COMPLEX EXTENSION OF THE REPRESENTATION OF THE SYMPLECTIC GROUP ASSOCIATED WITH THE CANONICAL COMMUTATION RELATIONS x)

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I. Introduction

As is well knwon, the Weyl form of the canonical commutation relations (CCR's) together with the Neumann's theorem leads to a strongly continuous representation of the real symplectic group. This representation has been studied by Bargmann [1] with the carrier space taken as Hilbert space with a reproducing kernel. In this case the operators onto which the group elements are mapped are describable as kernel operators, the kernel being a continuous function of the group elements. By considering the real symplectic group as a subgroup of the complex symplectic group it is then possible to formally extend the kernel function to a function of the elements of the complex symplectic group. The question then is what are the properties of the operators which these extended kernel functions define, in particular when are the operators bounded. It has been shown by Kramer, Moshinsky and Seligman [2] that in the case of the group Sp(2,$\mathbb{C}$) not all of these operators are bounded but those that are define a subsemigroup of Sp(2,$\mathbb{C}$). We shall show that this is also the case for Sp(2n,$\mathbb{C}$) for arbitrary n.

To discuss more explicitly the representation which we wish to extend we require a few facts about the Hilbert space employed by Bargmann. Let μ be the measure on $\mathbb{C}^n$ defined by the weighting function $z \mapsto \pi^{-n} \exp(-\|z\|^2)$ and for fixed n let

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$\mathcal{G} = \mathcal{L}^2(\mu, \mathbb{C})$ be the Hilbert space of complex valued functions square integrable with respect to this measure. With $\mathcal{E}$ denoting the space of all entire functions on C^n the Hilbert space $\mathcal{F} = \mathcal{E} \cap \mathcal{G}$ has a reproducing kernel $K^r(z, w) = e^{\bar{z}\cdot w}$ where $\bar{z}\cdot w = \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n$ i.e. for all $f \in \mathcal{F}$ the relation $f(z) = \int_{\mathbb{C}^n} K^r(z,w) f(w)\, d\mu(w)$ obtains. The Hilbert space $\mathcal{F}$ has the remarkable property that each bounded operator on $\mathcal{F}$ has a kernel, i.e. if T is bounded on $\mathcal{F}$ then $(Tf)(z) = \int_{\mathbb{C}^n} K(z,w) f(w)\, d\mu(w)$. Related to the existence of a reproducing kernel is the existence of a family e_a, $a \in \mathbb{C}^n$ of elements of $\mathcal{F}$ with the property that $\langle e_a, f \rangle = f(a)$ for all f in $\mathcal{F}$. These vectors, the principal vectors, are given explicitly by $e_a(z) = \exp(\bar{a}\cdot z)$. They are complete and the subspace $\mathcal{D}$ generated by the principal vectors is dense in $\mathcal{F}$. This property is extremely useful in checking the continuity of mappings into $\mathcal{F}$.

Recall that the complex symplectic group $H = Sp(2n, \mathbb{C})$ is the group of invertible isometries of the antisymmetric form $\{\ ,\ \}$ on C^{2n}, $\{z, w\} = \sum_{i=1}^{n} (z_i w_{i+n} - z_{i+n} w_i)$. Denoting an element of H by the matrix $h = \begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix}$ the H adjoint of h is given by $h^\# = \begin{pmatrix} {}^t\rho & -{}^t\mu \\ -{}^t\nu & {}^t\lambda \end{pmatrix}$. The symplectic property of h is then $h^{-1} = h^\#$, i.e. $h^\# h = h h^\# = 1$. The real symplectic group G_0 is defined similarly by restricting the form $\{\ ,\ \}$ to $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Rather than working directly with G_0 it is convenient to work with the isomorphic group $G = WG_0W^{-1}$ where $W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. The group G is sometimes called the complex form of the real symplectic group (not to be confused with H) and is equal to the intersection of H with U(n,n). The condition that $g \in$ G is that g have the form $g = \begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{pmatrix}$ where λ and μ satisfy $\lambda^* \lambda - \mu^* \mu = 1$ and ${}^t\lambda \bar{\mu} = \mu^* \lambda$ where ${}^t\lambda$ denotes the transpose of λ. With these definitions the representation of G (and therefore also of G_0) studied by Bargmann is defined as follows. To each element $g \in G$ we assign the kernel K_g,

