{1}{2}\int\phi\,\phi_{,i}\tilde{T}^{4i}\,dV = \frac{d}{dt}(G^2\mu(m_1^2 + m_2^2)/r^2) \quad .$$

We now have to convert the last term, $-\tfrac{1}{2}\int h_{44,i}\tilde{T}^{4i}dV$, into a time derivative and the work will be done. In order to do this, we need the expression for h_{44} to order $1/c^4$. The exact expression for h_{44} is given by the integral

$$h_{44} = -4G\int\frac{\left[S_{44} - \tfrac{1}{2}S\right]}{|\underset{\sim}{r} - \underset{\sim}{r}'|}dV' \quad . \qquad (5.67)$$

The retarded quantity can be expanded in powers of $1/c$

to give (to the desired order)

$$h_{44} = -4G\int \frac{\frac{1}{2}(\tilde{T}^{44} + \tilde{T}^{ii}) + \frac{1}{2}(\tilde{X}_{44} + \tilde{X}_{ii})}{|\underset{\sim}{r} - \underset{\sim}{r}'|} dV' - $$
$$- G\frac{\partial^2}{\partial t^2}\int \tilde{T}^{44}|\underset{\sim}{r} - \underset{\sim}{r}'| dV' \quad . \qquad (5.68)$$

Consider the first term on the right side of equation 5.68. This is, to lowest order, ϕ , which is of order c^{-2}. Thus we need an expansion of $\tilde{T}^{44} + \tilde{T}^{ii} + \tilde{X}_{44} +$ $+ \tilde{X}_{ii}$ to higher order in $1/c$. If we break up $\tilde{T}^{44}$ into $\tilde{T}^{44}$(mass) and $\tilde{T}^{44}$(stress), we have that

$$\tilde{T}^{44}(\text{mass}) + \tilde{T}^{ii} = mc^2 + \frac{3}{2}mv^2 \quad .$$

The expression for $\tilde{T}^{44}$(stress) $+ \tilde{X}_{44} + \tilde{X}_{ii}$ can be found from the lowest order approximation to $\tilde{X}^{(2)}_{\mu\nu}$. Skipping the details of this, we get

$$\tilde{T}^{44}(\text{stress}) + \tilde{X}_{44} + \tilde{X}_{ii} = \frac{1}{32\pi G}\left[4\phi_{,i}\,\phi_{,i} + 6\phi\,\phi_{,ii}\right] .$$

Substituting these in the first term of equation 5.68 gives

$$h_{44} = \underset{(A)}{\phi} + \underset{(B)}{\frac{1}{2}\phi^2} - \underset{(C)}{\frac{1}{8\pi}\int\frac{\phi\,\phi_{,ii}}{|\underset{\sim}{r} - \underset{\sim}{r}'|}dV'} - \underset{(D)}{\frac{3Gm_2v_2^2}{|\underset{\sim}{r} - \underset{\sim}{r}_2|}} \quad .$$

The evaluation of the second term is straight forward. adding these two terms then gives h_{44} correct to order $1/c^4$. This result is

$$h_{44} = (A) + (B) + (C) + (D) +$$

$$+ \underset{(E)}{\frac{Gm_2\, \underset{\sim}{r}\cdot\dot{\underset{\sim}{v}}_2}{|\underset{\sim}{r} - \underset{\sim}{r}_2|}} + \underset{(F)}{\frac{Gm_2(\underset{\sim}{r}\cdot\underset{\sim}{v}_2)^2}{|\underset{\sim}{r} - \underset{\sim}{r}_2|^3}} \quad . \tag{5.69}$$

We label the terms and work with each individually.

(A) $\frac{1}{2}\int h_{44,i}\tilde{T}^{4i}dV = Gm_1m_2\left(\frac{\underset{\sim}{r}\cdot\underset{\sim}{v}_1}{r^3} - \frac{\underset{\sim}{r}\cdot\underset{\sim}{v}_2}{r^3}\right) = -\frac{d}{dt}\, Gm_1m_2/r$

(B) $-2G^2m_2^2m_1\underset{\sim}{r}\cdot\underset{\sim}{v}_1r^{-4} + 2G^2m_2m_1^2\underset{\sim}{r}\cdot\underset{\sim}{v}_2r^{-4} = \frac{d}{dt}\, G^2\mu(m_1^2 + m_2^2)/r^2$

(C) $\frac{1}{16\pi}\int\frac{\phi\phi_{,ii}[x_i - x_i']}{|\underset{\sim}{r} - \underset{\sim}{r}'|^3}\, dV' m_1 v_i = \frac{d}{dt}\, G^2\mu^2M/r^2$.

D, E, and F can be written as time derivatives in the same manner. Only the result will be quoted.

$$(D) + (E) + (F) = \frac{d}{dt}\left[\tfrac{1}{2}\, G\mu^2(\underset{\sim}{r}\cdot\underset{\sim}{v})^2/r^3 - 2\, G\mu^2v^2/r + \right.$$
$$\left. + \tfrac{3}{2}\, Gm_1m_2\mu/r^2\right] .$$

We also have the terms coming from the expansion of $\tilde{T}^{4i}$ taken with the lowest order $h_{44,i}$:

$$\tfrac{1}{2}\, h_{44,i}\; mv_i(\tfrac{1}{2}v^2 - \tfrac{1}{2}\phi) = \frac{d}{dt}\left[-\tfrac{1}{2}\,\frac{G\mu}{M^2}(m_1^3 + m_2^3)\,\frac{v^2}{r} + \right.$$
$$\left. + \tfrac{1}{2}\, G^2\mu(m_1^3 + m_2^3)M^{-1}r^{-2} - \tfrac{1}{2}\, G^2\mu(m_1^2 + m_2^2)/r^2\right] .$$

One thing remains to be done. We have expressed all of the terms on the right side of equation 5.66 as pure time derivatives. Thus we can write equation 5.66 in the form of an energy conservation.

$$\frac{d}{dt}\int(\tilde{T}^{44} + \tfrac{1}{2}\, h_{44}\tilde{T}^{44} + \tfrac{1}{2}\, h_{4i}\tilde{T}^{4i})dV + \frac{d}{dt}\Big\{-Gm_1m_2/r +$$

$$+ \tfrac{1}{2}\, G\mu^2(\underset{\sim}{r}\cdot\underset{\sim}{v})^2/r^3 \;-\; G\mu^2v^2/r \;-\; \tfrac{1}{2}\,\frac{G\mu}{M^2}(m_1^3+m_2^3)\frac{v^2}{r}\Big\} = 0. \qquad (5.70)$$

To calculate explicitly the terms $\tilde{T}^{44}$, $h_{44}\tilde{T}^{44}$ and $h_{4i}\tilde{T}^{4i}$, we must use the equation for the stress-energy tensor in terms of the velocities and potentials. This was given in equation 3.34. To the desired order we then get

$$\int\tilde{T}^{44}dV = m\Big[1 + \tfrac{1}{2}\, v^2 - \tfrac{1}{2}\, h_{44} - h_{4i}v_i - \tfrac{1}{2}\phi v^2 +$$

$$+ \tfrac{3}{8}\, v^4 - \tfrac{3}{4}\, v^2\phi + \tfrac{3}{8}\phi^2\Big] \quad .$$

This can then be used in equation 5.70 to eliminate all of the terms within the integral, converting them to pure time derivatives. One must also use the expansion of h_{44} given by equation 5.68. When all of this is done and carried out, one gets the energy conservation law

$$\frac{d}{dt}\Big\{\tfrac{1}{2}\, m_1v_1^2 + \tfrac{1}{2}\, m_2v_2^2 + \tfrac{3}{8}\, m_1v_1^4 + \tfrac{3}{8}\, m_2v_2^4 - Gm_1m_2/r +$$

$$+ \tfrac{1}{2}\, G^2m_1m_2M/r^2 + \tfrac{1}{2}\, G\mu^2(\underset{\sim}{r}\cdot\underset{\sim}{v})^2/r^3 + \qquad (5.71)$$

$$+ \tfrac{3}{2}\, G\mu(m_1^2 + m_2^2)v^2/Mr + \tfrac{7}{2}\, G\mu^2v^2/r\Big\} = 0 \; .$$

The quantity in the brackets is then the energy of the system. It is found to be equivalent to that found by Fock.

The calculation of the angular momentum integral is done in the same manner as the energy integral. The details are, however, much easier to do. We can write angular momentum conservation in the same form as equation 5.64.

$$\frac{d}{dt}\int\left[x_i\ \tilde{T}_j^{\ 4} - x_j\ \tilde{T}_i^{\ 4}\right]dV = \tag{5.72}$$
$$= \frac{1}{2}\int(x_i\ h_{\alpha\beta,j}\ \tilde{T}^{\alpha\beta} - x_j\ h_{\alpha\beta,i}\ \tilde{T}^{\alpha\beta})\ dV\ .$$

If we write out explicitly the terms in the right side of equation 5.72, we find that there can only be a contribution from h_{44}, and of the terms in h_{44} only the last one in equation 5.69 gives a contribution. This gives the term

$$\frac{d}{dt}\ G\mu^2 r^{-1}(v_i x_j - v_j x_i)\quad .$$

As in the energy radiation case, we must expand the left side of equation 5.72. When this is carried out one can write the angular momentum conservation in the form

$$\frac{d}{dt}\left\{\mu(x_i v_j - x_j v_i)\left[1 + \tfrac{1}{2}(m_1^3 + m_2^3)v^2/M^3 + \right.\right. \tag{5.73}$$
$$\left.\left. + 3\ G\ (m_1^2 + m_2^2)/Mr + 7G\mu/r\right]\right\} = 0\quad .$$

The quantity in the brackets is then the angular momentum of the system. This is also equivalent to that found by Fock.

Although Fock proceeds to solve these equations to give the perihelion advance, the author feels that a different method, analogous to the solution of the non-relativistic equations, is easier to understand. If we now use the transformation $u = 1/r$ on the energy and angular momentum integrals, we get from equations 5.71 and 5.73

$$E = \tfrac{1}{2}\mu(r^2\dot{\theta})^2(u'^2 + u^2) + \tfrac{3}{8}\mu\frac{m_1^3 + m_2^3}{M^3}h^4(u'^2 + u^2)^2 - Gm_1m_2u + \tfrac{1}{2}G^2m_1m_2Mu^2 + \tfrac{1}{2}G\mu^2h^2u(u')^2 + \tfrac{3}{2}G\mu\frac{m_1^2 + m_2^2}{M}h^2u(u'^2 + u^2) + \tfrac{7}{2}G\mu^2h^2u(u'^2 + u^2)$$

$$L = \mu(r^2\dot{\theta})\left[1 + \tfrac{1}{2}\frac{m_1^3 + m_2^3}{M^3}h^2(u'^2 + u^2) + 3G\frac{m_1^2 + m_2^2}{M}u + 7G\mu u\right] \quad . \tag{5.74}$$

From L we can solve for $(r^2\dot{\theta})$ and substitute this into the expression for E. In second order we consider that L is approximately $\mu r^2\dot{\theta}$, which can then be considered constant since errors made will be of order $1/c^4$. Then combining terms, we get for $\mu E/L^2$

$$\mu E/L^2 = \tfrac{1}{2}(u'^2 + u^2) - \tfrac{1}{8}h^2(m_1^3 + m_2^3)M^{-3}(u'^2 + u^2)^2 -$$

$$- G M u/h^2 - \frac{3}{2} GM\, u(u'^2 + u^2) - \frac{1}{2} G\mu u^3 +$$
$$+ \frac{1}{2} G^2 M^2 u^2 \quad . \qquad (5.75)$$

We now differentiate equation 5.75 with respect to θ and then divide by u'. If we then substitute the classical solution into the perturbation terms and expand the terms in a series of $\cos(n\theta)$, we get

$$u'' + u = GM/h^2 + A_o + A_1 \cos\theta + A_2 \cos 2\theta \quad ,$$

where A_1 is given by

$$A_1 = 6\, G^3\, M^3 e/h^4 \quad .$$

It is easy to show that one-half the coefficient of the $\cos\theta$ term is the perihelion shift per unit angle times GMe/h^2. Thus our result is that the perihelion shift is given by

$$\frac{\Delta\theta_o}{\theta} = \frac{3\, G^2 M^2}{h^2} = \frac{3\, G\, M}{a(1 - e^2)} \ , \qquad (5.76)$$

where $M = m_1 + m_2$. This result agrees with the calculation done in the limit that one mass is much larger than the other, and also agrees with the calculations done by Fock and Infeld.

