DeWitt Metrics in M-theory

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1 Introduction

In the Hamiltonian formulation of General Relativity, one considers the space of all metrics in a space-like hypersurface of spacetime, modulo diffeomorphisms. There is a natural metric in this space [1], and this deWit metric has hyperbolic signature, the 'timelike' direction coming from the 'wrong' sign for the kinetic term of the conformal factor in the Einstein-Hilbert action.

In M/string theory, this is the relevant metric when all spatial dimensions are compactified, a situation that seems natural to contemplate when considering the early Universe. In such an scenario, a classical approximation probably does not apply, and one needs to consider a quantum wave-function over moduli space, but in this work I am restricting to a classical analysis.

Some of the questions I want to address are the following: what are the implications of this causal structure on moduli space? Do time-dependent solutions typically correspond to chaotic motion in moduli space [2, 3]? Do time-dependent solutions tend to asymptote to some particular regions in moduli space? *e.g.* if we consider a type II time-dependent solution, that initially is $CY \times T^3$, with the CY being very stringy, say a Gepner point, will the size of the CY typically remain stringy, or will it generically evolve towards a large volume region?

Since I am considering motion in moduli space, I restrict to compactifications that preserve 8 or more supercharges. In what follows I will review compactications with 32 and 8 supercharges, and present some new results.

Let's start with some mathematical preliminaries. The moduli space $\mathcal{M}$ has some coordinates ϕ^I. Given a function $V(\phi)$ (for us, this will be the volume of the compactification), there are two natural metrics we can define

$$M_{IJ} = \frac{\partial^2 V(\phi)}{\partial \phi^I \partial \phi^J} \qquad G_{IJ} = -\frac{\partial^2 \log V(\phi)}{\partial \phi^I \partial \phi^J} \tag{1}$$

In the cases we consider, M_{IJ} is a Lorentzian metric, and G_{IJ} is Euclidean. We also define a hypersurface $\mathcal{M}_1$, given by $V(\phi) = 1$. The pull-backs of M_{IJ} and $-G_{IJ}$ on $\mathcal{M}_1$ coincide. Let's denote this common pull-back by g_{ij}. g_{ij} is an Euclidean metric.

We will be interested in time dependent solutions, $\phi^I(t)$. Using time-reparametrization invariance, time-dependent solutions can be easily shown to be geodesics in $\mathcal{M}_1$.

Next, we recall a theorem that provides sufficient conditions for this motion to be chaotic: if all sectional curvatures of g_{ij} are negative and if the volume of $\mathcal{M}_1$ (or a fundamental domain thereof) is finite, the geodesic motion is chaotic.

A particularly simple example of negative sectional curvatures is characterized by the following theorem [7]: M_{IJ} is a flat metric if and only if all sectional curvatures of g_{ij} are constant and equal to $-d^2/4$.

2 M-theory on T^d

Consider M-theory compactified on T^d. The U-duality algebra is E_d, which is finite for $d = 8$, affine for $d = 9$ and hyperbolic for $d = 10$. We focus on rectangular tori, and set $C = 0$. The moduli space is conveniently parameterized by the logs of the radii,

$$p_i = \log \frac{R_i}{L_p} \tag{2}$$

The Weyl group leaves invariant the following bilinear form

$$I = (9 - d) \sum_i p_i^2 + (\sum_i p_i)^2 \tag{3}$$

which has hyperbolic signature for $d = 10$. Since the moduli space has an obvious cone structure, it makes sense to discuss its structure for asymptotic regions. One can ask [4] for which rays, the asymptotic radii are such that all sizes are large in appropriate units (l_P, l_s^A, or l_s^B). Furthermore, if the description used is a string theory, we require that the string coupling is small. The region where these conditions apply is characterized by $p_1 + 2p_3 \geq 0$ where it is assumed that all p_i have been ordered from smaller to larger.

Furthermore, one has to take into account U-duality, since a compactification that naively has very small volume might be brought to a 'large volume' region after a set of U-dualities. For $d = 10$, it is argued in [4] that the points with $I > 0$ in the forward light-cone can be brought into the safe region, while those with $I < 0$ or the backwards light-cone can not.

We want to characterize the compactifications with $I = 0$ in the forward light-cone. One can argue that all such compactifications can be brought by a set of Weyl reflections to the form $(0, 0, \ldots, p_{10})$. This is a T^{10} where a T^9 has all radii of Planck size, i.e. the compactification has a collapsing (real) divisor.

For time-dependent solutions, $R_i(t)$, time reparametrization invariance imposes $I = 0$. An example of such solutions are Kasner metrics, $R_i(t) = t^{p_i}$, with the p_i satisfying $I = 0$. Cosmological billiards [3] provide a beautiful dynamical realization of the Weyl reflections, in terms of a succession of Kasner solutions. The motion can be projected to the intersection of the fundamental Weyl chamber and the $\mathcal{M}_1$ hypersurface. Since this is a space with finite volume and negative sectional curvatures, motion is chaotic.

3 M-theory with 8 supercharges

There are many compactifications that preserve 8 supercharges. In this talk we focus on 11D SUGRA compactified on $CY_3 \times T^4$, and the related IIA/IIB compactifications. Actually, we present results mostly related to the CY Kähler moduli alone.

The volume of the CY is given by $V(Y) = \frac{1}{3!} C_{IJK} Y^I Y^J Y^K$. The possible boundaries of the Kähler cone are [5],

i) A 2-cycle collapses to a point, and there is a flop transition. This leads to a new CY, whose Kähler cone can be adjoined to the original one.

ii) A 4-cycle collapses to a 2-cycle. The new CY is a copy of the original one, and so is the new Kähler cone.

iii) A 4-cycle collapses to a point. The 4-cycle is a (generalized) del Pezzo surface.

By adjoining all the Kähler cones obtained by transitions of type i) and ii), one obtains the partially enlarged Kähler cone, whose boundary is given by the type iii) boundaries, i.e. the collapse of divisors. Furthermore, one can show [6] that G_{IJ} and M_{IJ} behave regularly on transitions of type i) and ii), but develop a zero eigenvalue when a divisor collapses.

The picture that emerges for the Kähler moduli space is then quite similar to that of M-theory on T^{10}. The partially enlarged Kähler cone is given by the union of chambers, the Kähler cones, which generalize the Weyl chambers of the toroidal compactification. For each such chamber the asymptotic region is a large volume, in the appropriate units. A major difference is that now not all such chambers are physically equivalent. As for the case of T^{10}, the boundary of the partially enlarged Kähler cone appears when a divisor collapses.

Concerning time-dependent solutions, a natural question is whether motion is chaotic in the Kähler moduli space of a CY. To apply the theorem mentioned in the introduction, we need to check that all sectional curvatures of g_{ij} are negative, and that the fundamental domain inside the corresponding $\mathcal{M}_1$ hypersurface has finite volume. This second condition seems to apply in general. On the other hand, there are examples of CYs whose g_{ij} have some positive sectional curvatures [8]. For instance , some Weierstrass models over Hirzebruch surfaces $\mathbb{F}_n$, for $n = 0, 1, 2$. These models have been considered in the physics literature [9].

The question of which Calabi-Yaus present chaotic motion in their moduli space remains open. It would be nice to obtain a characterization based on the types of divisors of the Calabi-Yau, and their intersections.

References

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