Hida distribution construction of $P(\phi)_d$ ($d \geq 4$) indefinite metric quantum field models without BPHZ renormalization

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Abstract

By removing the divergent functions (i.e., the divegent Fynman graphs) a system of "Schwinger functions " which corresponds to a $(\Phi^4)_4$ Euclidean quantum field theory is constructed. The system of "Schwinger functions", which are defined through the Hida distributions, satisfies the property of OS 2) (Euclidean covariance), OS 4) (Symmetry) and OS 5) (Cluster property). It does not satisfy OS 3) (Reflection positivity). Then, for the system of "Schwinger functions "a possibility that it admits an analytic continuation to a system of "Wightman functions "satisfying the modified Wightman axioms is discussed.

1 Explanation of well known results on $d = 2$ by means of probabilistic words

Below, let $d = 4$ or $d = 2$ with an adequate understanding.
Let $\dot{W}$ be the random variable such that $\dot{W}(\omega) \in \mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$, $P-a.e.\ \omega \in \Omega$, and for each $\varphi \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R})$, $< \dot{W}, \varphi >_{\mathcal{S}',\mathcal{S}}$ is a real valued Gaussian random variable (*white noise on* $\mathbb{R}^d$) satisfying

$$E\left[< \dot{W}, \varphi >_{\mathcal{S}',\mathcal{S}}\right] = 0, \tag{1}$$

$$E\left[< \dot{W}, \varphi_1 >_{\mathcal{S}',\mathcal{S}} \cdot < \dot{W}, \varphi_2 >_{\mathcal{S}',\mathcal{S}}\right] = \int_{\mathbb{R}^d} \varphi_1(\mathbf{x})\, \varphi_2(\mathbf{x})\, d\mathbf{x},\ \forall \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}). \tag{2}$$

Let $J^{\frac{1}{2}}_{d=2}$ be the integral kernel of the pseudo differental operator on $\mathcal{S}(\mathbb{R}^2)$ such that $(-\Delta+1)^{-\frac{1}{2}}$ with $\Delta = \Delta_{d=2}$
the Laplace operator on $\mathbb{R}^2$. For φ, $f_j \in \mathcal{S}(\mathbb{R}^2 \to \mathbb{R})$, $j = 1, \ldots, n$, let

$$\phi(f_j) = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f_j(\mathbf{x})\, J^{\frac{1}{2}}_{d=2}(\mathbf{x}-\mathbf{y}) d\mathbf{x} \right) \dot{W}(\mathbf{y}) d\mathbf{y}, \tag{3}$$

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and

$$<:\phi^4_{d=2}:,\varphi> \tag{4}$$

$$= \int_{(\mathbb{R}^2)^4}\left\{\int_{\mathbb{R}^2}\varphi(\mathbf{x})\prod_{i=1}^{4}J^{\frac{1}{2}}_{d=2}(x-y_i)dx\right\}:\dot{W}(\mathbf{y}_1)\cdots\dot{W}(\mathbf{y}_4):\,d\mathbf{y}_1\cdots d\mathbf{y}_4. \tag{5}$$

Then,

$$e^{-\lambda<:\phi^4_{d=2}:,1_\Lambda>}\in\cap_{p\geq 1}L^p(\Omega;P), \tag{6}$$

and the Schwinger function for $d=2$

$$S_n(f_1,\ldots,f_n)\equiv\frac{1}{Z(\lambda;\Lambda)}E\left[\phi(f_1)\cdots\phi(f_n)\,e^{-\lambda<:\phi^4_{d=2}:,1_\Lambda>}\right], \tag{7}$$

is well defined.

2 Formulation for $d=4$

By changing the 2-dimensional space time Gaussian white noise process by 4-dimensional space time ones in the above discussion, and if we apply the same considerations to $(\Phi^4)_4$ quantum field model, then all the terms in $\left(\phi(f_1)\cdots\phi(f_n)\,(<:\phi^4_{d=4}:,1_\Lambda>)^k\right)$ will not have the right to be random variables.