VI. RADIATION in the RELATIVISTIC LIMIT

A. Electromagnetic Radiation

Before calculating the gravitational radiation in the relativistic limit, it is worthwhile reviewing the different results of the electromagnetic case since the same methods can be applied to the gravitational case. The analogy is not complete, however, since in the case of gravity, the contributions from the stresses must also be taken into account.

Let us first consider the radiation due to a single charge acting on itself. It has been noted before that the self radiation reaction force is given by

$$R_i = \frac{2}{3} e^2 \ddot{v}_i \tag{6.1}$$

in a system where the charge is at rest momentarily. Transforming this force to a system where the charge has arbitrary velocity yields the relativistically correct radiation reaction force

$$R_i = \frac{2}{3} e^2 \frac{1}{1 - v^2} \left\{ \ddot{v}_i + 3 \dot{v}_i (\underset{\sim}{v} \cdot \dot{\underset{\sim}{v}})/(1 - v^2) + \right.$$
$$\left. + \frac{v_i}{1 - v^2} \left[\underset{\sim}{v} \cdot \ddot{\underset{\sim}{v}} + 3(\underset{\sim}{v} \cdot \dot{\underset{\sim}{v}})^2/(1 - v^2) \right] \right\} \quad . \tag{6.2}$$

Thus in an arbitrary motion of the charge, the instantaneous power loss can be found by calculating $\underset{\sim}{R}\cdot\underset{\sim}{v}$. As pointed out before, this does not have to agree at all times with other methods of calculation, but need only agree over a time average.

One could also find the energy loss by calculating the Poynting flux through a large sphere surrounding the charge. The angular distribution of the radiation is then found to be

$$\frac{d^2E}{dt\,d\Omega} = -\frac{e^2}{4\pi}\,\frac{R^2\left\{\underset{\sim}{R} \times \left[(\underset{\sim}{R} - \underset{\sim}{v}R) \times \dot{\underset{\sim}{v}}\right]\right\}^2}{(R - \underset{\sim}{R}\cdot\underset{\sim}{v})^6} \tag{6.3}$$

in general, and

$$\frac{d^2E}{dt\,d\Omega} = -\frac{e^2}{4\pi}\,\frac{\dot{v}^2 \sin^2\theta}{(1 - v\cos\theta)^6}\;, \quad \underset{\sim}{v} \parallel \dot{\underset{\sim}{v}} \tag{6.4}$$

$$\frac{d^2E}{dt\,d\Omega} = -\frac{e^2}{4\pi}\,\dot{v}^2\,\frac{(1-v\cos\theta)^2-(1-v^2)\sin^2\theta\cos^2\varphi}{(1 - v\cos\theta)^6}\;, \quad \underset{\sim}{v} \perp \dot{\underset{\sim}{v}}\;. \tag{6.5}$$

At this point one usually converts the rate with respect to t to a rate with respect to the retarded time t' of the electron, where t' is given by

$$t' = t - |\underset{\sim}{R} - \underset{\sim}{r}(t')| \;,$$

where $\underset{\sim}{R}$ is the radius vector to the observing point, and $\underset{\sim}{r}$ is the vector position of the charge at the retarded time. The reason for this is that the quantities on the right side of equations 6.3-5 are explicit functions of θ, ϕ, and t', not t. Thus if we wanted to find the total radiation emitted in a finite time T, we would calculate

$$\int_{0+|\underset{\sim}{R}-\underset{\sim}{r}|}^{T+|\underset{\sim}{R}-\underset{\sim}{r}|}\left[\frac{d^2E}{dt\,d\Omega}\right]dt = \int_0^T \frac{d^2E}{dt\,d\Omega}\frac{dt}{dt'}dt' \equiv \int_0^T \frac{d^2E}{dt'\,d\Omega}\,dt' \quad .$$

The last integral is easily done when the motion is specified because the parameters will be explicit functions of t'. Also we want to find the total rate of emission of energy as the integral over solid angle. We could not integrate equations 6.3-5 as they now stand over all angles to find $\frac{dE}{dt}$ since v and $\dot{v}$ are functions of t', which is a function of θ, ϕ, and t. Thus the right hand side is not an explicit function of t. However if we consider t' as independent and calculate $\int \frac{d^2E}{dt'd\Omega}\,d\Omega$, we can explicitly evaluate the integrals and get

$$\frac{dE}{dt'} = -\frac{2}{3}e^2\,\frac{\dot{v}^2-(\underset{\sim}{v}\times\dot{\underset{\sim}{v}})^2}{(1-v^2)^3} \quad \text{in general;} \qquad (6.6)$$

$$\frac{dE}{dt'} = -\frac{2}{3}e^2\,\frac{\dot{v}^2}{(1-v^2)^3} \quad , \underset{\sim}{v}\parallel\dot{\underset{\sim}{v}} \;; \qquad (6.7)$$

$$\frac{dE}{dt'} = -\frac{2}{3}e^2\,\frac{\dot{v}^2}{(1-v^2)^2} \quad , \underset{\sim}{v}\perp\dot{\underset{\sim}{v}} \;. \qquad (6.8)$$

It is easy to show that $\int\frac{dE}{dt'}dt' = \int \underset{\sim}{R}\cdot\underset{\sim}{v}\,dt'$ by a simple integration by parts.

In the case of more than one charge, we also have to take into account the effects of the Lorentz force on one charge due to the fields produced by the others. For our considerations here, we will investigate a system of two charges moving in a circular orbit. Let us also assume that they have the same magnitude of charge, but that they may differ in sign. Using the standard L.W. potentials for a moving charge, one can find the relativistically correct force exerted on one charge by the other. If the charges are in opposite positions on the circular orbit, then θ is defined to be the angle between the lines joining the center of the orbit with the present position of one of the particles and its position at the time $t - r/c$ (see Figure 7). Since $a\theta = r\,v$, the scalar product of the force on one particle due to the other times the velocity is given as a function of θ :

$$\underset{\sim}{F}^{int}\cdot\underset{\sim}{v} = \frac{e^2\, a\,\omega^3}{(1 + v\,\sin\frac{1}{2}\theta)^3\theta^2}\Big[(-\sin\tfrac{1}{2}\theta + a\,\omega\cos\theta)\times$$

$$\times(1 - v^2) + \tfrac{1}{2}\,a\theta\omega\sin\theta + a^2\theta\omega^2\cos\tfrac{1}{2}\theta\Big]\,, \qquad (6.9)$$

where θ relates r and v through the implicit equations

$$(r/a)^2 = 2(1 + \cos\theta)$$

$$(r/a) = \theta/v \quad .$$

The total force acting on one of the charges is the

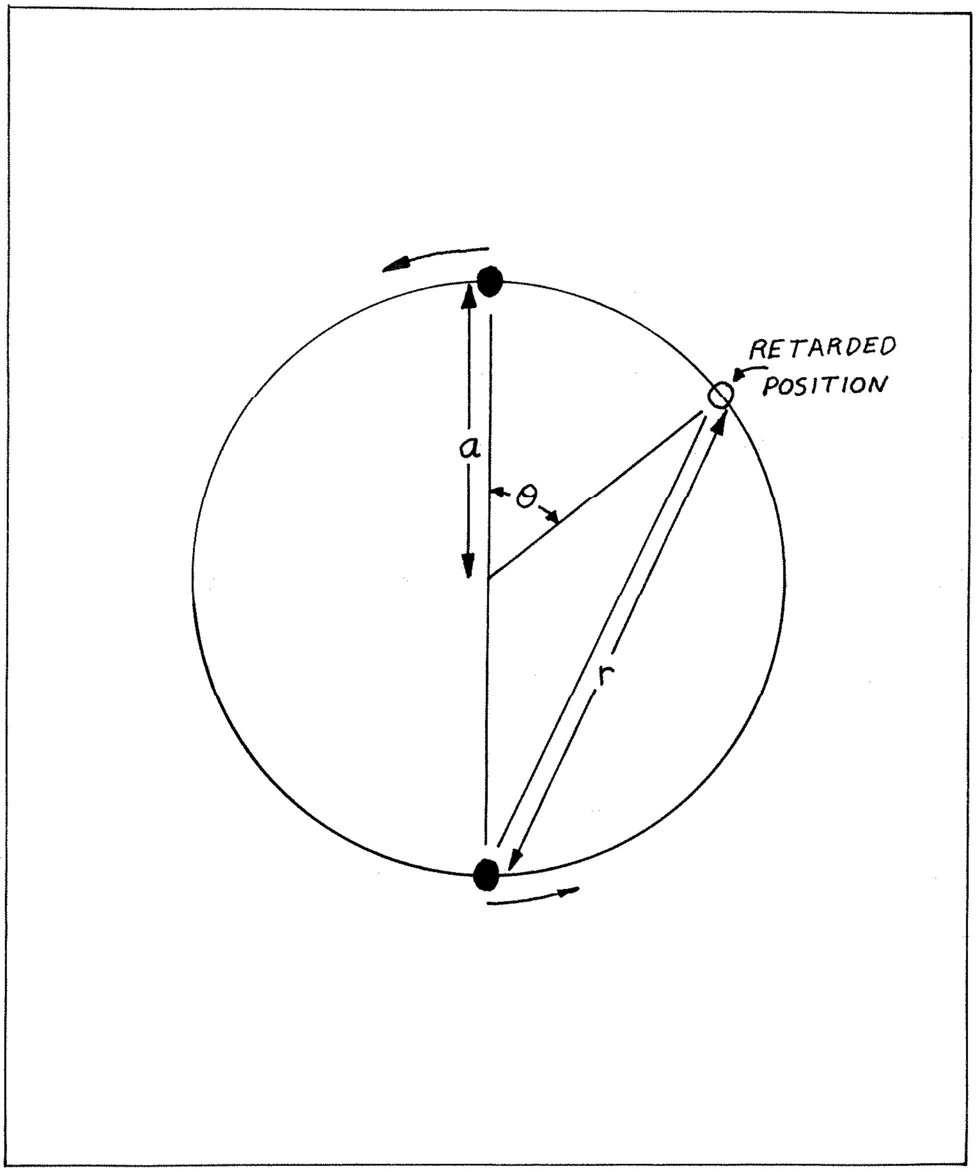


Figure 7. Diagram showing the parameters related to the retarded position.

sum of the radiation reaction force and the interaction force. We can see that in the extreme relativistic limit the interaction force can be neglected compared to the radiation reaction force. In the non-relativistic limit the two forces become comparable. In particular, taking the low velocity limit, we get

$$\underset{\sim}{R}\cdot\underset{\sim}{v} = -\frac{2}{3}\, e^2\, \dot{v}^2 \quad .$$

Since $\Theta^2 = 2\, v^2(1 + \cos\Theta)$, in the first approximation $\Theta \cong 2\, v$, and in the second approximation $\Theta \cong 2\, v - v^3$. Then

$$\underset{\sim}{F}^{int}\cdot\underset{\sim}{v} = \pm\frac{2}{3}\, e^2\omega^2\, v^2 = \pm\frac{2}{3}\, e^2\, \dot{v}^2 \quad . \qquad (6.10)$$

Thus the forces are equal (to order $(v/c)^3$) if the charges are of opposite sign and they cancel (to order $(v/c)^3$) if the charges are of the same sign. In the latter case, one would have to continue the expansion to order $(v/c)^5$ to get the first non-vanishing contributions. These results are, of course, to be expected. The first case is a case of a dipole source and the second is a case of a quadrupole source. In gravity we have an analogous situation except that all masses have the same sign and thus no dipole radiation is possible.

For non-circular motion, the results are similar.

The condition that the interaction force be much smaller than the self force is that

$$\left(|\underset{\sim}{R} - \underset{\sim}{r}(t')| - \underset{\sim}{v}(t')\cdot(\underset{\sim}{R} - \underset{\sim}{r}(t')\right)^{-1} \ll (1 - v^2)^{-1},$$

which is in general true for relativistic motion. Thus in the extreme relativistic case the particles radiate independently with negligible correlation effects.

This can also be seen by looking at the Poynting flux emitted by a system of charges. As is well known, the radiation fields are large only near the direction of motion of the charge at the retarded time. Thus if one of the charges is moving toward the observer and the other away, we get a contribution to $\frac{dE}{dt}$ of

$$\frac{dE}{dt} = \frac{f(v_1, \dot{v}_1, \theta, \phi)}{(1 - v_1\cos\theta)^6} + \frac{g(v_1, \dot{v}_1, v_2, \dot{v}_2, \theta, \phi)}{(1 - v_1\cos\theta)^3(1 + v_2\cos\theta)^3} + \frac{h(v_2, \dot{v}_2, \theta, \phi)}{(1 - v_2\cos\theta)^6}, \qquad (6.11)$$

where $|\underset{\sim}{v}_1| = |\underset{\sim}{v}_2|$, $\underset{\sim}{v}_1 = \underset{\sim}{v}_1(t_1')$ and $\underset{\sim}{v}_2 = \underset{\sim}{v}_2(t_2')$.

