13 Ion Optics and Beam Transport

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13.1 Introduction

Effective and efficient use of ion or electron accelerators requires a knowledge of how to control and manipulate beams of charged particles using electric and magnetic fields. Because this discipline shares many concepts acquired historically through the study of light, the names "electron optics" and "ion optics" came into early usage. Subsequently, the generic expression "beam transport" was coined, in part because transporting particle beams (of any kind) over long distances both inside and outside of accelerators has come to dominate the field. In this chapter, and depending on context, sometimes reference will be made to "optics" (especially if there is close analogy to conventional light optics), and at other times to "transport". No special distinction is intended. The word "ion" generally will mean any electrically charged particle moving (mostly) in vacuum, while "beam" will refer to collections of ions moving on average in the same direction.

At first blush, predicting particle trajectories within an electrostatic accelerator seems almost devoid of interest. In the uniform electric field of a basic accelerator tube, charged particles moving at low speeds ($v \ll c$) follow parabolic trajectories analogous to masses falling in a uniform gravitational field. The velocity component directed parallel to the tube axis undergoes constant acceleration, while transverse velocity components remain unchanged. Unfortunately, this much of the story is almost irrelevant. At each end of a tube, where the electric field strength changes, fringing fields create strong lenses [1] that dominate the overall optics. The difficulty of passing beams through the stronger of these lenses at the tube entrance without incurring substantial loss converts this to a problem of continuing interest. For the simplest accelerators it is enough to understand this much. But devices get added and machines grow in complexity. The largest electrostatic accelerators house elaborate beam transport systems that require careful study both for design and for efficient use. Outside the accelerator, the beam may have come from somewhere else (in the case of a tandem) and certainly has to go somewhere else. Both tasks require efficient coupling of the accelerator to external devices.

13.2 Methods

Throughout this discussion, the x axis points away from the reader into the page, the y axis lies within the page pointing toward the top and the z axis lies within the page pointing toward the right (longitudinal beam axis). It is common practice to select x, z to define the midplane of a dipole and, since dipoles often bend in the horizontal, x, z frequently is chosen to represent the horizontal plane. In axially symmetric systems, where the distinction between x and y disappears, an r, z frame may be substituted. The location of a particle moving from left to right along the z axis is described by (usually) small deviations from a reference particle representing the idealized center of the beam:

x displacement from beam axis in x direction
y displacement from beam axis in y direction
x' trajectory slope dx/dz in x, z plane
y' trajectory slope dy/dz in y, z plane
δp fractional deviation from central momentum
δE fractional deviation from central energy
δv fractional deviation from central velocity
δm fractional deviation from central mass
δz deviation from reference position along z
δt deviation from reference transit time
$\delta\varphi$ deviation from reference r.f. phase angle

Obviously δp, δE and δv are not independent; ordinarily, one is selected as a working coordinate (typically δp for all-magnetic systems and δE for all-electric systems) and the others calculated from it as needed. Likewise, for modulated beams, only one of δz, δt or $\delta\varphi$ is chosen to describe the longitudinal position of a particle relative to the center of a beam pulse or bunch. The differential mass variation, δm, is useful primarily for spectrometry and will not be pursued here. Mass selection (e.g. for AMS) is easily calculated for each discrete mass without introducing δm.

Six deviations (e.g. x, x', y, y', δp and δz) are commonly used to describe the location of a particle in a six-dimensional (6D) phase space volume centered on a reference particle. For dc beams, the longitudinal deviation δz is discarded but δp is retained to describe dispersion in magnetic dipoles (alternately, δE is retained to describe dispersion in electric dipoles, or δv in velocity filters). Acceleration changes x', y' and δp in a coherent way and once acceleration is accounted for, Liouville's theorem dictates that the total volume is conserved during beam transport even in the presence of aberration. Coherent aberrations such as sextupole (2nd order) or octupole (3rd order) in principle are reversible. In practice, although some corrections can be made, it usually proves impossible to force all of the misshapen genie back into the bottle because of the way aberrations warp the envelope of the phase space volume, making it effectively larger. Incoherent aberrations (caused, for example, by strippers) occur at the microscopic level and are equivalent to heating the beam; these are not reversible using corrective lenses but can be overcome in cyclic machines by procedures that cool the beam.

Typically, in two dimensions (e.g. x, x'), contours of equal intensity in a beam occupy areas that are roughly elliptical to start with [2,3]. The reasons for this are varied but, in simplest form, any beam that scrapes against multiple apertures tends to lose projecting corners and smooth toward an elliptical shape in phase space. For mathematical convenience, an ellipse (or ellipsoid) is usually substituted for less tractable polygons. An ellipse remains an ellipse under linear transformations and, because most common devices operate almost linearly for small deviations and do not cause mixing of x and y coordinates (an important exception being solenoid lenses that rotate the beam around the z axis), the 2D area is usually conserved. Suppose that such an ellipse is upright with semiaxes x and x' aligned to the coordinate frame. For area to be conserved, the product xx' must remain constant. This means that reductions in spot size at this location (called a beam waist) can only come at the expense of increased angles of incidence. The largest possible angle (x'_{max}) will be limited by the distance upstream to the nearest lens and the aperture at that lens. If the focused spot is to be made any smaller, then the upstream lens must be brought closer or given a larger working aperture; lens adjustments alone cannot overcome placement. Thus, the combination of physical layout and phase space content governs the smallest spot sizes available at a given location. Collimators (e.g. successive apertures, a tube or a channel) restrict both size and angle, thus defining the maximum area of an ellipse (or other figure) that can pass through [4]. Matching phase space content to a collimator is a much more stringent requirement than simply minimizing beam size [5].

Matrix analysis begins with equations that relate the coordinates of a particle as it leaves some part of a beam transport system to the coordinates at entry. To lowest order,

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \delta p \\ \delta z \\ 1 \end{pmatrix} = \begin{pmatrix} a_{x/x} & a_{x/x'} & a_{x/y} & a_{x/y'} & a_{x/\delta p} & a_{x/\delta z} & a_{x/1} \\ a_{x'/x} & a_{x'/x'} & a_{x'/y} & a_{x'/y'} & a_{x'/\delta p} & a_{x'/\delta z} & a_{x'/1} \\ a_{y/x} & a_{y/x'} & a_{y/y} & a_{y/y'} & a_{y/\delta p} & a_{y/\delta z} & a_{y/1} \\ a_{y'/x} & a_{y'/x'} & a_{y'/y} & a_{y'/y'} & a_{y'/\delta p} & a_{y'/\delta z} & a_{y'/1} \\ a_{\delta p/x} & a_{\delta p/x'} & a_{\delta p/y} & a_{\delta p/y'} & a_{\delta p/\delta p} & a_{\delta p/\delta z} & a_{\delta p/1} \\ a_{\delta z/x} & a_{\delta z/x'} & a_{\delta z/y} & a_{\delta z/y'} & a_{\delta z/\delta p} & a_{\delta z/\delta z} & a_{\delta z/1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ \delta p \\ \delta z \\ 1 \end{pmatrix}_0 . \tag{13.1}$$

The coefficients $a_{i/1}$ accommodate contributions that do not depend on incident coordinates.

