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MECHANICAL INSTABILITIES OF A SUPERCONDUCTING HELICAL STRUCTURE DUE TO RADIATION PRESSURE

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ABSTRACT

The condition for mechanical instability due to radiation pressure is derived and the electro-mechanical coupling factor for a helical structure is calculated. The results are discussed with respect to the requirements of a superconducting proton accelerator.

1. Introduction

At field levels of about 1 MV/m not only breakdown and decreasing of the Q-value but also loss of mechanical stability can become the limiting factor for operation of an accelerating structure.

Because of the high electric quality factor and the low rigidity of a superconducting helical structure it is evident, that mechanical instability can be more serious compared with a normalconducting structure.

Instabilities of this type have been reported previously by Karliner et al. [1] with a normal conducting single-cell cavity.

For a theoretical treatment two new aspects arise. Firstly, the mechanical structure is characterized by a distinct spatial distribution and secondly the approximation used by Karliner et al. [1] - mechanical eigenfrequency << electric bandwidth - is generally not satisfied.

A more general stability condition derived by the author [2] without the restricting assumption mentioned above will be reported and applied to calculate the threshold stability for a special set of mechanical and electric parameters.

2. Electromechanical coupling factor

To get a rough impression of the electromechanical system we will use two simple models, the sheath model [3,4] for the electric behaviour and a homogeneous "soft" rod [5] for the mechanical behaviour.

The gradient of the force acting on the wires can be obtained as

$$F'_\nu(z) = 2\pi a(\varepsilon_o E_z(\gamma_\nu a)\Delta E_r(\gamma_\nu a) + \mu_o H_r(\gamma_\nu a)\Delta H_z(\gamma_\nu a)) \qquad (2.1)$$

where Δ takes into account the discontinuity at the radius of the helix $r = a$, γ_ν is the radial wave number of the mode ν.

After solving the mechanical wave equation the deviation η_μ from the equilibrium position is determined by the differential equation [6]

$$\ddot{\eta}_\mu + \frac{\Omega_\mu}{Q_{m\mu}}\dot{\eta}_\mu + \Omega_\mu^2 \eta_\mu = -\frac{1}{m}\frac{\int_o^l \phi_\mu(z)F'_\nu(z,t)dz}{\int_o^l \phi_\mu^2(z)dz} \qquad (2.2)$$

where $\phi_\mu(z)$ is the mechanical normal mode μ, m the mass density, Ω_μ the mechanical angular frequency and $Q_{m\mu}$ the mechanical quality factor assumed to be very large compared with unity.

On the other hand the frequency shift due to adiabatic deformations of a metallic wall is according to Slater's formula

$$\frac{\Delta\omega}{\omega_\nu} = \frac{1}{2W_\nu}\int_{\Delta V}(\mu_o H^2 - \varepsilon_o E^2)dv \qquad (2.3)$$

where W_ν is the stored energy in the whole resonator and ΔV the perturbation volume.

If we assume that only one element of the wire is moved, while the others remain on their equilibrium position, the overall field, which is fairly well described by the sheath model, will also remain. The frequency shift caused by a distributed deviation $y_\mu(z,t) = \eta_\mu(t)\,\phi_\mu(z)$ in a first order approximation according to (2.3) is then given by

$$\left(\frac{\Delta\omega}{\omega_\nu}\right)_\mu = \frac{1}{2W_\gamma}\,\frac{\pi D^2}{4\tan\psi}\left(\frac{f(a+0) + f(a-0)}{2}\right)\int_o^l \phi_\mu(z)\,\frac{dq_\nu(z)}{dz}\,dz\cdot\eta_\mu(t) \qquad (2.4)$$

where the abbreviation $\frac{1}{2}(\mu_o H^2 - \varepsilon_o E^2) = f_\nu(r) q_\nu(z)$ has been used, ψ is the pitch angle, D the outer diameter of the wire and $\frac{1}{2}(f(a+0) + f(a-0))$ is the contribution of the integration in r-direction across the discontinuity at $r = a$, if $D << a$ holds. Looking at the right side of equation (2.4) it is obvious, that the maximum perturbation of the eigenfrequency arises, when the product of mechanical deviation and the gradient of the electric, respectively the magnetic field energy with sign reversed, reaches its maximum.