$$K_g(z,w) = \exp\left\{ \tfrac{1}{2} z \cdot \bar{\mu} \lambda^{-1} z - \tfrac{1}{2} \bar{w} \cdot \lambda^{-1} \mu \bar{w} + z \cdot {}^t\lambda^{-1} \bar{w} \right\} \tag{1}$$

and a complex number $\sigma_g = \det^{-1/2} \lambda$ (We adopt the convention that $z^{1/2} = |z| \exp(\tfrac{1}{2} i \operatorname{Arg} z)$. Bargmann has shown that K_g defines a bounded oprator S_g and that if we define $T_g = \sigma_g S_g$ the double valued mapping $g \mapsto (T_g, -T_g)$ is a double valued unitary representation of G. Of course this double valued representation can and should be viewed as a representation of the universal covering group $\tilde{G}$ (G is not simply connected).

To define an extension of the above representation to H it is necessary to extend both the mappings $g \mapsto K_g$ and $g \mapsto \sigma_g$. We are immediately forced by the appearance of λ^{-1} in (1) to restrict our attention to the set $\bar{H}_0 \subset \bar{H}$ consisting of those $h \in H$ such that λ is invertible. Then the formal replacement $\begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{pmatrix} \to \begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix}$ defines a mapping $h \mapsto K_g$ where K_h is given by

$$K_h(z,w) = \exp\left\{ \tfrac{1}{2} z \cdot \nu \lambda^{-1} z - \tfrac{1}{2} \bar{w} \cdot \lambda^{-1} \mu \bar{w} + z \cdot {}^t\lambda^{-1} \bar{w} \right\} \tag{2}$$

Some comments concerning the mapping $h \mapsto K_h$ are in order. Observe first that $K_h(\ , w)$, $K_h(\ , w)(z) = K_h(z,w)$ is an entire function. Suppose that it should happen that K_h defines a bounded operator S_h on $\mathcal{Y}$. It is readily computed that for a principal vector e_a , $S_h e_a = K_h(\ , a)$ whence $S_h(\mathcal{D}) \subset \mathcal{F}$. From the assumed continuity of S_h and the fact that $\mathcal{F}$ is closed in $\mathcal{Y}$ it follows that $S_h(\mathcal{F}) \subset \mathcal{F}$. This is highly fortuitous since it is somewhat easier to check boundedness on $\mathcal{Y}$ (which is an $\mathcal{L}^2$ space) than on $\mathcal{F}$.

It is instructive to consider K_g in the case where h is the diagonal matrix $\begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix}$, δ being diagonal. One then finds that S_k is bounded on $\mathcal{Y}$ (and therefore on $\mathcal{F}$) if and only if $\|\delta\| \geqslant 1$, where $\| \ \|$ is the operator norm. This example shows that while non-trivial extensions exist they are not guaranteed

II. Extending the Representation to a Subsemigroup of Sp(2n,$\mathbb{C}$)

In what follows it will be necessary to have conditions for the absolute convergence of integrals of the type

$$I = \int_{\mathbb{C}^n} \exp\left\{\tfrac{1}{2} z\cdot \gamma z + \tfrac{1}{2}\bar{z}\cdot\bar{\delta}\bar{z} + z\cdot \varepsilon \bar{z} + a\cdot z + \bar{b}\cdot \bar{z}\right\} d\mu(z) \tag{3}$$