Denote

$$\mathbf{x}\equiv(t,\overrightarrow{x})\in\mathbb{R}\times\mathbb{R}^3=\mathbb{R}^4,\qquad \xi\equiv(\tau,\overrightarrow{\xi})\in\mathbb{R}\times\mathbb{R}^3=\mathbb{R}^4.$$

Let

$$\mathcal{F}[\varphi](\xi)=\int_{\mathbb{R}^4}e^{-2\pi\sqrt{-1}\mathbf{x}\cdot\xi}\varphi(\mathbf{x})d\mathbf{x},\quad \mathcal{F}^{-1}[\varphi](\mathbf{x})=\int_{\mathbb{R}^4}e^{2\pi\sqrt{-1}\mathbf{x}\cdot\xi}\varphi(\xi)d\xi$$

For each $\epsilon>0$, let $j^{\frac{1}{2}}_\epsilon(\xi)$, $j^{\frac{1}{2}}(\xi)$, $j_\epsilon(\xi)$, and $j(\xi)$, resp., be the symbol of the pseudo differential operators, resp., such that

$$j^{\frac{1}{2}}_\epsilon(\xi)\equiv\left(|\xi|^2+1+\epsilon(|\xi|^2+1)^2\right)^{-\frac{1}{2}},\qquad j^{\frac{1}{2}}(\xi)\equiv\left(|\xi|^2+1\right)^{-\frac{1}{2}},$$

$$j_\epsilon(\xi)\equiv\left(|\xi|^2+1+\epsilon(|\xi|^2+1)^2\right),\qquad j(\xi)\equiv\left(|\xi|^2+1\right).$$

and define

$$(J^{\frac{1}{2}}_\epsilon\varphi)(\mathbf{x})=\mathcal{F}^{-1}(j^{\frac{1}{2}}_\epsilon\hat{\varphi})(\mathbf{x}),\qquad (J^{\frac{1}{2}}\varphi)(\mathbf{x})=\mathcal{F}^{-1}(j^{\frac{1}{2}}\hat{\varphi})(\mathbf{x}),$$

$$(J_\epsilon\varphi)(\mathbf{x})=\mathcal{F}^{-1}(j_\epsilon\hat{\varphi})(\mathbf{x}),\qquad (J\varphi)(\mathbf{x})=\mathcal{F}^{-1}(j\hat{\varphi})(\mathbf{x}).$$

Symbolically

$$J^{\frac{1}{2}}_\epsilon=\left(-\Delta_{d=4}+1+\epsilon(-\Delta_{d=4}+1)^2\right)^{-\frac{1}{2}},\qquad J^{\frac{1}{2}}=(-\Delta_{d=4}+1)^{-\frac{1}{2}},$$

$$J_\epsilon=\left(-\Delta_{d=4}+1+\epsilon(-\Delta_{d=4}+1)^2\right)^{-1},\qquad J=(-\Delta_{d=4}+1)^{-1},$$

where

$$\Delta_{d=4} \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \qquad \text{with} \quad \mathbf{x} = (t, x, y, z).$$

Let

$$< \phi(\omega), \varphi >_{\mathcal{S}', \mathcal{S}} = \int_{\mathbb{R}^4} \left(\int_{\mathbb{R}^4} \varphi(\mathbf{x})\, J^{\frac{1}{2}}(\mathbf{x} - \mathbf{y}) d\mathbf{x} \right) \dot{W}(\mathbf{y}) d\mathbf{y}, \ \text{for } \varphi \in \mathcal{S}'(\mathbb{R}^4 \to \mathbb{R}). \tag{8}$$

Also, define an $\mathcal{S}'(\mathbb{R}^4 \to \mathbb{R})$-valued random variable $\phi_\epsilon(\omega)$ such that

$$< \phi_\epsilon(\omega), \varphi >_{\mathcal{S}', \mathcal{S}} = \int_{\mathbb{R}^4} \left(\int_{\mathbb{R}^4} \varphi(\mathbf{x})\, J_\epsilon^{\frac{1}{2}}(\mathbf{x} - \mathbf{y}) d\mathbf{x} \right) \dot{W}(\mathbf{y}) d\mathbf{y}, \ \text{for } \varphi \in \mathcal{S}'(\mathbb{R}^4 \to \mathbb{R}), \tag{9}$$

and for $p \in \mathbb{N}$, define

$$<: \phi_\epsilon^p :, \varphi > = \int_{(\mathbb{R}^4)^p} \left\{ \int_{\mathbb{R}^4} \varphi(\mathbf{x}) \prod_{i=1}^p J_\epsilon^{\frac{1}{2}}(\mathbf{x} - \mathbf{y}_i) d\mathbf{x} \right\} : \prod_{j=1}^p \dot{W}(\mathbf{y}_j) : \prod_{j=1}^p d\mathbf{y}_j, \tag{10}$$

