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FUN WITH SU(3)

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The aim of this paper will be to demonstrate how many of the most important properties of SU(3) may be derived simply and directly using only the elements of group theory. (As I use the phrase, "the elements of group theory" consist of the contents of WIGNER's book [1]. However, we will need only a small subset of this.)

In the first part of the paper I will show how we can deduce many useful facts about the irreducible representations of SU(3) by simple tensorial methods. In particular, I will develop two algorithms, one for the isospin-hypercharge content of an irreducible representation, and one for the Clebsch-Gordan series, that I think are simpler and more efficient than most presented in the literature. In addition, they are very easy to derive. This is an important practical matter. There are many splendid methods of calculation whose use requires either an excellent memory or a copy of Hamermesh*; without these, all is lost. I hope that once you have seen me derive these algorithms, you will be able to derive them again, when the need to apply them occurs, on the back of an envelope, if necessary.

In the second part of the paper I will show how these methods can be used to show that the objects we have been manipulating are in fact a complete set of inequivalent irreducible representations. This is practice before theory, the reverse of the usual mathematical order, but nevertheless the most expedient way to do things here.

At the end I will make some remarks about the calculation of invariant couplings (Clebsch-Gordan coefficients). What I have to say will be neither very general nor (I am afraid) very impressive, but it fits in with the tensorial methods used in the other two parts. Perhaps one of you will be clever enough to figure out how to make the methods I will present here more general and more powerful.

Finally, I should apologize for the fact that this paper will contain so much mathematics and so little physics. This is not because I believe this is the way SU(3) should be taught, but only because the physics has already been covered in Prof. Lipkin's† paper.[1]

1. MATTERS OF PRACTICE

Tensor analysis for SU(n)

SU(n) is defined as the group of all unimodular unitary $n \times n$ matrices. This definition itself tells us one representation of the group. If we let x^i be a complex n-vector, on which the group acts in the following manner:

$$U : x^i \to U^i{}_j \, x^j$$

*M. Hamermesh, *Group Theory and its Application to Physical Problems*, Dover Publications, New York, 2012.

†H. J. Lipkin, "Some properties of mesons in SU(3)", pp. 325–330, preceding Coleman's in the same volume.

[1] Another good reference (indeed, the standard one) is the anthology edited by M. Gell-Mann and Y. Ne'eman, *The Eightfold Way*, Benjamin, New York, 1964.

then it is clear that the space of all x^i form a basis for a representation of U(n). A basis for another representation is formed by a set of vectors y_i which transform according to

$$U\colon y_i \to U_i{}^j y_j$$

where

$$U_i{}^j = \overline{U}^j{}_i$$

(We use an overbar throughout to indicate complex conjugation.)

The notation we have used, with its upper and lower indices, mimics that of ordinary tensor analysis. This mimicry is not deceptive, for, as a consequence of the unitarity of U,

$$U\colon x_i y^i \to x_i y^i$$

and thus, just as in ordinary tensor analysis, the summation of upper and lower indices is an invariant operation.

By the usual method — taking direct products — we can form, from these primitive objects, the spaces of all tensors of rank $(n+m)$, with n upper and m lower indices. Each of these spaces clearly forms the basis for a representation of U(n).

These representations are, however, not necessarily irreducible. If a_{ij} is a tensor of rank two, we may divide it into the sum of a symmetric and an antisymmetric part:

$$a_{ij} = a_{\{ij\}} + a_{[ij]}$$

This separation is invariant under the action of the group; thus we have divided the original representation space (of dimension n^2) into two invariant subspaces (of dimension $(n^2+1)/2$ and $(n^2-1)/2$, respectively). Likewise, if a^i_j is a mixed tensor, we may divide it into two parts,

$$a^i_j = \frac{1}{n}\,\delta_i{}^j a^k_k + \hat{a}^i_j$$

where

$$\hat{a}^i_i = 0$$

and again we have two invariant subspaces, in this case of dimension 1 and n^2-1. (Note that in this case we cannot make a further reduction by symmetrizing and antisymmetrizing, because the indices transform differently under the action of the group.)

One of the problems we set in the introduction was finding the irreducible representations of SU(3). Our general method of attack will be to (1) construct all tensors with a given number of upper and lower indices, (2) divide them invariantly into as many parts as we can, (3) discard the parts which we can show lead to representations equivalent to those obtained from tensors of lower rank, and (4) identify the remaining parts with (hopefully) new irreducible representations.

This method is, in principle, capable of generating all representations of SU(n), for any n. However, the combinatorics required to keep track of the irreducible tensors becomes formidable for n greater than 3, and other methods seem to be more efficient.

We will first tackle SU(2), to warm up.

The representations of SU(2)

It is a great convenience to introduce the antisymmetric two-index tensor ϵ^{ij}. Under the action of the group

$$U\colon \epsilon^{ij} \to (\det U)\,\epsilon^{ij}$$

but since U is unimodular, ϵ^{ij} is invariant. So is ϵ_{ij}, the corresponding entity with two lower indices. Given any tensor, we can use the ϵ tensors to raise and lower indices, just as the metric tensor is used in ordinary tensor analysis. In particular, we may write any tensor invariantly in terms of a tensor with all lower indices, and thus there is no representation of SU(2) induced on a general tensor that is not equivalent to one induced on .a tensor with only lower indices. Let

$$a_{i_1,\ldots,i_n}$$

be a tensor of this kind. We may divide a into two parts, which are respectively symmetric and antisymmetric under interchange of the first two indices. With the aid of ϵ_{ij} we may write the antisymmetric part in terms of a tensor of lower rank:

$$a_{[i_1,i_2]\ldots} = \epsilon_{i_1 i_2}\, b_{i_3\ldots}$$

Thus, we need only consider completely symmetric tensors. Thus we have (hopefully) an inequivalent irreducible representation of SU(2) associated with every space of completely symmetric tensors of a given rank, with all indices lower. (We adhere to an ancient convention, and call this rank $2s$.) We will call the representation $D^{(s)}$, or (s) for short. The dimension of (s) is the number of linearly independent tensors of the proper sort. Since the tensors are completely symmetric, and since the indices are allowed to assume only two values, this is the same as the number of ways $2s$ objects can be divided into two sets. That is to say,

$$\dim(s) = 2s + 1$$

a result which should be familiar.