Over a distance of length l in empty space, the particle position in the x, z plane drifts as $x = x_0 + lx'_0$ (and similarly for y, z), while the divergence "angle" does not change, keeping $x' = x'_0$. A thin lens reverses this; the divergence changes as $x' = \pm x_0/f_x + x'_0$, where f_x is the focal length (which need not be the same as f_y), while the instantaneous position remains unchanged at $x = x_0$. In matrix form, these two basic operations become the workhorses of beam transport analysis:

$$\begin{array}{ccc} \text{2D partition} & \text{drift} & \text{thin lens} \\ \begin{pmatrix} a_{x/x} & a_{x/x'} \\ a_{x'/x} & a_{x'/x'} \end{pmatrix}, & \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 \\ \pm 1/f_x & 1 \end{pmatrix} \end{array}. \tag{13.2}$$

The thin-lens matrix may be extended to describe thick lenses (asymptotically) as a single impulse or as a product of simpler components [5,6]:

$$\begin{aligned} &\begin{pmatrix} F_2/f_2 & \pm(F_1F_2 - f_1f_2)/f_2 \\ \pm 1/f_2 & F_1/f_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \pm\delta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & f_1/f_2 \end{pmatrix} \begin{pmatrix} 1 & \pm\delta_1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{13.3}$$

The $\pm$ signs select for divergent lenses (+) or convergent lenses (−). The subscripts 1 and 2 refer to the object and image sides, respectively. The focal lengths f_i and focal points (focal planes) F_i are normally positive. The differences $\delta_i = (F_i - f_i)$ are displacements of the principal planes from reference planes at points of entry and exit. The sum $\pm(\delta_1 + \delta_2)$ represents an effective (not necessarily actual) overall length. The signs in (13.3) have been altered from those of DiChio et al. [6] in order to preserve the beam transport convention that a convergent lens preceded by a drift matrix of length F_i produces a point-to-parallel focus, and followed by drift F_2 produces a parallel-to-point focus. If the determinant $f_1/f_2 < 1$, the 2D phase space area decreases (owing to acceleration); if $f_1/f_2 > 1$, the area increases (deceleration).

Continuous lensing action distributed over the length L of a component, such as occurs in quadrupole fields, may be represented for convergent (C) and divergent (D) lenses as follows [5]:

$$\begin{pmatrix} \cos\theta & (L/\theta)\sin\theta \\ -(\theta/L)\sin\theta & \cos\theta \end{pmatrix}_C = \begin{pmatrix} 1 & -\delta_C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_C & 1 \end{pmatrix} \begin{pmatrix} 1 & -\delta_C \\ 0 & 1 \end{pmatrix}, \tag{13.4}$$

$$\begin{pmatrix} \cosh\theta & (L/\theta)\sinh\theta \\ (\theta/L)\sinh\theta & \cosh\theta \end{pmatrix}_D = \begin{pmatrix} 1 & \delta_D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f_D & 1 \end{pmatrix} \begin{pmatrix} 1 & \delta_D \\ 0 & 1 \end{pmatrix}. \tag{13.5}$$

For a particle of energy qU moving parallel to the axis in an electric quadrupole having a pole potential V_a at radius a, or in a magnetic quadrupole having a gradient $\partial B_x/\partial y = -\partial B_y/\partial x$,

$$\theta_E = (L/a)(2V_a/E\rho)^{1/2},\ E\rho = \beta^2\gamma mc^2/q \xrightarrow{v \ll c} 2U\,, \tag{13.6}$$

$$\theta_B = L[(\partial B_x/\partial y)/B\rho]^{1/2},\ B\rho = \beta\gamma mc/q \xrightarrow{v \ll c} (2mU/q)^{1/2}\,, \tag{13.7}$$

$$1/f_C = (\theta/L)\sin\theta,\ F_C = f_C\cos\theta,\ \delta_C = f_C(\cos\theta - 1)\,, \tag{13.8}$$

$$1/f_D = (\theta/L)\sinh\theta,\ F_D = f_D\cosh\theta,\ \delta_D = f_D(\cosh\theta - 1)\,. \tag{13.9}$$

Note that (13.4) and (13.5) reflect (13.3) in mirror-symmetric form with $f_1 = f_2 = f$, $F_1 = F_2 = F$ and unit determinant. A quadrupole singlet lens is maximally astigmatic: convergent in one plane while divergent in the other. Combinations of alternating-gradient quadrupole lenses (doublets, triplets and higher multiplet combinations) provide double-focusing capability [5,7,8]. Uniform-field (flat-pole) magnetic dipoles focus continuously like a quadrupole lens in the bending plane. In the perpendicular plane, the motion is that of a pure drift of length $\rho\theta$:

$$\begin{pmatrix} x \\ x' \\ \delta p \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho^{-1}\tan\beta_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\rho^{-1}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ \rho^{-1}\tan\beta_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \delta p \end{pmatrix}_0 , \tag{13.10}$$

$$\begin{pmatrix} y \\ y' \\ \delta p \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\rho^{-1}\tan\beta_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ -\rho^{-1}\tan\beta_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ \delta p \end{pmatrix}_0 . \tag{13.11}$$

If the pole ends are rotated away from perpendicular to the beam axis by angles β_1 and β_2 then, to first order, equal and opposite thin lenses are created in the two planes. Positive rotation angles shorten the path within the dipole for particles having displacement $+x$. The third column in each matrix accounts for changes that occur when the particle momentum differs from the value that sets the bending radius ρ. Electric dipoles [9] and nonuniform magnetic dipoles [8] are only slightly more complicated. For other perspectives on dipoles and quadrupoles see [10–12].

13.2.1 Focal Constraints

Focusing requirements are satisfied by reducing one or more elements of a beam transport matrix to zero; for example, in the x,z plane,

$a_{x/x'} = 0$, focus point-to-point (final x independent of initial x')
$a_{x'/x} = 0$, focus parallel-to-parallel (final x' independent of initial x)
$a_{x/x} = 0$, focus parallel-to-point (final x independent of initial x)
$a_{x'/x'} = 0$, focus point-to-parallel (final x' independent of initial x')
$a_{x/\delta p} = 0$, dispersionless in space (final x independent of initial δp)
$a_{x'/\delta p} = 0$, dispersionless in angle (final x' independent of initial δp).

Note that unless axial symmetry is present, setting $a_{x/x'} = 0$ assures only a line focus in the x, y plane. A second constraint, $a_{y/y'} = 0$ in the y, z plane, is required to produce a double focus in both x and y (e.g. when using quadrupole lenses). Often two constraints are desired in one plane; for example, achieving $a_{x/\delta p} = a_{x'/\delta p} = 0$ eliminates (to first order) all dispersion introduced by upstream components [10]. Obviously, devices capable of effecting these goals must be represented in the beam transport matrix. Skills required for successful beam transport analysis include the selection, placement and adjustment of such controlling devices.