The integration over angles cannot be carried out explicitly with a trick as used before since the right side is a function of both $t_1'(\theta,\phi,t)$ and $t_2'(\theta,\phi,t)$. However, in the extreme relativistic case the second and third terms on the right side of equation 6.11 are clearly negligible compared with the first term,and thus $\frac{dE}{dt'}$ can be constructed as before, where t' now refers only to the charge moving toward the observer. Again we see the result that in the extreme relativistic approximation the charges radiate independently. Therefore, the results of the radiation from a single particle are valid also for many particles moving with $(1 - v^2) \ll 1$, providing the rates for all of the charges are added together.

B. Gravitational Radiation

We can proceed to apply the same methods in the case of gravitational radiation. The big complication here is in the calculation of the contributions of the stresses binding the system of masses. Unlike electromagnetism, where only moving charges are the sources of the radiation, gravity has as its source all energy and momentum present as well as the stresses involved. For this reason, we can only consider the case of an isolated system such that there are essentially no external forces acting on it, and cannot consider the case of radiation from one body moving arbitrarily. However, the radiation can be broken up into the contributions from the different sources present. We will deal first with the self force or radiation reaction force in the same manner as was done in the electromagnetic case. We might hope, in analogy with that case, that this would be the only important contribution in the relativistic limit.

The sources of the potentials are taken to be

$$T_{44} = \frac{\rho}{\sqrt{1 - v^2}} \quad ; \; T_{4i} = \frac{-\rho v_i}{\sqrt{1 - v^2}} \quad ; \; T_{ij} = \frac{\rho v_i v_j}{\sqrt{1 - v^2}} \; . \tag{6.12}$$

From these we obtain the potentials (analogous to the Lienard - Wiechert potentials)

$$\begin{aligned} \bar{h}_{44} &= \frac{-4 G m}{\sqrt{1 - v^2} \; s} \\ \bar{h}_{4i} &= \frac{+4 G m v_i}{\sqrt{1 - v^2} \; s} \\ \bar{h}_{ij} &= \frac{-4 G m v_i v_j}{\sqrt{1 - v^2} \; s} \\ \bar{h} &= \frac{-4 G m \sqrt{1 - v^2}}{s} \end{aligned} \tag{6.13}$$

where $s = |\underset{\sim}{R} - \underset{\sim}{r}| - (\underset{\sim}{R} - \underset{\sim}{r}) \cdot \underset{\sim}{v}, \quad \underset{\sim}{r} = \underset{\sim}{r}(t'), \; \underset{\sim}{v} = \underset{\sim}{v}(t')$.

In section IV.D. it was found that the equations of motion together with these potentials implied a gravitational radiation reaction force of

$$f_i = -\frac{11}{3} G m^2 \ddot{v}_i \quad . \tag{6.14}$$

In the non-relativistic limit this force was cancelled by the interaction force from the other masses much like the cancellation in the electromagnetic case for a system of two like charges. In the relativistic limit, one might assume, in analogy with electromagnetism, that the masses radiate independently. Then equation 6.14 would dominate in the calculation of the radiation in the relativistic limit. However, if we transform equation 6.14 to the relativistic case, we get

$$f_i = -\frac{11}{3} G m^2 \frac{1}{1 - v^2} \Big\{ \ddot{v}_i + 3\dot{v}_i(\underset{\sim}{v}\cdot\underset{\sim}{\dot{v}})/(1 - v^2) + \\ + \frac{v_i}{1 - v^2} \left[\underset{\sim}{v}\cdot\underset{\sim}{\ddot{v}} + 3(\underset{\sim}{v}\cdot\underset{\sim}{\dot{v}})^2/(1 - v^2) \right] \Big\} . \qquad (6.15)$$

If this were the only radiation reaction force acting on the mass, then this would imply that $\int \underset{\sim}{f}\cdot\underset{\sim}{v}\, dt > 0$, and thus that the body is gaining energy as a result of the self radiation. In section IV.A. it was shown that, over a time average, the energy of a system always decreases due to gravitational radiation. This was done assuming the coordinate condition $\bar{h}_{\mu\nu,\nu} = 0$. In section VII. we will show that the system must decrease in energy in any coordinate system or condition in which the field equations are consistent with the boundary condition $\bar{h}_{\mu\nu} \sim 1/r$ as $r \to \infty$. From this we can conclude that the interaction forces or the stresses or both are important in determining the radiation, since including only the self force part yields an answer with the wrong sign.

We may get the contribution of the interaction force from the linear force law. If $h \ll 1$, then we can write

$$f_i = \frac{m\, v^\alpha v^\beta}{\sqrt{1 - v^2}} \left[h_{\alpha i,\beta} - \tfrac{1}{2} h_{\alpha\beta,i} \right] \quad .$$

In the case of circular motion, $\underset{\sim}{f} \cdot \underset{\sim}{v} \propto (1 - v^2)^{-1}$, where the potentials $h_{\alpha\beta}$ give the additional factor of $(1 - v^2)^{\frac{1}{2}}$. The self force expression gives a power loss for circular motion of $\underset{\sim}{f} \cdot \underset{\sim}{v} \propto (1 - v^2)^{-2}$, so that for this case the interaction terms are of order $(1 - v^2)$ smaller than the self force terms. Even for the case of non-circular motion, if $(1 - \frac{\underset{\sim}{r} \cdot \underset{\sim}{v}}{r})$ does not reach a resonance, the interaction terms give a contribution $\propto (1 - v^2)^{-2}$ whereas the self force terms give a contribution $\propto (1 - v^2)^{-3}$. The resonance restriction is the same as found in the electromagnetic case. Thus any contribution which is to change the sign of our answer must come from the stresses.

Let us attempt to calculate the radiation by the Poynting flux method. The basic equation is

$$\frac{dE}{dt} = \int \tilde{X}^{(2)}_{4i}\, ds_i \quad . \tag{6.16}$$

$\tilde{X}^{(2)}_{4i}$ has been given previously. Over a time average, we found that

$$\int \frac{dE}{dt}\, dt = - \frac{R^2}{32\pi G} \iint dt\, d\Omega\; h_{\alpha\beta,4} \bar{h}_{\alpha\beta,4} \quad .$$

Then we can get the angular distribution

$$\frac{d^2E}{dt\, d\Omega} = - \frac{R^2}{32\pi G} \quad h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4} \quad . \qquad (6.17)$$

In the relativistic case, the $h_{\alpha\beta}$, given by equation 6.13, are functions of the angles θ and ϕ , and of the retarded time t', not the present time t. Therefore, let us change the variable of integration from t to t' to get

$$\int\frac{dE}{dt}\, dt = \iint\frac{d^2E}{dt d\Omega}\,\frac{dt}{dt'}dt'd\Omega = \iint\frac{d^2E}{dt'd\Omega}\, dt'd\Omega = \int\frac{dE}{dt'}dt' \quad ,$$

where

$$\frac{d^2E}{dt'd\Omega} = \frac{dt}{dt'}\,\frac{d^2E}{dt d\Omega} = - \frac{R^2}{32\pi G}\,\frac{dt}{dt'}\, h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4} \qquad ,(6.18)$$

and where the integration over angles of $\frac{d^2E}{dt'd\Omega}$ can be explicitly carried out. Thus the change in energy over one period can be found by either calculating the integral of the power loss with respect to the observer's time or the integral of the power loss with respect to the particle's own time. The two expressions $\frac{dE}{dt}$ and $\frac{dE}{dt'}$ are not related in any simple manner except in an integral over the time corresponding to one period of the motion. We cannot, for instance, say that

$$\frac{dE}{dt'} = \frac{dt}{dt'}\frac{dE}{dt}$$

because $\frac{dt}{dt'}$ is a function of the angles, whereas

both $\frac{dE}{dt}$ and $\frac{dE}{dt'}$ have been integrated over angles.
Thus we get that

$$\frac{d^2E}{dt'd\Omega} = - \frac{R^2}{32\pi G}\frac{dt}{dt'}\Big[\bar{h}_{44,4}\,\bar{h}_{44,4} - 2\,\bar{h}_{4i,4}\,\bar{h}_{4i,4} + + \bar{h}_{ij,4}\,\bar{h}_{ij,4} - \tfrac{1}{2}\,\bar{h}_{\sigma\sigma,4}\,\bar{h}_{\lambda\lambda,4}\Big]. \qquad (6.19)$$

Since we are interested in the fields $\bar{h}_{\alpha\beta,4}$ only for large R (wave zone), the following relations may be written

$$\frac{\partial}{\partial x_i} = -\frac{x_i}{R}\frac{\partial}{\partial t}\ ;\quad \frac{\partial s}{\partial t'} = -\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}}\ ;\quad \frac{dt}{dt'} = \frac{s}{R}\ .$$

Evaluating the fields gives

$$\bar{h}_{44,4} = -\frac{4GmR}{s^3(1-v^2)^{3/2}}\Big[\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}}(1-v^2) + \underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}\ s\Big]$$

$$\bar{h}_{4i,4} = +\frac{4GmR}{s^3(1-v^2)^{3/2}}\Big[v_i\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}}(1-v^2) + \dot{v}_i s(1-v^2) + + v_i\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}\ s\Big] \qquad (6.20)$$

$$\bar{h}_{ij,4} = -\frac{4Gm\ R}{s^3(1-v^2)^{3/2}}\Big[v_iv_j\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}}(1-v^2) + (v_i\dot{v}_j + + v_j\dot{v}_i)s(1-v^2) + v_iv_j\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}\ s\Big]$$

$$\bar{h}_{\sigma\sigma,4} = -\frac{4GmR}{s^3(1-v^2)^{\frac{1}{2}}}\Big[\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}}(1-v^2) - \underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}\ s\Big]\ .$$

The substitution of these fields into equation 6.19 yields

$$\frac{d^2E}{dt'd\Omega} = - \frac{GR^3m^2}{2\pi\, s^5}\Big[\tfrac{1}{2}(\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}})^2(1-v^2) - (\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}})(\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}})\ s\ -$$

$$- \frac{3}{2}\,(\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}})^2 s^2(1-v^2)^{-1} - 2\,\dot{v}^2 s^2\Big] \ . \qquad (6.21)$$

For $\underset{\sim}{v} \perp \dot{\underset{\sim}{v}}$, we get

$$\frac{d^2E}{dt'd\Omega} = - \frac{Gm^2\dot{v}^2}{4\pi(1-v\cos\theta)^5}\Big[(1-v^2)\sin^2\theta\cos^2\phi\ -$$

$$- 4\,(1 - v\cos\theta)^2\Big] \ . \qquad (6.22)$$

Integration over all θ and ϕ yields

$$\frac{dE}{dt'} = + \frac{11}{3}\,\frac{G\,m^2\,\dot{v}^2}{(1-v^2)^2} \ , \qquad (6.23)$$

which is in agreement with the result obtained from the radiation reaction force, but which is in disagreement with all physical intuition in that the energy of the system still seems to increase. For $\underset{\sim}{v} \parallel \dot{\underset{\sim}{v}}$, we get

$$\frac{d^2E}{dt'd\Omega} = \frac{G\,m^2\,\dot{v}^2}{4\pi(1-v\cos\theta)^5(1-v^2)}\Big[(1-v^2)\sin^2\theta\ +$$

$$+ 3(1- v\cos\theta)^2\Big] \ . \qquad (6.24)$$

Integrating this over all angles gives

$$\frac{dE}{dt'} = + \frac{11}{3}\,\frac{G\,m^2\,\dot{v}^2}{(1-v^2)^3} \ , \qquad (6.25)$$

which is also in agreement with the radiation reaction force calculation. It is also easy to show that the integral of equation 6.21 over all angles yields the general expression

$$\frac{dE}{dt'} = + \frac{11}{3} G m^2 \frac{\left[\dot{v}^2 - (\underset{\sim}{v} \times \dot{\underset{\sim}{v}})^2\right]}{(1-v^2)^3} , \qquad (6.26)$$

again in agreement with the previous results.

When we consider the case of more than one mass, we get contributions which appear as in equation 6.11. From this, one can again conclude that the interaction terms can be neglected in the relativistic limit. This, however, says nothing about the stress contributions.