An rf-structure operating near its resonant frequency can be regarded as an electric parallel resonant circuit with the lumped parameters R, L, C, respectively ω_ν-resonant frequency, Q_ν-loaded quality factor and Z_ν- shunt impedance known from theoretical calculations or measurements [7].

For the slowly varying complex amplitude of the accelerating voltage, which is defined by $E_{z\nu}(z,t)=E_{z\nu}(z)\mathrm{Re}(\vec{V}_\nu e^{j\omega_q t})$ we obtain the well-known differential equation of first order [1, 8]

$$\frac{2Q_\nu}{\omega_\nu}\dot{\vec{V}}_\nu + \left(1 + j\, 2Q_\nu\left(\frac{\omega_q - \omega_\nu}{\omega_\nu} + \left(\frac{\Delta\omega}{\omega_\nu}\right)_\mu\right)\right)\vec{V}_\nu = \vec{I}_\nu Z^{-1} \tag{2.5}$$

where $\vec{I}_\nu$ represents the vector sum of the generator and the beam current source without feedback from the electric field. $(\frac{\Delta\omega}{\omega_\nu})_\mu$ is the additional detuning caused by the interaction of the mechanical mode μ with the electric mode ν, while $\frac{\omega_q - \omega_\nu}{\omega_\nu}$ is the normal detuning adjustable by the generator frequency ω_q, which will be treated in the following discussion as the operating point parameter with the abbreviation $x = 2Q_\nu \frac{\omega_q - \omega_\nu}{\omega_\nu}$.

Inserting (2.1) into (2.2) and (2.4) into (2.5) we get two non-linear coupled differential equations. Now let us specify the mechanical boundary conditions. Generally the edges of the "soft" rod are confined by differential equations, which determine the coefficients A_μ, Φ_μ and the wave numbers k_μ of the general solution, or space harmonics, of the mechanical wave equation

$$\phi_\mu(z) = \sum_\mu A_\mu \sin(k_\mu z + \Phi_\mu) \tag{2.6}$$

In the simplest case with fixed ends at $z = 0$ and $z = l$, all Φ_μ vanish and the wave numbers are $k_\mu = \mu\frac{\pi}{l}$. The mechanical resonant frequencies are obtained by inserting k_μ into the dispersion relation

$\Omega_\mu = k_\mu c_m$, c_m being the velocity of sound along the helix. A better approach is to take into account the stiffness of the support. This leads to a phase shift $\Phi_\mu \neq 0$ and lowering of the mechanical eigenfrequencies.

Similar for the electric normal modes, taking into account end effects, the real field can expanded in normal modes

$$E,\ H_{z\nu}(z) = \sum_\nu B_\nu \cos(k_\nu z + \Phi_{z\nu})$$
$$E,\ H_{r\nu}(z) = \sum_\nu C_\nu \sin(k_\nu z + \Phi_{r\nu}) \qquad (2.7)$$

To understand the principal behaviour we shall confine ourselves to the ideal cases $\Phi_\mu = 0$, $\Phi_{z\nu} = 0$ and $\Phi_{r\nu} = 0$. In both equations (2.2) and (2.4) we obtain integrals of the type $\int_0^1 \sin \mu \frac{\pi z}{1} \sin 2\nu \frac{\pi z}{1}\, dz$ which vanishes for all combinations of μ and ν, excepted $\mu = 2\nu$. More general the integrals in (2.2) and (2.4) work together like a filter, which picks up preferably mechanical modes with a wave number near or equal twice the wave number of the electric mode.