Thus far, matrices have been assumed to operate on a single ray vector representing a single particle. For most purposes a beam of particles may be described using one ray for each dimension in phase space [12–14]. In two dimensions, the perimeter of a phase space ellipse may be traced as the parameter φ ranges from 0 to 2π using the following equation:

$$\begin{pmatrix} x(\varphi) \\ x'(\varphi) \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \mathbf{E} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} . \tag{13.12}$$

Columns in the beam ellipse matrix $\mathbf{E}$ comprise individual ray vectors, which must be selected for orthogonality; the most convenient starting choice is an upright ellipse (or ellipsoid) having all diagonal elements nonzero ($e_{ii} \neq 0$) and all off-diagonal elements zero. Desired conditions which apply to the beam as a whole (e.g. waist, minimum or size) may be achieved by applying constraints to the $\mathbf{E}$ matrix after it has been acted upon by a beam transport matrix. The parameter φ serves as an analytic tool for locating extrema and as a graphical tool for outlining ellipses; it is not required during beam transport calculations.

13.2.2 Focus, Waist or Minimum?

Confusion often arises from casual statements to the effect that lenses are used to bring the beam to a focus. A true focus is applicable for some purposes (notably microscopy and spectrometry), but far more commonly a beam is "focused" to a spot of minimum size or (less often) to a waist.

Focus: a location where rays that initially shared the same object position (point) or angle (parallel) arrive similarly correlated at an image; thus we have point-to-point, parallel-to-point, point-to-parallel and parallel-to-parallel foci, which are convenient for beam transport calculations because they are geometrically determined and independent of the beam (ignoring space charge). The familiar point-to-point focus ordinarily coincides with neither waist nor minimum; however, a combined point-to-point and parallel-to-parallel focus guarantees waist-to-waist transmission.

Waist: a place where the beam envelope momentarily is parallel as it passes through a local minimum in size; an upright beam ellipse. A waist is not the smallest size that can be obtained at this location; increasing the strength of the nearest upstream lens to draw the waist upstream will produce a smaller spot (a minimum) where the waist had been. A waist is of interest because of its reflection symmetry (upstream and downstream beam envelopes are mirror images) and because small displacements (as might be caused by fluctuating lens strength) produce only second-order changes in beam size. Very small waists are always near to a point focus and to a minimum; very large waists are near to a parallel focus (an angular minimum).

Minimum: the smallest size the beam can be made (in x, y or both) at a specific location using available controls. It is always possible to reduce the beam size to a minimum even if a waist or a focus is unattainable. A calculated minimum simulates the best "focus" that operators observe on beam viewers and scanners. Minimum size or minimum angle is accessible using (from among various possibilities) the ellipse–matrix formalism (i.e. an ellipse, not necessarily upright, projecting either the smallest spatial extent or the smallest angular extent at this location).

13.3 Single-Stage Accelerators

The heart of a single-stage electrostatic accelerator is the accelerator tube, whose nonrelativistic matrix description may be approximated as follows (see Sect. 13.6 for derivation, relativistic forms and other refinements):

$$\mathbf{A} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{1}{f_2} & 1 \end{pmatrix}}_{\text{lens}} \underbrace{\begin{pmatrix} 1 & \frac{2L}{R+1} \\ 0 & \frac{1}{R} \end{pmatrix}}_{\text{field}} \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{pmatrix}}_{\text{lens}} = \underbrace{\begin{pmatrix} -\frac{R-3}{2} & \frac{2L}{R+1} \\ -\frac{3(R+1)(R-1)^2}{8LR^2} & \frac{3R-1}{2R^2} \end{pmatrix}}_{\text{simple tube model}} . \quad (13.13)$$

L is the length of the accelerator tube; $R = v/v_0 = (U/U_0)^{1/2}$ is the ratio of exit to entrance velocities (a velocity increase being assumed here for purposes of illustration); $f_1 = 4U_0L/(U - U_0) = 4L/(R^2 - 1)$ is the focal length of the (assumed circular and convergent) entrance lens; and $f_2 = 4UL/(U - U_0) = 4LR^2/(R^2 - 1)$ is the focal length of the corresponding exit lens.

Suppose that, starting from an object located a distance p upstream from this tube, a focus is desired at a distance q beyond the exit of the tube. The model for this is

$$\begin{aligned} \mathbf{B} = \begin{pmatrix} b_{x/x} & b_{x/x'} \\ b_{x'/x} & b_{x'/x'} \end{pmatrix} &= \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_{x/x} + qa_{x'/x} & a_{x/x'} + pa_{x/x} + q(a_{x'/x'} + pa_{x'/x}) \\ a_{x'/x} & a_{x'/x'} + pa_{x'/x} \end{pmatrix} . \quad (13.14) \end{aligned}$$

For a point-to-point focus, the final position of any point in the focal plane is independent of initial angle; therefore $b_{x/x'} = 0$, which leads to

$$p = -\frac{a_{x/x'} + qa_{x'/x'}}{a_{x/x} + qa_{x'/x}} = \frac{4L/(R+1) + q(3R-1)/R^2}{(R-3) + (3/4)(q/L)(R+1)(R-1)^2/R^2} . \quad (13.15)$$

While this ignores many details, it illustrates clearly the problem of picking an object distance p. If $q \geq 0$ (i.e. a focus downstream from the tube exit), then for $R = 1$, $p < 0$, but for $R > 3$, $p > 0$; therefore, between $R = 1$ and $R = 3$ the denominator passes through zero and p makes a transition from a virtual object infinitely far downstream to a real object infinitely far upstream. Lenses alone cannot easily correct for this. The ideal solution would be a lens coincident with the tube entrance whose strength can be varied to compensate for the changing focal length of the entrance aperture lens. The so-called "gridded lens" used in some tandem accelerators does just this by nullifying the natural aperture lens with a wire mesh grid while substituting a long-focal-length gap lens in its place [15]. Most single-stage accelerators keep R approximately constant by varying the injection energy. An acceleration gap (often the ion source extractor or a downstream gap lens) located where the beam is small permits the beam energy to be changed without significant movement of the object point as seen by the tube. A less common variant is to change L by shorting out portions of the accelerator tube as a coarse form of adjustment, narrowing the dynamic range of R and f_1, and thereby permitting the injection energy to be kept high and more nearly constant [16].

13.4 Tandem Accelerators

A tandem electrostatic accelerator [17] consists of two acceleration stages that share a single high-voltage terminal but are separated by a charge-changing stripper. Negative ions injected into the low-energy (LE) stage accelerate to the terminal, change to positive at the stripper and accelerate a second time through the high-energy (HE) stage back to ground potential. Most tandems have a straight-through geometry; a few have been "folded" into a single-column structure by including a 180° reversing magnet in the terminal [18–21]. Except for some large machines equipped with mid-column lenses, the LE optics of tandems are characterized by (13.14). A focus (ultimately, a beam minimum) is desired at the stripper, only a short distance beyond the LE tube exit. Depending on the LE velocity gain ratio R_{LE}, the natural object location for this focus can range from far outside the accelerator to distressingly near the LE tube entrance.