Since we have already shown that the energy of the system must decrease with time, one might ask why we have obtained an answer which implies an energy increase. The answer lies in the fact that our sources do not include the stresses and thus fail to satisfy $S_{\mu\nu,\nu} = 0$. This does not necessarily imply that the potentials $\overline{h}_{\mu\nu}$ do not satisfy the coordinate condition $\overline{h}_{\mu\nu,\nu} = 0$. Thus we would like to see what the divergence is, if it is not zero, and to see how stress contributions might be added so that the potentials can be made to have zero divergence. If potentials can be defined which satisfy $\overline{h}_{\mu\nu,\nu} = 0$ in the wave zone, then the previous proof implies that we will get a net energy loss from the system, as we would expect.

In the non-relativistic case, one could not define potentials which depend only on the sources ρ, ρv_i, and $\rho v_i v_j$, and which also satisfy the divergence condition $\bar{h}_{\mu\nu,\nu} = 0$, even to lowest order in v/c. Although $\bar{h}_{44,4} - \bar{h}_{4i,i} = 0$ would be satisfied, $\bar{h}_{4i,4} - \bar{h}_{ij,j} = 0$ would not, since $\widetilde{X}_{ij}$, the stress contribution, is of the same order of magnitude as the components $\rho v_i v_j$. Thus we would certainly not hope to obtain correct expressions for the radiation from those potentials alone.

In the extreme relativistic case, where we ignore lower orders of $(1 - v^2)^{-1}$, one can try again to define potentials only by the mass terms. Let us then calculate their divergenc to see to what extent the coordinate condition $\bar{h}_{\mu\nu,\nu} = 0$ is satisfied. Let $4\,G\,m = 1$. Then we get

$$\bar{h}_{44,4} - \bar{h}_{4i,i} = - \frac{\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}}{(1-v^2)^{3/2} s} \qquad (6.27)$$

$$\bar{h}_{4i,4} - \bar{h}_{ij,j} = \frac{\dot{v}_i}{(1-v^2)^{\frac{1}{2}} s} + \frac{v_i\,\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}}{(1-v^2)^{3/2} s} \quad . \qquad (6.28)$$

The divergence is not zero in either case. However, the divergence of $\bar{h}_{\mu\nu}$ is related to $\bar{h}_{\mu 4,4}$ through the order of magnitude expression

$$\bar{h}_{\mu\nu,\nu} \sim \frac{S}{R}\,\bar{h}_{\mu 4,4} \sim (1 - v\cos\theta)\,\bar{h}_{\mu 4,4} \quad .$$

In an angular distribution the factor $(1 - v\cos\theta)$ causes one less factor of $(1 - v^2)$ in the denominator after the integration over angles is carried out. Also, assuming that the radiation is peaked in the forward direction, which our formulae seem to imply, the divergence of $\bar{h}_{\mu\nu}$ will be of order $(1 - v^2)$ smaller in the forward direction than the potentials or fields used to find the energy radiation. Although these arguments would ordinarily be enough to justify the neglect of the divergence, in this case they are not. There is cancellation of the higher powers of $(1-v^2)^{-1}$ so that the divergence actually contributes in the same order as the fields. We can see an example of this also from a consideration of the fields $\bar{h}_{\sigma\sigma,4}$. Even though $\bar{h}_{\sigma\sigma,4} \sim (1-v^2)\,\bar{h}_{44,4}$, it contributes to the radiation in the same order as $\bar{h}_{44,4}$.

The calculation of the stress parts of the fields appears to be very difficult in general. We may, however, get potentials which have zero divergence without explicit knowledge of the stresses. If we assume that the radiation of a given mass only depends on the parameters describing the motion and position of that mass, then we may add a corrective term to $\bar{h}_{\mu\nu}$ such that the new $\bar{h}_{\mu\nu}$ will have zero divergence in the wave zone. This is not as arbitrary as it might seem. Let the superscript m refer to the matter parts and the superscript s refer to the stress parts. Then we require that $\bar{h}^{s}_{\mu\nu}$ satisfy

$$\bar{h}^{m}_{\mu 4},_4 - \bar{h}^{m}_{\mu i},_i = - \bar{h}^{s}_{\mu 4},_4 + \bar{h}^{s}_{\mu i},_i \quad .$$

If $\bar{h}^{s}_{44},_4 \equiv \phi$, which will be an undetermined function, then all of the other components of $\bar{h}^{s}_{\mu\nu},_4$ in the wave zone will be determined from the divergence of $\bar{h}^{m}_{\mu\nu}$, equations 6.27 and 6.28, and the requirement that $\bar{h}^{s}_{\mu\nu}$ be symmetric in μ an ν . This yields, uniquely,

$$\bar{h}^{s}_{44},_4 = \phi$$

$$\bar{h}^{s}_{4i},_4 = - n_i\phi + n_i \frac{\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}}{(1-v^2)^{3/2}s}$$

$$\begin{aligned}\bar{h}^{s}_{ij},_4 = {} & n_i n_j \phi - n_i n_j \frac{\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}}{(1-v^2)^{3/2}s} - \\ & - (1-v^2)^{-\frac{1}{2}} s^{-1} \left[n_i \dot{v}_j + n_j \dot{v}_i - n_i n_j (\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}})/R \right] \\ & - \frac{\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}}{(1-v^2)^{3/2}s} \left[n_i v_j + n_j v_i - n_i n_j (\underset{\sim}{R}\cdot\underset{\sim}{v})/R \right] \quad .\end{aligned} \tag{6.29}$$

$$\bar{h}^{s}_{\sigma\sigma},_4 = \frac{\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}}{(1-v^2)^{3/2}s} + \frac{\underset{\sim}{R}\cdot\dot{\underset{\sim}{v}}}{(1-v^2)^{\frac{1}{2}}sR} + \frac{\underset{\sim}{v}\cdot\dot{\underset{\sim}{v}}\ \underset{\sim}{R}\cdot\underset{\sim}{v}}{(1-v^2)^{3/2}sR} \quad .$$

When the complete fields, $\bar{h}_{\mu\nu},_4 = \bar{h}^{m}_{\mu\nu},_4 + \bar{h}^{s}_{\mu\nu},_4$, are substituted in the energy flux, $h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4}$, one finds that the resultant flux is independent of the arbitrary function ϕ . This then yields for the energy radiated

$$\frac{d^2E}{dt'd\Omega} = -\frac{R^2}{32\pi G}\frac{dt}{dt'}\Big\{\tfrac{1}{2}(n_in_j\bar{h}_{ij,4})^2 - 2n_in_j\bar{h}_{im,4}\bar{h}_{jm,4} + \bar{h}_{ij,4}\bar{h}_{ij,4} + (n_in_j\bar{h}_{ij,4})\bar{h}_{mm,4} - \tfrac{1}{2}(\bar{h}_{mm,4})^2\Big\}. \quad (6.30)$$

If we now break up $\bar{h}_{\mu\nu}$ into the mass and stress terms in equation 6.30, we find that the stress fields drop out. Thus in the radiation given by equation 6.30, we can replace $\bar{h}_{ij,4}$ by $\bar{h}^m_{ij,4}$. Thus these stress fields, which changed the sign of the answer when the $h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4}$ form of the energy flux was used, play no role when the energy flux is reduced to terms with only spatial components. Of course, now the energy will be guaranteed to decrease, since the form of equation 6.30 was all that was needed to show that the energy change must be negative.

If we evaluate the angular distribution of equation 6.30, we find

$$\frac{d^2E}{dt'd\Omega} = -\frac{Gm^2}{4\pi}\frac{\sin^4\theta\ \dot{v}^2\ v^2}{(1-v\cos\theta)^5(1-v^2)^3}\Big[(1-v^2) + (1-v\cos\theta)\Big]^2, \quad \underset{\sim}{v} \parallel \dot{\underset{\sim}{v}} \qquad (6.31)$$

$$\frac{d^2E}{dt'd\Omega} = -\frac{Gm^2}{4\pi}\frac{\sin^2\theta\,\dot{v}^2}{(1-v\cos\theta)^5(1-v^2)}\Big\{4(1-v\cos\theta)^2v^2 +$$

$$+\cos^2\phi\Big[(1-v^2)^2 - 2(1+v^2)(1-v\cos\theta)^2 + \qquad (6.32)$$

$$+(1-v\cos\theta)^4\Big]\Big\} \quad , \ \underset{\sim}{v} \perp \dot{\underset{\sim}{v}} \quad .$$

In integrating equations 6.31 and 6.32 over angles, we should remember that we want only the lowest order part since we already have neglected the interaction radiation, which is of order $(1-v^2)$ smaller than the above. In the extreme relativistic approximation, $\sin^2\theta$ can then be written

$$\begin{aligned}\sin^2\theta' &= v^2(1-\cos^2\theta) + (1-v^2)\sin^2\theta' \\ &= -(1-v^2) + 2(1-v\cos\theta) - (1-v\cos\theta)^2 + \\ &\qquad + (1-v^2)\sin^2\theta \\ &\cong -(1-v^2) + 2(1-v\cos\theta)\end{aligned}$$

since, in the integration over angles, a factor $(1-v\cos\theta)$ just brings a factor $(1-v^2)$ into the final answer. Thus $\sin^2\theta \sim (1-v^2)$ and $(1-v\cos\theta) \sim (1-v^2)$. Then the integral of equations 6.31 and 6.32 over all angles becomes (for v=1)

$$\frac{dE}{dt'} = -\frac{1}{3}\frac{Gm^2\dot{v}^2}{(1-v^2)^3}\left[1 + 6\ln\left(\frac{1+v}{1-v}\right)\right] \ , \ \underset{\sim}{v} \parallel \dot{\underset{\sim}{v}} \qquad (6.34)$$

$$\frac{dE}{dt'} = -\frac{7}{3}\frac{Gm^2\dot{v}^2}{(1-v^2)^2} \quad , \ \underset{\sim}{v} \perp \underset{\sim}{v} \quad . \qquad (6.35)$$

Equation 6.35 is similar to equation 6.23, with the exception of the constant in front. One can also see that it is strikingly similar to that found in the electromagnetic case with the exception of the constant in front and the substitution of Gm^2 for e^2. Equation 6.34 is more confusing. The logarithmic term arises because we have a term like

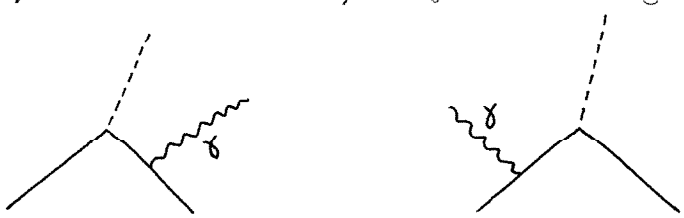
$$\int_{-1}^{1} \frac{\sin\theta\, d\theta}{(1-v\cos\theta)} \quad .$$

If we neglect the slow variation of the logarithm, we see that the form is the same as found in equation 6.25, except for some constant factor. However, for $v \cong 1$, the logarithmic term dominates and we get a different behaviour than before. In order to understand this better, let us examine the classical limit of the same quantum mechanical derivation.

