# PERTURBATION OF BISEMIGROUPS AND TRANSPORT THEORY

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A strongly continuous bisemigroup $E(t)$ on a Banach space $X$ is a function from $R\backslash\{0\}$ to the space of bounded operators $L(H)$ such that:
(i) $E(t)E(s)=\pm E(t+s)$ if $sgn(s)=sgn(t)=\pm$ and $E(t)E(s)=0$ if $sgn(s)=-sgn(t)$,
(ii) $E(\cdot)$ is strongly continuous,
(iii) $\Pi_+ + \Pi_- = I$,
where $\Pi_\pm = s\text{-}\lim(\pm E(t))$ as $\pm t \downarrow 0$, are the separating projectors for the bisemigroup. An operator $S$ is the generator of the bisemigroup $E(S;t)$ if $E(S;t)= \pm \exp(-tS)\Pi_\pm$, $\pm t>0$ and the separating projectors define invariant subspaces for $S$. The bisemigroup (B) will be called bounded, holomorphic (H), strongly decaying (SD) or exponentially decaying (ED) if the semi-groups $\pm E(t)\Pi_\pm$, $\pm t>0$, on $Ran\Pi_\pm$, have the respective property.

EDB (i.e. the resolvent of the generator contains a band around the imaginary axis) arise in situations where the notion exponential dichotomy is relevant, e.g. in system theory [1], or in the linearization of a dynamical system close to an invariant manifold [2]. SD or ED HB arise in the theory of stationary one dimensional transport processes [3),4),5),6),7)].

Suppose (for simplicity) that $S$ is a self-adjoint operator on a Hilbert space $H$ and zero is not an eigenvalue of $S$. It is immediate that $S$ generates a SDHB and the separating projectors are exactly the +/- spectral projectors for $S$. We ask the question: what are sufficient conditions a perturbation $S^x=S-SB$ to generate a SDHB? Note that this question is far more subtle that the problem of perturbing semigroups because in our case one has to find separating projectors for $S^x$, but even when $S$ is such a nice operator (self adjoint) there is no apriory reason why $S^x$ should have invariant subspaces.

Consider the operator valued function $\mathfrak{k}(\cdot)=E(S;\cdot)SB$. For a H-valued function $\psi(\cdot)$, define $(\mathfrak{L}\psi)(t)=\int ds\, \mathfrak{k}(t-s)\psi(s)$, $-\infty<s<\infty$. If the operator $(I+\mathfrak{L}^x)=(I-\mathfrak{L})^{-1}$ exists, then one checks that the perturbed bisemigroup $E^x$

is given by $E^{\times}(t)=((I+\mathcal{L}^{\times})E)(t)=E(t)+(k^{\times}*E)(t)$. In checking that $(I-\mathcal{L})$ is invertible, a key role plays the Bouchner-Phillips theorem [8),9),10)] (a noncommutative generalization of a lemma of Wiener[11)]). In order to apply the BP theorem we have to view $k$ as an element of an appropriate Banach algebra of operator valued functions, integrable in a certain sense, with multiplication given by convolution. Then the BP theorem asserts that $(I-\mathcal{L})$ is invertible iff its symbol $W(\lambda)$ is invertible on the extended imaginary axis. It is immediate to check that $W(\lambda)=I-\int\exp(\lambda t)k(t)dt=(\lambda-S)^{-1}(\lambda-S^{\times})$. Thus, if the operator $S^{\times}$ has no spectrum on the imaginary axis, $(I-\mathcal{L})$ is invertible, and so $S^{\times}=S-SB$ generates a bisemigroup.