13.4.1 Tandem Low-Energy Stage

Although the basics of electrostatic accelerator optics were known [22], tandems raised new challenges which had yet to be resolved. Typically, a lens outside the pressure vessel was used to get as much beam as possible into the first stage, but optical matching was alarmingly poor. A struggle ensued to improve the coupling of external sources to the tandem LE stage. The alternatives were to make efficient use of the entrance aperture lens, to alter it or to neutralize its effects. Table 13.1 offers a sampling of options;

Table 13.1. Types of beam coupling to a tandem LE stage

Type	Object	Requirements/comments
(a) Mid-column lens	Zero	Variable lens located in LE column
(b) Gap lens	Short	Used in many single-stage accelerators
(c) Constant gradient	Short	Parts of tube and column shorted
(d) Constant "Q"	Medium	Injection energy varied to match
(e) Gridded lens	Medium	Wire mesh grid on LE tube entrance
(f) Divergent lens	Medium	Requires grid in strong lens
(g) Step gradients	Medium	Gradient increased with energy
(h) Upstream lens(es)	Variable	Limited range of good matching
(i) Three-stage	Long	Ion source in injector terminal

some variants may utilize more than one entry. *Zero* object displacement is achieved by focusing the injected beam to a crossover at the LE tube entrance (effectively neutralizing the strong entrance aperture lens) and then using a mid-column lens to refocus at the stripper [23–25]. A *short* object displacement is the closest the object point approaches the LE tube entrance at the highest attainable terminal voltages; adjustments are made to the injection energy (b) or to the tube gradient (c) to keep this point fixed (or within a relatively narrow range) as terminal voltage is lowered. *Medium* displacement is achieved by injecting with energies that are kept proportional to the terminal voltage and that range into the hundreds of keV (d); by cancelling entrance fringing fields with a plane mesh grid that serves as one electrode of a weaker but still convergent gridded lens (e); by preceding the aperture lens with a strong divergent lens (for high terminal voltages), which converts to a convergent lens by reversal of polarity (for low terminal voltages) but which also requires the the use of a grid (f); or by progressing in stages from lower to higher gradients (g). *Variable* object displacement (h) is the normal outcome when nothing has been done to alter the LE entrance aperture lens or stabilize it against changes in R_{LE}. By introducing more than one lens between the ion source and the tandem accelerator, a wider range of object lengths can be accommodated than with a single lens. *Long* (possibly negative) displacements are attained by injecting with another accelerator (i) that contributes substantially to the negative-ion energy and usually results in $R_{LE} < 2$. The relative merits of various of these possibilities have been explored by [16, 26–28].

Negative-ion injectors (NIIs) vary widely in design but tend to fall into two broad categories: those in which negative ions reach ground potential immediately after extraction (or after a charge-exchange process), and those mounted on an insulated high-voltage platform that delivers ions to ground potential through a "preacceleration" stage. In the first category, lenses built into the NII may be adequate to provide a focus outside; otherwise, the outgoing beam diverges. In the second category, the preacceleration stage usually is operated much like a tandem HE stage since no penalty is incurred by crossing the beam near the preaccelerator tube entrance (often at a mass-selection aperture) and allowing it to diverge coming out. Outside, divergent beams are refocused by one or more external lenses, which may be axially symmetric electrostatic einzel lenses or electric or magnetic quadrupole lenses. Besides offering greater focal strength, quadrupoles also provide astigmatic correction for beams that emerge from the NII without axial symmetry as a consequence, for example, of passing through a mass-selection dipole. (However, from this distant upstream location, it is difficult to correct for astigmatism generated within the tandem by, for example, slot-shaped tube apertures in the LE stage.) Examples of studies of NII optics are found in [28–31].

13.4.2 Tandem High-Energy Stage

If, as is usually the case, the injection energy may be ignored, then the velocity gain ratio for the HE stage is simply

$$R_{HE} = [(V_T + qV_T)/V_T]^{1/2} = (1+q)^{1/2}\,, \tag{13.16}$$

where V_T is the terminal voltage and q is a positive charge state of interest produced by stripping. The focal length for a circular aperture at the HE tube entrance is approximately

$$f_{1HE} = 4U_0 L_{HE}/(U - U_0) = 4L_{HE}/(R_{HE}^2 - 1) = 4L_{HE}/q\,. \tag{13.17}$$

Thus, the problem of varying focal length reappears at the entrance to the HE stage as a consequence of different charge states produced by stripping rather than of changes in terminal voltage. The center of the charge state distribution (the most probable charge state) does rise with energy, initially as $E^{1/2}$ and progressively more slowly at higher energies [32]. Typically, f_{1HE} considerably exceeds the distance from stripper to HE tube; consequently, a convergent lens preceding the HE tube entrance offers improved transmission and becomes essential when a controlled focus is desired at a second stripper installed in the HE column. Additionally, if the HE tube begins with slot-shaped apertures, the astigmatism provided by a quadrupole matching lens can help compensate for aperture lens astigmatism.

Austerity prevails in small tandems, where gas strippers or a combination of gas and foil strippers may be the only components in the terminal that act on the beam. At the opposite extreme, the largest tandems have been equipped with a cornucopia of components, including devices that select one charge state from the many that emerge from a stripper. Noteworthy examples are the Daresbury vertical tandem [33], designed originally with offset LE and HE tubes linked by paired magnetic dipoles [25] and redesigned for axial geometry by changing to a displaced magnetic-quadrupole charge state selector [34]; the Oak Ridge, folded tandem [24], in which the 180° reversing magnet performs charge selection, with electrostatic quadrupole lenses providing focusing and matching [35]; and the VIVITRON [36], in which a displaced quadrupole charge selector and matching lens were integrated into one electrostatic package that could be shoehorned into a comparatively short horizontal terminal with the charge selection aperture relocated farther downstream at a second stripper position inside the HE column [37].

13.5 Transporting the Beam to Targets

If the beam emerging from an electrostatic accelerator is not deposited directly on a target, then it is likely that it will undergo a change in angle during passage through one or more beam-bending, switching and/or analyzing dipoles. The ability to change beam direction is, by itself, a valuable utility, and multiport switching magnets facilitate directing the beam to more than one target station. In addition, dipoles perform the useful and oftentimes indispensable function of filtering out contaminant beams that originate in the ion source or arise later because of charge-changing processes. However, the first dipole encountered by a beam after acceleration is likely to be used to convert variations in beam momentum (magnetic dipole) or energy (electric dipole) into spatial displacements that are detected at slits and fed back to a system which corrects for errors in the accelerating voltage. If the angle of bend is small then the dipole acts as a relatively weak lens (see Sect. 13.2), and the beam can be focused directly through the dipole onto regulating slits. As previously mentioned, some accelerators (usually single-stage) have the capability to do this using internal lenses. More typically, however, an external lens is required. In either case, the dipole should be located close to the exit of the accelerator to minimize magnification at the slits. Conversely, if the angle of bend is relatively large, as is the case for the widely used double-focusing 90° analyzer, the beam must first be focused at the object point of the dipole; the dipole then acts as a strong lens to refocus onto the image slits. For this arrangement to work well with an external lens, the large-angle dipole must be located considerably farther away from the accelerator and a more powerful lens provided in order to achieve an intermediate object point crossover. To obtain sufficient focal strength, the external lenses usually must be quadrupole doublets or triplets. A quadrupole triplet will preserve more of the axial symmetry in an initially symmetric beam than will a doublet, but often the properties of a doublet are better matched to requirements. Either lens can produce a double focus even if the upstream object is mildly astigmatic, but a doublet magnifies more in one perpendicular plane than in the other (typically by about 2:1), and smaller magnification in the bending plane usually provides a better match to aperture constraints, as well as being highly desired to increase the resolution of the analyzing dipole.