First it is useful to reconcile the quantum mechanical statements about electromagnetic radiation with the classical radiation laws. For instance, one says quantum mechanically that the amplitude A of the radiation goes like $\sin\theta/(1-v\cos\theta)$. One might imply that the power radiated (which goes like A^2) has an angular distribution of $\sin^2\theta/(1-v\cos\theta)^2$. In the case of $\underset{\sim}{v} \parallel \dot{\underset{\sim}{v}}$, the classical power loss has an angular distribution of $\sin^2\theta/(1-v\cos\theta)^5$, which is more peaked than A^2 would imply. If we follow the formalism of Feynman[1], then we can write the probability of a photon of frequency ω being emitted into frequency interval $d\omega$ and solid angle $d\Omega$ in a scattering of a charge from state 1 to state 2 as

$$dP = \frac{e^2 d\omega\, d\Omega}{\pi\,\omega} \left[\frac{p_2 \cdot e}{p_2 \cdot q/\omega} - \frac{p_1 \cdot e}{p_1 \cdot q/\omega}\right]^2 \qquad (6.36)$$

where, to lowest order, only these diagrams dominate:



Choosing e space-like and transverse, we have that

$$p_2 \cdot e = p_2 \sin\theta_2 \; ; \; p_2 \cdot q/\omega = E_2(1 - v_2\cos\theta_2) \quad .$$

Thus

$$dP \propto \frac{1}{\omega} \left[\frac{v_2 \sin\theta_2}{(1 - v_2\cos\theta_2)} - \frac{v_1 \sin\theta_1}{(1 - v_1\cos\theta_1)}\right]^2 \qquad (6.37)$$

When θ_1 and θ_2 are different, then we have a peaking in the forward direction of both θ_1 and θ_2 with amplitude $\sin\theta_i/(1 - v_i\cos\theta_i)$. However, to get the classical case, say $\underset{\sim}{v} \parallel \dot{\underset{\sim}{v}}$, we need the limit as $\theta_1 \longrightarrow \theta_2$, keeping $|\underset{\sim}{v}_1 - \underset{\sim}{v}_2| \equiv \Delta v$ fixed and small. Thus

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[1] R. P. Feynman, Quantum Electrodynamics (W. A. Benjamin, Inc., 1961), p. 110.

$$dP \propto \frac{1}{\omega}\left[\frac{(v+\Delta v)\sin\theta}{(1-(v+\Delta v)\cos\theta)} - \frac{v\sin\theta}{(1-v\cos\theta)}\right]^2$$

$$\propto \frac{1}{\omega}\left[\frac{\Delta v\sin\theta}{(1-v\cos\theta)^2}\right]^2$$

so that the energy emission (multiplying by the energy of each quantum emitted yields an expression proportional to ω^2) is proportional to

$$dE \propto \frac{(\Delta v)^2\sin^2\theta}{(1-v\cos\theta)^4} \quad .$$

This is still not the angular distribution of $\frac{d^2E}{dt'd\Omega}$. However this is because we are not asking the same question in both cases. Classically if we want to find the energy emitted at frequency ω in $d\omega$, we would Fourier transform the fields

$$E(\omega) = \int E(t)e^{i\omega t}\,dt = \int \frac{f(t')}{\left(1-\frac{\underline{r}\cdot\underline{v}(t')}{r}\right)^3}\,e^{i\omega t}\,dt \quad .$$

Changing the variable of integration from t to t'yields

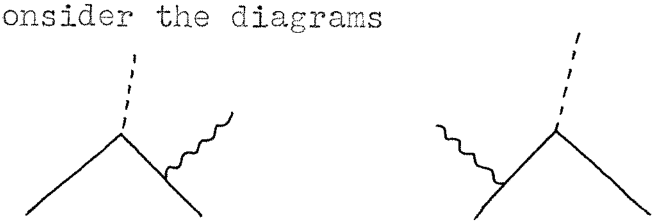
$$E(\omega) = \int \frac{f(t')}{\left(1-\frac{\underline{r}\cdot\underline{v}(t')}{r}\right)^2}\,e^{i\omega(t'+r(t'))}\,dt' \quad .$$

Since the transformed fields are proportional to $(1-v\cos\theta)^{-2}$ and not $(1-v\cos\theta)^{-3}$ as before, the energy emission is therefore proportional to $(1-v\cos\theta)^{-4}$, now in agreement with the quantum mechanical result. The other details can be easily worked out to show the agreement with the quantum calculations. $(\Delta v)^2$ can then be replaced by $\dot{v}^2$ in the time dependent form of the angular distribution, if a large number of collisions with infinitesimal deflections are considered.

In gravity, e is replaced by the tensor polarization and the matter stress-energy-momentum becomes the source. Thus the probability of emission is proportional to

$$\frac{p \cdot e \cdot p}{p \cdot q/\omega} \quad .$$

If we consider the diagrams



(6.38)

we get in the same way as electromagnetism that for $\underset{\sim}{v} \parallel \underset{\sim}{v}$,

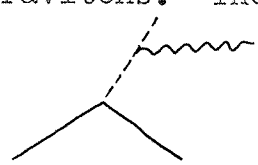
$$\frac{d^2E}{d\omega d\Omega} \propto \left[\frac{v_2^2 \sin^2\theta_2}{(1-v_2{}^2)^{\frac{1}{2}}(1-v_2\cos\theta_2)} - \frac{v_1^2 \sin^2\theta_1}{(1-v_1{}^2)^{\frac{1}{2}}(1-v_1\cos\theta_1)}\right]^2 . \tag{6.39}$$

By the same arguments as in the electromagnetic case, this implies that

$$\frac{d^2E}{dt'd\Omega} \propto \frac{\dot{v}^2 v^2 \sin^4\theta}{(1-v\cos\theta)^5(1-v^2)^3}\left[(1-v^2) + (1-v\cos\theta)\right]^2 \quad ,$$

which was found classically. This also suffers from the logarithmic defect in the integration over angles.

There is another diagram which will also contribute to the radiation in the classical limit. We see that energy and momentum must be conserved at the vertex of the diagrams in equation 6.38. Thus the particle propagating away from the vertex must carry away energy ΔE and momentum Δp. The actual quantum mechanical process involves integrals over these quantities. In the classical limit we take ΔE and ΔP to have the values necessary to conserve energy and momentum. The propagating particle will also act as a radiator of gravitons. Thus the diagram



must also be included. It produces an amplitude to emit a graviton of

$$A \propto -\frac{(\Delta p)^2 \sin^2\theta}{\Delta E - \Delta p \cos\theta} \quad .$$

If ΔE and Δp are found in terms of Δv, the A becomes

$$A \propto -\frac{(\Delta v)\sin^2\theta}{(1-v^2)^{3/2}(v-\cos\theta)} \quad , \qquad (6.40)$$

so that we can combine it with the amplitudes found before. Since

$$(v - \cos\theta)^{-1} = (1 - v\cos\theta)^{-1} + 0\left[(1-v^2)^0\right] \quad ,$$

the total amplitude becomes

$$A \propto \frac{\Lambda v \sin^2\theta}{(1-v\cos\theta)^2(1-v^2)^{\frac{1}{2}}} .$$

Translating this back to the time dependent form yields

$$\frac{d^2E}{dt'd\Omega} = -\frac{G}{4\pi}\frac{m^2\,\dot{v}^2\,\sin^4\theta}{(1-v^2)(1-v\cos\theta)^5} , \quad \underset{\sim}{v} \parallel \dot{\underset{\sim}{v}} . \qquad (6.41)$$

This has no logarithmic term in the integration over angles. In fact, integration over angles gives

$$\frac{dE}{dt'} = -\frac{2}{3}\,\frac{G\,m^2\,\dot{v}^2}{(1-v^2)^3} , \quad \underset{\sim}{v} \parallel \dot{\underset{\sim}{v}} , \qquad (6.42)$$

which is the same as the electromagnetic result if Gm^2 replaces e^2. The formula for the radiation for arbitrary $\underset{\sim}{v}$, $\dot{\underset{\sim}{v}}$, would then become

$$\frac{dE}{dt'} = -\frac{2}{3}\,\frac{G\,m^2[\dot{v}^2 - (\underset{\sim}{v} \times \dot{\underset{\sim}{v}})^2]}{(1-v^2)^3} \qquad (6.43)$$

It should be noted that this result is subject to some question. There might be other diagrams which become important, since if we assume that $(1-v^2) \ll 1$, the $h_{\mu\nu}$ may not be small compared to 1 and higher order stress terms may enter in. These types of terms would almost certainly be dependent on the kind of stresses binding the system, and the radiation for each individual case would have to be found separately. The previous results are probably as close as one can come to an energy loss equation without specifying the whole system under consideration.

VII. RADIATION without COORDINATE CONDITIONS

In section IV.B. we saw that the energy of a system must decrease as a result of the radiation of gravitational waves if the coordinate condition $\bar{h}_{\mu\nu,\nu} = 0$ was used. We will now relax this restriction and show that the result is valid for any coordinate system or condition providing the boundary condition $h_{\mu\nu} \sim 1/r$ as $r \rightarrow \infty$ is satisfied. In addition, in the non-relativistic approximation, the value must also be the same, namely,

$$\frac{dE}{dt} = -\frac{G}{5}\left[\dddot{Q}_{ij}\,\dddot{Q}_{ij} - \frac{1}{3}\,\dddot{Q}_{kk}\,\dddot{Q}_{mm}\right] . \qquad (7.1)$$

The expanded field equations are

$$\bar{h}_{\mu\nu,\lambda\lambda} - \bar{h}_{\mu\lambda,\lambda\nu} - \bar{h}_{\nu\lambda,\lambda\mu} + \delta_{\mu\nu}\,\bar{h}_{\sigma\lambda,\sigma\lambda} =$$
$$= -16\pi G\left[\tilde{T}^{\mu\nu} + \tilde{X}^{(2)}_{\mu\nu} + \tilde{X}^{(3)}_{\mu\nu} + \cdot\cdot\cdot\right] . \qquad (7.2)$$

The only contribution to the energy flux across a large sphere must come from the $\tilde{X}^{(n)}_{\mu\nu}$. If the asymptotic behaviour of $h_{\mu\nu}$ is $\sim 1/r$, then only $\tilde{X}^{(2)}_{\mu\nu}$ will contribute to the energy flux across this surface.

Let us consider asymptotic forms for $\bar{h}_{\mu\nu}$ of

$$\text{I.} \quad \bar{h}_{\mu\nu} = \frac{f(t - r)}{r}$$

$$\text{II.} \quad \bar{h}_{\mu\nu} = \frac{f(t)}{r}$$

$$\text{III.} \quad \bar{h}_{\mu\nu} = \frac{\text{const.}}{r} \quad .$$

Form III. has the property that $\bar{h}_{\mu\nu;i} \sim 1/r^2$ and $\bar{h}_{\mu\nu,4} = 0$, so its derivatives do not count in finding the radiation. In forms I. and II., f(x) is assumed to be a function such that $|f(x)| \approx |f'(x)| \approx |f''(x)|$. An example of such a function is f(x) = sin(x). In type II., $\bar{h}_{\mu\nu;i} \sim 1/r^2$, but $\bar{h}_{\mu\nu;4} \sim 1/r$. In type I., $\bar{h}_{\mu\nu;i} = - n_i \bar{h}_{\mu\nu;4}$, where $\bar{h}_{\mu\nu;4} \sim 1/r$. From symmetry we shall assume that all of the $\bar{h}_{4i}$ have the same asymptotic behaviour and that all of the $\bar{h}_{ij}$ also have the same asymptotic behaviour. However, we shall consider cases where $\bar{h}_{44}$, $\bar{h}_{4i}$, and $\bar{h}_{ij}$ have different asymptotic forms.

The expanded field equations, equation 7.2, are valid in any coordinate system since the non-linear field equations are invariant under arbitrary coordinate transformations. Since in the far zone $\tilde{X}^{(2)}_{\mu\nu} \sim 1/r^2$ for large r, the asymptotic forms of $\bar{h}_{\mu\nu}$ must satisfy the following conditions in order to be consistent with the field equations:

$$\bar{h}_{\mu\nu,\lambda\lambda} - \bar{h}_{\mu\lambda,\lambda\nu} - \bar{h}_{\nu\lambda,\lambda\mu} + \delta_{\mu\nu}\,\bar{h}_{\sigma\lambda,\sigma\lambda} \lesssim 1/r^2 \quad (7.3)$$

$$\bar{h}_{\sigma\sigma,\lambda\lambda} + 2\,\bar{h}_{\sigma\lambda,\sigma\lambda} \lesssim 1/r^2 \quad . \qquad (7.4)$$

It has been shown that the energy defined by the field equations is unique and has a time rate of change of

$$\frac{dE}{dt} = \int \widetilde{X}^{(2)}_{4i}\, ds_i \quad . \qquad (7.5)$$

Equation 3.13, giving $\widetilde{X}^{(2)}_{\mu\nu}$, can be reduced by the consistency conditions, equations 7.3 and 7.4, to give the only part which can contribute to order $1/r^2$.

$$\widetilde{X}^{(2)}_{\mu\nu} = -(32\pi G)^{-1}\Big\{ h_{\alpha\beta,\mu}h_{\alpha\beta,\nu} + 2\,h_{\mu\alpha,\beta}h_{\nu\alpha,\beta} -$$

$$- 2\,h_{\mu\alpha,\beta}h_{\nu\beta,\alpha} + 2\,h_{\alpha\beta}\,h_{\alpha\beta,\mu\nu} + 2\,h_{\alpha\beta}\,h_{\mu\nu,\alpha\beta} -$$

$$- 2\,h_{\alpha\beta}\,h_{\mu\alpha,\beta\nu} - 2\,h_{\alpha\beta}\,h_{\nu\alpha,\beta\mu} + \delta_{\mu\nu}[\quad]\Big\}. \qquad (7.6)$$

Since we are interested only in $\widetilde{X}^{(2)}_{4i}$, we shall forget about the $\delta_{\mu\nu}$ piece.