The representations of SU(3)

Now let us apply the same techniques to SU(3). The invariant ϵ-tensors now have three indices, and thus cannot be used to raise and lower indices. Thus we have to work with tensors that have both lower and upper indices:

$$a^{i_1\cdots i_n}_{j_1\cdots j_m}$$

However, we may still use the ϵ-tensors to write the antisymmetric part of any tensor in terms of a tensor of lower rank:

$$a^{[i_1,i_2]\cdots}_{\ldots} = \epsilon^{i_1 i_2 k}\, b^{\ldots}_{k\ldots}$$

and therefore we need only consider tensors completely symmetric in both their upper and lower indices. By similar reasoning, we need only consider traceless tensors.

We will define the representation induced on the space of all traceless, completely symmetric tensors with n upper indices and m lower indices as $D^{(n.m)}$, or simply (n, m), for short. Eventually, we will show that this family of representations forms a complete set of inequivalent irreducible representations. However, it is more expedient to first extract some properties of these representations that are of practical importance, and only afterwards to prove their completeness, inequivalence, and irreducibility. Therefore, for the time being, we will simply call them IR's. If you want to think of "IR" as an acronym for "irreducible representation", you are welcome (indeed, you are encouraged) to do so, but in fact, we shall not use this property until we have proved it.

Note that a simple consequence of our definitions is that

$$\overline{(n,m)} = (m,n)$$

where the overbar indicates complex conjugation.

Dimension of the IR's

To calculate the dimension of an IR is a straightforward exercise in combinatorics. The space of all completely symmetric tensors with n upper indices and m lower indices can be decomposed into the space of all symmetric tensors and to a space of tensors equivalent to the traces. (By "the traces" we mean those tensors which are obtained by summing one upper index with one lower index.) The space of the traces is equivalent to the space of all completely symmetric tensors with $(n-1)$ upper indices and $(m-1)$ lower indices. In representation language,

$$(n,0)\otimes(0,m)=(n,m)\oplus[(n-1,0)\otimes(0,m-1)]$$

Taking the dimensions of both sides

$$\dim(n,m)=\dim(n,0)\times\dim(0,m)-[\dim(n-1,0)\times\dim(0,m-1)]$$

By arguments similar to those we used in the discussion of SU(2),

$$\dim(n,0)=\tfrac{1}{2}(n+2)(n+1)=\dim(0,n)$$

and therefore,

$$\dim(n,m)=\tfrac{1}{2}(n+1)(m+1)(n+m+2)$$

There is an alternative method of designating representations, much used in the literature, in which a representation is labelled by its dimension. Since (n,m) and (m,n) have the same dimension, they are distinguished by labelling a representation by its dimension if n is greater than m, and by its dimension with an overbar if m is greater than n. Thus

$(1,0)$	is often called	3
$(0,1)$	"	$\overline{3}$
$(1,1)$	"	8
$(3,0)$	"	10
$(2,2)$	"	27, etc.

(This is still not totally unambiguous, since $\dim(4,0)=\dim(2,1)=15$, but it suffices for practical purposes.)

Isospin and hypercharge

Merely to state that the strong interactions are invariant under a given higher symmetry group is not sufficient information to construct a physical theory. We also have to know how isospin rotations and hypercharge rotations, the old familiar symmetries of strong-interaction physics, are embedded in the group. The most convenient way of specifying this for SU(3) is by giving the isospin and hypercharge transformation properties of the fundamental representation $(1,0)$.

$(1,0)$ is a three-dimensional representation. Therefore, when we restrict SU(3) to the isospin group SU(2), it can decompose in only three ways: into the sum of three isosinglets, into an isosinglet and an isodoublet, or into an isotriplet. The first case is mathematically impossible, for it implies that all the elements of SU(2) are inside the identity element of SU(3). The third case is mathematically possible, but physically uninteresting: if the fundamental triplet contains only integral isospin, then all the IR's (which are made from direct products of fundamental triplets) would contain only integral isospins, which is not very satisfactory for explaining nature.

Thus only the second possibility remains. We will choose our basis in unitary space so that the $(1,0)$ representation looks like

$$\begin{pmatrix} q^0 \\ q^{+1/2} \\ q^{-1/2} \end{pmatrix}$$

where we have labelled the basis vectors by the appropriate eigenvalue of I_z. All that remains is to assign the hypercharge.

In order that the hypercharge differences between observed particles be integers, it is necessary that the hypercharge difference between the singlet and the doublet be of magnitude one. We choose the hypercharge of the doublet to be the greater. (This is just a matter of convention, although it does not appear to be so at first glance. If we were to choose the opposite assignment, the representation $(0,1)$ would have our original assignment. Thus the structure of all calculations would be the same, except that (n,m) would everywhere be replaced by (m,n). This only has to do with our conventions about how we write things on paper, not with what goes on in the world. Formally, this degree of freedom corresponds to the existence of an outer automorphism of SU(3).)

Thus, if the hypercharge of the singlet is y, the hypercharge of the doublet is $1+y$. Now, if hypercharge rotations are to be a subgroup of SU(3), the hypercharge itself must be a generator of SU(3). The generators of unimodular matrices are traceless,[2] and therefore, if Y is the 3×3 hypercharge matrix, then

$$\operatorname{Tr} Y = 3y + 2 = 0$$

which means

$$y = -\tfrac{2}{3}$$

(If we had assigned a different value to y, we could not have put the hypercharge rotations inside SU(3). However, we could still have put them inside U(3), which is just as good a symmetry group for the purposes of physics. Unlike the choice connected with the previous degree of freedom, a different choice for y would make a real difference in the physics of the world. However, it would not affect what we plan to calculate here, the isospin and hypercharge assignments of the observed particles. The reason is that all observed particles may, in principle, be constructed from baryons and antibaryons. The baryons are assigned to the representation (1,1), and the hypercharge assignments within this representation do not depend on y.)

