Tetrade Form of Brinkman's Conditions

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Abstract

The integrability conditions for the equations determining conformal transformation of Einstein's spaces are given in compact and convenient form. The case when Riemannian spaces have the Stackel form is considered.

1 Introduction

Let V_n and $\tilde{V}_n$ be a n - dimensional Riemannian spaces with the metric tensors $g_{\alpha\beta}(x)$, $\tilde{g}_{\alpha\beta}(x)$ respectively and

$$\tilde{g}_{\alpha\beta}(x) = g_{\alpha\beta}(x) \exp 2\omega(x), \qquad \alpha, \beta, \gamma, \delta = 1, ..., n > 3. \tag{1}$$

These spaces V_n and $\tilde{V}_n$ are called the conformal ones. The relation (1) defines the conformal transformation $V_n \to \tilde{V}_n$. Function $\omega(x)$ is called the conformal factor.

The problem of conformal map arises when the problem of integration of the equations for the massless fields (wave equation, Weyl's equation etc.) is studied. Moreover the conformal map is known as a method of constraction of new exact solutions of the Einstein's equations [1].

That is why the problem of investigation of Riemannian spaces conformal to Einstein's spaces has attracted considerable interest. For the first time this problem was formulated by Brinkman [2]. Let us recall the main Brinkman's results. Let $\tilde{V}_n$ be an Einstein's space

$$\tilde{R}_{\alpha\beta} = \Lambda \tilde{g}_{\alpha\beta}, \qquad \Lambda = \text{const}, \qquad \tilde{R} = n\Lambda \tag{2}$$

here $\tilde{R}_{\alpha\beta}, \tilde{R}_{\alpha\beta\gamma\delta}, \tilde{R}$ are components of Ricci tensor , Rieman tensor and scalar curvature respectively for the space $\tilde{V}_n$, and $R_{\alpha\beta}, R_{\alpha\beta\gamma\delta}, R$ are those tensors for the space V_n. $\quad -R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} = \Gamma^{\gamma}_{\alpha\beta,\gamma} - \Gamma^{\gamma}_{\alpha\gamma,\beta} + \ldots$).

We denote:

$$\omega_{\alpha\beta} = \omega_{,\alpha;\beta} - \omega_{,\alpha}\omega_{,\beta} + \frac{1}{2} g_{\alpha\beta}(\nabla\omega)^2, \qquad (\nabla\omega)^2 = g^{\alpha\beta}\nabla_\alpha(\omega)\nabla_\beta(\omega), \tag{3a}$$

$$T_{\alpha\beta} = \frac{1}{n-2}\left(R_{\alpha\beta} - \frac{R g_{\alpha\beta}}{2(n-1)}\right), \qquad W = \frac{1}{2}(\nabla\omega)^2 - \frac{\Lambda}{2(n-1)} \exp 2\omega \tag{3b}$$

where $\omega_{,\alpha}$ are the partial derivatives and $\omega_{;\alpha} \equiv \nabla_\alpha \omega$ are the covariant derivatives in V_n. It is easy to show that

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} + (n-2)\omega_{\alpha\beta} + g_{\alpha\beta} g^{\gamma\delta}\omega_{\gamma\delta},$$

$$\tilde{R} = (R + 2(n-1) g^{\gamma\delta}\omega_{\gamma\delta}) \exp(-2\omega).$$

We can write (2) in the form

$$\omega_{,\alpha;\beta} - \omega_{,\alpha}\omega_{,\beta} + W g_{\alpha\beta} + T_{\alpha\beta} = 0. \tag{4}$$

Brinkman has shown that integrability conditions of eq. (4) have the form

$$\omega_{,\delta} C^{\delta}{}_{\alpha\beta\gamma} = S_{\alpha\beta\gamma} \tag{5}$$

$$S_{\alpha\beta\gamma} \equiv T_{\alpha\gamma;\beta} - T_{\alpha\beta;\gamma}$$

where $C_{\alpha\beta\gamma\delta}$ are the components of Weyl tensor.