We then see that (cf. [FelMagRivSé], [CRiv])

$$(<: \phi_\epsilon^4 :, 1_\Lambda >)^k = \sum_{G' \in A'(k;\epsilon,\Lambda)} G' + \sum_{G \in A(k;\epsilon,\Lambda)} G, \tag{11}$$

where $A'(k;\epsilon,\Lambda)$ is a set of random variables such that for $G'(k;\epsilon,\Lambda) \in A'(k;\epsilon,\Lambda)$ there exists an $n \in \mathbb{N} \cup \{0\}$ (in fact $0 \leq n \leq 4k$) and

$$\lim_{\epsilon \downarrow 0,\, \Lambda \uparrow \mathbb{R}^4} E\left[\left(\phi(f_1) \cdots \phi(f_n)\, G'(k;\epsilon,\Lambda) \right) \right] \ \text{diverges}, \quad \forall f_j \in \mathcal{S}(\mathbb{R}^4),\ j = 1, \ldots, n$$

and $A(k;\epsilon,\Lambda)$ is a set of random variables such that for $G(k;\epsilon,\Lambda) \in A(k;\epsilon,\Lambda)$

$$\lim_{\epsilon \downarrow 0,\, \Lambda \uparrow \mathbb{R}^4} E\left[\left(\phi(f_1) \cdots \phi(f_k)\, G(k;\epsilon,\Lambda) \right) \right] \ \text{converges},$$

$$\forall f_j \in \mathcal{S}(\mathbb{R}^4), \quad j = 1, \ldots, n, \quad \forall n \in \mathbb{N} \cup \{0\}, \tag{12}$$

By using (12) we can define a *Hida distribution*:

$$G = \lim_{\epsilon \downarrow 0,\, \Lambda \uparrow \mathbb{R}^4} G(k;\epsilon,\Lambda). \tag{13}$$

Definition 1 [Tensor Product]
Let F and G be the random variables in $\cap_{p \geq 1} L^p(\Omega; P)$ defined by the multiple stochastic integral with respect to $\dot{W}$ such that

$$F = \int_{(\mathbb{R}^4)^n} f(\mathbf{x}_1, \ldots, \mathbf{x}_n) : \prod_{k=1}^n \dot{W}(\mathbf{x}_k) : d\mathbf{x}_1 \cdots d\mathbf{x}_n, \tag{14}$$

$$G = \int_{(\mathbb{R}^4)^m} g(\mathbf{x}_1, \dots, \mathbf{x}_m) : \prod_{k=1}^{m} \dot{W}(\mathbf{x}_k) : d\mathbf{x}_1 \cdots d\mathbf{x}_n. \tag{15}$$

The *tensor product* $F \otimes G$ of F and G is defined by

$$F \otimes G = \int_{(\mathbb{R}^4)^{n+m}} f(\mathbf{x}_1, \dots, \mathbf{x}_n) g(\mathbf{x}_{n+1}, \dots, \mathbf{x}_{n+m}) : \prod_{k=1}^{n+m} \dot{W}(\mathbf{x}_k) : d\mathbf{x}_1 \cdots d\mathbf{x}_{n+m}. \tag{16}$$

As a usual notation by [GrotStreit], the notion expressed by $F \otimes G$ above is equivalent with the Wick product $F \diamond G$. ■

