

Diploma Thesis

On Theories of
Neutrino Oscillations

A Summary and Characterisation
of the Problematic Aspects

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“Liebe Radioaktive Damen und Herren!

*...bin ich angesichts ... des kontinuierlichen
beta-spektrums auf einen verzweifelten Ausweg
verfallen um ... den Energiesatz zu retten.
Nämlich die Möglichkeit, es könnten elektrisch
neutrale Teilchen, die ich Neutronen nennen
will, in den Kernen existieren.*

...

*Ich gebe zu, dass mein Ausweg vielleicht von
vornherein wenig wahrscheinlich erscheinen
wird, weil man die Neutronen, wenn sie
existieren, wohl schon längst gesehen hätte.
Aber nur wer wagt, gewinnt*

...

Also liebe Radioaktive, prüfet, und richtet...”

Wolfgang Pauli, 1930

Motivation

Neutrino physics is one of the most interesting and vividly discussed topics in high-energy physics today. Especially the question whether the neutrinos can oscillate or not (i. e. different neutrinos can change into each other) gave rise to a huge number of experiments to actually observe these oscillations. At least since the results from the SUPER KAMIOKANDE [A⁺04, H⁺06] and the SNO experiment [A⁺01, A⁺02a, A⁺02b] are published, it is widely believed that neutrino oscillations (NO) are an experimentally verified fact. However, the first hint has already been found in 1964 when the HOMESTAKE experiment [Dav64] discovered the solar-neutrino problem. That is, the number of measured electron neutrinos from the sun is by a factor of 2-3 less than the number of neutrinos predicted by the standard solar model (SSM).

Since within the standard model (SM) of particle physics the neutrinos are massless, and consequently cannot oscillate, their measurement shows that new physics beyond the SM exists. And indeed nowadays the experiments on NO are important to measure the unknown parameters of the SM and its minimal extensions. In particular, these unknown parameters are the neutrino masses and the entries in the neutrino mixing matrix.

From all the measurements made to discover NO one should think that the theory behind NO is well established and understood. But surprisingly this is not the case. The first who mentioned the idea of NO, though he assumed neutrino-antineutrino oscillations, was PONTECORVO in 1957 [Pon57, Pon58]. A few years later MAKI, NAKAGAWA and SAKA were the first to consider oscillations between the electron and the muon neutrino [MNS62]. Then it took around 20 years before KAYSER in 1981 showed that the up to that point used plane-wave approximation cannot hold for oscillating neutrinos and he proposed a wave packet treatment [Kay81], which then has again not been discussed for around 10 years. In the early 90s the discussion on the theoretical description of NO finally started with several seminal papers. First, GIUNTI, KIM and LEE explicitly calculated the oscillation probability for the neutrinos in a wave packet model [GKL91] and then showed that the state vectors used for the quantum mechanical description are, in general, ill-defined [GKL92]. In 1993 they published together with LEE a calculation of the probability in a quantum field theoretical framework without using state vectors for the neutrinos [GKLL93]. And finally, in 1995 BLASONE and VITIELLO showed that the description of mixed particles in quantum field theory (QFT) yields unexpected problems for the interpretation of neutrinos as particles. By only using exact—without perturbation—QFT methods they calculated an oscillation probability which differs significantly from the other results [BV95]. All these different approaches are even today still under discussion, but however under the assumption of relativistic neutrinos which have tiny mass-squared differences, all approaches give the same result. Thus, the theoretical discussion on the right description of the neutrinos does not spoil the experimental results, because today we are only able to measure ultra-relativistic neutrinos whose energy is at least a few orders

of magnitude higher than their mass.

The aim of this thesis is on the one hand to summarise the different theoretical approaches to NO and on the other hand to point out the important and critical points in their argumentation. In particular, the different approaches are the *standard plane-wave approximation* in chapter 3, the *internal wave-packet model* in chapter 4, the *external wave-packet model* in chapter 5, the *weak-process states* in chapter 7 and the *Blasone-Vitiello approach* in chapter 8.

Nevertheless, there are some points connected to NO which go beyond the scope of this thesis and will not be discussed. That are for example the question whether neutrinos are DIRAC or MAJORANA particles, whether the lepton sector of the SM breaks \mathcal{CP} -invariance and the determination of the mass hierarchy and the absolute mass scale for neutrinos.

Chapter 1

A Short Introduction to Neutrino Physics

Neutrino physics was born in 1930 when PAULI wrote his famous letter to the participants of the conference on radioactivity in Tübingen (see the quote on page v). In this letter he predicted the existence of a new particle in order to explain the continuous energy-spectrum of the electrons measured in β -decays. He first called the new particles *neutrons* but this name was later changed by FERMI into *neutrinos* to distinguish it from the *neutron* discovered by CHADWICK in 1932. Since the usual conservation laws on energy, angular and linear momentum, and charge should not be violated in the β -decay $n \rightarrow p + e^- + \bar{\nu}_e$, the neutrinos were predicted to be electrically-neutral spin-1/2 particles with a mass that is small compared to the electron mass. Until today three different types of neutrinos have been discovered, that are the *electron neutrino* ν_e (1956 by COWAN and REINES), the *muon neutrino* ν_μ (1962 by STEINBERGER, SCHWARTZ and LEDERMAN) and the *tauon neutrino* ν_τ (2000 with the DONUT experiment). This number of different flavours is exactly the one which we expect from the measurement of the Z -width made at LEP. This measurement predicts a number of 2.994 ± 0.012 [Y+06] neutrino flavours with a mass less than half the Z -mass.

1.1 Fermi's Theory

The first field-theoretical description of the β -decay was published in 1934 by FERMI [Fer34a, Fer34b]. He used the QED interaction Lagrangian, which couples an electron current to the photon field, and replaced it by a current-current term which couples a neutron-proton current $\bar{n}\gamma^\mu p$ to a neutrino-electron current $\bar{\nu}_e\gamma^\mu e$. Therefore, he used a four-fermion point-interaction without considering a messenger particle. The interaction Lagrangian can then be written as

$$\mathcal{L}_{\text{FERMI}} := -\frac{G_F}{\sqrt{2}}(\bar{n}\gamma^\mu p)(\bar{\nu}_e\gamma_\mu e), \quad (1.1)$$

where γ^μ are the usual γ -matrices and $G_F = 1.16637(1) \cdot 10^{-5} \text{ GeV}^{-2}$ [Y+06] is the *Fermi constant*. The factor $1/\sqrt{2}$ is due to historical reasons. Since the constant is numerically small compared to the other coupling constants, for example the fine-structure constant which is $\alpha \approx 1/137$, the name *weak interactions* is justified. This weakness together with the fact that neutrinos are electrically neutral explains why the measurement of neutrinos is so complicated.

During the years FERMI's theory was changed according to new observations. First it was noticed in the 50s that weak interactions are parity violating interactions. This was finally implemented in the theory by using not only *vector current* (V) but also *axial vector currents*

(A) in the form V-A. The Lagrangian can then be written as

$$\mathcal{L}_{\text{WEAK}} = -\frac{G_F}{\sqrt{2}} (\bar{n}\gamma^\mu(1 - g_A\gamma^5)p)(\bar{\nu}_e\gamma_\mu(1 - \gamma^5)e), \quad (1.2)$$

where $g_A = -(1.2573 \pm 0.0028)$ [KP93] is the nucleon axial vector coupling constant and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. This theory extended to other particles is consistent with all low energy experiments made until today [KP93].

A further improvement of the theory was done after it was recognised that the neutron and proton are not the fundamental fields but build out of quarks. Then, in the Lagrangian (1.2) the neutron can be replaced by the *up* quark and the proton by the *down* quark:

$$\mathcal{L}_{\text{WEAK}} = -\frac{G_F}{\sqrt{2}} (\bar{u}\gamma^\mu(1 - \gamma^5)d)(\bar{\nu}_e\gamma_\mu(1 - \gamma^5)e). \quad (1.3)$$

Note that the factor g_A disappeared in this Lagrangian, because its appearance in (1.2) is due to strong interactions of the nucleons. The Lagrangian (1.3) is a first step in the direction of the SM which we will describe next. The reason why we need a further improved theory for the weak interactions, although it is consistent with all low-energy experiments, is the non-renormalisability of the Lagrangian (1.3). This can be seen from the mass dimension of the current-current operator, which is 6. Thus, the coupling constant is of dimension -2, which leads to a breakdown of the theory for energies above $\sim G_F^{-1/2} \sim 300\text{GeV}$. Beneath this energy scale the Fermi theory can be considered as a effective theory.