2 Compact Form of Integrability Conditions

Let us transform eq. (5) in the next manner. From Bianchi identities

$$R_{\alpha\beta\gamma\delta;\sigma} + R_{\alpha\beta\delta\sigma;\gamma} + R_{\alpha\beta\sigma\gamma;\delta} = 0. \tag{6}$$

one can obtain

$$R^{\sigma}{}_{\alpha\beta\gamma;\sigma} = R_{\alpha\beta;\gamma} - R_{\alpha\gamma;\beta}. \tag{7}$$

Using eqs. (3) one can present the right hand part of the eq. (7) in the form

$$R_{\alpha\beta;\gamma} - R_{\alpha\gamma;\beta} =$$

$$(n-2)\left[\left(T_{\alpha\beta} + \frac{Rg_{\alpha\beta}}{2(n-1)(n-2)}\right)_{;\gamma} - \left(T_{\alpha\gamma} + \frac{Rg_{\alpha\gamma}}{2(n-1)(n-2)}\right)_{;\beta}\right] =$$

$$= (n-2)\left(T_{\alpha\beta;\gamma} - T_{\alpha\gamma;\beta} + \frac{g_{\alpha\beta}R_{,\gamma} - g_{\alpha\gamma}R_{,\beta}}{2(n-1)(n-2)}\right) =$$

$$= (n-2)\left(-S_{\alpha\beta\gamma} + \frac{g_{\alpha\beta}R_{,\gamma} - g_{\alpha\gamma}R_{,\beta}}{2(n-1)(n-2)}\right). \tag{8}$$

It follows from (8) that

$$R^{\sigma}{}_{\alpha\beta\gamma;\sigma} = -(n-2)S_{\alpha\beta\gamma} - \frac{1}{2(n-1)}(R_{,\gamma}g_{\alpha\beta} - R_{,\beta}g_{\alpha\gamma}). \tag{9}$$

From the identities (6) it follows that

$$(R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}R)_{;\alpha} = 0.$$

Hence

$$T^{\alpha}{}_{\beta;\alpha} = \frac{1}{2(n-1)}R_{,\beta}. \tag{10}$$

In terms of $T_{\alpha\beta}$ Weyl tensor can be written as

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\gamma}T_{\beta\delta} - g_{\alpha\delta}T_{\beta\gamma} + g_{\beta\delta}T_{\alpha\gamma} - g_{\beta\gamma}T_{\alpha\delta}, \tag{11}$$

then

$$C^{\alpha}{}_{\beta\gamma\delta;\alpha} = R^{\alpha}{}_{\beta\gamma\delta;\alpha} + T_{\beta\delta;\gamma} - T_{\beta\gamma;\delta} + g_{\beta\delta}T^{\alpha}_{\gamma;\alpha} - g_{\beta\gamma}T^{\alpha}_{\delta;\alpha}. \tag{12}$$

Using eqs. (5) and (10) one can take out from (12) next correlation

$$C^{\alpha}{}_{\beta\gamma\delta;\alpha} = R^{\alpha}{}_{\beta\gamma\delta;\alpha} + S_{\beta\gamma\delta} - \frac{1}{2(n-1)}(g_{\beta\gamma}R_{,\delta} - g_{\beta\delta}R_{,\gamma}). \tag{13}$$

That is why

$$C^{\delta}{}_{\alpha\beta\gamma;\delta} = -(n-3)S_{\alpha\beta\gamma} \tag{14}$$

and Brinkman's conditions can be presented in the form

$$\omega_{,\delta}C^{\delta}{}_{\alpha\beta\gamma} = -\frac{1}{(n-3)}C^{\delta}{}_{\alpha\beta\gamma;\delta},$$

finally

$$\nabla_{\delta}\left(C^{\delta}{}_{\alpha\beta\gamma}\exp{(n-3)\omega}\right) = 0. \tag{15}$$

If dimension of the space V_n equals to 4, eq. (15) has the form

$$\nabla^{\delta}\left(C_{\delta\alpha\beta\gamma}\exp\omega\right) = 0. \tag{16}$$

3 Tetrade Form of Integrability Conditions

To simplify the conditions (16) we use the null complex tetrade formalism (notation used here is the same as in [3]). Let us choose the vectors of the tetrade in the form

$$e_{A\alpha} = (l_{\alpha}, n_{\alpha}, m_{\alpha}, \bar{m}_{\alpha}), \qquad e^{\alpha}_{A} = (l^{\alpha}, n^{\alpha}, m^{\alpha}, \bar{m}^{\alpha}),$$

$$e_{A\alpha}e^{\alpha}_{B} = \eta_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \text{A,B,C,D} = 1,2,3,4. \tag{17}$$

Let us denote

$$C_{ABCD} = e^{\alpha}_{A}e^{\beta}_{B}e^{\gamma}_{C}e^{\delta}_{D}C_{\alpha\beta\gamma\delta}\ , \qquad {}_{,A} \equiv e^{\alpha}_{A}\nabla_{\alpha}.$$

Then conditions (16) take a form

$$\left(e^{\alpha}_{A}e^{\beta}_{B}e^{\gamma}_{C}e^{\delta}_{D}C^{ABCD}\exp\omega\right)_{;\alpha} = 0.$$