Definition 2 [simplified definition of Hida-distributions]
For $f^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{S}((\mathbb{R}^4)^n)$, $n \in \mathbb{N}$ ($f^{(0)}$ is understood as a constant), a random variable φ defined by

$$\varphi = \sum_{n=0}^{\infty} \int_{(\mathbb{R}^4)^n} f^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) : \prod_{k=1}^{n} \dot{W}(\mathbf{x}_k) : d\mathbf{x}_1 \cdots d\mathbf{x}_n,$$

is said to be in $(\mathcal{S})_r$ for $r \in \mathbb{N} \cup \{0\}$ if

$$\|\varphi\|^2_{L^2(\Omega,P),r} \equiv \sum_{n=0}^{\infty} n! \left\| (\prod_{k=1}^{n} (-\Delta_{\mathbf{x}_k} + |\mathbf{x}_k|^2 + 1)^r) f^{(n)} \right\|^2_{L^2((\mathbb{R}^4)^n} < \infty, \tag{17}$$

where $\Delta_{\mathbf{x}_k} \equiv \dfrac{\partial^2}{\partial^2 t_k} + \dfrac{\partial^2}{\partial^2 x_k} + \dfrac{\partial^2}{\partial^2 y_k} + \dfrac{\partial^2}{\partial^2 z_k}$ for $\mathbf{x}_k = (t_k, x_k, y_k, z_k) \in \mathbb{R}^4$.
$\|\varphi\|^2_{L^2(\Omega,P),r}$ is equivalent with

$$E\left[\Big(\sum_{n=0}^{\infty} \int_{(\mathbb{R}^4)^n} (\prod_{k=1}^{n} (-\Delta_{\mathbf{x}_k} + |\mathbf{x}_k|^2 + 1)^r) f^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) : \prod_{k=1}^{n} \dot{W}(\mathbf{x}_k) : d\mathbf{x}_1 \cdots d\mathbf{x}_n \Big)^2 \right].$$

We say that a sequence of random variables $\{G_\epsilon\}_{\epsilon > 0}$ defines a Hida distribution $G \in (\mathcal{S})_{-r}$ if there exists a constant $K < \infty$ and

$$|E\left[G_\epsilon \cdot \varphi\right]| \le K \|\varphi\|_{L^2(\Omega,P),r}, \qquad \forall \varphi \in (\mathcal{S})_r, \tag{18}$$

and the limit exists for $\forall \varphi \in (\mathcal{S})_r$:

$$\lim_{\epsilon \downarrow 0} E\left[G_\epsilon \, \varphi\right] \tag{19}$$

The following continuous linear functional G on $(\mathcal{S})_r$ is called as a Hida distribution in $(\mathcal{S})_{-r}$:

$$< G, \varphi >= \lim_{\epsilon \downarrow 0} E\left[G_\epsilon \, \varphi\right] \tag{20}$$

■

Remark 1 i) If the defining sequence $\{G_\epsilon\}_{\epsilon>0}$ of a *Hida distribution* G is composed by elements of n-*times* multiple stochastic integrals, then

$$E\left[G_\epsilon \cdot \Big(\int_{(\mathbb{R}^4)^m} f(\mathbf{x}_1,\ldots,\mathbf{x}_m) : \prod_{k=1}^{m} \dot{W}(\mathbf{x}_k) : d\mathbf{x}_1\cdots d\mathbf{x}_m\Big)\right] = 0, \tag{21}$$

for any $f \in \mathcal{S}(\mathbb{R}^m)$ with $m \neq n$.
ii) Since,

$$\begin{aligned}
&\left\|(\prod_{k=1}^{n}(-\Delta_{\mathbf{x}_k}+|\mathbf{x}_k|^2+1)^r)f\right\|_{L^2((\mathbb{R}^4)^n)}\\
&= \left\{\int_{(\mathbb{R}^4)^n}\left((\prod_{k=1}^{n}(|\mathbf{x}_k|^2+1)^{-2}\,(|\mathbf{x}_k|^2+1)^2(-\Delta_{\mathbf{x}_k}+|\mathbf{x}_k|^2+1)^r)f\right)^2 d\mathbf{x}_1\cdots d\mathbf{x}_n\right\}^{\frac{1}{2}}\\
&\le \left(\sup_{\mathbf{x}_1,\ldots,\mathbf{x}_n}\left|(\prod_{k=1}^{n}(|\mathbf{x}_k|^2+1)^2(-\Delta_{\mathbf{x}_k}+|\mathbf{x}_k|^2+1)^r)f\right|\right)\cdot\left\|\prod_{k=1}^{n}(|\mathbf{x}_k|^2+1)^{-2}\right\|_{L^2((\mathbb{R}^4)^n)}\\
&\le K\cdot p_{m,k}(f), \qquad \forall f(\mathbf{x}_1,\ldots,\mathbf{x}_n)\in\mathcal{S}((\mathbb{R}^4)^n),
\end{aligned} \tag{22}$$