1.2 The Standard Model

In the early 60s mainly GLASHOW, SALAM and WEINBERG started to develop a gauge theory for the weak interactions. This so-called GSW-model is today a part of the SM of particle physics. The other part is QCD gauge theory for the strong interaction. Since both parts do not influence each other and we do not need the strong interactions in this thesis, we will only describe the GSW-model in more detail here.

Before doing so, we should introduce the terms *left-* and *right-handed* fields. These fields are eigenfields of the operator γ^5 and we can project them out of an arbitrary field ψ by means of the chirality operators $P_{L/R} := (1 \pm \gamma^5)/2$

$$\psi_L := \frac{1}{2}(1 - \gamma^5)\psi, \quad \psi_R := \frac{1}{2}(1 + \gamma^5)\psi.$$

The names left- and right-handed stem from the fact that for massless particles the chirality eigenfields are simultaneously eigenfields of the helicity. Whereas helicity is defined as the projection of the spin on the direction of the momentum of the particle, which is called *right-handed* if the projected spin is in the direction of the momentum and *left-handed* for the opposite direction. The helicity of a particle is only unique in the case of massless particles which travel with the speed of light. For massive particles it depends on the frame of the observer. In contrast, chirality is independent of the frame.

In the GSW-model the neutrinos are assumed to be massless and thus chirality and helicity are the same. However, from experiments we know that only neutrinos with left-handed helicity participate in the weak interactions. In other words, no one has seen a right-handed neutrino yet. This is the reason why right-handed neutrinos are absent in the GSW-model.

The crucial step for the theory of weak interactions is now to assume that the experimental fact for the neutrinos can be generalised to all other fermions. That is, we assume that only

left-handed fields interact weakly. It is important to note that this does not mean the right-handed components are absent as in the case of the neutrino, because they can still interact via the electromagnetic and strong interactions. The starting point for the GSW-model is the gauge principle, which means we start with a global symmetry and postulate that it is also a local symmetry. In the GSW-model this gauge group is $SU(2)_w \times U(1)_Y$, where $SU(2)_w$ is called the *weak isospin* and $U(1)_Y$ is called the *Hypercharge*. Additionally, we assume that the left-handed fields form doublets under the weak isospin transformation while the right-handed fields are singlets. The left-handed doublets for the leptons and quarks can be grouped in the following way:

$$\begin{aligned} L_{eL} &:= \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, & L_{\mu L} &:= \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, & L_{\tau L} &:= \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \\ Q_{1L} &:= \begin{pmatrix} u_L \\ d_L \end{pmatrix}, & Q_{2L} &:= \begin{pmatrix} c_L \\ s_L \end{pmatrix}, & Q_{3L} &:= \begin{pmatrix} t_L \\ b_L \end{pmatrix}. \end{aligned} \quad (1.4)$$

To get a shorter notation we can write the different lepton fields as

$$\begin{aligned} \nu_{eL}, & & \nu_{\mu L}, & & \nu_{\tau L} \\ \ell_{eL} &:= e_L, & \ell_{\mu L} &:= \mu_L, & \ell_{\tau L} &:= \tau_L, \\ \ell_{eR} &:= e_R, & \ell_{\mu R} &:= \mu_R, & \ell_{\tau R} &:= \tau_R, \end{aligned} \quad (1.5)$$

In the following we will not go through all the details of the model, but only state some important points before coming to the interesting terms for this thesis.

Using the above defined singlets and doublets we could write down the most general, renormalisable Lagrangian which is invariant under the gauge group $SU(2)_w \times U(1)_Y$. But this Lagrangian does not involve any mass terms for leptons, quarks and gauge bosons, because such terms would violate the gauge invariance. This cannot describe the weak interactions as we are measuring them in experiments. Because the interactions are extremely short ranged the messenger particles, that is the gauge bosons, have to be massive in contradiction to the QED where the photon is massless. In order to create these masses in a gauge invariant way we use the *Higgs mechanism*. That is, we spontaneously break the gauge symmetry by assuming a non-vanishing vacuum expectation value (VEV) for the Higgs boson field

$$\langle \Phi \rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

This VEV is chosen in a way to get a remaining unbroken $U(1)_Q$ gauge symmetry, which is the QED gauge group. The breaking can then be symbolised as

$$SU(2)_w \times U(1)_Y \rightarrow U(1)_Q.$$

After the symmetry breaking the important parts of the Lagrangian—the ones which contain neutrinos—read

$$\mathcal{L}_\nu = \mathcal{L}_{\text{KIN}}^\nu + \mathcal{L}_{\text{CC}}^\nu + \mathcal{L}_{\text{NC}}^\nu \quad (1.6)$$

with

$$\mathcal{L}_{\text{KIN}}^\nu = \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} i \not{\partial} \nu_{\alpha L}, \quad (1.7a)$$

$$\begin{aligned}\mathcal{L}_{\text{CC}}^\nu &= -\frac{g}{2\sqrt{2}} \sum_{\alpha=e,\mu,\tau} \left(\bar{\nu}_{\alpha L} \gamma^\mu \ell_{\alpha L} W_\mu^+ + \bar{\ell}_{\alpha L} \gamma^\mu \nu_{\alpha L} W_\mu^- \right) \\ &= -\frac{g}{4\sqrt{2}} \sum_{\alpha=e,\mu,\tau} \left(\bar{\nu}_{\alpha L} \gamma^\mu (1 - \gamma^5) \ell_{\alpha L} W_\mu^+ + \bar{\ell}_{\alpha L} \gamma^\mu (1 - \gamma^5) \nu_{\alpha L} W_\mu^- \right),\end{aligned}\quad (1.7b)$$

$$\begin{aligned}\mathcal{L}_{\text{NC}}^\nu &= -\frac{g}{\cos \theta_W} \sum_{\alpha=e,\mu,\tau} \bar{\nu}_{\alpha L} \gamma^\mu \nu_{\alpha L} Z_\mu \\ &= -\frac{g}{\cos \theta_W} \sum_{\alpha=e,\mu,\tau} \bar{\nu}_{\alpha L} \gamma^\mu (1 - \gamma^5) \nu_{\alpha L} Z_\mu,\end{aligned}\quad (1.7c)$$

where $\cos \theta_W$ is the *weak mixing angle* which defines the mixing between the gauge bosons A_μ and Z_μ , and g is the $\text{SU}(2)_w$ coupling constant. The first term is just the usual kinetic term for a massless fermion with the exception that there is no right-handed field. The second term describes the so-called *charged currents* (CC) which are nothing else than the V-A currents, which leads to the low energy theory of FERMI. If we derive the effective low-energy theory from \mathcal{L}_{CC} we would find the following connection between the *Fermi constant* G_F and the weak coupling constant g

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}, \quad (1.8)$$

where M_W is the mass of the W -bosons. The third term in (1.7) describes *neutral currents* (NC) which were not included in FERMI's theory. These interactions couple an antineutrino-neutrino current to the Z -boson. They are hard to measure, because usually the electromagnetic interactions dominate the NC-processes. However, they are important for the understanding of NO in matter.

A closer look on the terms (1.7) shows that they are invariant under three different global $U(1)$ transformations which change the lepton and neutrino fields according to

$$U(1)_\alpha : \quad \nu'_{\alpha L} := e^{iL_\alpha} \nu_{\alpha L} \quad \ell'_{\alpha L} := e^{iL_\alpha} \ell_{\alpha L}. \quad (1.9)$$

These symmetries are the manifestation of the lepton number conservation for each flavour separately, that is L_e , L_μ and L_τ are conserved in the weak interactions. Although we have not shown it explicitly, we should note that the same holds for the complete Lagrangian.

Since NO are per definition flavour changing processes we immediately see that NO can not be described in the GSW-model. Thus, we have to extend the theory.

1.3 Neutrino Mixing

As already noted in the Motivation the first who mentioned NO was PONTECORVO [Pon57, Pon58]. But he considered neutrino-antineutrino oscillations which we will not describe here. The first who mentioned NO in the form in which they are mainly considered today were MAKI, NAKAGAWA and SAKA [MNS62]. They assumed that the neutrinos are massive, and that the neutrinos which we observe are actually superpositions of neutrinos with different masses. Due to the different masses they evolve differently in time and space, which then leads to oscillations.