Well-regulated electrostatic accelerators produce very little energy spread in the beam; consequently, small fluctuations in beam position caused by dispersive devices usually can be ignored. But residual dispersion may be intolerable for some applications, such as providing beam to a booster accelerator. Every change in beam direction adds or subtracts some dispersion and, when required, dispersion can be removed by balancing opposing contributions. Often this is done using mirror-symmetric configurations of dipoles and lenses such that dispersed rays converge back in mirror image onto the axis; however, neither mirror imaging nor lenses nor identical angles of bend are requisite, but at least two dipoles (including the one that initiated the dispersion) and two controlled parameters (one of which may be mirror symmetry) will be required to cancel, to first order, both the spatial ($a_{x/\delta p}$) and the angular ($a_{x'/\delta p}$) dispersion terms. Dispersion-control lenses typically produce point-to-point foci between the centers of dipoles (the pivot points from which dispersed rays appear to fan out), whereas the conventional beam focus may need to avoid these points (especially if a dipole is to serve as an analyzer). Satisfying all requirements is possible but challenging; see, for example, [10].

13.6 Accelerator Tube Matrices

13.6.1 Axial Accelerator Tube Model

At the core of any electrostatic accelerator is the accelerator tube, an evacuated region containing a strong longitudinal electric field. Wherever the field changes, focusing or defocusing occurs and this must be accounted for. Exacting analyses [38] show that weak modulations of the field caused by the finite thickness of tube electrodes and other internal details have appreciable effect, but often these are ignored because perturbations of comparable magnitude in the field distribution during operation remain largely unknown. Ordinarily, a variable lens system makes up for such shortcomings in the calculations. Selection and placement of such lenses constitutes an important part of optical design. Only when critical components have been positioned for good beam control and matching does it become worthwhile to shift the emphasis from idealized models to more detailed studies.

In order to evaluate velocities and times, let the particle kinetic energy be qU, where U is the potential difference required to raise the energy of a particle having charge q from zero to its present value. The total particle energy [39] is

$$E = mc^2 + qU = mc^2(1 + qU/mc^2) = \gamma mc^2 , \qquad (13.18)$$

where m is the (rest) mass and c the speed of light. The following conversions are convenient to work with at low kinetic energies, where qU (typically measured in keV or MeV) is more likely to be known than the momentum or velocity:

$$\gamma = (1 - \beta^2)^{-1/2} = (\beta^2\gamma^2 + 1)^{1/2} = 1 + qU/mc^2 \,, \tag{13.19}$$

$$\beta = v/c = [(2qU/mc^2)(1 + (1/2)qU/mc^2)]^{1/2}/(1 + qU/mc^2) \,, \tag{13.20}$$

$$\beta\gamma = (\gamma^2 - 1)^{1/2} = [(2qU/mc^2)(1 + (1/2)qU/mc^2)]^{1/2} \,. \tag{13.21}$$

In a region of electric field **E**, the applied force is

$$\mathbf{F} = q\mathbf{E} = d\mathbf{p}/dt = d(\gamma m\mathbf{v})/dt = \gamma m\, d\mathbf{v}/dt + m\mathbf{v}\gamma^3 vc^{-2}\, dv/dt \,, \tag{13.22}$$

$$F_i = qE_i = \gamma m\, dv_i/dt + mv_i\gamma^3 vc^{-2}\, dv/dt, \quad i = x,\, y \,\text{or}\, z \,. \tag{13.23}$$

For purely axial acceleration, $F_x = F_y = 0$; therefore, taking $i = x$ to represent x or y, (13.23) separates into functions of v_x and v, leading to

$$dv_x/v_x = -\gamma^2 vc^{-2}\, dv = -d\gamma/\gamma \,, \tag{13.24}$$

$$\int_{v_{x0}}^{v_x} v_x^{-1}\, dv_x = -\int_{\gamma_0}^{\gamma} \gamma^{-1}\, d\gamma \,, \tag{13.25}$$

$$\log(v_x/v_{x0}) = \log(\gamma_0/\gamma) \,, \tag{13.26}$$

$$v_x = v_{x0}\gamma_0/\gamma, \quad \text{or} \quad \gamma m v_x = \gamma_0 m v_{x0} \,. \tag{13.27}$$

In the absence of transverse forces, the transverse momenta $\gamma\, mv_x$ and $\gamma\, mv_y$ are separately conserved but transverse velocities v_x and v_y are not (except in the low-velocity limit). However, the divergence changes from x'_0 to

$$\begin{aligned} x' = v_x/v_z = \gamma_0 v_{x0}/\gamma v_z &= (\gamma_0 v_0/\gamma v)(v/v_z)(v_{z0}/v_0)(v_{x0}/v_{z0}) \\ &= x'_0/R_{\beta\gamma} + O(x'^2, y'^2) \,, \end{aligned} \tag{13.28}$$

where

$$R_{\beta\gamma} = \beta\gamma/\beta_0\gamma_0 \xrightarrow{v \ll c} R = (U/U_0)^{1/2} \,. \tag{13.29}$$

Since $v_z/v = (1 - x'^2 - y'^2)^{1/2}$, and both x' and y' are presumed to be small, no significant error results from discarding higher-order terms and assuming $v_z/v \cong 1$. Note that, from (13.28), x' and y' depend only on x'_0 and y'_0, and not on the transverse displacements x_0 and y_0.

If separate versions of (13.23) for $i = x$, y and z are multiplied by v_i/v and summed, then

$$\begin{aligned} q[(v_x/v)E_x &+ (v_y/v)E_y + (v_z/v)E_z] \\ &= \gamma m[(v_x/v)\, dv_x/dt + (v_y/v)\, dv_y/dt + (v_z/v)\, dv_z/dt] \\ &\quad + \gamma^3 mc^{-2}(v_x^2 + v_y^2 + v_z^2)\, dv/dt \\ &= \gamma m\, dv/dt + \gamma^3 mv^2c^{-2}\, dv/dt = \gamma^3 m\, dv/dt \,. \end{aligned} \tag{13.30}$$