where $p_{m,k}(\cdot)$ is a semi-norm of $\mathcal{S}((\mathbb{R}^4)^n)$ such that

$$p_{m,k}(f) = \sum_{|\alpha|\le m}\sup_{\overline{\mathbf{x}}}(1+|\overline{\mathbf{x}}|^2)^k|D^\alpha f(\overline{\mathbf{x}})|, \tag{23}$$

with

$$\overline{\mathbf{x}} = (\mathbf{x}_1,\ldots,\mathbf{x}_n)\in(\mathbb{R}^4)^n, \qquad \mathbf{x}_k = (t_k,x_k,y_k,z_k)\in\mathbb{R}^4,$$

$$\alpha = (\alpha_1,\ldots,\alpha_n), \quad \alpha_k = (\alpha_{k1},\alpha_{k2},\alpha_{k3},\alpha_{k4}), \quad |\alpha| = \sum_{k=1}^{n}\sum_{i=1}^{4}\alpha_{ki},$$

$$D^\alpha = \prod_{k=1}^{n}\left(\frac{\partial}{\partial t_k}\right)^{\alpha_{k1}}\left(\frac{\partial}{\partial x_k}\right)^{\alpha_{k2}}\left(\frac{\partial}{\partial y_k}\right)^{\alpha_{k3}}\left(\frac{\partial}{\partial z_k}\right)^{\alpha_{k4}}.$$

By this, if a *Hida distribution* G is defined through a sequence $\{G_\epsilon\}_{\epsilon>0}$ of n-*times* multiple stochastic integrals, then it can be *identified* with an element of $\mathcal{S}'((\mathbb{R}^4)^n)$:
$\exists K<\infty$ and $\exists m,\,k\in\mathbb{N}$ that depend only on r and

$$\begin{aligned}
&|<G,\varphi>|\le K\cdot p_{m,k}(f)\\
&\mathcal{S}'((\mathbb{R}^4)^n)\ni G\ :\ \mathcal{S}((\mathbb{R}^4)^n)\ni f\longmapsto<G,\varphi>,
\end{aligned} \tag{24}$$

for

$$\varphi = \int_{(\mathbb{R}^4)^n} f(\mathbf{x}_1,\ldots,\mathbf{x}_n) : \prod_{k=1}^{n}\dot{W}(\mathbf{x}_k) : d\mathbf{x}_1\cdots d\mathbf{x}_n.$$

■

3 Well defined terms for $d = 4$ and the strategy of the consideration

Through the discussions in the previous sections, we can define

$$e^{-\lambda <:\phi_\epsilon^4:, 1_\Lambda>} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(k!)} \Big(<: \phi_\epsilon^4 :, 1_\Lambda > \Big)^k. \tag{25}$$

The equality holds for $P - a.e.\ \omega \in \Omega$, because both side of the equality are real valued random-variables, regardless they are integrable or not. The number of terms (graphs) of $\big(<: \phi_\epsilon^4 :, 1_\Lambda >\big)^k$, is estimated by $(k!)^2$:

$$\big(<: \phi_\epsilon^4 :, 1_\Lambda >\big)^k = \sum_{G \in \tilde{A}(k;\epsilon,\Lambda)} G, \tag{26}$$

where, each $G \in \tilde{A}(k;\epsilon,\Lambda)$ is a *tensor product of* multiple stochastic integrals, and in the sense of Fynman graph it is a graph with k-verticies, also the cardinarity of the set $\tilde{A}(k;\epsilon,\Lambda)$ is the order $(k!)^2$. By the notation of (11), $\tilde{A}(k;\epsilon,\Lambda)$ can be expressed by the direct sum

$$\tilde{A}(k;\epsilon,\Lambda) = A'(k;\epsilon,\Lambda) \oplus A(k;\epsilon,\Lambda). \tag{27}$$

For $r \geq 0$, let

$$\Lambda_r \equiv \left\{ (t,x,y,z) \in \mathbb{R}^4 \,\middle|\, \sqrt{t^2+x^2+y^2+z^2} \leq r \right\}. \tag{28}$$