Before we go on and explain how we can extend the GSW-model to describe such superpositions we will make some important comments on the terms *flavour* and *mass neutrinos*.

First of all we have to define the term *flavour*. By flavour we usually mean the quantum numbers corresponding to the three $U(1)$ symmetry groups (1.9), which we call either e , μ or

τ . This corresponds to the labeling of the three different generations of leptons in the SM. Now a flavour neutrino is a neutrino which has a definite flavour. Thus, the usual neutrinos ν_e , ν_μ and ν_τ are flavour neutrinos. The question is then how can we define the flavour of a neutrino by just measuring it? This question is not as trivial as it might sound. The point is that we cannot measure the neutrinos directly, because they do not interact electromagnetically. That means they do not leave a track of ionised particles in a detector as for example electrons would do. Thus, the only way to observe neutrinos is by observing the particles which decay into a neutrino and the ones which are produced in a neutrino interaction. Experimentally a neutrino interaction is always identified by the lepton in connection with which it is produced. For example in the β^- -decay $n \rightarrow p + e^- + \bar{\nu}$ this lepton is the electron, while in the π -decay $\pi^+ \rightarrow \mu^+ + \nu$ the lepton is the muon. Therefore, we can define the flavour of the neutrino by aligning it to the flavour of the corresponding lepton. In the β^- -decay this means the neutrino is an electron neutrino while it is a muon neutrino in the π -decay. This correspondence between the flavour neutrinos is the reason why in the GSW-model the $SU(2)_w$ doublet are chosen in the way (1.4).

After having defined the flavour neutrino we can go on to the mass neutrinos. Simply spoken a mass neutrino is defined as a neutrino with definite mass. In particular, that means the mass term in the Lagrangian is diagonal if it is written in terms of the fields that describe mass neutrinos. Since the GSW-model describes massless neutrinos the mass term is trivially diagonal and the flavour and mass neutrinos coincide, because the in the theory described neutrinos have both a definite flavour and a definite mass. This again shows that there are no NO possible in the SM, because we do not have a superposition of different mass neutrinos.

As a conclusion we see that we need massive neutrinos in order to get NO. In particular that means we have to add mass terms for the neutrinos to the GSW-model.

Since the neutrinos are electrically neutral each neutrino could, in principle, exists in either of two types. The first possibility is a *Dirac particles*, that is the particle and antiparticle are different. This is for example the case for all other leptons in the SM. But Dirac particle consists of left- and right-handed parts. This is in principle not a problem, because right-handed electrically neutral particles are singlets under $SU(2)_w$ transformations and thus would interact neither weakly nor electromagnetically nor strongly. Therefore, they are not detectable due to SM interactions. However, they have a gravitational interaction due to their mass. Such a particle is usually called *sterile* in order to distinguish it from the *active* particles. Thus, we could add as many right-handed neutrinos as we want without changing the interactions of the other particles. However, we will just add three right-handed neutrinos, one for each flavour: ν_{eR} , $\nu_{\mu R}$ and $\nu_{\tau R}$. The expected mass term for a Dirac neutrino would be

$$\begin{aligned} \mathcal{L}_{\text{MASS}}^D &:= -m\bar{\nu}_{\alpha L}\nu_{\alpha R} - m\bar{\nu}_{\alpha R}\nu_{\alpha L} \\ &= -m\bar{\nu}_{\alpha}^D\nu_{\alpha}^D \end{aligned} \quad (1.10)$$

with $\nu_{\alpha}^D := \nu_{\alpha L} + \nu_{\alpha R}$.

As we already said, there is another possibility for the neutrinos. That is, we consider the neutrino and the antineutrino as the same particle, which can be written as

$$\psi = \psi^c := \mathcal{C}\bar{\psi}^T \quad (1.11)$$

with the *charge conjugation matrix* $\mathcal{C} = i\gamma^2\gamma^0$. This kind of particles are usually called *Majorana particles*. The important point is that we do not need additional right-handed neutrinos for a Majorana particle, we can just define the Majorana field as

$$\nu_{\alpha}^M := \nu_{\alpha L} + \nu_{\alpha L}^c = \nu_{\alpha L} + \mathcal{C}\bar{\nu}_{\alpha L}^T.$$

The generic mass term for such a field reads

$$\begin{aligned}\mathcal{L}_{\text{MASS}}^M &:= -\frac{m}{2}(\overline{\nu_{\alpha L}^c}\nu_{\alpha L} + \overline{\nu_{\alpha L}}\nu_{\alpha L}^c) \\ &= -\frac{m}{2}(-\nu_{\alpha L}^T \mathcal{C}^\dagger \nu_{\alpha L} + \overline{\nu_{\alpha L}} \mathcal{C} \overline{\nu_{\alpha L}}^T) \\ &= -\frac{m}{2}\overline{\nu_{\alpha}^M}\nu_{\alpha}^M\end{aligned}\tag{1.12}$$

which has the same form as the Dirac term except for the factor of one half which avoids double counting, because the two fields are not independent.

However, there is an important difference between the two types of mass terms. That is, the Dirac term can be generated via the Higgs mechanism while this is in the usual way not possible for the Majorana term, because it would need an additional Higgs triplet which would lead to non-renormalisable terms in the Lagrangian. However, this would not be a big problem because today it is believed that the SM is just an effective theory of a more general theory. Moreover, the Dirac term can in principle be invariant under the lepton number transformations (1.9), while this is not possible for the Majorana term, which breaks this symmetry. This is due to the Majorana condition $\psi = \psi^c$ which is not compatible with (1.9). This breaking of the lepton number conservation can lead to interesting phenomena, for example the neutrinoless double- β -decay.

For simplicity we will discuss in this thesis only the case of Dirac neutrinos. Therefore, we add three right-handed neutrinos to our theory. This also allows us to introduce *Yukawa couplings* between the neutrinos and the Higgs in the unbroken Lagrangian, just in the same way as it is done for the leptons and quarks. After the spontaneous symmetry breaking these couplings yield the mass terms for the neutrinos:

$$\mathcal{L}_{\text{MASS}}^\nu = -\frac{v}{\sqrt{2}} \sum_{\alpha,\beta=e,\mu,\tau} \left(\overline{\nu_{\alpha L}} Y_{\alpha\beta}^\nu \nu_{\beta R} + h.c. \right) - \frac{v}{\sqrt{2}} \sum_{\alpha,\beta=1,2,3} \left(\overline{\ell_{\alpha L}} Y_{\alpha\beta}^\ell \ell_{\beta R} + h.c. \right).\tag{1.13}$$

Note that we also wrote down the term for the leptons. Here Y^ν and Y^ℓ are the Yukawa couplings summarised as a 3×3 matrix. These terms are just the Dirac mass terms for the neutrinos and leptons, but in the general case they are non-diagonal. Thus, we have to diagonalise them in order to find the fields that describe the mass neutrinos. This can be done by means of a bi-unitary transformation:

$$\begin{aligned}U_L^{\nu\dagger} Y^\nu U_R^\nu &= Y'^\nu & \text{with} & & Y'_{ij}{}^\nu &= y_i{}^\nu \delta_{ij}, \\ U_L^{\ell\dagger} Y^\ell U_R^\ell &= Y'^\ell & \text{with} & & Y'_{\alpha\beta}{}^\ell &= y_\alpha{}^\ell \delta_{\alpha\beta}.\end{aligned}\tag{1.14}$$

In order to rewrite the Lagrangian in terms of the mass fields we define

$$\begin{aligned}\nu_{iL} &:= \sum_{\beta=e,\mu,\tau} (U_L^{\nu\dagger})_{i\beta} \nu_{\beta L}, & \nu_{iR} &:= \sum_{\beta=e,\mu,\tau} (U_R^{\nu\dagger})_{i\beta} \nu_{\beta R}, \\ \ell_{\alpha L} &:= \sum_{\beta=e,\mu,\tau} (U_L^{\ell\dagger})_{\alpha\beta} \ell_{\beta L}, & \ell_{\alpha R} &:= \sum_{\beta=e,\mu,\tau} (U_R^{\ell\dagger})_{\alpha\beta} \ell_{\beta R}.\end{aligned}\tag{1.15}$$