A complete characterization of operators generating EDB was made in [1)]. They also proved a theorem for perturbation of EDB by finite rank perturbations. In [12)] a perturbation theorem was proved (though the term bisemigroup was not used) for EDB and perturbing operator B which is compact and satisfies the following regularity condition: $\mathrm{Ran}B\subset\mathrm{Ran}|S|^{-\alpha}$ for some $\alpha>0$. This condition assures that $k$ belongs to the space of Bochner integrable functions $L_1(\mathbf{R},L(H))$, for which the BP theorem holds. In trying to relax the regularity assumption one looks for weaker notions of integrability. Using the weak integrability of $k$ (which is always true), Feldman[13)] showed that $(I-\mathcal{L})$ is invertible on $L_2$ functions. But the $L_2$-setting is not appropriate for obtaining bisemigroup results. One has to view $\mathcal{L}$ as an operator on $C(\mathbf{R},H)$ instead. To get a better understanding of the different spaces of integrable functions it is useful to consider them as different tensor product spaces (see [14),15)]), e.g. $L_1(\mathbf{R},L(H))=L_1(\mathbf{R})\otimes_{\pi}L(H)$. It is known that $C(\mathbf{R},H)=C(\mathbf{R})\otimes_{\varepsilon}H$, thus the natural algebra of integrable operator valued functions with multiplication given by convolution is $\mathfrak{a}=L_1(\mathbf{R})\otimes_{\bar{\varepsilon}}L(H)$, where the norm is the operator norm on $\mathfrak{a}$ viewed as a subalgebra of the algebra of bounded operators on $C(\mathbf{R},H)$. One has the following:

**Theorem.** If S is self adjoint with no eigenvalue at zero and B is a compact operator such that $W(\lambda)$ is invertible on the extended imaginary axis then $S^{\times}$ generates a strongly decaying (exponentially decaying if the spectrum of S has a gap at zero) holomorphic bisemigroup. Moreover, for every t the difference $E(t)-E^{\times}(t)$ is a compact operator.

This theorem was proved in [4),5)] with one omission - we assumed that B is trace-class in order to prove that $\mathcal{L}$ is bounded in C(R,H). This gap was filled in [7)], see Lemma 2.1.

All the equations modeling different stationary, linear transport processes (radiative transfer, neutron transport, rarefied gas dinamics, etc.)[3)], when resticted to a spacially homogeneous media in a plane parallel, half-space geometry can be represented as the following abstract boundary value problem:

$$T d\psi(x)/dx = -A\psi(x)\ ,\quad \pm x>0;\quad Q_{\pm}\psi(0)=\varphi_{\pm};\quad \|\psi(x)\|=o(x^n)\ \text{ as } n\to\pm\infty,$$

where n is some positive integer, the +/- sign stands for the right/left half space problems, $\varphi_{\pm}$ is the incoming flux, $\psi(x)$ is a vector in a Hilbert space H, T is a self adjoint injective operator on H, $Q_{\pm}$ are the separating projectors for T, and the collision operator A typically has the form "identity plus a compact" or "Sturm-Liouville plus a compact". Suppose that $T^{-1}A$ is a generator of a SDHB with separating projectors $P_{\pm}$ . A solution of the BVP will take the form $E(T^{-1}A,x)h$ ($\pm x>0$ for the right/left half space problems) for some $h \in H$ such that $Q_{\pm}P_{\pm}h=\varphi_{\pm}$. Set $V=Q_+P_+ + Q_-P_-$ The BVP is uniquely solvable iff the albedo operator $E=V^{-1}$ exists. In case B=I-A is compact we obtain that $P_{\pm}-Q_{\pm}$ is compact $\Rightarrow$ I-V is compact $\Rightarrow$ V is Fredholm of index zero $\Rightarrow$ V is invertible iff KerV=0. One obtains that the BVP is uniquely solvable if KerA=0 and A is accretive. When KerA$\neq$0 one first shows that an $T^{-1}A$-invariant decomposition of H exists with one summand being the zero root linear manifold of $Z_0(T^{-1}A)$. On the compliment the analysis proceeds in the same fashion as for the KerA=0 case. The measures of nonuniqueness and nonexistance are given by the dimensions of certain subspaces of $Z_0(T^{-1}A)$. The case when A is self-adjoint and positive has a long history (for a complete reference to the literature see [3)]). A complete investigation of the unique solvability for the BVP in the case of nonsymmetric, accretive A was made in [4),6),7)].

When the collision operator A is unbounded the Fredholm alternative is not applicable and "weak" solutions in spaces larger than the initial space H are sought [3)]. In [4),16)] an elegant modification of the existing methods was presented. We propose to work in the closure of H with respect to the indefinite scalar product $(T\cdot,\cdot)$ - a Krein space.

Instead of the Fredholm alternative one uses some simple geometrical facts about maximal definite subspaces in a Krein space.

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