For each $k \in \mathbb{N} \cup \{0\}$, define $A_r(\epsilon,k)$, a subset of $A(k;\epsilon,\Lambda_r) \subset \tilde{A}(k;\epsilon,\Lambda_r)$ in (28), as follows:

$$A_r(\epsilon,k) \;=\; \text{the set with the elements } G \in \tilde{A}(k;\epsilon,\Lambda_r) \text{ (cf. (40)), each of which is a tensor product of stochastic integrals, and each component of the tensor product is identified with a connected Fynman graph who is in } \textit{type O} \text{ or } \textit{type I}. \tag{29}$$

Where, *type O* resp., and *type I* is defined as follows:
Type O is the set of graphs, each element of which satisfies the following:
(O-i) *each vertex of the graph has at least* 1 *free (open) leg,*
(O-ii) *there exists at least* 1 *vertex which has more than* 2 *free legs,*
(O-iii) *there exists a connected path passing through every vertex of the graph exactly onece,*
(O-iv) *in the graph there is no sub graph such that two vertices connect two or three legs each other.*
Type I is the set of graphs, each element of which satisfies the following:
(I) *between each vertex and the other vertex in the graph there exists exactly only one (directed) path that connects these two vericies, and each vertex of the graph has at least* 1 *free (open) leg.*

We would like to show that the limit

$$S_n(\phi(f_1),\ldots,\phi(f_n)) \equiv \lim_{r\to\infty}\left\{\lim_{\epsilon\downarrow 0} E\Big[\phi(f_1)\cdots\phi(f_n)\Big(\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\big(\sum_{G\in A_r(\epsilon,k)} G\big)\Big)\Big]\right\}, \tag{30}$$

for $f_j \in \mathcal{S}(\mathbb{R}^4 \to \mathbb{R}),\ j=1,\ldots,n,\ \ n\in\mathbb{N}\cup\{0\}$, defines a system of "*Schwinger functions*" $\{\mathcal{S}_n\}_{n\in\mathbb{N}\cup\{0\}}$ in a "*modified sense*" . *Because of (0.21), (O-i), (O-ii) and (I), since the sum is precisely a finite sum, and the expectation is clearly well-defined*:

$$\begin{aligned} &E\Big[\phi(f_1)\cdots\phi(f_n)\Big(\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\big(\sum_{G\in A_r(\epsilon,k)} G\big)\Big)\Big] \\ &\quad = E\Big[\phi(f_1)\cdots\phi(f_n)\Big(\sum_{k=0}^{n}\frac{(-\lambda)^k}{k!}\big(\sum_{G\in A_r(\epsilon,k)} G\big)\Big)\Big], \end{aligned} \tag{31}$$

and the limit of (30) can be taken within the framework of finite sum of *Hida distributions* (cf. Definition 2).

In fact the limit exists and we have Theorem's 4.1 and 4.2 below. To give the statements we prepare some notions and notations:
Denote the number of the elements of $A_r(\epsilon,k)$ by $N(A(k))$, and give an index to each $G\in A_r(\epsilon,k)$ to indicate it as $G_j(k;\epsilon,r),\ \ j=1,\ldots,N(A(k))$. Then,

$$A_r(\epsilon,k) = \Big\{G_j(k;\epsilon,r)\Big\}_{j=1,\ldots,N(A(k))}, \qquad k\in\mathbb{N}. \tag{32}$$

Recall that for $f_j\in\mathcal{S}(\mathbb{R}^4)$, $j=1,2,\ldots,$

$$\begin{aligned} \phi(f_1)\phi(f_2) &= :\phi(f_1)\phi(f_2): + E[\phi(f_1)\phi(f_2)] \\ \phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4) &= :\phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4): \\ &+ (:\phi(f_1)\phi(f_2):)\,E[\phi(f_3)\phi(f_4)] + (:\phi(f_1)\phi(f_3):)\,E[\phi(f_2)\phi(f_4)] \\ &+ (:\phi(f_1)\phi(f_4):)\,E[\phi(f_2)\phi(f_3)] + (:\phi(f_2)\phi(f_3):)\,E[\phi(f_1)\phi(f_4)] \\ &+ (:\phi(f_2)\phi(f_4):)\,E[\phi(f_1)\phi(f_3)] + (:\phi(f_3)\phi(f_4):)\,E[\phi(f_1)\phi(f_2)] \\ &+ \sum_{\text{distinguished } i_l\text{'s}} E[\phi(f_{i_1})\phi(f_{i_2})]E[\phi(f_{i_3})\phi(f_{i_4})], \end{aligned}$$

