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Efficient Calculation of One-Loop QCD Amplitudes

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Abstract

We present the outline of a new and efficient technique for the calculation of loop amplitudes in a gauge theory. The technique is based on the technology of four-dimensional heterotic strings. We display here the application to the calculation of the one-loop corrections to gluon-gluon scattering, in ordinary dimensional regularization. We find complete agreement with the previous Feynman diagram calculation of Ellis and Sexton.

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Perturbative calculations lie at the foundation of our understanding of physics at short distances. Of special importance are perturbative calculations in quantum chromodynamics, because of the present and likely future importance of hadron collider experiments to our understanding of the standard model and to our hope of uncovering what lies beyond.

Even at tree level, such calculations are quite difficult using the traditional techniques of Feynman diagrams. (The calculation of the *amplitude* for the scattering of two to six gluons, for example, would involve nearly 35000 Feynman diagrams and on the order of half a billion terms.) In recent years, however, several advances have made possible a variety of multi-jet calculations [1]. These include the decomposition of amplitudes into sums of gauge-invariant partial amplitudes multiplied by color trace factors; the use of the spinor helicity basis [2]; and the Berends-Giele recurrence relations [3]. The tree level color decomposition [4] and recurrence relations [5] emerge quite naturally from string theories.

In this letter, we present the outline of a new technique, based on string theory, for calculating loop amplitudes in massless gauge theories, and the results for some of the helicity amplitudes for the four-point amplitude. The details of the method as well as a discussion of certain technical issues will be given elsewhere. The helicity amplitudes presented below can be used to compute the $O(\alpha_s^3)$ corrections to unpolarized $gg \to gg$ scattering, first computed by Ellis and Sexton [6]; we find complete agreement for this quantity. In addition, the helicity amplitudes could be used to compute the $O(\alpha_s^3)$ corrections to polarized $gg \to gg$ scattering which have not been computed previously.

There are several aspects of the method which indicate its advantages over the conventional diagrammatic technology. In essence, the starting point — the string amplitude for n-gluon scattering — already sums up all Feynman diagrams, organizes them in a color decomposition, and performs all momentum integrals, leaving only integrals analogous to Feynman parameter integrals to be done. This bypasses all algebra associated with the large number of terms generated by the nonabelian gauge vertex factors. Furthermore, as the initial expression is a function solely of the external momenta, polarization vectors, and color indices (as well as Feynman parameters), it is very well suited to use of the spinor-helicity basis [2] which reduces the complexity of the amplitude enormously.

The color decomposition, which emerges from the string amplitude, organizes the full amplitude into a sum over certain permutations of color factors times partial amplitudes. Each partial amplitude is gauge-invariant (under on-shell gauge transformations) and contains contributions from many Feynman diagrams eliminating most of the large cancellations typical of Feynman diagram computations. Another aspect of the reorganization is the absence of extra Faddeev-Popov ghost diagrams, even though the string formalism is completely covariant.

The new formalism also lends itself to a richer set of consistency checks than does the conventional one. One has the usual checks: on gauge invariance; on unitarity; and on cancellation of infrared divergences against the soft and collinear divergences of $(n+1)$-point tree cross sections. In addition, the various gauge theory partial amplitudes are related via decoupling equations [7]; these are the one-loop version of the tree-level 'twist' or 'subcyclic' identities [4]. (Alternatively, these identities could be used to reduce the number of partial amplitudes that need be computed.)

Although the use of a string-like reorganization of the amplitude is by now standard in tree-level gauge theory computations, a number of technical complications might appear to impede the application of such a formalism to loop computations.

The most obvious issue is that of the massless spectrum of the string theory. All string states which couple to gauge bosons can circulate in loops; thus one needs control of the massless spectrum of the string model in order to obtain QCD amplitudes simply by taking the infinite-tension limit of the corresponding string amplitudes. The technologies for controlling the spectrum are precisely the four-dimensional string constructions [8,9]. With the Kawai-Lewellen-Tye (KLT) version of this technology, we have constructed examples of modular-invariant four-dimensional heterotic string theories [10] which contain a pure nonabelian gauge theory in the infinite-tension limit [11]. In fact, a consistent string is not really needed for practical computations, but it does serve as a guarantee that no extraneous problems enter to affect the results.

A related issue is the decoupling of massive states and unwanted massless states such as the graviton and dilaton. Aside from the dilaton, which requires a more subtle argument, the decoupling of unwanted states is automatic and straightforward.