If we rewrite the Lagrangians containing the neutrino part in terms of these new field operators we find

$$\mathcal{L}_{\text{KIN}}^\nu = \sum_{i=1,2,3} \overline{\nu_{iL}} i \not{\partial} \nu_{iL},\tag{1.16a}$$

$$\mathcal{L}_{\text{MASS}}^\nu = -\sum_{i=1,2,3} \frac{v y_i{}^\nu}{\sqrt{2}} \left(\overline{\nu_{iL}} \nu_{iR} + h.c. \right) - \sum_{\alpha=e,\mu,\tau} \frac{v y_\alpha{}^\ell}{\sqrt{2}} \left(\overline{\ell'_{\alpha L}} \ell'_{\alpha R} + h.c. \right).\tag{1.16b}$$

$$\mathcal{L}_{\text{CC}}^\nu = -\frac{g}{2\sqrt{2}} \sum_{i=1,2,3} \sum_{\alpha=e,\mu,\tau} \left(\bar{\nu}_{iL} \gamma^\mu (U_L^{\nu\dagger} U_L^\ell)_{i\alpha} \ell'_{\alpha L} W_\mu^+ + \bar{\ell}'_{\alpha L} (U_L^{\ell\dagger} U_L^\nu)_{\alpha i} \gamma^\mu \nu_{iL} W_\mu^- \right) \quad (1.16c)$$

$$\mathcal{L}_{\text{NC}}^\nu = -\frac{g}{\cos\theta_W} \sum_{i=1,2,3} \bar{\nu}_{iL} \gamma^\mu \nu_{iL} Z_\mu. \quad (1.16d)$$

In the following we abbreviate the masses for the neutrinos as $m_i = \nu y_i^\nu / \sqrt{2}$. As we can see, the neutrinos for the right-handed fields vanish and only the charged current term depends on the matrix combination

$$U_{\alpha i} = (V_L^{\ell\dagger} V_L^\nu)_{\alpha i} \quad (1.17)$$

which is usually called the PMNS-matrix after PONTECORVO, MAKI, NAKAGAWA and SAKA. Here, we already used the convention that flavour neutrinos gain a Greek index and mass neutrinos a Latin index.

By construction the lepton fields ℓ'_α are the ones with a definite flavour. Thus, we can simply ignore the primes. It is then convenient to define the flavour neutrino fields as

$$\nu_{\alpha L} = \sum_i U_{\alpha i} \nu_{iL}. \quad (1.18)$$

This is the mixing of the flavour and mass neutrino field operators we were looking for. Note that it is similar to the mixing in the quark sector, where $U_{\alpha i}$ is called CKM-matrix. Due to this correspondence we can simply adopt some information on the PMNS-matrix. For example in the case of Dirac neutrinos the $N \times N$ unitary matrix can be parameterised by

$$\frac{N(N-1)}{2} \quad \text{angles and} \quad \frac{(N-1)(N-2)}{2} \quad \text{complex phases.}$$

This follows from the parameterisation of a general $N \times N$ matrix. But here we can show that some of the phases are not physical and thus can be defined away. However, in the case of Majorana neutrinos we have to be a bit more careful, because in this case we cannot remove the same number of phases. In the end we are left with

$$\frac{N(N-1)}{2}$$

physical phases [GK07]. In particular, this means for three different Dirac neutrinos that the PMNS-matrix can be parameterised by three angles and one phase. One possible way to write the matrix in this case is

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \quad (1.19)$$

where $c_{ab} = \cos\theta_{ab}$ and $s_{ab} = \sin\theta_{ab}$. Since we will often use the two flavour case, we will also give the usual parameterisation for this case. For Dirac neutrinos we only need one angle and thus can write the matrix as

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \quad (1.20)$$

The absence of a phase implies that we will not have \mathcal{CP} -violation in the two flavour case [GK07]. Moreover, from (1.14) we can see that the non-diagonal mass matrix for two flavours is symmetric.

In conclusion, we have shown that we can extend the GSW-model in order to describe a mixing between flavour and mass neutrinos. How this mixing actually leads to NO will be the main issue of this thesis.

1.4 Neutrino Oscillation Experiments

Before we go on with the theoretical description of NO we should briefly explain how NO are actually measured in experiments. We will mainly follow [GK07] for the description of the experiments. In order to understand the results from the experiment we will anticipate some results from later chapters. The main quantity which shall be measured in experiments is the oscillation probability. For the two flavour case it is given by

$$\mathcal{P}(\alpha \rightarrow \alpha; L) = 1 - \sin^2(2\theta) \sin^2 \left(1.27 \frac{\Delta m^2 [eV^2] L [m]}{E [MeV]} \right), \quad (1.21a)$$

$$\mathcal{P}(\alpha \rightarrow \beta; L) = \sin^2(2\theta) \sin^2 \left(1.27 \frac{\Delta m^2 [eV^2] L [m]}{E [MeV]} \right), \quad (1.21b)$$

where Δm^2 is the difference of the squared masses

$$\Delta m^2 = m_i^2 - m_j^2.$$

The factor of 1.27 stems from the conversion between the different units. We see that for a given distance L between source and detector, and energy of the neutrinos, we can measure the mixing angle as well as the mass-squared difference. It is important to note that NO experiments can only measure relative and not absolute masses. From (1.21) we can define the oscillation length

$$L^{osc} = 2.47 \frac{E [MeV]}{\Delta m^2 [eV^2]}, \quad (1.22)$$

which gives the distance for a complete oscillation. The oscillation length yields an important constraint on the measurement conditions. That is, oscillations can only be measured if $L \sim L^{osc}$, because for $L \ll L^{osc}$ there are no oscillations and for $L \gg L^{osc}$ the oscillations are averaged out due to natural uncertainties for the neutrino energy. Thus, we have the condition

$$\frac{\Delta m^2 L}{2E} \sim 1. \quad (1.23)$$

All NO experiments can then be classified by their so-called Δm^2 *sensitivity* which is the value of Δm^2 that satisfy the observability condition for a given L and E . The sensitivities for different types of experiments are shown in table 1.1 According to their sensitivity the different experiments are traditionally classified into groups of *short-baseline* (SBL), *long-baseline* (LBL) and *very long-baseline* (VBL) experiments.

Types of Neutrino Oscillation Experiments Basically there are two different types of experiments

- *Appearance experiments*, which search for flavours that have not been present in the initial beam. They have the advantage to be very sensitive to rather small mixing angles.
- *Disappearance experiments*, which compare the measured number of neutrinos with the expected number.

Another important way to classify the different experiments is to group them according to the origin of the neutrinos. There are three groups

Type of experiment	L	E	Δm^2 sensitivity
Reactor SBL	$\sim 10\text{m}$	$\sim 1\text{MeV}$	$\sim 0.1\text{eV}^2$
Accelerator SBL (Pion DIF)	$\sim 1\text{km}$	$\gtrsim 1\text{GeV}$	$\gtrsim 1\text{eV}^2$
Accelerator SBL (Muon DAR)	$\sim 10\text{m}$	$\sim 10\text{MeV}$	$\sim 1\text{eV}^2$
Accelerator SBL (Beam Dump)	$\sim 1\text{km}$	10^2GeV	$\sim 10^2\text{eV}^2$
Reactor LBL	$\sim 1\text{km}$	$\sim 1\text{MeV}$	$\sim 10^{-3}\text{eV}^2$
Accelerator LBL	$\sim 10^3\text{km}$	$\gtrsim 1\text{GeV}$	$\gtrsim 10^{-3}\text{eV}^2$
ATM	$20\text{-}10^4\text{km}$	$0.5\text{-}10^2\text{ GeV}$	$\sim 10^{-4}\text{eV}^2$
Reactor VLB	$\sim 10^2\text{km}$	$\sim 1\text{MeV}$	$\sim 10^{-5}\text{eV}^2$
Accelerator VLB	$\sim 10^4\text{km}$	$\gtrsim 1\text{GeV}$	$\gtrsim 10^{-4}\text{eV}^2$
SOL	$\sim 10^{11}\text{km}$	$0.2\text{-}15\text{MeV}$	$\sim 10^{-12}\text{eV}^2$

Table 1.1: Types of neutrino oscillation experiments with their typical source-detector distance, energy, and sensitivity to Δm^2 (taken from [GK07]).