For accelerator tubes, E_z is the dominant term; therefore, let the left side of (13.30) be written as

$$\begin{aligned} &qE_z(v_z/v)[1 + (v_x/v_z)E_x/E_z + (v_y/v_z)E_y/E_z] \\ &= qE_z(v_z/v)(1 + x' \tan\theta_x + y' \tan\theta_y) = qkE_z \ , \end{aligned} \tag{13.31}$$

where $\theta_x = \arctan(E_x/E_z)$ and $\theta_y = \arctan(E_y/E_z)$ are fixed angles of inclination of fields with respect to the tube axis, $x' = v_x/v_z$ and $y' = v_y/v_z$ are the usual x and y divergences, and

$$k = (v_z/v)(1 + x' \tan\theta_x + y' \tan\theta_y) \tag{13.32}$$

adjusts for the orientation of a uniform electric field not aligned with z. Rearranging (13.30) leads to

$$dt = \gamma^3 m(qkE_z)^{-1}\, dv \ . \tag{13.33}$$

Although k is, in general, a function of velocity, the variation in k over an interval of acceleration may be small enough to allow k to be approximated as a constant; whereupon (13.33) can be integrated ([39], p. 139) from an initial state v_0, t_0 before acceleration to a final state v, t to yield the duration of acceleration

$$\begin{aligned} t - t_0 &= \int_{t_0}^{t} dt = \int_{v_0}^{v} \gamma^3 m(qkE_z)^{-1}\, dv \\ &= m(qkE_z)^{-1}(\gamma v - \gamma_0 v_0) = mc(qkE_z)^{-1}(\beta\gamma - \beta_0\gamma_0) \ . \end{aligned} \tag{13.34}$$

For axial acceleration, $\theta_x = \theta_y = 0$; consequently, from (13.32), $k = (v_z/v)$. But $v_z/v \longrightarrow 1$ when $O(x'^2, y'^2)$ is neglected; therefore, $k = 1$ will be assumed from here on for the axial case.

Because prior history is not of concern, the choice of t_0 is arbitrary (v_0 is not); therefore, let $t_0 = 0$, and extract from (13.34)

$$\gamma(t) = (\beta^2\gamma^2 + 1)^{1/2} = [(qE_z m^{-1} c^{-1} t + \beta_0\gamma_0)^2 + 1]^{1/2} \ . \tag{13.35}$$

An integration over x may be performed by using (13.27) in the form $dx/dt = v_x = v_{x0}\gamma_0/\gamma$, assisted by $dt = mcq^{-1}E_z^{-1}\, d(\beta\gamma)$ obtained by differentiating (13.34) with $k = 1$:

$$\begin{aligned} x - x_0 &= \int_{x_0}^{x} dx = v_{x0}\gamma_0 \int_0^t \gamma^{-1}(t)\, dt \\ &= v_{x0}\gamma_0 mcq^{-1}E_z^{-1} \int_{\beta_0\gamma_0}^{\beta\gamma} (\beta^2\gamma^2 + 1)^{-1/2}\, d(\beta\gamma) \\ &= \{v_{x0}v_{z0}v_0 L\gamma_0/[v_{z0}v_0 c(\gamma - \gamma_0)]\} \\ &\quad \times \log[(\beta\gamma + \beta^2\gamma^2 + 1)^{1/2}/(\beta_0\gamma_0 + \beta_0^2\gamma_0^2 + 1)^{1/2}] \\ &= [(x_0' L\beta_0\gamma_0)/(\gamma - \gamma_0)] \log[(\beta\gamma + \gamma)/(\beta_0\gamma_0 + \gamma_0)] \\ &\quad + O(x'^2, y'^2) \ , \end{aligned} \tag{13.36}$$

where terms of $O(x'^2, y'^2)$ are to be discarded and E_z has been replaced, using (13.19), by

$$E_z = (U - U_0)/L = mc^2 q^{-1} L^{-1} (\gamma - \gamma_0)\,. \tag{13.37}$$

For nearly all electrostatic-accelerator applications, a low-velocity approximation will suffice. Provided $\beta\gamma < 1$ (or, $\gamma - 1 = qU/mc^2 < 0.4$), the log function is expandable in powers of $\beta\gamma$ ([40], p. 45):

$$\begin{aligned}\log(\beta\gamma + \gamma) &= \log[\beta\gamma + (\beta^2\gamma^2 + 1)^{1/2}] \\ &= \beta\gamma - (1/2)(\beta^3\gamma^3/3) + [(1\times 3)/(2\times 4)]\,(\beta^5\gamma^5/5) \\ &\quad - [(1\times 3\times 5)/(2\times 4\times 6)]\,(\beta^7\gamma^7/7) + \ldots\end{aligned} \tag{13.38}$$

$$\begin{aligned}&\log[(\beta\gamma + \gamma)/(\beta_0\gamma_0 + \gamma_0)] \\ &\quad = (\beta\gamma - \beta_0\gamma_0)[1 - (1/6)(\beta^2\gamma^2 + \beta\gamma\beta_0\gamma_0 + \beta_0^2\gamma_0^2) + \ldots]\,.\end{aligned} \tag{13.39}$$

Keeping only leading $\beta\gamma$ terms in (13.36), discarding $O(x'^2, y'^2)$, substituting for $R_{\beta\gamma}$ from (13.29) and remembering that $\beta^2\gamma^2 = \gamma^2 - 1$ leads to the following reduction:

$$\begin{aligned}x - x_0 &\cong x'_0 L\beta_0\gamma_0(\beta\gamma - \beta_0\gamma_0)/(\gamma - \gamma_0) \\ &= x'_0 L(\gamma + \gamma_0)/(R_{\beta\gamma} + 1) \xrightarrow{v\ll c} 2x'_0 L/(R+1)\,.\end{aligned} \tag{13.40}$$

To summarize, from (13.28), (13.29), (13.36) and (13.40), axial acceleration is described (for either x or y) by the first-order (linear) matrix

$$\begin{aligned}\begin{pmatrix} a_{x/x} & a_{x/x'} \\ a_{x'/x} & a_{x'/x'} \end{pmatrix} &= \begin{pmatrix} 1 & [(L\beta_0\gamma_0)/(\gamma - \gamma_0)]\log[(\beta\gamma + \gamma)/(\beta_0\gamma_0 + \gamma_0)] \\ 0 & \beta_0\gamma_0/\beta\gamma \end{pmatrix} \\ &\xrightarrow{v\ll c} \begin{pmatrix} 1 & 2L/(R+1) \\ 0 & 1/R \end{pmatrix}\,.\end{aligned} \tag{13.41}$$

13.6.2 Inclined-Field Accelerator Tube Model

Axially symmetric accelerator tubes are more susceptible to electrical breakdown than are tubes in which a component of the field is perpendicular to the axis. One method for generating such fields is to cant the internal electrodes at an angle of order 10° to 15° to produce a substantial transverse field component [41–43]. For convenience, assume that the transverse coordinate system has been rotated about the beam axis until an electric-field exists only in the x, z plane and that perpendicular to this plane v_y is negligibly small. After acceleration, any such rotation will have to be reversed to realign to the original axes. If $v_\parallel$ and $v_\perp$ are components of initial velocity parallel and perpendicular to the electric-field vector $\mathbf{E}$, and θ is the angle between the electric field and the tube axis then, by transformation of coordinates,

$$v_{\perp 0} = v_{x0}\cos\theta - v_{z0}\sin\theta = v_{z0}\cos\theta(x_0' - \tan\theta)\,, \tag{13.42}$$

$$v_{\|0} = v_{x0}\sin\theta + v_{z0}\cos\theta = v_{z0}\cos\theta(1 + x_0'\tan\theta)\,. \tag{13.43}$$