From here one can lead

$$e^{\beta}_{B}e^{\gamma}_{C}e^{\delta}_{D}\left(C^{ABCD}\exp\omega\right)_{,A} + \left(e^{\alpha}_{A;\alpha}e^{\beta}_{B}e^{\gamma}_{C}e^{\delta}_{D} + e^{\alpha}_{A}e^{\beta}_{B;\alpha}e^{\gamma}_{C}e^{\delta}_{D} + \right.$$
$$\left. + e^{\alpha}_{A}e^{\beta}_{B}e^{\gamma}_{C;\alpha}e^{\delta}_{D} + e^{\alpha}_{A}e^{\beta}_{B}e^{\gamma}_{C}e^{\delta}_{D;\alpha}\right)C^{ABCD} = 0.$$

or

$$(C^A{}_{BCD}\exp\omega)_{,A} - (\gamma^N{}_A{}^A C_{NBCD} + \gamma^N{}_B{}^A C_{ANCD} + \\ + \gamma^N{}_C{}^A C_{ABND} + \gamma^N{}_D{}^A C_{ABCN}) = 0, \qquad \gamma^A{}_B{}^C = e^A_\alpha e^{\alpha;\beta}_B e^C_\beta. \tag{18}$$

Using the well known notations of Newman-Penrose formalism

$$\kappa = \gamma_{311}, \quad \rho = \gamma_{314}, \quad \varepsilon = (\gamma_{211} + \gamma_{341})/2,$$

$$\sigma = \gamma_{313}, \quad \mu = \gamma_{243}, \quad \gamma = (\gamma_{212} + \gamma_{342})/2,$$

$$\lambda = \gamma_{244}, \quad \tau = \gamma_{312}, \quad \alpha = (\gamma_{214} + \gamma_{344})/2,$$

$$\nu = \gamma_{242}, \quad \pi = \gamma_{241}, \quad \beta = (\gamma_{213} + \gamma_{343})/2.$$

and

$$D = l^\alpha \nabla_\alpha, \quad \Delta = n^\alpha \nabla_\alpha, \quad \delta = m^\alpha \nabla_\alpha, \quad \bar\delta = \bar m^\alpha \nabla_\alpha,$$

we present the conditions (16) in the form

$$\begin{aligned}
-\Psi_0\Delta\omega + \Psi_1\delta\omega + \delta\Psi_1 - \Delta\Psi_0 + \Psi_0(4\gamma - \mu) - 2\Psi_1(\beta + 2\tau) + 3\Psi_2\sigma = 0,\\
-\Psi_0\bar\delta\omega + \Psi_1 D\omega + D\Psi_1 - \bar\delta\Psi_0 + \Psi_0(4\alpha - \pi) - 2\Psi_1(\varepsilon + 2\rho) + 3\Psi_2\kappa = 0,\\
-\Psi_1\Delta\omega + \Psi_2\delta\omega + \delta\Psi_2 - \Delta\Psi_1 + \Psi_0\nu + 2\Psi_1(\gamma - \mu) - 3\Psi_2\tau + 2\Psi_3\sigma = 0,\\
-\Psi_1\bar\delta\omega + \Psi_2 D\omega - \bar\delta\Psi_1 + D\Psi_2 + \Psi_0\lambda + 2\Psi_1(\alpha - \pi) - 3\Psi_2\rho + 2\Psi_3\kappa = 0,\\
\Psi_2\Delta\omega - \Psi_3\delta\omega - \delta\Psi_3 + \Delta\Psi_2 - 2\Psi_1\nu + 3\Psi_2\mu + 2\Psi_3(\tau - \beta) - \Psi_4\sigma = 0,\\
\Psi_2\bar\delta\omega - \Psi_3 D\omega - D\Psi_3 + \bar\delta\Psi_2 - 2\Psi_1\lambda + 3\Psi_2\pi + 2\Psi_3(\rho - \varepsilon) - \Psi_4\kappa = 0,\\
\Psi_3\Delta\omega - \Psi_4\delta\omega - \delta\Psi_4 + \Delta\Psi_3 - 3\Psi_2\nu + 2\Psi_3(2\mu + \gamma) + \Psi_4(\tau - 4\beta) = 0,\\
\Psi_3\bar\delta\omega - \Psi_4 D\omega + \bar\delta\Psi_3 - D\Psi_4 - 3\Psi_2\lambda + 2\Psi_3(2\pi + \alpha) + \Psi_4(\rho - 4\varepsilon) = 0.
\end{aligned} \tag{19}$$

