Cohomological methods in supermanifold theory † ‡

CLAUDIO BARTOCCI §
Mathematics Institute, University of Warwick,
Coventry CV4 7AL, England

UGO BRUZZO
Dipartimento di Matematica, Università di Genova,
Via L. B. Alberti 4, 16132 Genova, Italy

ABSTRACT

We introduce the basic elements of some cohomology theories that arise naturally in the context of supermanifold theory.

1. INTRODUCTION

As some recent literature clearly points out, a global geometric understanding of supermanifolds — apart from its own interest — should be signifying in applications to field and string theory, e.g. with reference to anomalies in supersymmetric field theories and superstring theories [1], or to provide a rigorous setting for superstring theory [2,3]. In order to reach such understanding, it is necessary to develop an analogue of differential topology in the category of supermanifolds.

The purpose of this talk is to introduce some basic tools, which are the very first foundations of a theory of that kind. To be precise, Section 2 recalls some preliminaries about supermanifolds, which are essentially intended according to De-Witt and Rogers [4,5,6], but with a slight modification of the structure sheaf, so as to avoid some disadvantages related with the previous definitions [7]. Section 3 is devoted to expound some facts concerning the cohomology of the structure sheaf of supermanifolds, and the cohomology of the complex of superdifferentiable forms. Contrary to the case of smooth manifolds, whose structure sheaf has no cohomology,

---

† Talk presented by the second author at the XVIth International Colloquium on Group Theoretical Methods in Physics, Varna (Bulgaria), 15–20 June 1987.

‡ Research partially supported by the National Group for Mathematical Physics of the Italian Research Council and by the Italian Ministry for Public Education through the research project "Geometria e Fisica".

§ Permanent address: Dipartimento di Matematica, Università di Genova, Via L. B. Alberti 4, 16132 Genova, Italy.

and in analogy with holomorphic manifolds, it turns out that the structure sheaf of a supermanifold has in general a non-trivial cohomology. This has the consequence that the cohomology of the complex of superdifferentiable forms of a supermanifold is not equivalent to the de Rham cohomology of the underlying smooth structure. However, in the particular case of DeWitt supermanifolds, the cohomology of the structure sheaf is proved to be trivial. Finally, in Section 4 we introduce a class of complex superbundles which extend, in some sense, the notion of ordinary complex vector bundles and we define some characteristic classes.

2. SUPERMANIFOLDS

Let B_L be the exterior algebra over $\mathbf{R}^L$, with L finite. B_L has a natural $\mathbf{Z}_2$-gradation $B_L = (B_L)_0 \oplus (B_L)_1$ and admits a decomposition $B_L = \mathbf{R} \oplus N_L \equiv \sigma(B_L) \oplus s(B_L)$, where the projections σ and s are respectively called *body* and *soul* map. The cartesian product B_L^{m+n} is naturally a $\mathbf{Z}_2$-graded B_L-module, whose even part is $(B_L)_0^m \times (B_L)_1^n \equiv B_L^{m,n}$. A body map $\sigma^{m,n} : B_L^{m,n} \to \mathbf{R}^m$ is defined by setting $\sigma^{m,n}(x^1 \dots x^m, y^1 \dots y^n) = (\sigma(x^1) \dots \sigma(x^m))$.

On $B_L^{m,n}$, endowed with its vector space topology, we introduce, in compliance with Rogers [6], a sheaf of B_L-valued functions. We fix two positive integers L and L', such that $L - L' \geq n$, and we regard $B_{L'}$ as immersed in B_L. For any open set $U \subset \mathbf{R}^m$, we define a map

$$Z_{L',L} : C[U; B_{L'}] \to C[(\sigma^{m,0})^{-1}(U); B_L]$$

between the $B_{L'}$-valued smooth functions over U and the B_L-valued smooth functions on $(\sigma^{m,0})^{-1}(U) \subset B_L^{m,0}$, explicitly given by

$$Z_{L',L}(f)(x^1 \dots x^m) = \sum_{i_1 \dots i_m = 0}^{L} \frac{1}{i_1! \dots i_m!} (\partial_1^{i_1} \dots \partial_m^{i_m} f)_{|\sigma^{m,0}(x^1 \dots x^m)} \times$$

$$\times\, s(x^1)^{i_1} \dots s(x^m)^{i_m} . \tag{1}$$

$Z_{L',L}$ is shown to be injective. The ring $\mathcal{GH}((\sigma^{m,n})^{-1}U)$ of GH^∞ functions of even and odd variables consists by definition of elements of the form

$$F(x^1 \dots x^m, y^1 \dots y^n) = F_0(x^1 \dots x^m) + \sum_{\substack{1 \leq k \leq n \\ 1 \leq \alpha_1 < \dots < \alpha_k \leq n}} F_{\alpha_1 \dots \alpha_k}(x^1 \dots x^m)\, y^{\alpha_1} \dots y^{\alpha_k}$$

where F_0, $F_{\alpha_1 \dots \alpha_k}$ belong to $Z_{L',L}(C[U; B_{L'}])$.