Evaluating the integrals (2.2) and (2.4) we finally get the static detuning due to radiation pressure for the helical structure using the relation $\mu = 2\nu$ for optimum coupling:

$$\left(\frac{\Delta\omega}{\omega_\nu}\right) = \frac{\pi D^2}{4} \frac{\tilde{\sigma}(\nu\pi \frac{a}{1})}{\tan\psi} \frac{\varepsilon_o}{2} E_\nu^2 \frac{1}{4f} \ ; \quad \tan\psi = \frac{1}{2\pi a w} \qquad (2.8)$$

f is the spring constant with $f = \frac{(D^4 - d^4)G1}{64\ a^3 w}$, w -number of windings, G-torsion modulus, d - inner diameter of the wire. $\tilde{\sigma}$ is a sensitivity function depending on the mode shown in fig. 1 for a typical range of normalized wave numbers $\nu\pi\frac{a}{1}$. It should be mentioned, that $\tilde{\sigma}$ is essentially positive as pointed out by Karliner et al. [1] and further that $\tilde{\sigma}$ shows only a weak dependence on the radius b of the outer conductor for reasonable values of b (b/a > 2).

$$\tilde{\sigma} = + I_o^2 \left(1 + \frac{I_o K_o}{I_1 K_1}\right) \left(\frac{I_1}{I_o} - \frac{K_1}{K_o}\right)^2 \qquad (2.9)$$

I, K are the modified Bessel functions of the order 0 and 1 with the argument $\nu\pi \frac{a}{l}$ and with $b \rightarrow \infty$. $\tilde{\sigma}$ has a minimum of about 3.0 at $a \simeq \lambda/4$. If the guide wavelength λ is greater or smaller than twice the helix diameter, the sensitivity becomes larger. Both, the magnetic and the electric fields lower the resonant frequency, but the contribution of the electric field to (2.4) can be neglected.

Leaving all mode indices and normalizing the standing wave amplitude to the resonant value $\vec{e} = \frac{\vec{E}}{E_q}$ we get the static equation

$$(1 + j\,(x + p\,|\vec{e}|^2))\vec{e} = 1 \qquad (2.10)$$

where the normalized value $p = 2\,Q_\nu(\frac{\Delta\omega}{\omega_\nu})$ with $E_\nu = |\vec{E}_q|$ can be called the electromechanical coupling factor.

The static resonance curve for the normalized power $|\vec{e}|^2$ as shown in fig. 2 becomes unstable, when at the left point of inflection with $|\vec{e}|^2 = \frac{3}{4}$ the slope becomes negative, which occurs, if p becomes greater than 1.54 (see curve 2 and 4 in fig. 2). This can be due to either a high field strength or a high Q-value as illustrated in the curves 2, 3 and 4, 5 of fig. 2. The broken lines in curve 1 of fig. 2 indicates the limits of static stability. The same hysteresis effect is shown by the phase curve while the frequency locus in the impedance plane is the same as without radiation pressure effect, only the detuning marks are changed.

3. Dynamical Instability

The question of total stability for this nonlinear system is not quite easy to answer. To begin with, we will investigate only small deviation from the steady state. This small signal stability is only a necessary but not a sufficient condition for total stability. From a practical point of view the total stability will be of a certain interest only in connection with a control system. Firstly, we will discuss the principal behaviour of the uncontrolled system, if the electric quality factor is increasing.

With the normalized detuning q caused by radiation pressure we obtain from (2.2) and (2.5) the system of differential equations

$$\ddot{q} + 2\,a\dot{q} + b^2 q = b^2 p\,|\vec{e}|^2 \qquad (3.1a)$$

$$\dot{\vec{e}} + (1 + j(x + q))\,\vec{e} = 1 \qquad (3.1b)$$

where the time is normalized with the electric decay time $\tau = \frac{2Q}{\omega}$, a is the ratio of the electric and mechanical decay time $a = \frac{Q}{Qm} \cdot \frac{\Omega}{\omega}$ and b is the product of the electric decay time and the mechanical angular frequency $b = 2Q\,\frac{\Omega}{\omega}$.

It should be noted, that the following investigations are applicable for any structure.