Here $\Psi_0 \dots \Psi_4$ are the tetrade components of Weyl tensor:

$$\Psi_0 = -C_{1313}, \Psi_1 = -C_{1213}, \Psi_2 = -C_{1342}, \Psi_3 = -C_{1242}, \Psi_4 = -C_{2424}.$$

4 Conformal Null Stackel Spaces

To demonstrate the effectiveness of the obtained relations we consider the problem of classification of the conform Stackel's spaces satisfying the Einstein's equations

$$\tilde R_{\alpha\beta} = \Lambda \tilde g_{\alpha\beta}, \qquad \Lambda = \text{const}.$$

Let us remind the main statements of the theory of Stackel spaces (see for example [4–6]).

The space $\tilde V_n$ is called the conformal Stackel one if the equation

$$\tilde g^{ij} S_{,i} S_{,j} = 0 \tag{20}$$

can be integrated by the method of complete separation of variables. In this case the privileged coordinate set $\{u^i\}$ exists where $\tilde g^{ij}$ has the form

$$\tilde g^{ij} = \exp(-2\omega) g^{ij}, \qquad g^{ij} = \sum_\nu (\phi^{-1})^\nu_4 h^{ij}_\nu(u^\nu), \qquad \phi^\nu_\mu = \phi^\nu_\mu(u^\mu),$$

$$h^{\mu\tau}_\nu = \delta^\mu_\nu \delta^\tau_\nu h^{\nu\nu}_\nu, \qquad h^{\mu p}_\nu = \delta^\mu_\nu\, h^{\nu p}_\nu,$$

$$p, q = 1, \dots, N, \qquad \mu, \nu, \tau = N+1, \dots, 4$$

with fixed number N, $0 < N \le 4$. One can verify that quantities

$$\underset{p}{Y}{}^i = \delta^i_p \tag{21}$$

satisfy the Killing equations

$$\underset{p}{Y}{}_{(i\,;\,j)} = 0.$$

Let us calculate the number N_0:

$$N_0 = N - rank\, \| \underset{p}{Y}{}^i \underset{q}{Y}{}_i \|.$$

If the signature of V_n has the form $(+,-,-,-)$ then N_0 can be equal to 0 or to 1. The Stackel spaces are called the non null ones when $N_0 = 0$ and the null ones when $N_0 = 1$. We present all types of null Stackel spaces for all sets $(N.N_0)$.

1. (3.1) – type

$$g^{ij} = g^{ij}(u^3), \qquad g^{33} = 0.$$

2. (2.1) – type

$$g^{ij} = \frac{1}{\Phi}\begin{pmatrix} a_3(u^3) & b_3(u^3) & 1 & 0 \\ b_3(u^3) & c_2(u^2) + c_3(u^3) & -f_2(u^2) & 0 \\ 1 & -f_2(u^2) & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\Phi = \phi_2(u^2) + \phi_3(u^3).$$

3. (1.1) – type

$$g^{ij} = \frac{1}{\Phi}\begin{pmatrix} W & W^1 & 0 & 0 \\ W^1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 W^2 & 0 \\ 0 & 0 & 0 & \varepsilon_3 W^3 \end{pmatrix},$$

$$W^1 = t_2(u^2) - t_3(u^3), \qquad W^2 = t_1(u^1) - t_3(u^3), \qquad W^3 = t_2(u^2) - t_1(u^1),$$

$$W = l_2(u^2)W^2 + l_3(u^3)W^3,$$

$$\Phi = \phi_1(u^1)W^1 + \phi_2(u^2)W^2 + \phi_3(u^3)W^3, \qquad \varepsilon_\nu = \pm 1.$$

The main result of the section is

Theorem. Let us consider null conformal Stackel spaces which are not conformally flat. If these spaces are Einstein's ones then they admit the Killing vectors (21). *Proof.* We give the proof for the type (2.1). Others types are considered in the similar manner.

Let us introduce the vectors of complex isotrop tetrade

$$l_i = (0,0,1,0), \qquad n_i = (1,0,-a_3/2,0), \qquad m_i = \frac{1}{\sqrt{2G}}(f_2, 1, -\Omega, i\sqrt{G}).$$

$$\Omega = a_3 f_2 + b_3, \qquad G = -\det g^{ij} = -(a_3 {f_2}^2 + 2b_3 f_2 + c_2 + c_3), \qquad P = \ln G.$$