where γ and δ are $n\times n$ complex symmetric matrices, ε is an $n\times n$ selfadjoint matrix, and $a, b \in \mathbb{C}^n$. Using the standard technique of reducing convergence to that of a Laplace integral we have found that the integral in (3) is absolutely convergent if and only if

$$1 - \bar{\varepsilon} > 0 \tag{4a}$$

and

$$1 - \bar{\varepsilon} - \tfrac{1}{4}(\gamma + \delta)(1-\varepsilon)^{-1}(\bar{\gamma} + \bar{\delta}) > 0 \tag{4b}$$

where >0 (resp. $\geqslant 0$) denotes positive definiteness (resp. positiveness). We are interested in the value of the integral only in the case where $\varepsilon = 0$. This integral has been computed by Itzykson [3] and is given by

$$\begin{aligned} I = {} & \det{}^{-1/2}(1-\gamma\bar{\delta})\,\times \\ & \times \exp\Big\{\tfrac{1}{2} a\cdot\bar{\delta}(1-\gamma\bar{\delta})^{-1}a + \tfrac{1}{2}\bar{b}\cdot(1-\gamma\bar{\delta})^{-1}\gamma b\,+ \\ & \qquad + b\cdot(1-\gamma\bar{\delta})^{-1}a \Big\} \end{aligned} \tag{5}$$

where the sign of $\det(1-\gamma\bar{\delta})^{-1/2}$ is obtained by analytic continuation.

The possibility of extending the representation to a semigroup rests on

Lemma 1. Suppose that for $h_1, h_2 \in H$ the kernels K_{h_1} and K_{h_2} define operators S_{h_1} and S_{h_2} which are bounded on $\mathcal{Y}$ and therefore on $\mathcal{F}$. Then $K_{h_1 h_2}$ defines a bounded operator $S_{h_1 h_2}$ on $\mathcal{F}$ and

$$S_{h_1 h_2} = \tau_{h_1, h_2} S_{h_1} S_{h_2} \tag{6}$$

where

$$\tau_{h_1, h_2} = \det^{-1/2}(1 - \lambda_1^{-1}\mu_1 \nu_2 \lambda_2^{-1}) \tag{7}$$

The proof of the above lemma is carried out by direct computation using the properties of the integral given by (3) discussed above and the symplectic property of h_1 and h_2. It follows immediately from this lemma that if $\mathcal{B}$ denotes the set of all $h \in H$ such that K_h defines a bounded operator on $\mathcal{F}$ then $\mathcal{B}$ is a semigroup.

Our starting point in determining the semigroup $\mathcal{B}$ is the determination of those $h \in H_0$ such that K_h defines a Hilbert Schmidt (HS) operator on $\mathcal{Y}$. It follows easily that in this case K_h also defines an HS operator on $\mathcal{F}$. Since the product of two HS operators is HS the set $\mathcal{S}$ of all such h is also a semigroup.

In the $\mathcal{L}^2$ space $\mathcal{Y}$ the condition that a kernel K defines an HS operator is equivalent to

$$\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |K(z,w)|^2 d\mu(z)\, d\mu(w) < \infty \tag{8}$$

This condition is fairly easy to compute. Using the symplectic nature of h we find that $\bar{K}_h(w,z) = K_{h^\#}(z,w)$ where $h^\# = \begin{pmatrix} \lambda^* & -\nu^* \\ -\mu^* & \rho^* \end{pmatrix}$. By Lemma 1 and Fubini's theorem we find that the condition expressed by (8) is equivalent to

$$\int_{\mathbb{C}^n} K_{h^\#}(w,z)\, K_h(z,w)\, d\mu(z) < \infty \tag{9a}$$

and

$$\int_{\mathbb{C}^n} K_{h^\# h}(w,w)\, d\mu(w) < \infty \tag{9b}$$

The conditions expressed by (9) may now be explicitly computed using (4). Let $u : H_0 \to \mathfrak{S}$, where $\mathfrak{S}$ denotes the set of all $2n \times 2n$ selfadjoint matrices, be defined by

$$u(h) = \begin{pmatrix} \alpha(h)-1 & \gamma(h) \\ \gamma^*(h) & 1-\beta(h) \end{pmatrix} \tag{10}$$

and where

$$\alpha(h) = \lambda^*\lambda - \nu^*\nu \quad (11a) \qquad \beta(h) = \rho^*\rho - \mu^*\mu \quad (11b)$$

and

$$\gamma(h) = \lambda^*\mu - \nu^*\rho \tag{11c}$$

The conditions (9) and therefore the condition that $h \in \mathcal{S}$ may then be stated as $u(h) > 0$