- *Solar neutrino experiments* These experiments measure the neutrinos produced in the fusion reactions in the core of the sun, which are only electron neutrinos. Due to the large distance between source and detector these experiments are sensitive to extremely small values of Δm^2 . Important examples for solar neutrino experiments are HOMESTAKE, KAMIOKANDE, GALLEX and SNO. Most of these experiments are disappearance experiments and their results contribute mainly to the measurements of Δm_{12}^2 and θ_{12} .
- *Atmospheric neutrino experiments* Due to the cosmic radiation which interacts with the upper atmosphere a huge number of pions is produced in this region. The pions then decay into muon neutrinos, which can be measured either coming from above or coming from below after passing the earth. The important representants for these kinds of experiments are KAMIOKANDE, SUPER KAMIOKANDE and MINOS. They mainly measure the effects on Δm_{23}^2 and θ_{23} .
- *Reactor and accelerator neutrino experiments* The neutrinos for these kinds of experiments are produced either in nuclear reactors as products of β -decay of the fission products or in accelerators as decay products of pion or muon beams. The sensitivity for these experiments varies over the whole spectrum and they can be used to measure all three angles and two mass differences. The main experiments are CHOOZ, K2K and KAMLAND.

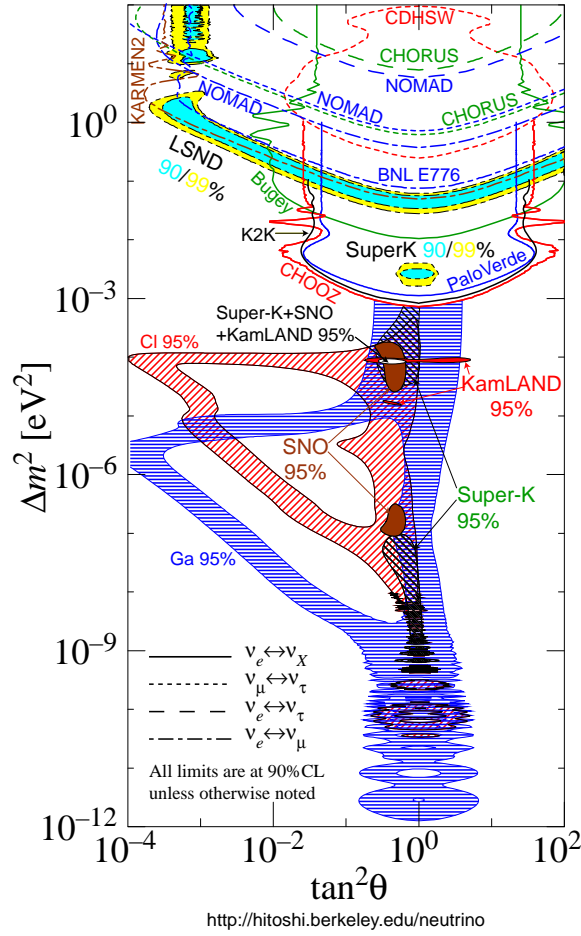


Figure 1.1: The combined results of all NO experiments. The shaded areas are inclusion regions while the lines define exclusion regions. In particular, the regions above these lines are excluded. The best fit results are the white area above for Δm_{23}^2 and θ_{23} and the red area below for Δm_{12}^2 and θ_{12} .

Experimental Results The results from all important NO experiments are combined in figure 1.1. The results for the best fit come mainly from SUPER KAMIOKANDE, SNO and KAMLAND. Numerically the results are given by the PARTICLE DATA GROUP [Y⁺06] as

$$\begin{aligned} \sin^2(2\theta_{12}) &= 0.86_{-0.04}^{+0.03} & \Delta m_{21}^2 &= (8.0_{-0.3}^{+0.4}) \cdot 10^{-5} \text{eV}^2 \\ \sin^2(2\theta_{23}) &> 0.92 & \Delta m_{32}^2 &= 1.9 - 3.0 \cdot 10^{-3} \text{eV}^2 \\ \sin^2(2\theta_{13}) &< 0.19, \text{CL} = 90\% \end{aligned}$$

Chapter 2

Oscillations in Quantum Mechanics

In this chapter we will derive the oscillation probability in the framework of quantum mechanics. We will do this in a quite general way in order to easily compare the plane-wave approximation which we consider in chapter 3 and the Gaussian wave-packet model in chapter 4.

2.1 State Vectors for Flavour Neutrinos

Up to now we only considered the mixing of flavour and mass neutrinos in terms of the field operators (1.18). But for the description in quantum mechanics we need the state vectors that should describe the flavour neutrinos. The usual way to get a relation for the mixing in terms of state vectors is to just assume the relation

$$|\nu_\alpha\rangle := \sum_i U_{\alpha i}^* |\nu_i\rangle. \quad (2.1)$$

Equivalently, the same can be done for the anti-particle state vectors, which is defined as

$$|\bar{\nu}_\alpha\rangle := \sum_i U_{\alpha i} |\bar{\nu}_i\rangle. \quad (2.2)$$

The definitions of the state vectors as in (2.1) and (2.2) are the most common ones and used in almost any textbook (see e. g. [MP91, KP93, FY03, GK07]) and most papers dealing with neutrino oscillations in quantum mechanics. We should note here that this assumption is not without problems. In particular in chapter 6 we will discuss this choice in more detail.

By comparing (2.1) and (2.2) we can already see that the difference between the treatment of neutrinos and anti-neutrinos is just a complex conjugation of the PMNS-matrix. Therefore, in the following only the treatment of neutrinos will be done in detail, while at the end all formulas can be rewritten for the anti-neutrinos by just complex conjugating all appearing PMNS-matrices.

Before we actually start with the calculation we should make some comments on the interpretation of the flavour state vectors (2.1) and (2.2). The usual way is to say that $|\nu_\alpha\rangle$ describes a state with one flavour neutrino, but this leads to the further question of the interpretation of a particle that does not have a well-defined mass. These kind of particles provide some very basic problems in the experimental handling as well as in the theoretical description. First of all, the behaviour of neutrinos reflects the usual particle-wave duality of quantum mechanics. That is, the production and detection processes are localised in a small space region and can

be regarded as particle processes, while the oscillation behaviour and, in particular, the superposition of mass neutrinos to a flavour neutrino are only understandable in the wave picture. Thus, whenever in this thesis a superposition of mass neutrinos is mentioned, one should bear in mind this duality.

Compared to a particle with well-defined mass, a flavour neutrino has a more complicated energy-momentum dependence. While in the first case the definite mass implies also a definite energy and momentum, which can, in principle, be measured at the same time if the particle is considered to be free, this is not possible for a flavour neutrino without a definite mass. This comes from the superposition of different mass neutrinos. If the energy and momentum of a flavour neutrino are measured with a high precision, which can be done by precise measurement of the corresponding leptons, the dispersion relation $m = \sqrt{E^2 - \mathbf{p}^2}$ implies a specific mass. In other words, the precise measurement picks out one of the mass neutrinos and in turn destroys the information on the superposition. However, as will be seen later, flavour changing is still possible in this case, while space-time dependent oscillations are ruled out. This comes from the remaining incoherent mixing. From this measurement problem one can obtain some relations that have to be obeyed by the production and detection processes in order to allow the measurement of oscillations. These are

$$\sigma_E > |E_i - E_j| \quad \text{and} \quad \sigma_p > |p_i - p_j|,$$

where σ_E and σ_p are the uncertainties of the energy and momentum of the flavour neutrino, respectively, while E_i and p_i are the energy and momentum of the i -th mass neutrino. These conditions were first mentioned by Kayser in 1981 [Kay81] and we will describe them in more detail in chapter 4. The use of uncertainties already shows, that a flavour neutrino has to be described as a wave packet with uncertainties σ_E and σ_p and not as a plane wave. However, since the plane wave treatment is the most easiest one and allows some first views on the theory, it will be described in the next chapter while the wave packet treatment is postponed to chapter 4.