Isospin-hypercharge decompositions

We want to determine how an IR decomposes into irreducible representations of the isospin-hypercharge subgroup when we restrict SU(3) to that subgroup. We will label representations of the subgroup by their isospin and hypercharge, thus:

$$(i)^Y$$

We already know one decomposition from the preceding section:

$$(1,0) \to \left(\tfrac{1}{2}\right)^{1/3} \oplus (0)^{-2/3}$$

[2]This comes from differentiating the famous formula

$$\det A = \exp \operatorname{Tr} \ln A$$

From this formula it is trivial to calculate the decomposition of $(n,0)$ since this is constructed by forming the completely symmetric product of n factors of $(1,0)$. Thus,

$$(n,0) \to \left(\frac{n}{2}\right)^{n/3} \oplus \left(\frac{n-1}{2}\right)^{(n/3)-1} \oplus \cdots \oplus (0)^{-2n/3}$$

Likewise,

$$(0,m) \to \left(\frac{m}{2}\right)^{-m/3} \oplus \left(\frac{m-1}{2}\right)^{-(m/3)+1} \oplus \cdots \oplus (0)^{2m/3}$$

From these formulae we may graphically (see Table I) represent the decomposition of $(n,0) \otimes (0,m)$. We form an $(m+1)$ by $(n+1)$ rectangular array, and label each column by a term in the first decomposition and each row by a term in the second. Inside each box we put the product of the factor associated with the row and the factor associated with the column. (To calculate this requires only the ability to add hypercharges and to multiply representations of the rotation group.) The total content of the array is then the desired isospin-hypercharge decomposition.

TABLE I

GRAPHICAL REPRESENTATION OF THE DECOMPOSITION OF $(n,0) \otimes (0,m)$

	$\left(\frac{n}{2}\right)^{n/3}$	$\left(\frac{n-1}{2}\right)^{(n/3)-1}$	$\cdots$
$\left(\frac{m}{2}\right)^{-m/3}$			
$\left(\frac{m-1}{2}\right)^{-(m/3)+1}$			
$\vdots$			

The table shows such an array. (The shading will be explained momentarily.)

Likewise, in the same manner, we may decompose $(n-1,0) \otimes (0,m-1)$. The relevant primary decompositions are

$$(n-1,0) \to \left(\frac{n-1}{2}\right)^{(n-1)/3} \oplus \cdots$$

and

$$(0,m-1) \to \left(\frac{m-1}{2}\right)^{-(m-1)/3} \oplus \cdots$$

However, as far as calculating the product goes, we could with no error subtract $\frac{2}{3}$ from all the hypercharges occurring in the first series and add the same amount to all the hypercharges in the second series. But then the decomposition of this product is revealed as just the content of the shaded portion of the original array.

But since

$$(n,0) \otimes (0,m) = (n,m) \oplus [(n-1,0) \otimes (0,m-1)]$$

TABLE II

DECOMPOSITION OF $(2,2)$

	$(1)^{2/3}$	$\left(\frac{1}{2}\right)^{-1/3}$	$(0)^{-4/3}$
$(1)^{-2/3}$	$(2)^0 \oplus (1)^0 \oplus (0)^0$	$\left(\frac{3}{2}\right)^{-1} \oplus \left(\frac{1}{2}\right)^{-1}$	$(1)^{-2}$
$\left(\frac{1}{2}\right)^{1/3}$	$\left(\frac{3}{2}\right)^{1} \oplus \left(\frac{1}{2}\right)^{1}$		
$(0)^{4/3}$	$(1)^2$		

this means that the decomposition of (n,m) is nothing but the content of the unshaded portion of the array, the border. In Table II, we have calculated the decomposition of $(2,2)$ by this method.**

A common (although extremely awkward) method of displaying the isospin-hypercharge decomposition of a representation of SU(3) is to plot a graph, in which the vertical axis is Y and the horizontal axis is I_z. If a state occurs in a representation with a given (Y, I_z) assignment, a dot is placed on the graph in the appropriate location. If two such states occur, the dot is circled. If three, the dot is circled twice, etc. Such a graph is called a weight diagram. Our result may be used to construct the weight diagram of (2,2) shown in Fig. 1.

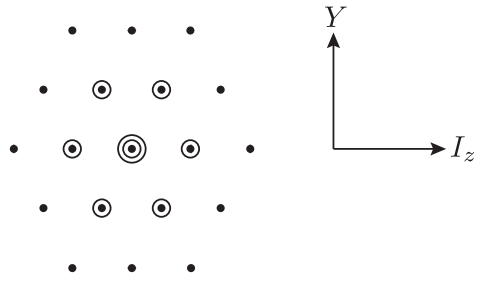

Figure 1

Weight diagram of $(2,2)$

It is a good exercise to use this method to construct the weight diagrams for $(1,1)$, $(3,0)$, and $(4,1)$, all of which are of physical interest. ($(1,1)$ is usually identified with the baryons and mesons; (3, 0) with the $\frac{3}{2}^+$ resonances in baryon-meson scattering. It has recently been proposed that some of the higher resonances may be part of (4,1).)

The Clebsch-Gordan series

I now want to present a method for decomposing the direct product of two IR's into a direct sum of IR's, The method proceeds in two steps: First we decompose the direct product of two IR's

**In Table II, the exponents in the left-hand column read $(1)^{-1/3}$, $(\frac{1}{2})^{2/3}$, and $(0)^{5/3}$, respectively; and those in the top row read $(1)^{1/3}$, $(\frac{1}{2})^{-2/3}$, and $(0)^{-5/3}$, respectively.

into a direct sum of certain special reducible representations, which will be defined below. Then we decompose the special reducible representations into a direct sum of IR's.