We are able now to define the sheaf $\mathcal{GH}$ of GH^∞ functions over $B_L^{m,n}$, which is a sheaf of $\mathbf{Z}_2$-graded $B_{L'}$-algebras whose sections over an arbitrary open set $W \subset B_L^{m,n}$ are $\mathcal{GH}(W) \equiv \mathcal{GH}((\sigma^{m,n})^{-1}\sigma^{m,n}(W))$. This rather involute definition means that, according to eq. (1), GH^∞ functions are naturally defined on "tubes" over open sets in $\mathbf{R}^m$, and that a GH^∞ function on a generic open set W is obtained by restricting to W a function defined on a tube passing through W. The partial derivatives of a GH^∞ function are uniquely determined via Taylor expansion [6].

Finally, we introduce the sheaf $\mathcal{G}$ of $\mathbf{Z}_2$-graded B_L-algebras by letting $\mathcal{G} = \mathcal{GH} \otimes_{B_{L'}} B_L$, which eliminates some awkward characteristics of GH^∞ functions [7].

DEFINITION 1 An (m,n) dimensional supermanifold is a Hausdorff, second-countable topological space, that admits an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \varphi_\alpha : U_\alpha \to B_L^{m,n}\}$ such that the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are $\mathcal{G}$ maps.† □

A remarkable class of supermanifolds are the so-called *DeWitt supermanifolds* [4,5]. An (m,n) supermanifold is DeWitt iff it has an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that $\varphi_\alpha(U_\alpha) = \sigma^{m,n}(\varphi_\alpha(U_\alpha)) \times N_L^{m,n}$; in other words, the images of the charts are tubes over open sets in $\mathbf{R}^m$. A DeWitt supermanifold M can be regarded as a locally trivial smooth bundle $\Phi : M \to M_0$ over a smooth m-manifold M_0 (called the *body* of M), having typical fibre $N_L^{m,n}$.

3. COHOMOLOGY

The purpose of this section is to state some results about the cohomological properties of supermanifolds. Firts of all, we study the cohomology of the structure sheaf $\mathcal{G}$ of a supermanifold M. To this aim, we shall use the Čech cohomology $\{\check{H}^k(M,\mathcal{G}),\, k \in \mathbf{N}\}$, which is equivalent to any other sheaf cohomology since the underlying topological space is paracompact, as follows from the definition of supermanifold.

In the case of the sheaf $\mathcal{C}$ of smooth functions on a smooth manifold, it is well known that the cohomology of $\mathcal{C}$ vanishes, as $\mathcal{C}$ is a fine sheaf (in fact, it admits smooth partitions of unity). On the contrary, the sheaf $\mathcal{G}$ is not soft, viz. it is not possible in general to extend a section of $\mathcal{G}$ over a closed subset to all of the space. Since the space is paracompact, this implies that $\mathcal{G}$ is not even fine, so that

† This definition is slightly different from the one we used elsewhere, see e.g. Refs. 8-10. The reason of our present choice is that it is simpler and it is equivalent to the other as far as cohomology is concerned; for details, see Ref. 7.

the existence of superdifferentiable partitions of unity is precluded in the category of supermanifolds.

On the basis of these preliminary remarks, we expect that in general, given a supermanifold M with structure sheaf $\mathcal{G}$, $\check{H}^k(M,\mathcal{G}) \neq 0$ for $k > 0$ (in other words, that $\mathcal{G}$ is not acyclic). This is true, as the example at the end of this section shows. Nevertheless, the following theorem holds (for the proof, we refer to [9]).

THEOREM 1 The structure sheaf of a DeWitt supermanifold M is acyclic. □

We can also introduce the analogue of the ordinary de Rham cohomology, as the cohomology of the complex [3,8]

$$\mathcal{G}(M) \xrightarrow{d} \mathcal{G}^1(M) \xrightarrow{d} \mathcal{G}^2(M) \xrightarrow{d} \dots$$

where $\mathcal{G}^p$ is the sheaf of superdifferentiable p-forms on M and d the restriction to superdifferentiable forms of the usual Cartan's differential. We denote this "super" de Rham cohomology by $H^*_{SDR}(M)$. It is natural to wonder which is its relationship with the ordinary de Rham cohomology $H^*_{DR}(M)$ of M regarded as a smooth manifold.