Let us first consider the case where $\bar{h}_{44}$, $\bar{h}_{4i}$, and $\bar{h}_{ij}$ all have the asymptotic form of type III. Then $\bar{h}_{\mu\nu,\lambda\lambda} = 0$ to order $1/r$, and the conditions given by equations 7.3 and 7.4 become

$$\bar{h}_{\alpha\lambda,\lambda\beta} + \bar{h}_{\beta\lambda,\lambda\alpha} \lesssim 1/r^2 \qquad (7.7)$$

$$\bar{h}_{\sigma\lambda,\sigma\lambda} \lesssim 1/r^2 \quad . \qquad (7.8)$$

The time average of equation 7.5 is now considered as before. Since for type III. solutions, $\frac{\partial}{\partial x_i} = - n_i \frac{\partial}{\partial t}$ to lowest order in 1/r, any derivatives can be integrated by parts. The reduced consistency relations, equations 7.7 and 7.8, can then be applied to reduce equations 7.5 and 7.6 to

$$\int \frac{dE}{dt}\, dt = \iint \tilde{X}^{(2)}_{4i}\, ds_i dt = -(32\pi G)^{-1} \iint ds_i dt \Big\{ - h_{\alpha\beta,4} \bar{h}_{\alpha\beta,i} + $$
$$+ 3\, \bar{h}_{\beta\sigma,\sigma} \big[\bar{h}_{4\beta,i} + \bar{h}_{i\beta,4} \big] \Big\} . \qquad (7.9)$$

The first term is what was found before. To show how one works with these consistency relations, we shall explicitly show that the second term is zero.

$$\iint_{\odot} ds\, dt\, n_i\, \bar{h}_{\beta\sigma,\sigma} \big[\bar{h}_{4\beta,i} + \bar{h}_{i\beta,4} \big] =$$
$$= \iint_{\odot} ds\, dt\, n_i \big[-n_i \bar{h}_{\beta\sigma,\sigma} \bar{h}_{4\beta,4} - \bar{h}_{\beta\sigma,\sigma 4} \bar{h}_{i\beta} \big] =$$
$$= \iint_{\odot} ds\, dt \big[\bar{h}_{\beta\sigma,\sigma 4} \bar{h}_{4\beta} + n_i \bar{h}_{\sigma 4,\sigma\beta} \bar{h}_{i\beta} \big] =$$
$$= \iint_{\odot} ds\, dt \big[-\bar{h}_{4\sigma,\sigma\beta} \bar{h}_{4\beta} + n_i \bar{h}_{\sigma 4} \bar{h}_{i\beta,\sigma\beta} \big] =$$
$$= \iint_{\odot} ds\, dt \big[-\bar{h}_{4\sigma,\sigma 4} \bar{h}_{44} + n_i \bar{h}_{4\sigma,\sigma 4} \bar{h}_{4i} - n_i \bar{h}_{\sigma 4} \bar{h}_{\sigma\beta,\beta i} \big]$$
$$= (0) + (0) + \iint_{\odot} ds\, dt\, \bar{h}_{\sigma 4} \bar{h}_{\sigma\beta,\beta 4} = 0 \quad .$$

Therefore, regardless of coordinate conditions (none was used),

$$\int \frac{dE}{dt}\, dt = -(32\pi G)^{-1} \iint_{\odot} ds\, dt\, h_{\alpha\beta,4} \bar{h}_{\alpha\beta,4} \quad , \qquad (7.10)$$

which demonstrates that the form of the gravitational radiation is independent of the coordinate system or conditions which one may use.

We can now reduce equation 7.10 to terms involving only spatial components of $h_{\mu\nu}$.

$$\oint\!\!\int h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4}ds\ dt = \oint\!\!\int\Big[-\tfrac{1}{2}\ \bar{h}_{44,44}\bar{h}_{44} + 2\ \bar{h}_{4i,44}\bar{h}_{4i} + \\ + \bar{h}_{ij,4}\bar{h}_{ij,4} - \bar{h}_{kk}\bar{h}_{44,44} - \tfrac{1}{2}(\bar{h}_{kk,4})^2\Big]ds\ dt \quad . \quad (7.11)$$

Making use of equations 7.7 and 7.8 yields

$$\oint\!\!\int h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4}ds\ dt = \iint\Big[-\tfrac{1}{2}\ \bar{h}_{4i,i4}\bar{h}_{44} + 2\ \bar{h}_{ij,j4}\bar{h}_{4i} + \\ + \bar{h}_{ij,4}\bar{h}_{ij,4} - \bar{h}_{kk}\bar{h}_{4i,4i} - \tfrac{1}{2}(\bar{h}_{kk,4})^2\Big]ds\ dt \quad . \quad (7.12)$$

By repeated use of equations 7.7 and 7.8 and integrations with respect to time, we can calculate an expression which does not have any 4 components. If we then evaluate the flux in a system in which $n_1 = n_2 = 0,\ n_3 = 1$, we get

$$\frac{d^2E}{ds\ dt} = -(32\pi G)^{-1}\Big[\tfrac{1}{2}(\bar{h}_{11,4}-\bar{h}_{22,4})^2+2(\bar{h}_{12,4})^2\Big]. \quad (7.13)$$

Thus the form of the angular distribution is the same as before. In addition, this shows that the energy of a system always decreases due to gravitational radiation if the $\bar{h}_{\mu\nu}$ all have the same asymptotic dependence, $f(t - r)/r$. If we were to have vanishing radiation, we would have to have $\overline{h}_{11,4} = \overline{h}_{22,4}$ and $\overline{h}_{12,4} = 0$ in the wave zone, looking in the 3-direction. If we anticipate the result that in the N.R. limit, $\overline{h}_{ij,4}$ is a tensor which has no angular dependence, we would then have that the radiation vanishes only when the quantity $\overline{h}'_{ij,4} = \overline{h}_{ij,4} - \frac{1}{3}\delta_{ij}\,\overline{h}_{kk,4} =$ $= 0$. Thus if the radiation vanishes in one system of coordinates, it vanishes in all systems of coordinates; likewise, if the radiation is non-zero in one system, it must be non-zero in all systems.

Instead of an asymptotic form of $f(t - r)/r$ for $h_{\mu\nu}$, we can also consider the forms $f(t)/r$ and const./r distributed among the $\overline{h}_{\mu\nu}$. The details are straight forward. One substitutes the combinations into the asymptotic field equations and sees if there is any inconsistency. It is found that these equations allow only three types of solutions.

$$\overline{h}_{44}\,,\ \overline{h}_{4i},\ \overline{h}_{ij} \sim f(t - r)/r$$

$$\overline{h}_{4i}\,,\ \overline{h}_{ij} \sim f(t - r)/r\ ;\ \overline{h}_{44} \sim \text{const.}/r$$

$$\overline{h}_{ij} \sim f(t - r)/r\ ;\quad \overline{h}_{44}\,,\ \overline{h}_{4i} \sim \text{const.}/r\,.$$

Each of these cases gives a radiation given by equation 7.13, and thus for all three cases the system must lose energy.

In the non-relativistic approximation, one can explicitly solve for $\bar{h}_{ij,4}$ in terms of the mass distribution of the source. If we assume that $\bar{h}_{44}$ is much larger than $\bar{h}_{4i}$ and $\bar{h}_{ij}$, and that time derivatives are small compared to spatial derivatives so that retardation effects can be neglected near the source, then the field equations with $\mu, \nu = 4, 4$ become

$$\bar{h}_{44,ii} = 16\pi G\, S_{44} \quad . \tag{7.14}$$

To lowest order, S_{44} is just the mass density and thus, for a point mass,

$$\bar{h}_{44} = -4\, G\, m/r \quad . \tag{7.15}$$

With the use of this expression for $\bar{h}_{44}$, the field stresses can be found to order $(v/c)^0$ and thus also the S_{ij}. The integral over space of this part of the S_{ij} is simply given:

$$\int S_{ij}\, dV = \tfrac{1}{2}\, \ddot{Q}_{ij} \quad , \tag{7.16}$$

where Q_{ij} is the moment of inertia tensor of the system

$$Q_{ij} = \sum_a m_a\, x_i^a\, x_j^a \quad .$$

The equation for $\bar{h}_{ij}$ is given by equation 7.2 with $\mu, \nu = i, j$. This can be rewritten

$$\bar{h}_{ij,\lambda\lambda} = -16\pi G\, S_{ij} + \bar{h}_{i\lambda,\lambda j} + \bar{h}_{j\lambda,\lambda i} + + \delta_{ij}\bar{h}_{\sigma\lambda,\sigma\lambda} \;. \tag{7.17}$$

The last three terms must have an asymptotic radius dependence of $1/r^2$ by equation 7.3 and our three allowed solutions. However, this and the $1/r^3$ part is of higher order in v/c than we are calculating, and thus the only part of the three terms which can count is that proportional to $1/r^4$, which is localized at the sources. Similarly, the only part of S_{ij} which counts is that localized at the sources. Simplifying equation 7.17 by the assumption that retardation effects can be neglected near the masses, we can then write for the $1/r$ part of $\bar{h}_{ij}$

$$\bar{h}_{ij} = -\frac{2\,G\,\ddot{Q}_{ij}(t-r)}{r} + \frac{4G}{r}\left[\int dV\,(\bar{h}_{i\lambda,\lambda j} + + \bar{h}_{j\lambda,\lambda i} + \delta_{ij}\bar{h}_{\sigma\lambda,\sigma\lambda})\right]_{t-r} \;. \tag{7.18}$$

The last term on the right side vanishes by integrations by parts with respect to spatial derivatives and the use of the equation for $\bar{h}_{44}$ close in.

We then have the same $\bar{h}_{ij,4}$ as was found in the gauge $\bar{h}_{\mu\nu,\nu} = 0$, and since the formulae for the radiation were found to be the same, we find a total radiation emitted of

$$\frac{dE}{dt} = -\frac{G}{5}\left[\dddot{Q}_{ij}\,\dddot{Q}_{ij} - \frac{1}{3}\,\dddot{Q}_{kk}\,\dddot{Q}_{mm}\right]. \qquad (7.19)$$

This result is now true for any non-relativistic system in which retardation effects can be ignored and in which the potentials $\sim 1/r$ for large r.

Infeld states that the gravitational radiation can be made whatever we wish by choosing the coordinate system appropriately[1]. We will now consider to what extent this is true. Let us first consider his statement that if $\bar{h}_{4\lambda,\lambda} = 0$ and $\bar{h}_{ik,k} = 0$, the radiation vanishes. The field equations, with this choice of coordinates, can be written

$$-\bar{h}_{44,ii} = -16\pi G\, S_{44}$$

$$-\bar{h}_{4i,jj} = -16\pi G\, S_{4i}$$

$$\bar{h}_{ij,\lambda\lambda} - \bar{h}_{i4,4j} - \bar{h}_{j4,4i} - \delta_{ij}\bar{h}_{44,44} = -16\pi G\, S_{ij}. \qquad (7.20)$$

We note that we now have ∇^2 acting on $\bar{h}_{44}$ and $\bar{h}_{4i}$ instead of $\Box$. In the far zone $\bar{h}_{44}$ and $\bar{h}_{4i}$ thus have an asymptotic form of const./r. Because of the $\Box$ operating on $\bar{h}_{ij}$, $\bar{h}_{ij}$ will have the form f(t-r)/r. It has been shown that this yields a negative energy

[1]L. Infeld and J. Plebanski, Motion and Relativity (Pergamon Press, 1960), Chap. VI.

emission and indeed, in the non-relativistic case, the energy emission is the same as for any other system of coordinates which yields one of the asymptotic forms for $\bar{h}_{44}$, $\bar{h}_{4i}$, and $\bar{h}_{ij}$. This contradicts Infeld's statement that the radiation must always vanish in this particular system of coordinates, because we get a vanishing radiation only when $\bar{h}'_{ij,4} = 0$, that is when we expect it to vanish, and negative otherwise.

Let us examine Infeld's arguments in this case. Assume that $h_{\mu\nu} \to 0$ as $r \to \infty$ and $S_{4i} \sim 1/r^2$. Then equation 7.20 becomes $\bar{h}_{4i,jj} = A_i/r^2$ for large r. A_i is a function of angles and time. Infeld expands A_i in terms of x_j/r, i.e.

$$\bar{h}_{4i,jj} = \frac{A_i^o}{r^2} + \frac{A_i{}^j x_j}{r^3} + \cdots \qquad (7.21)$$

which has the solution

$$\bar{h}_{4i} = A_i \ln(r) + A_i{}^j x_j/r + \cdots \ . \qquad (7.22)$$

Thus since as $r \to \infty$, $h_{4i} \to \ln(r) + f(\theta,\varphi)$, this violates the assumption that $h_{\mu\nu} \to 0$ for $r \to \infty$, and thus he concludes that this implies that there can be no terms in S_{4i} of order $1/r^2$ and hence no radiation.