We shall denote the special reducible representations by $D^{(n,n';m,m')}$ — or for brevity, simply by $(n,n';m,m')$. The representation $(n,n';m,m')$ is defined as that representation which has for its basis the set of all tensors with $(n+n')$ upper indices and $(m+m')$ lower indices, that are completely symmetric among the first n upper indices, completely symmetric among the last n' upper indices, completely symmetric among the first m lower indices, completely symmetric among the last m' lower indIces, and traceless. Roughly speaking, $(n,n';m,m')$ may be thought of as the direct product of (n,m) and (n',m') with all traces removed, but without any symmetrization.

It is a simple matter to decompose the direct product of IR's into special reducible representations. We merely separate out all tensors that can be obtained by contracting, in all possible ways, indices from the set of n with indices from the set of m', and indices from the set of n' with indices from the set of m. That is to say,

$$\begin{aligned}(n,m)\otimes(n',m') = (n,n';m,m') &\oplus (n-1,n';m,m'-1)\\ &\oplus (n,n'-1;m-1,m')\\ &\oplus (n-1,n'-1;m-1,m'-1)\oplus\cdots\end{aligned}$$

The process terminates whenever we run out of indices to contract; that is, whenever a zero appears in the series on the right. In more compact form,

$$(n,m)\otimes(n',m') = \sum_{i=0}^{\min(n,m')}\ \sum_{j=0}^{\min(n',m)} (n-i,n'-j;m-j,m'-i)$$

where the summation sign indicates a direct sum.

We now wish to decompose one of our special reducible representations into direct sums of IR's. In the language of tensors, we want to decompose an arbitrary tensor from the basis of $(n,n';m,m')$ into a sum of linear combinations of completely symmetric traceless tensors. Let us begin with the upper indices. Let

$$T^{i_1\cdots i_n\, i_{n+1}\cdots i_{n+n'}}_{j_1\cdots j_m\, j_{m+1}\cdots j_{m+m'}}$$

be an arbitrary tensor of the type under discussion. Let us choose a pair of upper indices; with no loss of generality they may be i_1 and i_{n+1}. We may write the tensor as the sum of two tensors, one of which is symmetric under interchange of these indices, and the other of which is antisymmetric. Using the ϵ tensor, we may write the antisymmetric part in terms of a tensor of lower rank,

$$S^{i_2\cdots i_n\, i_{n+2}\cdots}_{kj_1\cdots} = \epsilon_{ki_1i_{n+1}}\, T^{i_1 i_2\cdots i_n\, i_{n+1}\, i_{n+2}\cdots}_{j_1\cdots}$$

The surprising fact, which enormously simplifies the whole reduction, is that this tensor is already completely symmetric in its lower indices.

Proof: For example, let us take the indices j_1 and j_{m+1}. We prove the tensor is symmetric under interchange of these indices by showing that their contraction with the ϵ-tensor vanishes,[‡]

$$\begin{aligned}\epsilon^{rj_1j_{m+1}} S^{i_1\cdots i_n\, i_{n+2}\cdots}_{kj_1\cdots} &= \epsilon^{rj_1j_{m+1}}\epsilon_{ki_1i_{n+1}}\, T^{i_1\cdots}_{j_1\cdots}\\ &= \Big(\delta^r_k\delta^{j_1}_{i_1}\delta^{j_{m+1}}_{i_{n+1}} - \delta^{j_1}_k\delta^r_{i_1}\delta^{j_{m+1}}_{i_{n+1}} + \text{ cyclic permutations}\Big)\times T^{i_1\cdots}_{j_1\cdots}\end{aligned}$$

[‡]The factor $\epsilon_{ki_1i_{n+1}}$ was missing in this equation.

But by the tracelessness of T, the right-hand side of this equation is zero. Similar arguments work for any pair of indices.

Thus the symmetrization is very simple. We may remove pairs of upper indices, adding a lower index whenever we do so; or, alternatively, we may remove pairs of lower indices, adding an upper index whenever we do so. But we can never remove both a pair of upper indices and a pair of lower indices, for once we have removed a pair of upper (lower) indices, the tensor is already completely symmetric in its lower (upper) indices. The process terminates when we run out of indices. Returning from the basis space to the representation, we may write the decomposition in compact form:

$$(n,n';m,m') = (n+n',m+m') \oplus \sum_{i=1}^{\min(n,n')} (n+n'-2i, m+m'+i) \oplus \sum_{j=1}^{\min(m,m')} (n+n'+j, m+m'-2j)$$

where the summation sign again represents the direct sum.

To demonstrate the efficiency of this method, we conclude with two examples. All arithmetic is shown.

Example 1: $(1,1) \otimes (1,1)$

$$(1,1) \otimes (1,1) = (1,1;1,1) \oplus (1,0;0,1) \oplus (0,1;1,0) \oplus (0,0;0,0)$$

$$\begin{aligned}(1,1;1,1) &= (2,2) \oplus (0,3) \oplus (3,0)\\ (1,0;0,1) &= (1,1)\\ (0,1;1,0) &= (1,1)\\ (0,0;0,0) &= (0,0)\end{aligned}$$

The desired decomposition is the sum of all the terms on the right. If we use the notation on p. 4 to write this sum in terms of the representations' dimensions, we find

$$8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 1$$

This is a familiar (and useful) result. It tells us the number of independent amplitudes for the scattering of two octets into two octets (eight, if time reversal imposes no further restrictions), the number oi independent Yukawa couplings for antibaryon-baryon-pseudoscalar meson (two), the number of amplitudes for the decay of a $\frac{3}{2}^+$ resonance into baryon and pseudoscalar meson (one), etc.