THEOREM 2 [8] The assumption $\check{H}^k(M,\mathcal{G}^p) = 0$ for $0 \leq p \leq q-1$ and $1 \leq k \leq q$ implies $H^k_{SDR}(M) \simeq H^k_{DR}(M) \otimes B_L$ for $1 \leq k \leq q$. □

This result, in conjuction with Theorem 1, enables us to relate the SDR cohomology of a DeWitt supermanifold M to the DR cohomology of its body M_0. Indeed, M and M_0 have the same DR cohomology, since the fibre of the body fibration is a contractible space.

COROLLARY 1 [8] If M is a DeWitt supermanifold, then

$$H^*_{SDR}(M) \simeq H^*_{DR}(M) \otimes B_L \simeq H^*_{DR}(M_0) \otimes B_L\,. \qquad \square$$

It is interesting to observe that SDR cohomology is not a topological invariant, but only a superdifferentiable one, as the following example shows.

Example. Let us consider the topological space $M = T \times \mathbf{R}^2$, where T is the torus $S^1 \times S^1$. It is possible to endow M with two inequivalent supermanifold atlases $\mathcal{A}_1$ and $\mathcal{A}_2$, such that $(M,\mathcal{A}_1) \equiv M_1$ is a DeWitt supermanifold with body $(M_1)_0 \simeq T$, and $(M,\mathcal{A}_2) \equiv M_2$ is not DeWitt (for details, see [8]). By explicit computation one obtains

$$H^1_{SDR}(M_1) \simeq H^1_{DR}(T) \otimes B_L \simeq B_L \oplus B_L\,; \qquad H^1_{SDR}(M_2) \simeq B_L\,.$$

By force of Theorem 2, this example shows also that, if $\mathcal{G}$ is the structure sheaf of M_2, we must have $\check{H}^1(M_2,\mathcal{G}) \neq 0$.

4. SUPERBUNDLES AND CHARACTERISTIC CLASSES

In this section, the materials so far expounded are used to extend the notion of smooth complex vector bundles on ordinary manifolds and to introduce the preliminaries of a theory of characteristic classes in the category of supermanifolds. †

By super fibre bundles E on a supermanifold M we mean a triple (E, π, M), where E is a supermanifold and $\pi : E \to M$ is a superdifferentiable surjective map.

DEFINITION 2 A complex (m,n) super vector bundle E on a supermanifold M is a locally trivial super fibre bundle (E, π, M), having typical fibre $C_L^{m+n} \equiv B_L^{m+n} \otimes \mathrm{C}$ and structure group $GL(m,n;\mathrm{C})$. □

In the previous definition, $GL(m,n;\mathrm{C})$ is the (super Lie) group [11,10] of even automorphisms of the graded C_L-module C_L^{m+n}. Moreover, we shall denote by $\mathcal{F}E$ the sheaf of sections of E, regarded as a sheaf of graded C_L-modules.

Analogously to the ordinary theory, operations between graded C_L-modules that are expressed by even morphisms can be extended to operations between super vector bundles. In particular, given E_1, E_2 on M, we denote by $Hom(E_1, E_2)$ the bundle whose typical fibre at $x \in M$ consists of all even morphisms $(E_1)_x \to (E_2)_x$. We can suitably define the notion of connection and curvature on a complex (m,n) super vector bundle (E, π, M).

DEFINITION 3 A connection Δ is an even morphism

$$\Delta : \mathcal{F}E \to \mathcal{F}Hom(T^{\mathrm{C}}M, E)$$

(where $T^{\mathrm{C}}M$ is the complexification of the graded tangent bundle of M [10]) satisfying

$$\Delta(fs) = df \cdot s + f\Delta(s)$$

for all sections s of E and for all $(C_L)_0$-valued superdifferentiable functions f on M. □

The *curvature* of Δ, defined as $F_\Delta = [\Delta, \Delta]$, determines an element of $\Gamma(Hom(T^{\mathrm{C}}, End(E)))$, denoted by the same symbol. Thus, on an open set $U \subset M$ over which E trivializes, F_Δ is represented by a matrix of two-forms on U. It is easily shown that F_Δ verifies the Bianchi identity, i.e. $\Delta F_\Delta = 0$. Moreover, it is possible to prove that, for any positive integer k, Str $(F_\Delta)^k$ does not depend on the trivialization, thus defining a global two-form on M, which, as a consequence of the

† A different approach, where Weil polynomials are used to introduce characteristic classes of principal super fibre bundles, was developed in Ref. 10.

Bianchi identity, is closed (for the definition of the supertrace Str see Ref. 12). So Str $(F_\Delta)^k$ determines a cohomology class

$$s_k(E) = \left[\text{Str } (\frac{i}{2\pi} F_\Delta)^k \right] \in H^{2k}_{SDR}(M) \otimes \mathbf{C}$$

which is independent of the connection Δ. This last fact can be proved with the same arguments used in Ref. 10 in the case of principal super fibre bundles.