2.2 The Quantum Mechanical Description of Neutrinos

In the last section we showed how the mixing of flavour and mass neutrinos can be described in terms of state vectors (cf. (2.1)). This result will be used in the present section in order to find the description of the space-time dependence of a flavour neutrino in quantum mechanics, which then can be used to describe the oscillation behaviour. The notation (which follows roughly [KP93]) in this chapter is slightly extended compared to the usual one found in most publications. That is, the description of the degrees of freedom for the space-time as well as the energy-momentum dependence and the degree of freedom that characterises the neutrino species shall be factorized. This is analogous to the usual quantum mechanical description of a spin-1/2 particle, where the general wave function can be factorised into a spin-independent wave function and a spin vector

$$\Psi(\mathbf{x}, \sigma) = \begin{pmatrix} \psi_{\uparrow}(\mathbf{x}) \\ \psi_{\downarrow}(\mathbf{x}) \end{pmatrix} = \psi_{\uparrow}(\mathbf{x}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_{\downarrow}(\mathbf{x}) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which then leads to a factorisation of the Hilbert space into two parts

$$\mathcal{H} = L^2 \otimes \mathbb{C}^2.$$

For the neutrinos this kind of factorisation is (in this simple manner) only possible for the mass neutrinos, because the space-time and energy-momentum degrees of freedom for a

flavour neutrino are a priori unknown. However, the full Hilbert space \mathcal{H} for the neutrinos can be written as a direct product of the space for the dynamical degrees of freedom $\mathcal{H}_d = L^2$ and a space which describes the mass degree of freedom $\mathcal{H}_m = \mathbb{C}^{N_m}$:

$$\mathcal{H} := \mathcal{H}_d \otimes \mathcal{H}_m = L^2 \otimes \mathbb{C}^{N_m}. \quad (2.3)$$

The already introduced state vectors $|\nu_i\rangle$ shall be used as a basis for \mathcal{H}_m , where we additionally assume that they are orthonormalised, while the elements of \mathcal{H}_d will be denoted by $|\psi_i\rangle$, where the index i indicates the mass dependence of the dynamics. A general neutrino state $|N_i\rangle \in \mathcal{H}$ can then be defined as

$$|N_i\rangle := |\psi_i\rangle \otimes |\nu_i\rangle. \quad (2.4)$$

In this chapter, the state vectors $|\psi_i\rangle$ will not be specified. However, in the next chapters we will choose them to describe either the dynamics of plane waves or wave packets.

As already mentioned, a similar simple relation cannot be written down for a flavour neutrino. Nevertheless, by comparing (2.1) and (2.4) it is possible to define a general flavour-neutrino state as

$$\begin{aligned} |N_\alpha\rangle &:= \sum_i U_{\alpha i}^* |N_i\rangle \\ &= \sum_i U_{\alpha i}^* |\psi_i\rangle \otimes |\nu_i\rangle, \end{aligned} \quad (2.5)$$

After the state vectors have been defined, we can analyse their time development. In order to do so, the vector $|N_i\rangle$ has to be expanded in a basis whose energy dependence is known. One possibility for this basis are the momentum eigenstates $|\mathbf{p}\rangle$, which are—according to the dispersion relation $E = \sqrt{\mathbf{p}^2 + m^2}$ —also energy eigenstates if the mass is given. Hence, inserting a complete set of momentum eigenstates in (2.5) yields

$$\begin{aligned} |N_\alpha\rangle &= \sum_i U_{\alpha i}^* \int d^3p |\mathbf{p}\rangle \langle \mathbf{p} | \psi_i \rangle \otimes |\nu_i\rangle \\ &= \sum_i U_{\alpha i}^* \int d^3p \psi_i(\mathbf{p}) |\mathbf{p}\rangle \otimes |\nu_i\rangle, \end{aligned} \quad (2.6)$$

where $\psi_i(\mathbf{p})$ is the momentum space wave functions for the neutrino with mass m_i :

$$\psi_i(\mathbf{p}) := \langle \mathbf{p} | \psi_i \rangle. \quad (2.7)$$

Actually, for (2.6) to be formally correct, one needs to have an additional operator that acts in the Hilbert space \mathcal{H}_m . However, this shall be a unity operator and left implicit here as well as in the following steps.

Assuming the neutrino to be produced at $t = t_P$ (P stands for production) with a given flavour α fixes the initial condition for the time dependence to $|N_\alpha(t_P)\rangle = |N_\alpha\rangle$. The state vector which describes the neutrino at time $t \geq t_P$ can then be obtained by means of the time evolution operator $U(t - t_P) = \exp[-iH(t - t_P)]$, where H is the Hamilton operator, which is assumed to be time-independent, of the system:

$$\begin{aligned} |N_\alpha(t)\rangle &= e^{-iH(t-t_P)} |N_\alpha(t_P)\rangle \\ &= \sum_i U_{\alpha i}^* \int d^3p \psi_i(\mathbf{p}) e^{-iE_i(\mathbf{p})(t-t_P)} |\mathbf{p}\rangle \otimes |\nu_i\rangle. \end{aligned} \quad (2.8)$$

Here, we used the fact that the momentum states $|\mathbf{p}\rangle$ are also eigenstates of the Hamilton operator with energy E_i , which are different for each mass state and given by the above mentioned dispersion relation

$$E_i(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_i^2}. \quad (2.9)$$

From (2.8) it follows, as one would naively expect, that the time development of a flavour neutrino is described by a superposition of the time developments of the mass neutrinos.

The next step on our way to a full description of a flavor neutrino will be the discussion of the space dependence of the neutrino. This is important, because the experiments on neutrino oscillations are localised in space and the oscillations are connected with the distance between production and detection. Thus, the theoretical description should reflect this dependence. In the same manner as in the case of the time dependence, the vector $|N_\alpha(t_P)\rangle$ should be expanded in a basis whose spatial dependence is known. This will be the position eigenstates $|\mathbf{x}\rangle$. Inserting a complete set of these eigenvectors and additionally assuming the neutrino to be produced at $\mathbf{x} = \mathbf{x}_P$, which results in an additional phase, yields

$$\begin{aligned} |N_\alpha(t)\rangle &= \sum_i U_{\alpha i}^* \int d^3p d^3x \psi_i(\mathbf{p}) e^{-iE_i(\mathbf{p})(t-t_P)} |\mathbf{x}\rangle \langle \mathbf{x}|\mathbf{p}\rangle \otimes |\nu_i\rangle \\ &= \sum_i U_{\alpha i}^* \int \frac{d^3p d^3x}{(2\pi)^{3/2}} \psi_i(\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_P) - iE_i(\mathbf{p})(t-t_P)} |\mathbf{x}\rangle \otimes |\nu_i\rangle \\ &= \sum_i U_{\alpha i}^* \int d^3x \psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P) |\mathbf{x}\rangle \otimes |\nu_i\rangle. \end{aligned} \quad (2.10)$$

In the last step the wave function of a neutrino with mass m_i in position space was introduced. It is given by the Fourier transformation of the wave function in momentum space

$$\psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P) := \int \frac{d^3p}{(2\pi)^{3/2}} \psi_i(\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_P) - iE_i(\mathbf{p})(t-t_P)}. \quad (2.11)$$

The variables in the parenthesis are not all of the same kind. While \mathbf{x} and t are real variables, are \mathbf{x}_P and t_P placeholders for the initial conditions due to the experiment.

In conclusion, the result (2.10) can be summarised as follows: A general flavour-neutrino is described by a state vector $|N_\alpha(t)\rangle$ which is a superposition of vectors that describe full mass-neutrinos. The dynamics of this flavour neutrino is then given by the dynamics of each of the mass neutrinos, which can be described by a wave function either in momentum or in position space.

2.3 The General Oscillation Formula

In this section we will use the results from the previous section in order to derive the formula which reproduces the measured effects of neutrino oscillations, or in other words, the probability that a neutrino, which is produced at the space-time point (\mathbf{x}_P, t_P) with a flavour α , is detected as a neutrino with flavour β at the space-time point (\mathbf{x}_D, t_D) . The amplitude that corresponds to this probability can be written as

$$\mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T) := \int dt \langle N_\beta^D(t) | N_\alpha(t) \rangle, \quad (2.12)$$

where we introduced the abbreviations $\mathbf{L} = \mathbf{x}_D - \mathbf{x}_P$ and $T = t_D - t_P$ as a simplification. The D in the bra denotes the dependence on the detection process. The explicit form of this state vector will be given below. Furthermore, the integration over t is unusual and not found in papers which only deal with the plane wave approximation. However, we will use a general notation in this thesis, which allows the computation in terms of plane waves as well as wave packets.