Example 2: $(2,2) \otimes (3,0)$

$$(2,2) \otimes (3,0) = (2,3;2,0) \oplus (2,2;1,0) \oplus (2,1;0,0)$$

$$\begin{aligned}(2,3;2,0) &= (5,2) \oplus (3,3) \oplus (1,4)\\ (2,2;1,0) &= (4,1) \oplus (2,2) \oplus (0,3)\\ (2,1;0,0) &= (3,0) \oplus (1,1)\end{aligned}$$

In the alternative notation,

$$27 \otimes 10 = 81 \oplus 64 \oplus 35 \oplus \overline{35} \oplus 27 \oplus 10 \oplus \overline{10} \oplus 8$$

2. MATTERS OF PRINCIPLE

I now wish to show that the IR's we have been discussing do indeed form a complete set of inequivalent irreducible representations of SU(3). In order to do this, we need a theorem imported from Wigner. Let G be any compact group, and let the irreducible representations of G be arranged in a series, $D^{(0)}$, $D^{(1)}, \ldots$, with $D^{(0)}$ the trivial representation. Then any representation of G may be decomposed into a direct sum of irreducible representations. We will write this in the following way:

$$D = \oplus \sum n_i D^{(i)}$$

where n_i is the number of times $D^{(i)}$ occurs in the sum. The theorem we need is this:[3]

Theorem: In the decomposition of $\overline{D}^{(i)} \otimes D^{(j)}$, $n_0 = \delta_{ij}$.

Corollary: If, in the decomposition of $\overline{D} \otimes D$, $n_0 = 1$, D is irreducible.

Now we know how to decompose the direct product of any two IR's into a sum of IR's, from the algorithms of the preceding section. Thus, if we know which IR's contain $D^{(0)}$, we can use the corollary to check if the IR's are irreducible, and the theorem to check if they are inequivalent.

We will now prove the following

Lemma: Of all the IR's, only $(0, 0)$ contains $D^{(0)}$ in its decomposition, and it contains it once.

Proof: Let us assume an IR contains $D^{(0)}$. Then it must contain a hypercharge-zero isosinglet. It is a trivial consequence[4] of our algorithm for the hypercharge-isospin decomposition of an IR that only (m, m) contains a hypercharge-zero isosinglet, and further, that it only contains one such state. Thus any tensor component of such an IR which transforms like a hypercharge-zero isosinglet must be pure $D^{(0)}$. Such a component is

$$a^{11\ldots1}_{11\ldots1}$$

But this cannot be pure $D^{(0)}$, for there are evidently transformations of the group (for example, those which interchange the first and second axes and leave the third invariant) under which it is not invariant. The only exception to this argument occurs for (0,0).

Thus, to find how many times $D^{(0)}$ occurs in the decomposition of $(n, m) \otimes (n, m) = \overline{(m, n)} \otimes (n, m)$, we need only find how many times (0,0) occurs.[‡‡] But it is a trivial consequence[4] of our algorithm for the Clebsch-Gordan series that this only occurs once. Thus we have the following.

Theorem: (n, m) is an irreducible. representation of SU(3).

By the same token, to prove the inequivalence of the IR's we need only see how many times $(0, 0)$ occurs in the decomposition of $(n, m) \otimes (n', m') = \overline{(m, n)} \otimes (n', m')$.[***] But it follows trivially[4] from the Clebsch-Gordan algorithm that (0,0) only occurs in this series if $m = m'$ and $n = n'$. Thus

[3] Actually, this is not in Ref. [1] in precisely this form; however it is a trivial corollary of Schur's lemma and the fact that every representation of a compact group is equivalent to a unitary representation. (It can also be derived simply from the orthogonality relations.)

[‡‡] Originally, the previous equation read $(n, m) \otimes (n, m) = (m, n) \otimes (n, m)$.

[4] Prove it.

[***] Originally, this equation read $(n, m) \otimes (n', m') = (m, n) \otimes (n', m')$.

we have the following.

Theorem: (n,m) and (n',m') are equivalent only if $n=n'$ and $m=m'$.

To prove the completeness of the IR's, we need some more information from WIGNER [1]. For any compact group it is possible to define an integral over group space in such a way that

$$\int \overline{D}^{(i)}_{\alpha\beta}(g) D^{(j)}_{\gamma\delta}(g)\, dg = 0 \text{ (if } i \neq j)$$

where the Greek subscripts indicate matrix elements. Now, let us assume that there exists an irreducible representation of SU(3) which is not equivalent to any (n,m). Let us call this representation $D^{(?)}$. Then by assumption

$$\int \overline{D}^{(?)}_{\alpha\beta}(g) D^{(n,m)}_{\gamma\delta}(g)\, dg = 0$$

The representation $(1,0)$ has eight independent matrix elements. Let me call these z_i $(i = 1 \ldots 8)$. They form a set of co-ordinates in group space. The matrix elements of (0,1) are the $\overline{z}_i$. The equation above says that the matrix elements of $D^{(?)}$ are orthogonal to z_i and $\overline{z}_i$, that is to say, to all linear functions of z_i and $\overline{z}_i$. It also says that the matrix elements of $D^{(?)}$ are orthogonal to all the matrix elements of $(1,0)\otimes(1,0)$, $(1,0)\otimes(0,1)$, and $(0,1)\otimes(0,1)$, that is to say, to $z_i z_j$, $z_i\overline{z}_j$, and $\overline{z}_i\overline{z}_j$. In other words, these matrix elements are orthogonal to all polynomials of order two in the co-ordinates. In fact, by similar arguments, we can show that they are orthogonal to all polynomials in the co-ordinates, and therefore

$$D^{(?)}_{\alpha\beta}(g) = 0 \text{ for all } g$$

But this is a contradiction, because for any representation $D_{\alpha\beta}(1) = \delta_{\alpha\beta}$. Thus we have our final

Theorem: The IR's form a complete set of inequivalent irreducible representations of SU(3).