In a coordinate system aligned to $\mathbf{E}$,

$$\begin{aligned} X'_{\perp 0} &= (v_\perp/v_\|)_0 = (x_0' - \tan\theta)/(1 + x_0'\tan\theta) \\ &= x_0'/\cos^2\theta - \tan\theta + O(x'^2, y'^2)\;. \end{aligned} \tag{13.44}$$

In the $\perp, \|$ system, $E_\| = |\mathbf{E}|$ and $E_\perp = E_y = 0$; therefore, (13.23) transforms into

$$F_\perp = qE_\perp = \gamma m\, dv_\perp/dt + mv_\perp\gamma^3 vc^{-2}\, dv/dt\;, \tag{13.45}$$

which, as in (13.24) through (13.27), separates into functions of $v_\perp$ and v that can be integrated to yield the conservation of transverse momentum,

$$v_\perp = v_{\perp 0}\gamma_0/\gamma, \quad \text{or} \quad \gamma m v_\perp = \gamma_0 m v_{\perp 0}\;. \tag{13.46}$$

After acceleration, the transformation back to v_x and v_z is

$$v_x = v_\perp\cos\theta + v_\|\sin\theta\;, \tag{13.47}$$

$$v_z = -v_\perp\sin\theta + v_\|\cos\theta\;. \tag{13.48}$$

Using (13.44) through (13.48), the ratio v_x/v_z may now be written as

$$\begin{aligned} x' &= (v_x/v_z) \\ &= [(v_\perp/v_{\|0})\cos\theta + (v_\|/v_{\|0})\sin\theta]/[-(v_\perp/v_{\|0})\sin\theta + (v_\|/v_{\|0})\cos\theta] \\ &= [X'_{\perp 0} + R_\|\tan\theta]/[R_\| - X'_{\perp 0}\tan\theta] \\ &= x_0' R_\|(1 + \tan^2\theta)^2/(R_\| + \tan^2\theta)^2 \\ &\quad + (R_\| - 1)\tan\theta/(R_\| + \tan^2\theta) + O(x'^2, y'^2)\;, \end{aligned} \tag{13.49}$$

where, based on the central ray trajectory, for which $v_{x0} = x_0' = 0$ in (13.42) and (13.43), the ratio of momenta parallel to the electric vector before and after acceleration is

$$\begin{aligned} R_\| &= \gamma\, m v_\|/\gamma_0 m v_{\|0} \\ &= (\gamma/\gamma_0)[v^2/(v_{z0}^2\cos^2\theta) - (v_{z0}^2\sin^2\theta)/(v_{z0}^2\cos^2\theta)]^{1/2} \\ &= R_{\beta\gamma}[1 + (1 - \beta_0^2/\beta^2)\tan^2\theta]^{1/2} + O(x'^2, y'^2) \\ &\xrightarrow{v \ll c} R[1 + (1 - R^{-2})\tan^2\theta]^{1/2}\;. \end{aligned} \tag{13.50}$$

Referring to (13.32) and applying (13.46), the correction factor k in the $v_\perp, v_\|$ plane becomes

$$k_\| = v_\|/v = (1 - v_\perp^2/v^2)^{1/2} = (1 - \gamma_0^2 v_{\perp 0}^2/\gamma^2 v^2)^{1/2}\;, \tag{13.51}$$

which is neither constant nor small, since typically $v_\perp/v \cong 1/4$. As a consequence, $k_\|(v)$ must be included inside the velocity integral in (13.34) when calculating the transit time, as follows:

$$\begin{aligned}
t - t_0 &= \int_{t_0}^{t} dt = mq^{-1}E_\|^{-1} \int_{v_0}^{v} \gamma^3 k_\|^{-1} dv \\
&= mq^{-1}E_\|^{-1} \int_{\gamma_0 v_0}^{\gamma v} (\gamma^2 v^2 - \gamma_0^2 v_{\perp 0}^2)^{-1/2} \gamma v \, d(\gamma v) \\
&= mcq^{-1}E_\|^{-1}[(\beta^2\gamma^2 - \gamma_0^2 v_{\perp 0}^2/c^2)^{1/2} - (\beta_0^2\gamma_0^2 - \gamma_0^2 v_{\perp 0}^2/c^2)^{1/2}] \\
&= mcq^{-1}E_\|^{-1}[\zeta - \zeta_0] \, ,
\end{aligned} \tag{13.52}$$

where

$$\begin{aligned}
\zeta &= \beta_\|\gamma = (\beta^2\gamma^2 - \gamma_0^2 v_{\perp 0}^2/c^2)^{1/2} \\
&= (\beta^2\gamma^2 - \beta_{\perp 0}^2\gamma_0^2)^{1/2} = (\gamma^2 - 1 - \beta_{\perp 0}^2\gamma_0^2)^{1/2} \, ,
\end{aligned} \tag{13.53}$$

$$\gamma(\zeta) = (\zeta^2 + 1 + \beta_{\perp 0}^2\gamma_0^2)^{1/2} \, , \tag{13.54}$$

$$dt = mcq^{-1}E_\|^{-1} \, d\zeta \, . \tag{13.55}$$

Guided by (13.36), an integration over $x_\perp$ may be performed using $dx_\perp/dt = v_\perp = v_{\perp 0}\gamma_0/\gamma$ from (13.46), assisted by (13.42) as well as ζ, $\gamma(\zeta)$ and dt from above:

$$\begin{aligned}
x_\perp - x_{\perp 0} &= \int_{x_{\perp 0}}^{x_\perp} dx_\perp = v_{\perp 0}\gamma_0 \int_0^t \gamma^{-1}(t) \, dt \\
&= v_{\perp 0}\gamma_0 mcq^{-1}E_\|^{-1} \int_{\zeta_0}^{\zeta} (\zeta^2 + 1 + \beta_{\perp 0}^2\gamma_0^2)^{-1/2} \, d\zeta \\
&= [(x_0' - \tan\theta)L\beta_0\gamma_0 \cos^2\theta/(\gamma - \gamma_0)] \\
&\quad \times \log[(\zeta + \gamma)/(\zeta_0 + \gamma_0)] + O(x'^2, y'^2) \, ,
\end{aligned} \tag{13.56}$$

where, using (13.37), $E_\|$ has been replaced by

$$E_\| = E_z/\cos\theta = (U - U_0)/(L\cos\theta) = mc^2(\gamma - \gamma_0)/(qL\cos\theta) \, . \tag{13.57}$$

Note that x_0' is not cleanly separated out in (13.56), because ζ is a function of $\beta_{\perp 0}$, which, in (13.42), contains x_0'.