Relativistic quantum-mechanical theories in four dimensions with massless particles display infrared divergences in on-shell amplitudes with fixed particle number [12]. This is of course true of string theories as well, since these divergences have a physical origin. As a consequence, four-dimensional strings are ill-defined (even before taking the infinite-tension limit). For practical computations, we require a regularization scheme that has a known connection with the standard field-theory regularization of both infrared and ultraviolet divergences. For this purpose, we have developed a string version of ordinary dimensional regularization [13] and of dimensional reduction [14], based on the work of Brink, Green, and Schwarz [15]. At the four-point level, the use of dimensional regularization allows for a direct comparison to the results of Ellis and Sexton [6].

In order to compute a physical S-matrix element from a Green function, one must multiply by a factor of the square root of the wavefunction renormalization Z_A for each external leg. In field theory, one would simply compute the value of the two-point function. In the string-based formalism, this issue appears subtle because the formulation is on shell; and in the on-shell Polyakov formulation the two-point function vanishes identically. Furthermore, higher-point Polyakov amplitudes[†] are ill-defined as they contain factors of 0/0 when loops are isolated on external legs. We have performed a detailed analysis of these issues [16] and have shown the consistency of a particular prescription [17] for handling them. We may also note that the string ambiguities are ultimately irrelevant to practical calculations using a dimensional regularization scheme for all divergences, since the ultraviolet and infrared divergences cancel, leaving a vanishing result for loops on external legs. (The same cancellation is also used as a prescription in field theory computations [18]; in the new formalism, it is straightfoward to prove its consistency by demanding gauge invariance of the amplitude.)

While knowledge of string theory is important in resolving the technical issues outlined above, for practical computations one can rely on a set of rules presupposing ignorance of string theory.

The starting point of the computation of $gg \to gg$ is the one-loop N-gluon string amplitude, as given in refs. [16]. Schematically it has the form

$$\begin{aligned} &\mathcal{N}\begin{pmatrix}\text{color}\\ \text{charges}\end{pmatrix} \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int \left(\prod_{\substack{i=1\ldots N\\ m=1\ldots 4}} d\theta_{i,m}\right) \int \left(\prod_{l=1}^{N} d^2\nu_l\right) \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} \mathcal{Z}^{\vec{\alpha}}_{\vec{\beta}}(\tau) \\ &\quad\times \prod_{i<j} \exp\Bigg[\lambda k_i \cdot k_j G_B(\nu_{ij}) - \theta^2_{\bullet,\bullet}\delta\begin{pmatrix}\text{color}\\ \text{indices}\end{pmatrix} G_F(\nu_{ij}) \\ &\qquad - \theta^2_{\bullet,\bullet}\left\{\lambda k_i\cdot k_j, i\sqrt{\lambda}k_i\cdot\varepsilon_j, \varepsilon_i\cdot\varepsilon_j\right\}\left\{G_F(\overline{\nu}_{ij}), \dot{G}_B(\overline{\nu}_{ij})\right\} + \theta^4_{\bullet,\bullet}\varepsilon_i\cdot\varepsilon_j\,\ddot{G}_B(\overline{\nu}_{ij})\Bigg] \end{aligned} \tag{1}$$

where τ is the complex modular parameter describing the torus which is the one-loop world-sheet of the string, the ν_i are the Koba-Nielsen variables [19] describing the locations of the external gluon vertex operators on the world sheet ($\nu_{ij} = \nu_i - \nu_j$), and the ε_i are the ordinary gluon polarization vectors. The vectors of rational numbers $\vec{\alpha}$ and $\vec{\beta}$ describe the choices of world-sheet boundary conditions for the world-sheet fermions. One must sum over these boundary conditions with the KLT coefficients $C^{\vec{\alpha}}_{\vec{\beta}}$. The fermionic Green functions G_F (and fermionic contributions to the partition function $\mathcal{Z}^{\vec{\alpha}}_{\vec{\beta}}(\tau)$) depend on the choices of these world-sheet boundary conditions, while the bosonic Green functions G_B (and bosonic contributions to the partition function) are independent of the the boundary conditions (in the fermionic formulation of superstrings [8]).

Performing the integrations over the Grassmann parameters $\theta_{i,m}$ leads to a result which is multilinear in the polarization vectors. It immediately gives a color-decomposed form [7]: a sum

[†] With fewer than maximal number of spacetime supersymmetries

of terms, where each term consists of three pieces: a color factor — one or two traces of products of color charge matrices T^a (times left-mover Green functions), a kinematic tensor — a product of dot products of polarization vectors and momenta (times right-mover Green functions), and a kinematic core (consisting of the exponentiated bosonic Green functions $\exp(\sum \lambda k_i \cdot k_j G_B(\nu_{ij}))$). It is convenient, and possible, to integrate by parts with respect to the $\overline{\nu}_i$ so as to remove all appearances of double derivatives of the bosonic Green's functions. The amplitude then has a uniform positive power of the inverse string tension λ sitting in front.