3. INVARIANT COUPLINGS: AN AWKWARD MATTER

The last subject I am going to talk about is the construction of invariant couplings. (For trilinear interactions, this is equivalent to the problem of constructing the Clebsch-Gordan coefficients.) As I said in the introduction, I know of no methods here as simple and powerful as the ones discussed in the first section. However, there are some special tricks based on tensorial methods which are useful for a very restricted class of problems (which, through the grace of God, turns out to include many cases of physical interest). My main aim here will be to discuss these; however, I would like to begin with a general discussion, to place these methods within a wider framework.

The problem of Cartesian components

Let us suppose we wish to couple three octets of fields. Just to be definite, let us make them baryon (denoted by ψ), antibaryon ($\overline{\psi}$) and meson (ϕ). The Clebsch-Gordan algorithm tells us that there are only two invariant couplings, and a moment's thought shows what they must be:

$$\overline{\psi}^i_k \psi^j_i \phi^k_j \quad \text{and} \quad \overline{\psi}^i_k \psi^k_j \phi^j_i$$

(Of course, if these are really baryon and meson fields, ψ is an octet of Dirac bispinors, and there should be a γ_5 in the above expressions to conserve parity. However, we will ignore degrees of

freedom associated with space-time transformation properties here, and treat all fields as if they were scalars, for the sake of simplicity.) In a sense, this expression solves the problem completely, for it explicitly gives the invariant coupling in terms of the fields. Unfortunately, it gives the coupling in terms of the Cartesian components of the fields (ϕ_1^1, ϕ_2^1, etc.), and for practical applications, we need the coupling in terms of isospin-hypercharge eigenstates (π^+, K^0, etc.).

Thus, in this formulation, the whole problem of constructing invariant couplings reduces to the problem of expressing the Cartesian components of SU(3) tensors in terms of hypercharge-isospin eigenstates. If I could present a simple algorithm for constructing such expressions, the problem would be completely solved. Regrettably, I know of no such algorithm. However, we can always construct such expressions by what are essentially cut-and-try methods; the main portion of the subsequent exposition will be devoted to an explanation of these techniques. They are definitely not of the back-of-the-envelope class; to do calculations, you need tables, which is unfortunate. However, with this way of looking at the problem you only need one table per representation, while with the more usual technique, you need one table per triplet of representations. So there is some gain.

As before, we will begin by discussing SU(2), where things are simple.

SU(2) again

Let us begin with a fundamental doublet of fields, which we will call p and n, and which we will write as a column vector,

$$N = \begin{pmatrix} p \\ n \end{pmatrix}$$

The conjugate doublet of antiparticle fields transforms like a row vector,

$$\overline{N} = (\overline{p}, \overline{n})$$

But of course, with the aid of the ϵ-tensor, we may write this as a column vector also:

$$\begin{pmatrix} \overline{n} \\ -\overline{p} \end{pmatrix}$$

(The minus sign appears because of the antisymmetry of the ϵ-tensor.)

Next, let us consider a triplet of fields (the pions, for example). We may represent this triplet in three equivalent ways, as tensors of the three forms

$$\phi_{ij},\ \phi^{ij},\ \phi^i_j$$

We will choose the third way, and write the components in the form of a 2×2 matrix:

$$\begin{pmatrix} \phi_1^1 & \phi_2^1 \\ \phi_1^2 & \phi_2^2 \end{pmatrix}$$

Symmetry for one form implies tracelessness for the other:[††]

$$0 = \epsilon^{ij}\phi_{ij} = \phi^i_i$$

Thus

$$\phi_1^1 = -\phi_2^2$$

[††]The first word in this sentence was originally "Antisymmetry".

It is clearly an invariant condition to demand that the matrix ϕ be Hermitian, and we shall impose such a condition here. (There is nothing wrong with anti-Hermitian matrices, but they would describe sets of six real fields — perfectly suitable for the Σ hyperons, but not for the pions.) The component ϕ_2^1 carries charge $+1$, and we will normalize the matrix by demanding that it be the π^+ field. Then Hermiticity and tracelessness determine everything to within one free real parameter, α:

$$\phi = \begin{pmatrix} \alpha\pi^0 & \pi^+ \\ \pi^- & -\alpha\pi^0 \end{pmatrix}$$

This is determined by looking at the invariant-mass term in the Lagrangian. This is $\phi_j^i\phi_i^j$, and we want to normalize our fields so that it will be the sum of three squares with coefficients one.[5] Thus,

$$\tfrac{1}{2}\phi_j^i\phi_i^j = \pi^+\pi^- + \alpha^2\pi^0\pi^0 = \pi^+\pi^- + \tfrac{1}{2}\pi^0\pi^0$$

which means

$$\alpha = \pm\sqrt{\tfrac{1}{2}}$$

The sign ambiguity is just a matter of phase conventions. We choose the plus sign.

One advantage of writing these fields as row vectors, column vectors, and matrices is that the ordinary operations of matrix multiplication (which just involve, with these conventions, the summation of upper and lower indices) are invariant under the action of the group. Thus the usual pion-nucleon interaction may be written as

$$\mathscr{L} = \tfrac{1}{2}\,\mathrm{Tr}\,[\partial_\mu\phi\,\partial^\mu\phi] + \overline{N}(i\gamma^\mu\partial_\mu - m)N + g\overline{N}\gamma_5\phi N + \lambda\,\mathrm{Tr}\,[\phi^4]$$

where Tr represents the ordinary matrix trace.