We define

$$ch(E) = \sum_{k=0}^{\infty} \frac{1}{k!} s_k(E)$$

(the sum is actually finite) to be the *Chern character* of the complex (m,n) super vector bundle E. Notice that $s_0(E) = m - n$. To show that this definition is actually a healthy one we prove that, as in the classical theory, there exists a relationship between $s_1(E)$ and a (suitably defined) first Chern class $c_1(E)$ in the rank-one case.

Let E be a complex (1,0) super vector bundle on M, i.e. an *even complex line superbundle* [9]. The structure group of E is $GL(1,0;\mathbf{C}) \equiv (C_L)_0^*$, that is, the abelian group of all the elements in $(C_L)_0$ with non-vanishing body. In this case, the curvature F_Δ of a generic connection Δ determines directly a global closed two-form on M, which coincides with Str (F_Δ). By a standard argument, E is in a one-to-one correspondence with an element (again denoted by E) of $\check{H}^1(M, \mathcal{I}^*)$, where $\mathcal{I}^*$ is the sheaf of $(C_L)_0^*$-valued superdifferentiable functions on M. Using the cohomology sequence induced by the exact sequence of sheaves over M

$$0 \to \mathbf{Z} \xrightarrow{\text{incl}} \mathcal{I} \xrightarrow{\text{Exp}} \mathcal{I}^* \to 0$$

(where $\mathcal{I}$ is the sheaf of $(C_L)_0$-valued superdifferentiable functions on M and Exp(f) $\equiv \exp(2\pi i f)$), we can associate with E an element $c_1(E) \in \check{H}^2(M, \mathbf{Z})$, called the first Chern class of E [9].

In order to state the relationship between $s_1(E)$ and $c_1(E)$, we need to introduce some further maps. Let us consider the exact sequences of sheaves over M

$$0 \to B_L \xrightarrow{\text{incl}} \mathcal{G} \xrightarrow{d} \mathcal{Z}^1 \to 0, \qquad 0 \longrightarrow \mathcal{Z}^1 \xrightarrow{\text{incl}} \mathcal{G}^1 \xrightarrow{d} \mathcal{Z}^2 \to 0,$$

where $\mathcal{Z}^k$ is the sheaf of B_L-valued superdifferentiable k-forms which are closed under d. The two exact sequences yield connecting morphisms

$$\alpha : H^2_{SDR}(M) \equiv \frac{\check{H}^0(M, \mathcal{Z}^2)}{\text{Im}\, \check{H}^0(M, \mathcal{G}^1)} \to \check{H}^1(M, \mathcal{Z}^1),$$

$$\beta : \check{H}^1(M, Z^1) \to \check{H}^2(M, B_L)\,.$$

Then we have [9]

THEOREM 3 Let E be a complex line superbundle on M that admits a connection $\bar{\Delta}$ whose curvature form $F_{\bar{\Delta}}$ is pure imaginary. Then we have

1) $s_1(E) \in H^2_{SDR}(M)$;

2) $\beta \circ \alpha(s_1(E)) = \mu(c_1(E))$,

where μ is the map $\check{H}^2(M, \mathbf{Z}) \to \check{H}^2(M, B_L)$ induced by the inclusion of sheaves $\mathbf{Z} \to B_L$.

Proof. This result is proved as in the ordinary theory [13]. $\square$

REFERENCES

1. L. Bonora, P. Pasti, M. Tonin. *Nucl. Phys.* **B286**, 150 (1987), and references therein.
2. M. B. Green, J. H. Schwarz. *Phys. Lett.* **B136**, 367 (1984).
3. J. M. Rabin. *Commun. Math. Phys.* **108**, 375 (1987).
4. B. De Witt. *Supermanifolds*, Cambridge Univ. Press, London 1984.
5. A. Rogers. *J. Math. Phys.* **21**, 1352 (1980).
6. A. Rogers. *Commun. Math. Phys.* **105**, 375 (1986).
7. C. Bartocci, U. Bruzzo. *Some remarks on the differential-geometric approach to supermanifolds* (in preparation).
8. C. Bartocci, U. Bruzzo. *J. Math. Phys.* **28**, 2363 (1987).
9. C. Bartocci, U. Bruzzo. *Cohomology of the structure sheaf of real and complex supermanifolds.* Preprint Dipartimento di Matematica Università di Genova 12/1987.
10. C. Bartocci, U. Bruzzo, G. Landi. *Chern-Simons forms on principal super fibre bundles.* Preprint SISSA 109/87/FM, Trieste (1987).
11. A. Rogers. *J. Math. Phys.* **22**, 939 (1981).
12. V. Rittenberg, M. Scheunert. *J. Math. Phys.* **19**, 713 (1978).
13. P. Griffiths, J. Harris. *Principles of algebraic geometry,* Wiley, New York 1978.