If this proof were correct, this would also deny the existence of electromagnetic radiation or any other type of radiation. For example, if we calculate the electromagnetic radiation from a system, we find a flux proportional to $1/r^2$. However the previous argument shows that this would generate an $\bar{h}_{4i}$ which goes to infinity for large r, contradicting our assumption that space becomes flat for large r. Therefore therefore there can be no electromagnetic radiation.

Let us examine this question closer. We found that the asymptotic radiation field depends only on $\bar{h}_{ij}$, but $\bar{h}_{ij} = \bar{h}_{ij}(t-r)$. Thus if the coefficients A_i have a time dependence, it must be of the form $A_i(t-r)$, so that we would then have

$$\bar{h}_{4i,jj} = A_i(t-r)/r^2 \tag{7.23}$$

which we could satisfy by a solution of the form

$$\bar{h}_{4i} = B_i(t-r)/r^2 \tag{7.24}$$

where $A_i = B_i''$. Therefore we get trouble only if there is a constant part of A_i, independent of t-r. Therefore we must consider the solution to

$$\bar{h}_{4i,jj} = \text{const.}/r^2 \quad . \tag{7.25}$$

If we take the time integral of the radiation of a periodic system, we find a ΔE which is independent of time. We know that this is not strictly true. As a system loses energy, the parameters describing the system change, and consequently so does the radiation. We have neglected this variation in calculating the radiation. A better approximation would be to let the parameters of the system have a slowly varying non-periodic time dependence. Then the flux would also be modified by a slowly varying amplitude which must also be a function of t-r. Thus what we thought was a term like const./r^2 is really a term like $f(t-r)/r^2$, where for all practical purposes, the change in $f(t-r)$ in one period may be neglected. With this consideration, we can satisfy equation 7.25 for large r with a solution of the form $\bar{h}_{4i} = g(t-r)/r^2$. Thus the author does not feel that there is any inconsistency between Infeld's coordinate conditions and a non-zero radiation.

The next question involves coordinate transformations. We know that the field equations are invariant under arbitrary coordinate transformations. Under such a transformation, we have that the field equations become

$$\bar{h}_{\mu\nu;\lambda\lambda} - \bar{h}_{\mu\lambda;\lambda\nu} - \bar{h}_{\nu\lambda;\lambda\mu} + \delta_{\mu\nu}\bar{h}_{\sigma\lambda;\sigma\lambda} + W_{\mu\nu} =$$
$$= -16\pi G\, S_{\mu\nu} + W_{\mu\nu} \quad , \qquad (7.26)$$

where $W_{\mu\nu} = \sum_{n=2}^{\infty} W_{\mu\nu}^{(n)}$. We may give two definitions of gravitational radiation. (1). We may define the radiation to be the $1/r^2$ part of the right hand side of equation 7.26. (2). If we rewrite equation 7.26 as

$$\bar{h}_{\mu\nu;\lambda\lambda} - \bar{h}_{\mu\lambda;\lambda\nu} - \bar{h}_{\nu\lambda;\lambda\mu} + \delta_{\mu\nu}\bar{h}_{\sigma\lambda;\sigma\lambda} + 16\pi G\, S_{\mu\nu} = 0 \ , \tag{7.27}$$

then we may define the radiation to be the part of the expression on the left side of equation 7.27 which is of higher than first order in $h_{\mu\nu}$ and is proportional to $1/r^2$. Under a coordinate transformation (1) is not invariant while (2) is invariant. Definition (2) is appealing as it takes care of any discussion about the change of the field stresses from one system of coordinates to another. However, since definition (1) is the one mostly used, let us consider the effect of a coordinate transformation on the radiation defined by (1). We assume a time average as before and consider coordinate changes $\propto 1/r$. Then since

$$\bar{h}_{\alpha\beta} \to \bar{h}_{\alpha\beta} + \eta_{\alpha,\beta} + \eta_{\beta,\alpha} - \delta_{\alpha\beta}\eta_{\sigma,\sigma} + 0(1/r^2) \quad ,$$

we have that

$$\bar{h}_{\alpha\beta;4}\bar{h}_{\alpha\beta;4} - \tfrac{1}{2}\bar{h}_{\lambda\lambda;4}\bar{h}_{\sigma\sigma;4} \longrightarrow \text{same} + 2\,\bar{h}_{\alpha\beta;4}\big[\eta_{\alpha,\beta} + \eta_{\beta,\alpha} - \delta_{\alpha\beta}\eta_{\sigma,\sigma}\big]_{;4} + 2\,\bar{h}_{\sigma\sigma;4}\eta_{\lambda,\lambda 4} + (\eta_{\alpha,\beta} + \eta_{\beta,\alpha} - \delta_{\alpha\beta}\eta_{\sigma,\sigma})_{,4}(\eta_{\alpha,\beta} + \eta_{\beta,\alpha} - \delta_{\alpha\beta}\eta_{\sigma,\sigma})_{,4} - 2\,\eta_{\sigma,\sigma 4}\eta_{\lambda,\lambda 4} =$$

$$= \text{same} + 4\,\bar{h}_{\alpha\beta,4}\,\eta_{\alpha,\beta 4} + 2\,\eta_{\alpha,\beta 4}\,\eta_{\alpha,\beta 4}. \qquad (7.28)$$

Thus it appears as if there is a change. However, we must remember that the field equations were consistent with $X_{4i} \lesssim 1/r^2$ only for the asymptotic forms $f(t-r)/r$ and const./r. Therefore, for consistency, we must choose $\eta_\alpha \sim f(t-r)/r$ or $\eta_{\alpha,\beta} \sim$ const./r. Under this type of transformation, $\int dt \int ds_i \tilde{X}_{4i}^{(2)}$ remains unchanged.

If we had chosen $\eta_4 = f(t)/r$, we would have a change which is non-zero. But we saw that this would give a contradiction to the assumption that $X_{4i} \lesssim 1/r^2$. Infeld's argument in this case rest on allowing such coordinate transformations. For the above reasons, the author does not believe his conclusion that we may make the radiation whatever we please by choosing the coordinate transformation appropriately.

Summarizing, we can say that the energy of an arbitrary system must decrease as a result of the radiation of gravitational waves. This is true for any coordinate system or coordinate condition where the metric is asymptotically flat at infinity and where the energy flux does not diverge as $r \to \infty$. In addition to causing a net decrease of energy of the system, the energy radiation has been shown to be form invariant. In the non-relativistic approximation, where the solution for $\bar{h}_{ij,4}$ is known independent of the coordinate system, the energy radiation is just the familiar result, equation 7.19.

APPENDIX

The Uniqueness of the Gravitational Field Stresses

We have seen that the field equations predict a definite expression for the total stress-energy-momentum of the system. If this is broken up in the following form,

$$S_{\mu\nu} = \left[\tilde{T}^{\mu\nu} + \tilde{X}^{(2)}_{\mu\nu} + \tilde{X}^{(3)}_{\mu\nu} + \cdot \cdot \cdot \right] \quad ,$$

then the field equations give a unique $\tilde{X}^{(2)}_{\mu\nu}$ when the matter tensor $\tilde{T}^{\mu\nu}$ is specified. One might ask, since the expression for $\tilde{X}^{(2)}_{\mu\nu}$ is rather complicated, if there are other expressions for $X^{(2)}_{\mu\nu}$ which would serve our purpose as well and yet not be in contradiction to present experimental evidence. One might also wish to have these new stresses come from a principle or method that determines them, in order that one could not be accused of picking them out of thin air. Previous work on this same subject has been carried out by Feynman[1] and Huggins[2].

Aside from the obvious requirements that the total energy, momentum and forces be correctly given

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[1]R. P. Feynman, lecture notes, California Institute of Technology (unpublished)

[2]E. Huggins, Ph. D. thesis, California Institute of Technology (1962)

in the non-relativistic approximation, the only experimental result which one has is the perihelion shift of a body moving in an elliptical orbit around another body under the influence of gravity. How this may be affected by different choices of the field stresses is seen below.

One may separate the expression for $X^{(2)}_{\mu\nu}$ into two parts: one, the part which is of order c^0, and two, the remainder, which is, for all practical purposes, unobservable. For a non-relativistic system, we have seen that only $\bar{h}_{44}$ is large, and that $h_{11} \approx h_{22} \approx h_{33} \approx$ $\approx h_{44} = \phi = \frac{1}{2}\bar{h}_{44}$. Then only space-like derivatives of ϕ will be in the observable part of $X^{(2)}_{\mu\nu}$. In particular, in the expression for $X^{(2)}_{44}$ or $X^{(2)}_{ii}$, we will have only terms of the form $A\ \phi_{,i}\,\phi_{,i} + B\ \phi\ \phi_{,ii}$. We may also get terms of the form $\phi\phi_{,ii}$ from the expansion of the matter tensor in powers of $h_{\mu\nu}$. These are needed to give the correct energy and will, in the following, be implicitly thrown into the stress terms. Then we have that the energy depends only on the difference A - B.

Consider now, however, the exact expression for h_{44} for a stationary mass:

$$h_{44} = -\ 4G\int\frac{T'^{44} + X'_{44} - \frac{1}{2}(T'^{\alpha\alpha} + X'_{\alpha\alpha})dV'}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \quad .$$

Let us approximate $X_{\mu\nu}$ by $X^{(2)}_{\mu\nu}$. One can easily work out the formula for h_{44} in terms of ϕ for a mass at rest using the $\tilde{X}^{(2)}_{\mu\nu}$ given by general relativity to get

$$h_{44} = \phi + \tfrac{1}{2}\phi^2 \quad ,$$

and it is this number $\frac{1}{2}$ which is necessary to give the correct perihelion shift.

Let us now consider some other expression for the stresses. This may have the form $C\,\phi_{,i}\phi_{,i} + D\,\phi\phi_{,ii}$, where $C - D = A - B$. In particular, the observable components of $X_{44} - \frac{1}{2}X_{\mu\mu} = \frac{1}{2}X_{44} + X_{ii}$ will have to be of the form

$$\tfrac{1}{2}\left[X_{44} + X_{ii}\right]_{\text{orig.}} + E\left[\phi_{,ii}\phi + \phi_{,i}\,\phi_{,i}\right] \quad . \qquad \text{(A1)}$$

There fore we will get a new h'_{44} related to the old h_{44} by

$$h'_{44} = h_{44} - 4GE\int \frac{\phi_{,i}\phi_{,i} + \phi\,\phi_{,ii}}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\, dV' \quad .$$

Integrating by parts the $\phi_{,i}\phi_{,i}$ term, we get for the integral

$$-\int \phi_{,i}\phi\,\frac{\partial}{\partial x'_i}\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\, dV' \text{ and inte-}$$

grating a second time by parts gives

$$\int \phi\,\phi_{,i}\,\frac{\partial}{\partial x'_i}\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\, dV' + \int \phi^2\,\nabla'^2\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\, dV' \quad ,$$

so that $\int \phi\,\phi_{,i}\,\frac{\partial}{\partial x'_i}\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\, dV' = 2\pi\phi^2(r)$, and thus

$$h'_{44} = h_{44} + 8\pi G E \phi^2 \quad . \qquad (A2)$$

We may therefore get any perihelion shift we want by just changing E. This gives us an additional requirement that any expression for the stresses must satisfy.

To give an example at this point that we are not dealing with quantities as they normally appear, we look at the principle that if we add to a given stress-energy tensor a symmetric quantity which is a divergence and has no divergence, the same physics follows. Consider the expression

$$\begin{aligned} X'_{\mu\nu} &= h_{\alpha\beta,\mu} h_{\alpha\beta,\nu} + h_{\alpha\beta} h_{\alpha\beta,\mu\nu} - \\ &\quad - \delta_{\mu\nu}\left[h_{\alpha\beta,\gamma} h_{\alpha\beta,\gamma} + h_{\alpha\beta} h_{\alpha\beta,\gamma\gamma}\right] . \end{aligned} \qquad (A3)$$

We see that $X'_{\mu\nu} = \left[\delta_{\nu\lambda} h_{\alpha\beta,\mu} h_{\alpha\beta} - \delta_{\mu\nu} h_{\alpha\beta,\lambda} h_{\alpha\beta}\right]_{,\lambda}$, and also $X'_{\mu\nu,\nu} = 0$. For the trace of $X'_{\mu\nu}$ and X'_{44} we get

$$\begin{aligned} X'_{\lambda\lambda} &\cong -3\left[h_{\alpha\beta,\gamma} h_{\alpha\beta,\gamma} - h_{\alpha\beta} h_{\alpha\beta,\gamma\gamma}\right] \\ X'_{44} &\cong -\left[h_{\alpha\beta,\gamma} h_{\alpha\beta,\gamma} - h_{\alpha\beta} h_{\alpha\beta,\gamma\gamma}\right] \quad , \end{aligned}$$

so that $E \neq 0$ in equation A1, and we get a different perihelion shift.