SU(3) octets: trilinear couplings

By exactly the same arguments as we used above, we can write the (1,1) representation of SU(3) as a 3×3 matrix. To identify this matrix with the physical fields, all we need do is study its transformation properties under the SU(2) subgroup. (I remind you that this is the set of all transformations that leave the first index invariant.) This determines everything to within normalization factors, which are determined by the same argument we used in the SU(2) case. Thus we find for the octet of baryon fields,

$$\psi = \begin{pmatrix} \frac{2}{\sqrt{6}}\Lambda & \Xi^- & -\Xi^0 \\ p & -\frac{1}{\sqrt{6}}\Lambda + \frac{1}{\sqrt{2}}\Sigma^0 & \Sigma^+ \\ n & \Sigma^- & -\frac{1}{\sqrt{6}}\Lambda - \frac{1}{\sqrt{2}}\Sigma^0 \end{pmatrix}$$

The minus sign for the Ξ doublet arises because we are writing it as a row vector. (See the discussion for the nuclear doublet above.) Likewise, we may write a matrix for the pseudoscalar meson octet[§]

$$\phi = \begin{pmatrix} \frac{2}{\sqrt{6}}\eta & K^- & \overline{K}^0 \\ K^+ & -\frac{1}{\sqrt{6}}\eta + \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ K^0 & \pi^- & -\frac{1}{\sqrt{6}}\eta - \frac{1}{\sqrt{2}}\pi^0 \end{pmatrix}$$

[5] We don't have to do this of course; we could choose α to be any non-zero real number, and it would all cancel out of any calculation when we were finished with the renormalizations. This is, however, the traditional and convenient choice; it makes the symmetry between the three pions manifest.

[§] The matrix element ϕ_3^3 originally was equal to $\frac{1}{\sqrt{6}}\eta - \frac{1}{\sqrt{2}}\pi^0$; the trace of this matrix must equal zero.

Note that ϕ is Hermitian. The corresponding minus sign does not arise here. This is because the usual convention is to define $(K^-, \overline{K}^0)$ as the anti-particles of (K^+, K^0); the particles which form an isotopic doublet with proper phase relations are $(\overline{K}^0, -K^-)$. The two minus signs cancel.

Just as before, the ordinary operations of matrix algebra are invariant under the action of the group. Thus the two invariant couplings discussed on page 11 may be written as

$$\mathrm{Tr}\,(\overline{\psi}\psi\phi) \quad \text{and} \quad \mathrm{Tr}\,(\overline{\psi}\phi\psi)$$

Actually, it is conventional to consider not these two couplings, but the two linear combinations

$$\mathrm{Tr}\,(\overline{\psi}\,\{\psi,\phi\}) \quad \text{and} \quad \mathrm{Tr}\,(\overline{\psi}\,[\phi,\psi])$$

These are called d-type and f-type coupling, respectively.

SU(3) octets: quadrilinear couplings

We may also use these matrices to study the invariant coupling of four octets. This is useful in the analysis of meson-baryon scattering, and also in the analysis of nonleptonic hyperon decays, using octet spurions. Let us represent the four octets by four 3×3 traceless matrices, A_i, $(i = 1 \ldots 4)$.

The possible invariants fall into two classes. There are those of the form

$$\mathrm{Tr}\,(A_1A_2A_3A_4)$$

There are twenty-four permutations of the four A's, but due to the invariance of the trace under cyclic permutations, only six of these lead to distinct invariants. Secondly, there are also invariants of the form

$$\mathrm{Tr}\,(A_1A_2)\,\mathrm{Tr}\,(A_3A_4)$$

Here only three permutations lead to distinct invariants.

Thus we have nine invariants in all. Unfortunately, we know that there are only eight independent couplings of four octets. (This was one of the consequences of the first example given of the Clebsch-Gordan algorithm.) This is an apparent contradiction; we can only escape if there is a linear relation among our nine matrix invariants. Indeed there is such a relation; it is

$$\sum_{\text{six distinct perms}} \mathrm{Tr}\,(A_1A_2A_3A_4) \;=\; \sum_{\text{three distinct perms}} \mathrm{Tr}\,(A_1A_2)\,\mathrm{Tr}\,(A_3A_4)$$

for any four 3×3 traceless matrices. This identity was no doubt known to Cayley; however, it was first shown to me by BURGOYNE [2], and I will embarrass him by calling it Burgoyne's identity.

Proof: Let A be any 3×3 matrix, with eigenvalues a_1, a_2, a_3. A satisfies its own characteristic equation

$$(A - a_1)(A - a_2)(A - a_3) = 0$$

Let us write the coefficients of the powers of A in terms of invariants. We then obtain

$$A^3 - (\mathrm{Tr}\,A)A^2 - \tfrac{1}{2}\left[\mathrm{Tr}\,(A^2) - (\mathrm{Tr}\,A)^2\right]A - (\det A) = 0$$

Multiplying this equation by A and taking the trace, we find

$$\mathrm{Tr}\,(A^4) = \tfrac{1}{2}\left(\mathrm{Tr}\,(A^2)\right)^2$$

if A is traceless. Now let $A = \sum_i \lambda_i A_1$, and let us extract the coefficient of $\lambda_1\lambda_2\lambda_3\lambda_4$ from the above equation. We then obtain Burgoyne's identity.

This identity has an amusing consequence. Let us consider quartic self-couplings of the meson. At first we would think there would be two independent couplings, $[\mathrm{Tr}\,(\phi^2)]^2$ and $\mathrm{Tr}\,(\phi^4)$. However, the identity shows that these are not independent. Thus the only invariant coupling is

$$[\mathrm{Tr}\,(\phi^2)]^2 = \left(\sum_{i=1}^{8} \phi_i^2\right)^2$$

where ϕ_i is a set of eight orthogonal real fields (for example, π_x, π_y, π_z, $K^{\pm}$, K^0, $\overline{K}^0$, and η). But this is invariant not only under SU(3), but also under the much larger group SO(8). So if we try to do a low energy calculation in which we only consider the self-interactions of the pseudoscalar mesons, we will find many "accidental" degeneracies in the output.