Since no HS operator is unitary we see that $\mathcal{S} \cap G = \emptyset$. Hence a semigroup larger than $\mathcal{S}$ is required. Recall that whenever a topological semigroup is embedded in a Lie group the closure of that semigroup is again a semigroup. It turns out that $\mathcal{S}^-$, the closure of $\mathcal{S}$, contains G and is in fact equal to the semigroup $\mathcal{B}$.

It is possible here only briefly to indicate how the above assertions may be verified. First, using the fact that the set of all positive $2n \times 2n$ matrices is closed in $\mathfrak{S}$ and the continuity of u one shows that if $h \in \mathcal{S}^-$ then $u(h) \geqslant 0$. The reverse inequality is then established by showing that every neighborhood of the identity of H contains a point h_1 such that $u(h_1) > 0$ and then using the equation $u(h_1 h) = h_1^* u(h) h_1 + u(h_1)$ and the fact that H is a homogeneous space. One has then established that $\mathcal{S}^-$ is the set of all those $h \in H_0$ such that $u(h) \geqslant 0$. Since $u(g) = 0$ for all $g \in G$ it follows that $G \subset \mathcal{S}^-$. It remains to show that for all $h \in \mathcal{S}^-$, K_h defines a bounded operator on $\mathcal{F}$. To do this one proceeds as follows. Denoting the operator defined by K_h as S_h

we compute, using (4), that for any principal vector e_a , $S_h e_a$ is defined and has finite norm. Then for an element $v = \Sigma_i \xi_i e_{a_i}$ of $\mathcal{D}$ we find that $\| S_h v \|$ is finite if and only if $h \in \mathcal{S}^-$ and in that case $\| S_h v \|^2 \leqslant \tau_{h\#h} \| v \|$.

The above results permit us to define a representation of the semigroup $\mathcal{B}$ which maps semigroup elements onto contraction operators on $\mathcal{F}$. Let $\sigma_g = \det^{-1/2} \lambda$ and define $T_h = \sigma_h S_h$. Since $\tau_{h\#h} = |\det \lambda|^2 \det^{-1/2} \alpha(h)$ we find that $\| T_h \|^2 \leqslant |\sigma_h|^2 \tau_{h\#h} =$ $= \det^{-1/2} \alpha(h)$. But $h \in \mathcal{S}^-$ implies that $u(h) \geqslant 0$ and hence by (10) $h \in \mathcal{S}^-$ implies $\alpha(h) \geqslant 1$ and hence that $\det \alpha(h) > 1$. Thus T_h is a contraction operator for each $h \in \mathcal{S}^-$. With this normalization we find that $T_h T_{h'} =$ $= (\sigma_h \sigma_{h'} \tau_{hh'} / \sigma_{hh'}) T_{hh'} = \pm T_{hh'}$ by virtue of the symplectic nature of h and h' . Thus if we assign to each $h \in \mathcal{B} = \mathcal{S}^-$ the pair $(T_h, -T_h)$ we obtain a double valued representation of $\mathcal{B}$. Since our normalization agrees with the original normalization of Bargmann it follows that when restricted to G this representation becomes unitary and is in fact the representation given by Bargmann. Hence the original representation has been extended.