Then

$$D = \partial_0 - f_2\partial_1, \Delta = \frac{a_3}{2}\partial_0 + (\Omega - a_3 f_2/2)\partial_1 + \partial_2, \delta = -\sqrt{G/2}\partial_1 - \frac{i}{\sqrt{2}}\partial_3. \tag{22}$$

and

$$\kappa = \varepsilon = \rho = \sigma = 0,$$

$$\alpha = \beta = \frac{1}{4\sqrt{2}}(-\dot{f}_2/\sqrt{G} + iP_{,3}),$$

$$\mu = -P_{,2}/4, \qquad \nu = -i\dot{a}_3/2\sqrt{2}, \qquad \gamma = -i\Omega_{,3}/4\sqrt{G}, \tag{23}$$

$$\tau = \pi = \alpha + \bar{\alpha}, \qquad \lambda = 2\gamma + \mu.$$

The components of Weyl tensor have the form

$$\Psi_0 = \Psi_1 = 0,$$

$$\Psi_2 = \frac{1}{24G}[G(-2P_{,33} + {P_{,3}}^2) - 4{\dot{f}_2}^2 + 6iP_{,3}\dot{f}_2\sqrt{G}], \tag{24}$$

$$\Psi_3 = \frac{1}{8\sqrt{2}G}[\sqrt{G}(2\Omega_{,33} - 3\Omega_{,3}P_{,3} - 3P_{,2}\dot{f}_2 + 2\ddot{f}_2) + i(-4\Omega_{,3}\dot{f}_2 - 2P_{,23}G + P_{,2}P_{,3}G)],$$

$$\Psi_4 = \frac{1}{8G}[G(2\ddot{a}_3 + \dot{a}_3 P_{,3} + 2P_{,22} - {P_{,2}}^2) + 4{\Omega_{,3}}^2 + i\sqrt{G}(2\dot{a}_{,3}\dot{f}_2 + 4\Omega_{,23} - 6\Omega_{,3}P_{,2})].$$

Let us consider eqs. (19). Using (22), (23) and (24) one can present them in the form

$$\Psi_2 D\omega = 0 \quad ,$$

$$\Psi_2\delta\omega + \delta\Psi_2 - 3\Psi_2\tau = 0 \quad ,$$

$$\Psi_2\bar{\delta}\omega - \Psi_3 D\omega + \bar{\delta}\Psi_2 + 3\Psi_2\pi = 0 \quad , \tag{25}$$

$$\Psi_2\Delta\omega - \Psi_3\delta\omega - \delta\Psi_3 + \Delta\Psi_2 + 3\Psi_2\mu + 2\Psi_3(\tau - \beta) = 0 \quad ,$$

$$\Psi_3\Delta\omega - \Psi_4\delta\omega - \delta\Psi_4 - 3\Psi_2\nu + 2\Psi_3(2\mu + \gamma) + \Psi_4(\tau - 4\beta) = 0 \quad ,$$

$$\Psi_3\bar{\delta}\omega - \Psi_4 D\omega + \bar{\delta}\Psi_3 - 3\Psi_2\lambda + 2\Psi_3(2\pi + \alpha) = 0 \quad .$$

One has to consider next cases:

1. $\Psi_2 = 0,\ D\omega \neq 0.$ 2. $\Psi_2 \neq 0,\ D\omega = 0,$ 3. $\Psi_2 = D\omega = 0.$

1. $\Psi_2 = 0,\ D\omega \neq 0.$

From (25) it follows that $\Psi_3 = \Psi_4 = 0 \Longrightarrow \tilde{V}_4$ is conformally flat space.

2. $\Psi_2 \neq 0,\ D\omega = 0.$

From (23) and (25) it follows that

$$\Psi_2(\delta + \bar{\delta})\omega + (\delta + \bar{\delta})\Psi_2 = 0.$$

Since from (22)

$$\delta + \bar{\delta} = -\sqrt{2G}\partial_1 \quad ,$$

then $\omega_{,1} = 0$ and from $D\omega = 0$ it follows $\omega_{,0} = 0$. Therefore $\underset{p}{Y}^i = \delta^i_p$ $(p = 0, 1)$ are Killing vectors.

3.$\Psi_2 = D\omega = 0$.

From $\Psi_2 = 0$ it follows that $f_2 = 0$. Then

$$\omega = \omega(u^1, u^2, u^3), \qquad \tau = \pi = 0, \qquad \alpha = -\bar{\alpha}. \tag{26}$$

Substituting (26) into eqs. (25) one can find

$$\omega = l_2(u^2)x^1 + \tilde{\omega}(u^2, u^3).$$

From Einstein's equations it follows that $l_2 = 0 \Longrightarrow \underset{p}{Y}^i = \delta^i_p$ are the Killing vectors.

The theorem is proved.

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