To understand the usage of this notation, it is important to note that a realistic detector is not a point-like object, which is switched on for exactly one point in time, but it has spatial and temporal spread. The spatial spread is just the spatial uncertainty of the particle that actually detects the neutrino, while the temporal spread comes from the fact, that a detector measures for a finite—non-zero—time-interval. These uncertainties of the detection process are described by the state vector $|N_\beta^D(t)\rangle$, which can be written—in terms of either the momentum or the position space wave functions, $\psi_j^D(\mathbf{p}, t)$ or $\psi_j^D(\mathbf{x}', \mathbf{x}_D, t, t_D)$, respectively—as

$$\begin{aligned} |N_\beta^D(t)\rangle &= \sum_j U_{\beta j}^* \int \frac{d^3 p d^3 x'}{(2\pi)^{3/2}} \psi_j^D(\mathbf{p}, t) e^{i\mathbf{p}\cdot(\mathbf{x}' - \mathbf{x}_D) - iE_j(\mathbf{p})(t - t_D)} |\mathbf{x}'\rangle \otimes |\nu_j\rangle \\ &= \sum_j U_{\beta j}^* \int d^3 x' \psi_j^D(\mathbf{x}', \mathbf{x}_D, t, t_D) |\mathbf{x}'\rangle \otimes |\nu_j\rangle. \end{aligned} \quad (2.13)$$

The wave function in momentum space can be explicitly time-dependent since it should describe the temporal uncertainty of the detection process. The integration over t in the amplitude is then just the calculation of the overlap of the neutrino and “detector wave packets”.

We can re-obtain the case of a point-like detection process by using the following wave function

$$\psi_j^D(\mathbf{x}', \mathbf{x}_D, t, t_D) := \delta(\mathbf{x}' - \mathbf{x}_D) \delta(t - t_D), \quad (2.14)$$

which then reduces (2.13) to

$$|N_\beta^D(t)\rangle = \delta(t - t_D) |\mathbf{x}_D\rangle \otimes |\nu_\beta\rangle. \quad (2.15)$$

In the general case the calculation of the amplitude yields

$$\begin{aligned} \mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T) &= \int dt \left(\sum_j U_{\beta j} \int d^3 x' \psi_j^{D*}(\mathbf{x}', \mathbf{x}_D, t, t_D) \langle \mathbf{x}' | \otimes \langle \nu_j | \right) \\ &\quad \cdot \left(\sum_i U_{\alpha i}^* \int d^3 x \psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P) |\mathbf{x}\rangle \otimes |\nu_i\rangle \right) \\ &= \sum_i U_{\beta i} U_{\alpha i}^* \int dt d^3 x \psi_i^{D*}(\mathbf{x}, \mathbf{x}_D, t, t_D) \psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P), \end{aligned} \quad (2.16)$$

where we used the normalization of the position eigenstates and the orthogonality of the mass-neutrino states.

The remaining step in order to get a measurable quantity is the calculation of the probability

that corresponds to this amplitude. That is,¹

$$\begin{aligned}
\mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T) &:= |\mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T)|^2 \\
&= \sum_i |U_{\beta i}|^2 |U_{\alpha i}^*|^2 \left| \int dt d^3x \psi_i^{D*}(\mathbf{x}, \mathbf{x}_D, t, t_D) \psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P) \right|^2 \\
&\quad + 2\text{Re} \left[\sum_{i < j} U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \int dt d^3x \psi_i^{D*}(\mathbf{x}, \mathbf{x}_D, t, t_D) \psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P) \right. \\
&\quad \left. \cdot \int dt' d^3x' \psi_j^D(\mathbf{x}', \mathbf{x}_D, t', t_D) \psi_j^*(\mathbf{x}', \mathbf{x}_P, t', t_P) \right]. \tag{2.17}
\end{aligned}$$

This is the neutrino oscillation formula, which is general in the sense, that the wave functions for the mass neutrinos and the detection process are not specified at this point.

2.4 Remarks on the Oscillation Formula

In this section, some remarks on the oscillation formula (2.17) will be given. These points can already be obtained in the general case without knowing the specific wave functions.

1. The probability should be normalized in the following way

$$\sum_{\alpha} \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T) = \sum_{\beta} \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T) = 1, \tag{2.18}$$

which ensures that the neutrino can be found as one of the flavour states at any time.

2. As already mentioned in section 2.1, the difference between the treatment of neutrinos and anti-neutrinos is obtained by replacing U by U^* and vice versa, which follows from (2.1) and (2.2). From (2.17) it then follows that the relation

$$\mathcal{P}(\bar{\beta} \rightarrow \bar{\alpha}; \mathbf{L}, T) = \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T) \tag{2.19}$$

is satisfied. This is nothing but the manifestation of the $\mathcal{CP}\mathcal{T}$ -invariance of the theory.

3. If the PMNS-matrix is real, which is for example the case in \mathcal{CP} -invariant theories (see, chapter 1), the probability satisfies two different relations (cf. (2.17)):

$$\mathcal{P}(\bar{\alpha} \rightarrow \bar{\beta}; \mathbf{L}, T) = \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T) \tag{2.20}$$

and

$$\mathcal{P}(\beta \rightarrow \alpha; \mathbf{L}, T) = \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T). \tag{2.21}$$

The first relation shows that neutrino and anti-neutrino probabilities are equal, which is nothing but the manifestation of the \mathcal{CP} -invariance, whereas the second relation shows the invariance under interchanging the initial and final state: This is the \mathcal{T} -invariance.

¹Note: There is a misprint in [KP93] on p.135, where the last sum reads $\sum_{i \neq j}$ and not $\sum_{i < j}$.

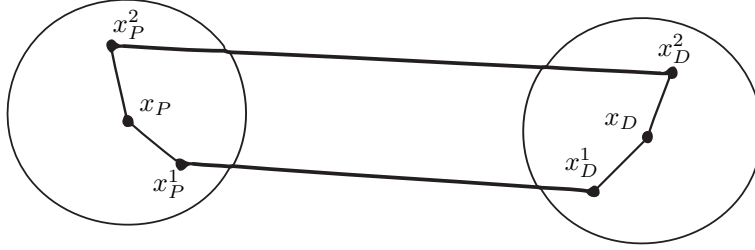


Figure 2.1: Depiction of the situation for interference at different points.

4. Up to now we said nothing about the differences between Dirac and Majorana neutrinos. The only relevant differences here are the possible additional \mathcal{CP} -phases in the PMNS-matrix for the Majorana neutrinos (see chapter 1). However, if the PMNS-matrix is replaced by

$$U_{\alpha i} \rightarrow e^{i\delta_\alpha} U_{\alpha i} e^{-i\delta_i}$$

the phase of each matrix entry is changed separately. Therefore, adding the Majorana \mathcal{CP} -phases is a special case of this replacement. But due to the complex conjugations in (2.17), all these additional phases vanish and thus it is not possible to distinct between Dirac and Majorana neutrinos only by measuring the oscillation probability.

5. In the calculation of the oscillation probability (2.17) we explicitly took into account possible uncertainties in the production and detection process. This gives rise to the question whether all mass neutrinos are produced and detected at the same space-time points or not. Especially Kiers, Nussinov and Weiss [KNW96, KW98] pushed these discussion by showing that it should be possible to measure interference between spatially separated wave packets if they arrive the detector within its temporal uncertainty. Giunti and Kim [GK01] then considered the interference conditions at different space-time points in detail. In the following we will briefly summarise these considerations. The setup for the calculation of Giunti and Kim is shown in figure 2.1, where for simplicity only two mass neutrinos are shown. The points labeled in this figure shall be understood as pairs of space and time coordinates, that is, for example $x_P = (\mathbf{x}_P, t_P)$. The dashed circles around x_P and x_D symbolises the uncertainties of the source and the detector, $\sigma_{x_{P/D}}$ and $\sigma_{t_{P/D}}$, respectively. The production and detection points of the neutrinos ν_1 and ν_2 are labeled by $x_{P/D}^1$ and $x_{P/D}^2$. These points must lie inside the uncertainties, thus

$$|\mathbf{x}_{P/D}^i - \mathbf{x}_{P/D}| < \sigma_{x_{P/D}} \quad \text{and} \quad |t_{P/D}^i - t_{P/D}| < \sigma_{t_{P/D}}.$$