The field theory limit is simply the limit $\lambda \to 0$. In the case of the four-point function, the amplitude contains an over-all factor of λ^2; thus in order to extract a non-vanishing contribution we must extract two poles in λ. There are two sources of such poles. One is a pinch of Koba-Nielsen variables $\nu_i - \nu_j \to 0$. In this limit we obtain contributions of the form

$$\int d^2\nu |\nu|^{-2-\lambda k_i \cdot k_j/\pi} \sim -\frac{2\pi^2}{\lambda k_i \cdot k_j} . \tag{2}$$

The other is the large $\mathrm{Im}\,\tau$ region where we obtain

$$\int d\,\mathrm{Im}\,\tau \; (\mathrm{Im}\,\tau)^p e^{-\lambda A\, \mathrm{Im}\,\tau} \sim \frac{\Gamma(p+1)}{(\lambda A)^{p+1}} \tag{3}$$

where the power of p depends on how many unpinched Koba-Nielsen variables remain. It is only in these limits that we need the expansions of the Green function and partition function. Here we shall not display the explicit form of these expansions, but will instead give the behavior of typical combinations of Green functions which occur in the amplitude.

Consider first the color-charge factors (described by the left-movers). For each ordering of the $\mathrm{Im}\,\nu_i$ there is a distinct contribution, each one of which is trivial. For example, with the ordering $\mathrm{Im}\,\nu_1 \leq \mathrm{Im}\,\nu_2 \leq \mathrm{Im}\,\nu_3 \leq \mathrm{Im}\,\nu_4 = \mathrm{Im}\,\tau$ one finds in the gauge theory limit

$$Z_L G_F(\nu_{21}) G_F(\nu_{32}) G_F(\nu_{43}) G_F(\nu_{14}) \longrightarrow -N_c, \qquad Z_L G_F(\nu_{21}) G_F(\nu_{31}) G_F(\nu_{43}) G_F(\nu_{24}) \longrightarrow 0 \tag{4}$$

where N_c is the number of colors and Z_L is the left-mover partition function.

The kinematic tensor (described by the right-movers) has a slightly richer structure but also simplifies in the gauge theory limit. For example, with the same ordering of $\mathrm{Im}\,\nu$'s given above one finds that

$$\begin{aligned} &Z_R G_B(\overline{\nu}_{21}) G_B(\overline{\nu}_{12}) G_F(\overline{\nu}_{43}) G_F(\overline{\nu}_{34}) \longrightarrow \frac{1}{2}(1-2x_{21})^2 \\ &Z_R G_B(\overline{\nu}_{21}) G_B(\overline{\nu}_{32}) G_B(\overline{\nu}_{43}) G_B(\overline{\nu}_{14}) \longrightarrow \frac{1}{2^4}(2-\delta_R\epsilon)(1-2x_{21})(1-2x_{32})(1-2x_{43})(1-2x_{41}) \end{aligned} \tag{5}$$

where $\delta_R = 1$ for ordinary dimensional regularization and $\delta_R = 0$ for dimensional reduction. (The sum over world-sheet boundary conditions with appropriate coefficients is included in these simplifications.) With $x_i \equiv \operatorname{Im}\nu_i/\operatorname{Im}\tau$, the $x_{ij} \equiv x_i - x_j$ are standard Feynman parameters.

The kinematic core results in an expression of the form

$$\int d\operatorname{Im}\tau\,(\operatorname{Im}\tau)^{1-\epsilon/2}\exp\left[\frac{\lambda}{2}\operatorname{Im}\tau\Big(s(G_B(\nu_{12})+G_B(\nu_{34}))+t(G_B(\nu_{14})+G_B(\nu_{23}))+u(G_B(\nu_{13})+G_B(\nu_{24}))\Big)\right]$$
$$\longrightarrow \Gamma(2-\epsilon/2)\Big(-\lambda(sx_1x_2+tx_2x_3+ux_1x_3+tx_1-tx_2)\Big)^{-2+\epsilon/2} \tag{6}$$

and is the same in both dimensional regularization and reduction.

Combining the color factor (4), the kinematic tensor factors (5), the kinematic core (6), and summing over the various terms then yields a Feynman parameterized form of the full amplitude. The evaluation of these final Feynman parameter integrals can then be done by standard methods.