Lastly, we point out that the above representation of $\mathcal{B}$ is strongly continuous. Owing to the existence of the principal vectors this is not too difficult to show. It follows from the fact that the representation is contractive that it is strongly continuous if and only if for all principal vectors the mapping $h \mapsto T_h e_a$ is continuous. This is not difficult to check.

In way of summary we can say the following. We began with a double valued, strongly continuous representation of G $\subset$ H and showed that there exists a semigroup $\mathcal{B}$, G $\subset \mathcal{B} \subset$ H with its interior $\mathcal{S}$ also a semigroup. The original representation of G extends to a double valued, strongly continuous representation of $\mathcal{B}$ by contraction operators on $\mathcal{F}$, $h \mapsto T_h$ and for $h \in \mathcal{S}$, T_h

is a Hilbert Schmidt operator.

III. The Algebraic Structure of the Semigroup

We begin by pointing out that the semigroup $\mathcal{B}$ has a simple geometric interpretation. If (,) denotes the U(n,n) inner product on $\mathbb{C}^{2n}$ then the set of all invertible mappings A such that $(Az, Az) \geqslant (z,z)$ for all $z \in \mathbb{C}^{2n}$ is easily seen to define a semigroup, the semigroup of all U(n,n) expansion operators. Using the condition $h \in \mathcal{B}$ if and only if $u(h) \geqslant 0$ together with (10) it may be shown that the intersection of this semigroup with H is precisely $\mathcal{B}$. Similarly the intersection of the semigroup obtained from the condition $(Az, Az) > (z,z)$ with H is $\mathcal{S}$. Recalling that $G = U(n,n) \cap H$ this is not unreasonable.

Using the above model and the Jordan decomposition we are then able to show that every $h \in \mathcal{B}$ may be decomposed as $h = g_1 \mathcal{b} g_2$ where $g_1, g_2 \in G$ and $\mathcal{b}$ is a direct sum of matrices of the form $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and $\lambda \cdot \mathbf{1}$ with $\lambda \geqslant 1$ and $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{B}$. It then follows that $\mathcal{B}$ is the semigroup generated by G and those matrices having the form of $\mathcal{b}$.

IV. Relation to the Canonical Commutation Relations

Define the unbounded operators Z_k and Y_k in $\mathcal{F}$ by

$$(Z_k f)(z) = z_k f(z) \quad \text{and} \quad (Y_k f)(z) = \frac{\partial f}{\partial z_k} \,.\text{ If}$$

$$z'_k = \Sigma_\ell g^{-1}_{k\ell} z_\ell \quad \text{and} \quad Y'_k = \Sigma_\ell g^{-1}_{k\ell} Y_\ell$$

for $g \in G$ then the fact that the original representation of G was associated with a representation of the CCR's may be expressed by

$$Y_k' = T_{h^{-1}} Y_k T_h \qquad (12a) \qquad \text{and} \qquad Z_k' = T_{h^{-1}} Z_k T_h \qquad (12b)$$

this having meaning only when the operators are applied to functions in dom (Z_k) $\cap$ dom (Y_k) . In attempting to extend (12) to the representation of $\mathcal{B}$ we encounter the problem that if $h \in \mathcal{B}$ then $T_{h^{-1}}$ is in general unbounded. Nevertheless, we are able to show the following. If T_h has the operator defined by $\sigma_{h^{-1}} K_{h^{-1}}$ as its left inverse then $T_{h^{-1}}$ has a dense domain which includes the domains of Z_k and Y_k and (12) holds for functions in this domain.

REFERENCES

[1] V. Bargmann, "Group Representations on Hilbert Spaces of Analytic Functions" in Analytic Methods in Mathematical Physics, (Gilbert and Newton, eds), Gordon and Breach, New York (1968)

[2] P. Kramer, M. Moshinsky and T. H. Seligman "Complex Extensions of Canonical Transformations in Physics" in Group Theory and its Applications (E.M. Loebl ed.) Academic Press, New York (1975)

[3] C. Itzykson, Commun. Math. Phys. (N.Y.) 6, 301 (1959).