The steady state will be signed with the index zero and the complex amplitude written as $\vec{e} = \alpha + j\beta$. The steady state relations are then

$$\alpha_o = \alpha_o^2 + \beta_o^2 \qquad \text{(circle in the complex plane)} \tag{3.2a}$$

$$q_o = p\alpha_o \tag{3.2b}$$

$$\beta_o = -\alpha_o(x_o + q_o) \tag{3.2c}$$

Assuming small deviations $u + jv = \vec{e} - \vec{e}_o$ and $\lambda = q - q_o$ from the steady state and neglecting all nonlinear terms we get the homogeneous linear system of differential equations

$$\begin{aligned} &\ddot{\lambda} + 2\,a\dot{\lambda} + b^2\lambda = 2\,b^2 p\,(\alpha_o u + \beta_o v) \\ &\dot{u} + u - (x_o + q_o)v - \beta_o\lambda = 0 \\ &\dot{v} + v + (x_o + q_o)u + \alpha_o\lambda = 0 \end{aligned} \tag{3.3}$$

with $\dot{\lambda} = \mu$, which is proportional to the mechanical velocity, we define a state vector $\underline{z} = \begin{pmatrix} u \\ v \\ \lambda \\ \mu \end{pmatrix}$. Now the system can be written as $\dot{\underline{z}} = \underline{A}\ \underline{z}$ (3.4) with the matrix

$$\underline{A} = \begin{pmatrix} -1 & -\frac{\beta_o}{\alpha_o} & +\beta_o & 0 \\ +\frac{\beta_o}{\alpha_o} & -1 & -\alpha_o & 0 \\ 0 & 0 & 0 & 1 \\ 2b^2p\alpha_o & 2b^2p\beta_o & -b^2 & -2a \end{pmatrix}$$

The solution of (3.4) is with $\underline{z}_i$-initial state at $t = 0$,

$$\underline{z} = \underline{z}_i \, e^{+\underline{A} t/\tau} \qquad (3.5)$$

The solution $\underline{z}$ is stable, if all real parts of the complex eigenvalues κ_i of the equation det $(\underline{A} - \kappa \underline{1}) = 0$ are less than zero. Inserting (2.5) and using (3.2) we obtain

$$(\kappa^2 + 2\, a\kappa + b^2)((\kappa + 1)^2 + (\frac{\beta_o}{\alpha_o})^2) = 2\, b^2\, p\beta_o \qquad (3.6)$$

At the point of resonance $\beta_o = 0$ the system is always stable, because the coupling, i.e. the right side of (3.6), is zero as in the case of small radiation pressure $p \to 0$.

For this forth order system we can apply successfully an algebraic stability criterion [9]. Using the Hurwitz criterion we have to calculate the coefficients of the power series in κ

$$a_o\kappa^4 + a_1\kappa^3 + a_2\kappa^2 + a_3\kappa + a_4 = 0 \qquad (3.7)$$

Stability is given, if the determinant of the matrix

$$\begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_o & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_o & a_2 & a_4 \end{pmatrix}$$

and the minor determinants indicated by broken lines are positive. The first two, i.e. a_1 and $(a_1a_2 - a_oa_3)$, are always positive.

The last condition is equivalent to $a_4 > 0$, that is exactly the condition for static stability treated in part 2.

We are left with the condition

$$a(4(a + 1)\ (\frac{a}{\alpha_o} + b^2) + (b^2 - \frac{1}{\alpha_o})^2) + 2\, b^2\, p\beta_o(a + 1)^2 > 0 \qquad (3.8)$$

Because the electromechanical coupling factor p is essentially positive and the real part α_o of the complex voltage is positive (see equ. (3.2a)) (3.8) is always valid, if the imaginary part β_o is positive, i.e. on the left side of the resonance durve (see equ. 3.2c).

On the right side, with the simplification a << 1 which is valid in most practical cases, we have

$$p < \frac{a}{2} \frac{4 + (b - \frac{1}{\alpha_o b})^2}{\sqrt{\alpha_o (1 - \alpha_o)}} \tag{3.9}$$

The absolute minimum of the right side of (3.9) is obtained for

$$b = \sqrt{2} \quad \text{and} \quad \alpha_o = \frac{1}{2} \quad \text{whence follows} \quad p < 4a \tag{3.10}$$

That means, if the electric Q-value is $Q = \frac{1}{\sqrt{2}} \frac{\omega}{\Omega}$, we can already get oscillations at a certain lower field level $\tilde{E}$, while the resonance curve will just tip over at the considerable higher field level $0.76 \sqrt{Q_m} \cdot \tilde{E}$.