In the four-point amplitude, there are 43 formally independent terms, the number of multilinear functions of the polarization vectors. Physically, however, they are redundant, because they are related by constraints of gauge invariance. The entire physical content is contained in the three helicity amplitudes $\mathcal{A}(++++)$, $\mathcal{A}(-+++)$, and $\mathcal{A}(--++)$. (Note that we use the convention that all momenta are outgoing, that is $k^0_{1,2} < 0$.) The spinor-helicity basis [2] chooses $\varepsilon^{(+)}_\mu(k;q) = \langle q_-|\gamma_\mu|k_-\rangle/\sqrt{2}\langle q_-|k_+\rangle$ and $\varepsilon^-_\mu(k,q) = \langle q_+|\gamma_\mu|k_+\rangle/\sqrt{2}\langle k_+|q_-\rangle$, where k is the gluon momentum, q is an arbitrary *reference* momentum such that $q^2 = 0$, $k\cdot q \neq 0$, and $|k_\pm\rangle$ is a Weyl spinor. A judicious choice of the reference momenta allows us to extract the physical information efficiently, by forcing many terms to vanish and by combining others. In the new formalism, the spinor helicity basis can be used immediately in the starting formula, equation (1).

At tree level, $\mathcal{A}(++++)$ and $\mathcal{A}(-+++)$ vanish. Furthermore, only the amplitude associated with a single trace term in the color decomposition is needed for computing the $O(\alpha_s^3)$ corrections. In computing these corrections in either dimensional reduction or the original 't Hooft-Veltman dimensional regularization scheme (in which only internal gluons are continued to $D = 4-\epsilon$ dimensions), one needs only the dispersive (real) parts of the partial amplitudes in the physical region,

$$\begin{aligned}
\mathrm{Disp}\, A_{1-\mathrm{loop}}(1^-,2^-,3^+,4^+) &= i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\frac{\Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{8\pi^2\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon/2} \\
&\times\left(-\frac{8}{\epsilon^2}-\frac{22}{3\epsilon}+\frac{11}{6}l_Q(\mu^2)+\frac{2}{\epsilon}\left(l_Q(s)+l_Q(t)\right)-l_Q(s)\,l_Q(t)+\frac{11}{6}l_Q(t)+\frac{\pi^2}{2}-\frac{32}{9}-\frac{\delta_R}{6}\right) \\
\mathrm{Disp}\, A_{1-\mathrm{loop}}(1^-,2^+,3^-,4^+) &= i\frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\frac{\Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{8\pi^2\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon/2} \\
&\times\left(-\frac{8}{\epsilon^2}-\frac{22}{3\epsilon}+\frac{11}{6}l_Q(\mu^2)+\frac{2}{\epsilon}\left(l_Q(s)+l_Q(t)\right)-\frac{(u^2-st)^2}{2u^4}\left(l_Q(s)+l_Q(t)\right)^2\right. \\
&+\frac{s^4+t^4}{2u^4}\left(l_Q^2(s)+l_Q^2(t)\right)-\frac{11u^2-3st}{6u^3}\left(t\,l_Q(s)+s\,l_Q(t)\right) \\
&\left.-\frac{st}{2u^3}\left(s\,l_Q(s)+t\,l_Q(t)\right)-\frac{st}{2u^2}+\frac{\pi^2}{2}-\frac{\pi^2}{2}\widehat{\Theta}(u)\frac{st(2u^2-st)}{u^4}-\frac{32}{9}-\frac{\delta_R}{6}\right)
\end{aligned} \tag{7}$$

where $\langle i\,j\rangle = \langle k_{i-}|k_{j+}\rangle$, s, t and u are the usual Mandelstam invariants, μ^2 is the renormalization scale, Q^2 is the factorization scale, $l_Q(x) = \ln|x/Q^2|$, $\widehat{\Theta}(x>0) = 1$, $\widehat{\Theta}(x<0) = 0$, and we have used the $\overline{\mathrm{MS}}$ renormalization prescription. The other relevant partial amplitudes can be obtained from these by a relabeling of external legs. The one-loop correction to the color-summed cross-section is then given by

$$2g^6(\mu^2)\left(\mu^2\right)^\epsilon N_c^3(N_c^2-1)\sum_{\substack{\sigma\in S_4/Z_4\\ \text{helicities}}} A^*_{\mathrm{tree}}(\sigma)\,\mathrm{Disp}\,A_{1-\mathrm{loop}}(\sigma)\,. \tag{8}$$

Ellis and Sexton [6] used a form of dimensional regularization where the external polarizations were also continued to D-dimensions; in order to compare our results we need to include the additional ϵ-helicity states [20] that arise in this case. We have computed these additional helicity amplitudes, and with them, find exact agreement with the results of ref. [6], equation (2.25). This agreement provides the first independent complete check on the latter calculation, in addition to verifying our understanding of regularization and renormalization within the string-derived formalism.

The inclusion of massless fermions at one-loop in the new formalism is straightforward. Although the technology has not yet been extended to higher loops or to massive fermions, we expect that such an extension can be devised.

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