and

$$:\phi(f_1)\cdots\phi(f_n) := \int_{(\mathbb{R}^4)^n}\prod_{j=1}^{n}(J^{\frac12}f_j)(\mathbf{x}_j) : \prod_{j=1}^{n}\dot{W}(\mathbf{x}_j) :\, d\mathbf{x}_1\cdots d\mathbf{x}_n. \tag{33}$$

By using the above, through a simple evaluation we have the following:

Lemma 3.1 *For each $r \geq 0$, $\epsilon > 0$ and $k \geq 1$, let $A_r(\epsilon, k)$ be the set of Fynman graphs (i.e., set of multiple stochastic integrals) defined by (29). Then, by using the expression (32), the following hold:*

$$N(A(k)) \simeq (k!)^{\frac{3}{2}}, \tag{34}$$

and by denoting the number of free legs of $G_j(k; \epsilon, r) \in A_r(\epsilon; k)$ by $N_f(G_j(k; \epsilon, r))$,

$$k + 2 \leq N_f(G_j(k; \epsilon, r)) \leq 4k, \qquad j = 1, \ldots, N(A(k)), \tag{35}$$

also (cf. (33))

$$E\left[\left(: \phi(f_1) \cdots \phi(f_n) :\right) G_j(k; \epsilon, r)\right] = 0, \quad \text{if } n \neq N_f(G_j(k; \epsilon, r)), \tag{36}$$

$$E\left[\left(\phi(f_1) \cdots \phi(f_n)\right) G_j(k; \epsilon, r)\right] = 0, \quad \text{if} \quad n - 2 < k, \tag{37}$$

moreover, there exists $M < \infty$ and

$$\left| E\left[\left(: \phi(f_1) \cdots \phi(f_n) :\right) G_j(k; \epsilon, r)\right] \right| \leq M^k (n!) \prod_{i=1}^{n} \left\| \hat{f}_i \right\|_{L^1(\mathbb{R}^4)}, \tag{38}$$

for $\forall f_i \in \mathcal{S}(\mathbb{R}^4 \to \mathbb{R})$, $i = 1, \ldots, n$, $\forall n \in \mathbb{N}$; $\forall G_j(k; \epsilon, r) \in A_r(\epsilon, k)$; $\forall k \in \mathbb{N}$, $\forall \epsilon > 0$, and $\forall r > 0$ (if $n \neq N_f(G_j(k; \epsilon, r))$, then "right hand side of (33)"$= 0$).

□

From Lemma 4.1, we immediately have

Theorem 3.1 *For each $k \in \mathbb{N}$ and $j = 1 \ldots, N(A(k))$, there exists a Hida distribution $G_j(k)$, which is a Fynman graph, such that*

$$\lim_{r \to \infty} \lim_{\epsilon \downarrow 0} G_j(k; \epsilon, r) = G_j(k) \qquad \text{with} \quad N_f(G_j(k)) = N_f(G_j(k; \epsilon, r)) \tag{39}$$

and the set $A_r(\epsilon, k)$ converges to a set of Hida distributions $A(k)$,

$$\lim_{r \to \infty} \lim_{\epsilon \downarrow 0} A_r(\epsilon, k) = A(k) = \{G_j(k)\}_{j=1,\ldots,N(A(k))}. \tag{40}$$