The trajectory has now been tracked in the $\perp, \|$ plane from the equipotential U_0, which crosses at $z = 0$, to the equipotential U, which crosses at $z = L$. Along the $x_\|$ axis, the original z axis has diverged by $-L\sin\theta$. To compensate for this, $L\sin\theta$ must be added to (13.56) to obtain the net displacement $\varDelta_\perp = L\sin\theta + x_\perp - x_{\perp 0}$, along $x_\perp$. Because equipotentials are not orthogonal to the z axis, the endpoint will not, in general, lie in the perpendicular plane $z = L$ which bounds the accelerator tube. Since this rudimentary model does not include details of transitions from one inclination angle to the next, the question of how a region of uniform inclined field should be terminated remains open. To continue acceleration until the equipotential plane intersecting $z = L$ is reached would change the final energy. To simply transform the $x_\perp, x_\parallel$ coordinates into x, z would leave $z_{final} \neq L$. To drift with divergence x', until $z = L$, has some physical justification but seems unnecessarily complicated. To project parallel to **E**, so that $x = \Delta_\perp / \cos\theta$, would be appropriate if the trajectories were essentially parallel to **E**, but that is unlikely because trajectories are deliberately programmed to remain, on average, close to the axis and thus more nearly parallel to z. Instead, the procedure chosen here is to let

$$x - x_0 = \Delta_\perp \cos\theta = (L\sin\theta + x_\perp - x_{\perp 0})\cos\theta\ , \tag{13.58}$$

implying $x_{\perp 0} = x_0/\cos\theta$. In effect, this slides the final result parallel to z, as necessary, in order to arrive at $z = L$. Virtues of this imperfect choice are that sections having different inclination angles may be joined without passing information from section to section and that contiguous sections having the same angle join without perturbation.

By a change of variable from ζ to Z, and substituting $\gamma(\zeta)$ from (13.54), the log function in (13.56) may be expanded in powers of Z for $Z_0 \leq Z < 1$, in the same way as $\beta\gamma$ in (13.38) and (13.39):

$$Z = \zeta(1 + \beta_{\perp 0}^2 \gamma_0^2)^{-1/2}\ , \tag{13.59}$$

$$\begin{aligned} &\log[(\zeta + \gamma)/(\zeta_0 + \gamma_0)] \\ &= (Z - Z_0)[1 - (1/6)(Z^2 + ZZ_0 + Z_0^2) + \ldots]\ . \end{aligned} \tag{13.60}$$

The leading term may be decomposed using (13.43), (13.50), (13.53), and (13.59) into

$$\begin{aligned} Z - Z_0 &= Z\zeta^{-1}(\zeta - \zeta_0) = Z\zeta^{-1}(\zeta^2 - \zeta_0^2)/(\zeta + \zeta_0) \\ &= Z\zeta^{-1}(\gamma^2 - \gamma_0^2)/[\beta_{\parallel 0}\gamma_0(R_\parallel + 1)] \\ &= (1 + \beta_{\perp 0}^2\gamma_0^2)^{-1/2}(\gamma^2 - \gamma_0^2) \\ &\quad /[\beta_0\gamma_0(v_z/v_0)\cos\theta(1 + x_0'\tan\theta)(R_\parallel + 1)]\ . \end{aligned} \tag{13.61}$$

Substituting (13.60) reduced to (13.61) into (13.56) and then that result into (13.58) yields the low-velocity approximation

$$\begin{aligned} x &\cong x_0 + L\sin\theta\cos\theta + [(x_0' - \tan\theta)L\beta_0\gamma_0\cos^3\theta(\gamma - \gamma_0)(\gamma + \gamma_0)] \\ &\quad /\{(\gamma - \gamma_0)(1 + \beta_{\perp 0}^2\gamma_0^2)^{1/2}[\beta_0\gamma_0\cos\theta(1 + x_0'\tan\theta)(R_\parallel + 1)]\} + O(x'^2, y'^2) \\ &\cong x_0 + L\sin\theta\cos\theta + [(x_0' - \sin\theta\cos\theta)L(\gamma + \gamma_0)] \\ &\quad /[(1 + \beta_{\perp 0}^2\gamma_0^2)^{1/2}(R_\parallel + 1)] + O(x'^2, y'^2) \\ &\xrightarrow{v \ll c} x_0 + L\sin\theta\cos\theta(R_\parallel - 1)/(R_\parallel + 1) + 2x_0'L/(R_\parallel + 1) \end{aligned} \tag{13.62}$$

This result suggests a compromise that will extract x_0' from within (13.56) for most applications. Let

$$H = [(\beta_0\gamma_0 \cos\theta)/(\gamma - \gamma_0)] \log[(\zeta + \gamma)/(\zeta_0 + \gamma_0)] \xrightarrow{v \ll c} 2/(R_\parallel + 1)\,, \quad (13.63)$$

with the proviso that H is to be evaluated for a reference trajectory having $x_0' = 0$. To balance against this, replace $x_0' \cos^2\theta$ in (13.56) with x_0', as happens in (13.62) for $\beta_{\perp 0}^2\gamma_0^2 \ll 1$. In matrix form,

$$\begin{pmatrix} a_{x/x} & a_{x/x'} & a_{x/1} \\ a_{x'/x} & a_{x'/x'} & a_{x'/1} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & LH & L\sin\theta\cos\theta(1-H) \\ 0 & \frac{R_\parallel(1+\tan^2\theta)^2}{(R_\parallel+\tan^2\theta)^2} & \frac{(R_\parallel-1)\tan\theta)}{R_\parallel+\tan^2\theta} \\ 0 & 0 & 1 \end{pmatrix}. \quad (13.64)$$

The third column, containing $a_{x/1}$ and $a_{x'/1}$ elements, is introduced to accommodate particle displacements from a predefined geometric axis that are independent of initial position and angle. It also may be used to describe beam steerers [12], displaced quadrupole charge selectors [11], and steering effects caused, for example, by misalignment (deliberate or otherwise) of beam transport components. The matrix (13.64) supersedes results presented by the author in [44]. Reversals of the inclination angle within a tube module are required to keep the cumulative $a_{x/1}$ from growing too large. It is also desirable to exit each module with $a_{x/1} = a_{x'/1} = 0$. Interesting examples of this art may be found in [42–51].

In the orthogonal y, z plane, transverse momentum is unchanged by the inclined field; therefore, adapting (13.28),

$$y' = v_y/v_z = \gamma v_y/\gamma v_z = \gamma_0 v_{y0}/\gamma v_z = y_0'/R_{\beta\gamma} + O(x'^2, y'^2)\,. \quad (13.65)$$

During acceleration, γv_y behaves the same as $\gamma v_\perp$. After adjustment for the fact that coordinate transformations are not required, (13.56) through (13.62) serve as guidelines for $y - y_0$:

$$\begin{aligned} y - y_0 &= \int_{y_0}^{y} dy \\ &= v_{y0}\gamma_0 \int_0^t \gamma^{-1}(t)\,dt = v_{y0}\gamma_0 mcq^{-1}E_\parallel^{-1} \int_{\zeta_0}^{\zeta} (\zeta^2 + 1 + \beta_{\perp 0}^2\gamma_0^2)^{-1/2}\,d\zeta \\ &= [(y_0' L\beta_0\gamma_0 \cos\theta)/(\gamma - \gamma_0)] \log[(\zeta + \gamma)/(\zeta_0 + \gamma_0)] + O(x'^2, y'^2) \\ &\xrightarrow{v \ll c} 2y_0' L/(R_\parallel + 1)\,. \qquad (13.66) \end{aligned}$$

Comparison of the above with (13.49) and (13.62) shows that, to within the approximations used in (13.64), the matrix for y is the same as (13.64), except that $a_{y/1} = a_{y'/1} = 0$.

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