In expanding the various forms of the field equations, we found that we got different $X^{(2)}_{\mu\nu}$ for each case, even though the sum $T^{\mu\nu} + X_{\mu\nu}$ was invariant. Thus specifying that $X^{(2)}_{\mu\nu}$ comes from the expansion of the field equations does not specify $X^{(2)}_{\mu\nu}$ because one must specify which matter tensor one is using in splitting up the $S_{\mu\nu}$. The field equations come from a variation with respect to $g_{\mu\nu}$ of the quantity

$$A = \int R\sqrt{-g}\, d^4\tau + \int g_{\mu\nu}\, T^{\mu\nu}\sqrt{-g}\, d^4\tau \quad ,$$

which then specifies that

$$\left[R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu} R\right]\sqrt{-g} = 8\pi G\; T^{\mu\nu}\sqrt{-g} \quad . \qquad \text{(A4)}$$

Thus we may ask if the requirement that $X^{(2)}_{\mu\nu}$ comes from the variation of an action uniquely determines it, if we are considering the matter tensor $\tilde{T}^{\mu\nu}$.

We want to have $X^{(2)}_{\mu\nu}$ have two h's and two derivatives; therefore we must consider terms in the action A like $\int h_{\alpha\beta,\gamma} h_{\alpha\beta,\delta} h_{\gamma\delta}\, d^4\tau$. We can find 16 independent terms of this type. Variation of this action results in a $X^{(2)}_{\mu\nu}$ with 16 undetermined parameters. If we use the condition that in the absence of matter, $T^{\mu\nu} = 0$, we have that $X^{(2)}_{\mu\nu}$ has zero divergence. This yields 34 equations that the 16 parameters obey. They are not all independent, but it turns out that all the parameters are determined up to a scaling factor in the action or stresses. If we then calculate the divergence of $X^{(2)}_{\mu\nu}$ in the presence of matter, we should find that this is the negative of the divergence of some matter tensor, because we want energy and momentum conservation. Calculating this, we find that

$$X^{(2)}_{\mu\nu,\nu} = \text{const.}[\alpha\beta,\mu]\tilde{T}^{\alpha\beta} \quad ,$$

when the first order field equations are used. Hence this $X^{(2)}_{\mu\nu}$ corresponds to the $\tilde{T}^{\mu\nu}$ discussed before. The remaining undetermined constant is then evaluated, and we then have the same $\tilde{X}^{(2)}_{\mu\nu}$ as was found from expanding equation A4.

It is to be noted that we have not used the assumption $\bar{h}_{\mu\nu,\nu} = 0$. If this coordinate condition is used after the variation of the action, then there are only 16 equations to be satisfied and not 34. Obviously we are not allowed to make this restriction in the action itself. With $\bar{h}_{\mu\nu,\nu} = 0$, we identically reproduce the reduced $\tilde{X}^{(2)}_{\mu\nu}$, equation 3.22. Thus we have found a uniquely defined $\tilde{X}^{(2)}_{\mu\nu}$ which corresponds to only one $\tilde{T}^{\mu\nu}$ (the correspondence given by the divergence condition) and none, say, which correspond to other forms of the matter tensor. We will now investigate to what extent there may be other choices.

One might assume that the correct gravitational stresses would be obtained by the same methods as in the determination of the electromagnetic stress tensor.

In addition to the variation of an action method, they could be:

a) Canonical construction of the stresses.

b) Restriction to first derivatives.

c) Building up the stresses from invariant fields.

d) Working from the equations of motion backwards to the divergence of the stresses.

e) Guessing.

a) Canonical.

We start with the Lagrangian of the first order equation

$$\mathcal{L} = -\tfrac{1}{2}\Big[h_{\mu\nu,\sigma}h_{\mu\nu,\sigma} - 2\,h_{\mu\nu,\nu}h_{\mu\sigma,\sigma} + 2\,h_{\mu\nu,\nu}h_{\sigma\sigma,\mu} - h_{\alpha\alpha,\mu}h_{\sigma\sigma,\mu}\Big] \quad , \qquad \text{(A5)}$$

from which Euler's equations yield the correct first order field equations in the absence of matter. We may use the canonical procedure to give an expression for the second order stresses(3). This, in the case of gravity, is not symmetric. Belinfante has given a method of symmetrizing any canonical stress tensor by adding a divergenceless quantity. However, there is no reason why this symmetrization is unique, and

[3] G. Wentzel, Quantum Theory of Fields (Interscience Publ., New York, 1949).

indeed we may add any amount of terms like equation A3 to the answer.

However, something can be gained of interest by this method. If we choose $\bar{h}_{\mu\nu,\nu} = 0$, then the solution of the canonical method plus Belinfante corrections breaks up into two pieces: a simple appealing expression for $X^{(2)}_{\mu\nu}$ and some divergenceless junk. This appealing piece is

$$X''^{(2)}_{\mu\nu} = \frac{A}{32\pi G}\left[h_{\alpha\beta,\mu}\bar{h}_{\alpha\beta,\nu} - \tfrac{1}{2}\delta_{\mu\nu}h_{\alpha\beta,\gamma}\bar{h}_{\alpha\beta,\gamma}\right] . \qquad (A6)$$

In the absence of matter this has zero divergence. In the presence of matter it is related to $\tilde{T}_{\mu}{}^{\nu}$, i.e.,

$$\tilde{T}_{\mu}{}^{\nu}{}_{,\nu} = \tfrac{1}{2}\, h_{\alpha\beta,\mu}\,\tilde{T}^{\alpha\beta} = -\, X''^{(2)}_{\mu\nu,\nu} \quad ; \; A = 1 \; .$$

So, one might say, this is a simplified version of the stresses obtained from the expansion of the field equations

$$\left[R_{\mu}{}^{\nu} - \tfrac{1}{2}\, g_{\mu}^{\nu}\, R\right]\sqrt{-g} = 8\pi G\, \tilde{T}_{\mu}{}^{\nu} . \qquad (A7)$$

This is not valid for two reasons:

i) The $X^{(2)}_{\mu\nu}$ obtained from equation A7 is not symmetric, but equation A6 is symmetric.

ii) The perihelion effect is wrongly given.

We could, of course, get the correct perihelion shift by adding an appropriate amount of equation A3, but then the simplicity is gone and, what is more important, the expression would probably give the wrong answers to problems of higher order which have not been experimentally verified.

b) First derivatives.

Let us use the condition that $\bar{h}_{\mu\nu,\nu} = 0$. Then we can get 10 different terms in $X^{(2)}_{\mu\nu}$ and thus 10 coefficients to solve for. The requirement that $X^{(2)}_{\mu\nu,\nu} = 0$ in the absence of matter yields 8 equations. We then get two types of solutions which may be superimposed: one which is the same as equation A6 and one which can be written

$$\overset{\prime\prime\prime}{X}{}^{(2)}_{\mu\nu} = \frac{B}{32\pi G}\Big[h_{\alpha\mu,\beta} h_{\alpha\beta,\nu} + h_{\alpha\nu,\beta} h_{\alpha\beta,\mu} + h_{\mu\nu,\alpha} h_{\sigma\sigma,\alpha} - h_{\mu\alpha,\beta} h_{\nu\alpha,\beta} - \tfrac{1}{2} h_{\mu\alpha,\nu} h_{\sigma\sigma,\alpha} - \tfrac{1}{2} h_{\nu\alpha,\mu} h_{\sigma\sigma,\alpha} - \tfrac{1}{2}\delta_{\mu\nu} h_{\alpha\beta,\gamma} h_{\alpha\gamma,\beta} - \tfrac{1}{8}\delta_{\mu\nu} h_{\sigma\sigma,\alpha} h_{\lambda\lambda,\alpha}\Big] . \quad \text{(A8)}$$

In the presence of matter,

$$\overset{\prime\prime\prime}{X}{}^{(2)}_{\mu\nu,\nu} \propto h_{\alpha\mu,\beta} \tilde{T}^{\alpha\beta} - \tfrac{1}{2} h_{\sigma\sigma,\alpha} \tilde{T}^{\alpha\mu} .$$

Consider the matter tensor given by

$$\overset{\prime}{T}{}^{\mu\nu} = (-g)\, T^{\mu\nu} \cong (1 + \tfrac{1}{2} h_{\sigma\sigma}) \tilde{T}^{\mu\nu} .$$

Then

$$\overset{\prime}{T}{}^{\mu\nu}{}_{,\nu} = \tilde{T}^{\mu\nu}{}_{,\nu} + \tfrac{1}{2} h_{\sigma\sigma,\nu} \tilde{T}^{\mu\nu} = -[\alpha\beta,\mu]\tilde{T}^{\alpha\beta} + \tfrac{1}{2} h_{\sigma\sigma,\alpha} \tilde{T}^{\mu\alpha} = -h_{\alpha\mu,\beta}\tilde{T}^{\alpha\beta} + \tfrac{1}{2} h_{\alpha\beta,\mu}\tilde{T}^{\alpha\beta} + \tfrac{1}{2} h_{\sigma\sigma,\nu}\tilde{T}^{\mu\nu}$$

so that if we choose A = 1, B = - 2, we have an $X_{\mu\nu}$ which corresponds to a $T^{\mu\nu}$ with only first derivatives. However the factor involving the perihelion shift is - 1 instead of $\frac{1}{2}$ as should be obtained.

c) Invariants.

In electromagnetism the basic field equations for A_μ are invariant if $A_\mu \longrightarrow A_\mu + \eta_{,\mu}$. Since in this case the fields are measurable quantities, they should also be invariant under the gauge transformation. Then we can only form the invariant tensor $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$. If the stresses are to be bilinear in the fields, we can form only

$$X_{\mu\nu} = A\, F_{\mu\alpha}\, F_{\nu\alpha} + B\, \delta_{\mu\nu} F_{\alpha\beta}\, F_{\alpha\beta} \quad ,$$

which, together with the correct divergence condition gives us A and B.

In gravity we look for combinations of $h_{\alpha\beta,\lambda}$ and see if there are any combinations which are invariant under $h_{\mu\nu} \rightarrow h_{\mu\nu} - \eta_{\mu,\nu} - \eta_{\nu,\mu}$. It turns out that there are none, even though the first order field equations are invariant under this transformation. Thus we can have no analogue for this method in gravity.

d) Equations of Motion.

Since in gravity there are no invariant fields, we cannot write a simple expression like

$$f_\mu = \{\text{fields}\}_{\mu\nu\lambda} \; T^{\nu\lambda}$$

where f_μ and $T^{\nu\lambda}$ are independent of the fields $h_{\alpha\beta,\lambda}$ or the potentials $h_{\mu\nu}$, and thus of the gauge chosen. Then we cannot extrapolate back to get a unique $X_{\mu\nu}$ since there are no unique equations of motion.

e) Guessing.

We have already seen that there are many possibilities for $X^{(2)}_{\mu\nu}$ and only one seems uniquely defined. It may be instructive to find out how much freedom one has in guessing to hit upon a given choice.

If we write all possible $X^{(2)}_{\mu\nu}$ terms (with the coordinate condition $\bar{h}_{\mu\nu,\nu} = 0$), we find that there are 21 possible terms. The divergence condition yields 16 equations for the 21 undetermined parameters. Thus we have 5 arbitrary constants and a great deal of freedom. Then we can certainly construct many solutions which give the correct perihelion shift and correspond to some $T^{\mu\nu}$, but there would be no feeling that any one of them would be the correct one.

One may also ask if there is much freedom in choosing a symmetrical divergenceless (identically divergenceless) $X_{\mu\nu}$ which comes from the divergence of a quantity. We have 17 possible terms in any $F_{\mu\nu\lambda}$ from which we get the $X_{\mu\nu}$ by the relation $X_{\mu\nu} = F_{\mu\nu\lambda,\lambda}$.

Imposing the condition $X_{\mu\nu,\nu} = 0$, even in the presence of matter, we get 16 equations as before. However there are many dependent equations, and we are left with additional pieces like equation A3. These will not change the energy and momentum, but will change the predicted perihelion shift.

We can thus conclude that only the variational principle gives the second order stresses uniquely and correctly.