Denote

$$\left\langle \left(: \phi(f_1) \cdots \phi(f_n) :\right), G_j(k) \right\rangle \equiv \lim_{r \to \infty} \lim_{\epsilon \downarrow 0} E\left[\left(: \phi(f_1) \cdots \phi(f_n) :\right) G_j(k; \epsilon, r)\right], \tag{41}$$

then there exists $M < \infty$ and

$$\left| \left\langle \left(: \phi(f_1) \cdots \phi(f_n) :\right), G_j(k) \right\rangle \right| \leq M^k (n!) \prod_{i=1}^{n} \|(-\Delta_{d=4} + 1)^2 f_i\|_{L^2(\mathbb{R}^4)}, \tag{42}$$

$$\Big|\Big\langle\big(:\phi(f_1)\cdots\phi(f_n):\big),\,G_j(k)\Big\rangle\Big| \le M^k(n!)\prod_{i=1}^{n}p_{4,2}(f_i), \tag{43}$$

for $\forall f_i \in \mathcal{S}(\mathbb{R}^4 \to \mathbb{R}),\ i=1,\ldots,n,\ \forall n\in\mathbb{N};\ \ \forall G_j(k)\in A(k);\ \ \forall k\in\mathbb{N}$
(if $n \neq N_f(G_j(k))$*, then "right hand side of (42) and (43)"*$=0$*), where* $p_{m,k}(f)$ *is the semi-norm defined by (23).*

□

By the above theorem, we can set the following Definition.

Definition 3 Let $A(0)=\{1\}$, namely, $G_1(0)=1$ and $N(A(0))=1$. For each $\lambda\in\mathbb{Z}$ let $\{S_n^\lambda\}_{n\in\mathbb{N}\cup\{0\}}$, be a system of Schwartz distributions such that

$$S_0^\lambda = S_1^\lambda = 0, \tag{44}$$

$$\begin{aligned} < S_n^\lambda\,,\, f_1\otimes\cdots\otimes f_n > &= \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\left(\sum_{j=1}^{N(A(k))}\big\langle\big(\phi(f_1)\cdots\phi(f_n)\big),\,G_j(k)\big\rangle\right)\\ &= \sum_{k=0}^{n-2}\frac{(-\lambda)^k}{k!}\left(\sum_{j=1}^{N(A(k))}\big\langle\big(\phi(f_1)\cdots\phi(f_n)\big),\,G_j(k)\big\rangle\right)\\ &\quad \text{for}\quad f_i\in\mathcal{S}(\mathbb{R}^4),\ i=1,\ldots,n,\ n\in\mathbb{N}. \end{aligned} \tag{45}$$

□

For the statements of the next theorem, we recall the OS (Osterwalder-Schrader) axioms (cf., e.g., [Si1]):
It is said that a set of functions $\{S_n\}$ is a *system of Schwinger functions* with *Osterwalder-Schrader axioms* OS'1-5, if it satisfies
OS 1') (Temperedness + Analytic continuity)

$$S_n^p\in\mathcal{S}'_{\neq}(\boldsymbol{R}^{dn}),\qquad \overline{S_n^p(f)}=S_n^p(\theta f^*),$$

and S_n^p is a Laplace transform of some $M_n\in\mathcal{S}'(({\boldsymbol{R}_+^d})^{n-1})$, precisely

$$\begin{aligned} S_n^p(x_1^0,\vec{x}_1;\ldots;x_n^0,\vec{x}_n) &= \int_{({\boldsymbol{R}_+^d})^{n-1}} M_(\tau_1,\xi_1;\ldots;\tau_{n-1},\xi_{n-1})\\ \times\exp\Big\{\sum_{j=1}^{n-1}&\big(\sqrt{-1}\xi_j\cdot(\vec{x}_{j+1}-\vec{x}_j)-\tau_j(x_{j+1}^0-x_j^0)\big)\Big\}d\xi_1 d\tau_1\cdots d\xi_{n-1}d\tau_{n-1}; \end{aligned} \tag{46}$$

OS 2) (Euclidean covariance);
OS 3) (Reflection positivity);

OS 4) (Symmetry);
OS 5) (Cluster property).

Theorem 3.2 *Let $\{S_n^\lambda\}_{n\in\mathbb{N}\cup\{0\}}$ be the system of Schwartz distributions defined by Definition 3, then it is a system of "modified Schwinger functions" in the sense that it satisfies OS 2), OS 4) and OS 5).*

□

Remark 2.

We have an affirmative rigorous result on the analyticity OS 1') for $\{S_n^\lambda\}_{n\in\mathbb{N}\cup\{0\}}$. It will be announced in forthcomming papers.

■

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