For b << 1 we have the approximate condition

$$p < \frac{a}{2b^2} \frac{1}{\alpha_o^2 \sqrt{\alpha_o (1 - \alpha_o)}} \tag{3.11}$$

The system has maximum sensitivity or in other words the lowest threshold field, if the right side of (3.11) has a minimum. This occurs in agreement with Karliner et al. [1] at $\alpha_o = 5/6$ and leads to $p < 1.93\ a/b^2$. The threshold field is therefore $\tilde{E} \sim \frac{1}{\sqrt{Qm} \cdot Q}$.

In the case b >> 1, i.e. electric decay time large compared with the time of a mechanical period, for a given b we can always find an operating point α_o, respectively a detuning $x_o = - p\alpha_o + \sqrt{\frac{1-\alpha_o}{\alpha_o}}$, which vanishes the quadruture term on the right of (3.9). This occurs for $\alpha_o = \frac{1}{b^2}$. At these minima the condition $p \overset{\sim}{<} 2\ ab$ should hold. The threshold field in this case is $\tilde{E} \sim \sqrt{\frac{Q}{Qm}}$ for $\alpha_o \sim \frac{1}{Q^2}$.

From a practical point of view operation far from the resonance is not very interesting, because the generator power must increase by a factor of $\frac{1}{\alpha_o}$ to produce the desired field at the axis, but one has to take care of this effect because of possible shock excitations caused by external perturbations especially during start-up.

If we tune the generator near to resonance, we get a threshold field $\tilde{E} \sim \frac{Q}{\sqrt{Qm}}$ for $\alpha_o \lesssim 1$. The situation is illustrated in fig. 3 where the right side of (3.9), i.e. the threshold radiation pressure $\tilde{E}^2$ is plotted against $b = 2\,Q\,\frac{\Omega}{\omega}$, divided by its maximum value $\tilde{E}^2_{opt}$ at $b = \sqrt{2}$ and $\alpha_o = \frac{1}{2}$. The heavy line is the envelope due to variation of α_o.

4. Example and Conclusion

Now we shall discuss the relations derived above with respect to a superconducting low energy accelerator presented at this conference[10].

Typical dimensions of the electrical strong coupled helical pieces in the region of 0.75 to 5 MeV are: radius a 3 - 4 cm, length l 7-17 cm, number of windings W 7 - 13, outer diameter of the wire 0.6 cm, inner diameter of the wire 0.45 cm with an electric fundamental frequency of 90 MHz.

For niobium this leads to a mean mechanical fundamental frequency of about 45 Hz. Because of $\mu = 2\nu$ (see part 2) for the lowest mode $\nu = 1$ we get a frequency ratio $\frac{\Omega}{\omega} \simeq 10^{-6}$.

The loaded quality factor of the resonator with high beam loading is in the case of optimal power matching [8] approximately $Q \simeq \frac{\text{reactive power of the resonator}}{\text{power absorbed by the beam}}$ and therefore independent of the unloaded quality factor.

For a designed energy gain of 2 MeV/m and a proton current of some 100 µA we obtain with the geometry factor $Z/Q \simeq 10$ kΩ/m a loaded Q of the order of magnitude of 10^6. Since $\frac{\Omega}{\omega} \cong 10^{-6}$ we can expect as shown in part 3 even for this low Q-value maximum excitation of vibrations due to radiation pressure (see fig. 3).

According to (2.8) for the geometry given above we get a coupling factor $p \simeq Q \cdot 10^{-4}$ with the field level of the standing wave of about 5 MV/m. Inserting in (3.9) we see, that the desired field exceeds the threshold field by a factor of more than 100, if a mechanical Q-value of $10^3 - 10^4$ is assumed.