A mixed notation

Actually, for many purposes, it is unnecessary to reduce an SU(3)-invariant interaction to an expression which explicitly involves isospin-hypercharge eigenstates; it suffices to reduce the coupling to a sum of SU(2)-invariant interactions. This can easily be done within our framework. For example, instead of explicitly writing down the matrix for the baryon fields, we can display the decomposition in the following way:

$$\begin{aligned}
\psi_1^1 &= \tfrac{2}{\sqrt{6}}\,\Lambda \\
\psi_a^1 &= \Xi_a \\
\psi_1^a &= N^a, \text{ and} \\
\psi_b^a &= \Sigma_b^a - \tfrac{1}{\sqrt{6}}\,\delta_b^a \Lambda
\end{aligned}$$

where $a, b = 2, 3$, and the terms on the right are the appropriate SU(2) tensors. If we insert this (and the corresponding expression for the mesons) in the invariant coupling, we will obtain the desired result.

This procedure is evidently insensitive to the convention one adopts for phases within an SU(2) multiplet. This is an advantage because there are two such conventions which are widely used in th e literature. The convention we have used here is the one that is common in elementary particle physics and the one which is used in most of the earlier SU(3) literature. There is another convention, which is designed to agree with the standard angular momentum convention, and which is used in de SWART's tables [3] and in some of the later literature. In this convention, the field conjugate to π^+ is minus π^-.

Using the methods we have explained here, we can construct similar decompositions for any representation. (But the process is tedious.) For example, the decomposition for the (3,0) representation is

$$\begin{aligned}
\psi^{111} &= \Omega \\
\psi^{11a} &= \tfrac{1}{\sqrt{3}}\,\Xi^a \\
\psi^{1ab} &= \tfrac{1}{\sqrt{3}}\,\Sigma^{ab}, \text{ and} \\
\psi^{abc} &= \Delta^{abc}
\end{aligned}$$

where Ξ and Σ label the states with the same isospin-hypercharge assignments as the corresponding baryons, Ω is a $(0)^{-2}$ state, and Δ a $(\frac{3}{2})^1$ multiplet, and where we have omitted components obtainable from the ones we have listed by trivial permutations of indices.

The Gell-Mann–Okubo mass formula

There is one particular class of Clebsch-Gordan coefficients that is known in simple closed form for any representation. This is the set of coefficients that enters into the famous Gell-Mann–Okubo mass formula. The fundamental physical assumption of the formula (which we shall not discuss here) is that the effect of the medium-strong interactions is such that the effective mass Lagrangian for baryons (the mass-squared Lagrangian for mesons) transforms like the sum of a unitary singlet and a unitary octet. The unitary singlet merely gives a contribution to the common mass of all the particles in a unitary multiplet; it is the unitary octet that is responsible for the splitting. Since the medium-strong interactions conserve hypercharge and isospin, only one component of the octet, the 11 component, can enter into the formula. It follows from our Clebsch-Gordan algorithm that there are at most two ways of coupling an octet to the product of an IR and its conjugate; therefore there are at most three independent coefficients in the mass formula: one for the common mass and two for the splitting.

The problem is to calculate explicitly the two splitting terms. It was first solved by Okubo, using a very lengthy method. We shall give here a short proof due to SMORODINSKI [4].

For any representation of SU(3), the eight matrices which form the generators of the group transform, under the action of the group, like an octet. Therefore we can arrange them in a 3×3 matrix (a matrix of matrices),

$$G = \begin{pmatrix} -\beta Y & \star & \star \\ \star & \frac{1}{2}\beta\, Y + \frac{1}{\sqrt{2}} T_z & T_+ \\ \star & T_- & \frac{1}{2}\beta\, Y - \frac{1}{\sqrt{2}} T_z \end{pmatrix}$$

The T's are the usual isospin generators, Y is the hypercharge, β is an unknown real coefficient, and the blank spaces $(\star)$ are occupied by strangeness-changing generators in which we are not interested. To determine β, we observe that those transformations which mix the first and third axes and leave the second invariant do not change the electric charge. Electric charge is a generator of the group, and it is evident that the only member of the matrix of generators which is left invariant by these transformations is the 22 entry. Therefore this must be proportional to the electric charge. Since

$$Q = T_z + \tfrac{1}{2} Y$$

this means that

$$\beta = \tfrac{1}{\sqrt{2}}$$

Now, we want to find objects which transform like a mixture of octet and singlet, that is to say, like a matrix. One such object is clearly

$$G^1_1 = -\tfrac{1}{\sqrt{2}} Y$$

But since matrix operations are invariant, any matrix function of G will also transform like a mixture of octet and singlet. In particular, let us consider the cofactor matrix. (The cofactor matrix is the matrix composed of the determinants of the minors of a matrix; it occurs in the standard expression

for the inverse of a matrix.)

$$\begin{aligned}(\operatorname{cof} G)^1_1 &= G^2_2 G^3_3 - G^2_3 G^3_2 \\ &= \tfrac{1}{8}Y^2 - \tfrac{1}{2}T_z^2 - T_+T_- \\ &= \tfrac{1}{2}\left(\tfrac{1}{4}Y^2 - \mathbf{T}^2\right)\end{aligned}$$

Thus the three terms in the mass formula for any multiplet are

$$m = a + bY + c\left(\tfrac{1}{4}Y^2 - T(T+1)\right)$$

This is for fermions; as 1 said earlier, the rule of the game is that one uses squared masses for bosons.[6] For the most common boson multiplets (e. g., the pseudoscalar mesons), there is a further simplification: The multiplet is turned into itself by charge conjugation, and thus the coefficient of the term linear in the hypercharge must vanish.

REFERENCES

[1] WIGNER, E. P., *Group Theory*, Academic Press, New York, 1959.
[2] BURGOYNE, N., private communication (via S. L. Glashow).
[3] de SWART, J. J., *Rev. Mod. Phys.* **35** (1963) 916–939. This paper is reprinted, see footnote 1.
[4] SMORODINSKI, I. Ya., private communication.

[6] You will not find an explanation of this here (nor anywhere else for that matter).