To overcome this instability one needs a fast frequency feedback system [11]. For the operation of a helical rf-unit in a linear accelerator each tank must be tuned to a common frequency. Investigations on the requirement of the system due to radiation pressure are still in progress.

A serious additional task for the frequency control system is the damping of frequency detuning caused by vibrations of the support system of the helix. Under typical working conditions we observed relativ frequency shifts of about 10^{-6} with a modulation frequency up to 100 Hz, which can increase by a factor of 10 due to shock exciting. In order to give sufficient damping the frequency control loop should have a unity gain at minimum 5 kHz. This seems to be a very hard requirement for a mechanical tuning system. At the moment we investigate a ferrit system housing in an external coaxial cavity coupled strongly to the helical tank.

Finally it should be mentioned, that tuning of the structure also can be done by phasing the transmitter working as a reactance tube. In this case one needs a generator power of about 4 kW/m (uncontrolled relative frequency shift times reactive power) for the parameters given above. This is 20 times the power absorbed by a 100 μA beam, but only twice for a 1 mA beam. Therefore such a system would be economic only at high beam currents.

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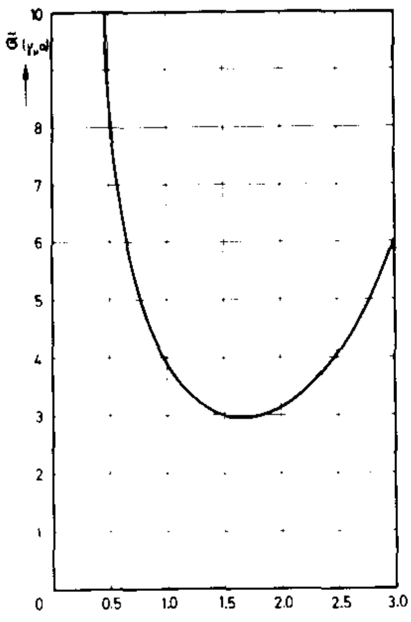

FIG. 1

Fig. 1

Sensitivity function $\tilde{\sigma}$ (proportional to the electromechanical coupling factor p) plotted against $\gamma_\nu a$ (γ_ν - radial wave number, a - helical radius)

Fig. 2

Power in the resonator (normalized on the value in resonance) α_o and phase angle ψ due to radiation pressure plotted against the frequency shift

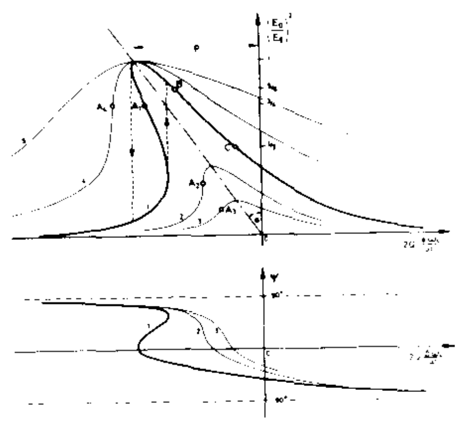

FIG 2

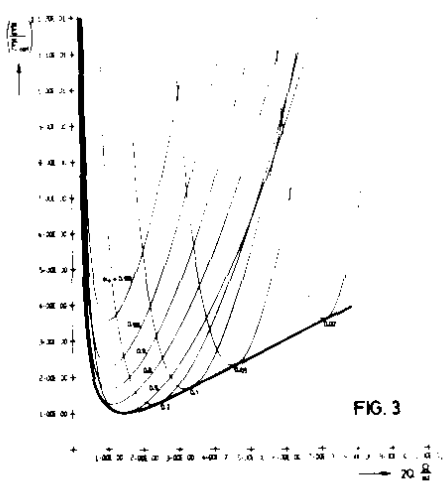

FIG. 3

Fig. 3

Threshold field energy (normalized on the minimum value at $b = \sqrt{2}$ and $\alpha_o = \frac{1}{2}$) plotted against mechanical frequency Ω(normalized on the electric decay time) with operating point parameter α_o (part of the resonant power)