The important point for the production and detection process is the coherence of the mass-neutrino waves, which implies a well-defined phase relation between the different waves. Therefore, we can introduce initial and final phases $e^{i\phi_{P/D}^i}$ for each mass neutrino separately. Since we are free to define a global overall phase, the initial and final phases can be related to specific points. For convenience, these are the production and detection point, x_P and x_D , because they are the only ones known in an actual experiment. Hence, the arguments can be written as

$$\phi_{P/D}^i = i\mathbf{p} \cdot (\mathbf{x}_{P/D}^i - \mathbf{x}_{P/D}) - iE_i(\mathbf{p})(t_{P/D}^i - t_{P/D}). \quad (2.22)$$

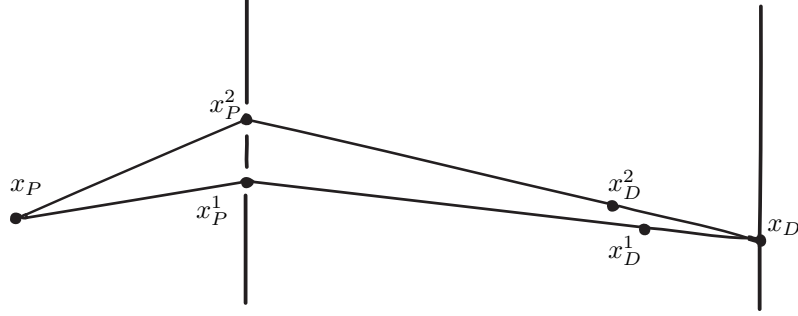


Figure 2.2: The double slit experiment as an analog to interference at different point.

In order to take these phases into account in the derivation of the oscillation formula, the state vectors $|N_\alpha(t)\rangle$ (2.10) and $|N_\beta^D(t)\rangle$ (2.13) have to be rewritten to

$$|N_\alpha(t)\rangle = \sum_i U_{\alpha i}^* \int \frac{d^3p d^3x}{(2\pi)^{3/2}} \psi_i(\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_P^i) - iE_i(\mathbf{p})(t-t_P^i)} e^{i\phi_P^i} |\mathbf{x}\rangle \otimes |\nu_i\rangle \quad (2.23)$$

and

$$|N_\beta^D(t)\rangle = \sum_j U_{\beta j}^* \int \frac{d^3p d^3x'}{(2\pi)^{3/2}} \psi_j^D(\mathbf{p}, t) e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x}_D^j) - iE_j(\mathbf{p})(t-t_D^j)} e^{i\phi_D^j} |\mathbf{x}'\rangle \otimes |\nu_j\rangle. \quad (2.24)$$

The first change concerns the production and detection points in the first exponentials, which are now different. The second change is the implementation of the initial and final phases for each mass neutrino. By comparing (2.22) with (2.23) and (2.24) we immediately see that the new state vectors reduce to the old ones and all dependences on $x_{P/D}^i$ disappear.

This result is not surprising, since we can also obtain it in a more hand-waving way. That is, the interference at different space-time points is analogous to the usual double-slit interference experiment. In this experiment, a wave is produced at one point (which in our case would be x_P) then propagates through two separated slits (at x_P^1 and x_P^2) and finally interferes with itself at the detector (at x_D). If we now assume two additional points (x_D^1 and x_D^2), which lie somewhere on the two paths behind the slits, the situation is the same as for the neutrinos (cf. figure 2.2). In the double-slit experiment the two paths are coherent, but the points on the paths are in general spatially and temporally separated. Thus, if we concentrate on the part of the diagram lying in-between these points and forget about x_P and x_D , the double-slit case can be regarded as the creation of a coherent superposition at different space-time points which is then detected at different points. The initial phases at the production are then just the phase differences between the production points and x_P , while the final phases are the phase differences between the detection points and x_D . Therefore, it is obvious that the whole phase difference between the two waves is the same as it would be in the case of a production at x_P and a detection at x_D and not at the intermediate points.

This argumentation allows the usage of equal production and detection points in the theoretical treatment even if this might not be the case in the real world [GK01, Giu02b, Giu04b].

6. The last remark concerns a basic problem of the mixing of states as represented in (2.1), which is noted by some authors (e. g. [Zra98, Beu03]). That is, in non-relativistic quantum mechanics the superposition of states with different masses is forbidden due to the BARGMANN superselection rule [Bar54]. The usual argumentation is as follows: Non-relativistic quantum mechanics should be invariant under Galilean transformations. If such a transformation acts on a wave function—in the form of a transformation to another system and then back to the original system—it results in a phase that depends on the mass of the state. It then follows that the transformation of a superposition of different mass states acquires a relative phase depending on the mass difference. This relative phase would, in principle, be measurable. In order to avoid this problem, one usually impose a superselection rule, which simply forbids such superpositions. However, neutrinos are relativistic and thus they are not described by the non-relativistic Schrödinger equation but by the relativistic Dirac equation. Therefore, the invariance group is the Lorentz rather than the Galilean group. But, the action of a Lorentz transformation on a state vector that satisfies the Dirac equation does not yield a mass-dependent phase factor and hence no relative phase for a superposition will occur. Furthermore, it was shown that the relative phase is just the non-relativistic residue of the usual twin paradox of special relativity and thus there is no reason to wonder about its appearance [Gre01].

Chapter 3

Neutrinos as Plane Waves

In this chapter we will approximate the mass neutrinos as plane waves. The simple treatment in the first section yields a probability which is not satisfactory from an experimental point of view, because it contains an unmeasurable time-dependence. This will be changed in the second section, where different assumptions are used to convert the temporal into a spatial dependence. The discussion of these assumptions will be done in the third section. Finally, the last section contains a few remarks on the discussed plane-wave treatment.

3.1 The General Plane Wave Solution

As already mentioned in the previous chapter, the flavour neutrinos are described as superpositions of mass neutrinos whose space-time and energy-momentum dependence are stored in wave functions $\psi_i(\mathbf{p}) = \langle \mathbf{p} | \psi_i \rangle$. In this chapter these wave functions will be considered as plane waves with definite momenta \mathbf{p}_i , or in other words, the $|\psi_i\rangle$ are momentum eigenstates. Hence, the wave functions in momentum space are given by delta functions

$$\psi_i(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}_i). \quad (3.1)$$

Using the Fourier transform (2.11), then yields the wave functions in position space

$$\psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P) = \frac{1}{(2\pi)^{3/2}} \exp [i\mathbf{p}_i \cdot (\mathbf{x} - \mathbf{x}_P) - iE_i(t - t_P)], \quad (3.2)$$

which are the mentioned plane waves. The dispersion relation for the energy E_i is the same as in (2.9), but this time the momentum has the definite value \mathbf{p}_i . Thus,

$$E_i = \sqrt{\mathbf{p}_i^2 + m_i^2}. \quad (3.3)$$

In order to derive the oscillation probability for the plane-wave treatment, we additionally have to fix the description of the detection process. Since plane waves do not have a spatial uncertainty, we will assume a point-like detection process, given by (2.14) and (2.15), here. That is, the detection shall take place at one well-defined space-time point (\mathbf{x}_D, t_D) . Under this assumption the general probability (2.17) reduces to

$$\begin{aligned} \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T) &= \frac{1}{(2\pi)^3} \sum_i |U_{\beta i}|^2 |U_{\alpha i}^*|^2 \\ &+ \frac{2}{(2\pi)^3} \text{Re} \left[\sum_{i < j} U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \exp [i(\mathbf{p}_i - \mathbf{p}_j) \cdot \mathbf{L} - i(E_i - E_j)T] \right]. \end{aligned} \quad (3.4)$$

This probability consists of two parts. The first part, which does not depend on space and time, is the transition probability which we would get if the superposition of the mass neutrinos were incoherent. This would for example be the case if the flavour neutrinos were not a superposition but a statistical mixture of mass neutrinos with probabilities given by the PMNS-matrix. The second part is the actual oscillation part that includes the coherent effects and therefore gives rise to oscillations. Since a real neutrino beam always has an energy-momentum spread, the second term will be washed out if the distance or time is large enough and no oscillations can be measured in this region. However, even if this is the case, the transition probability is not zero due to the incoherence term. A more precise estimation for this will be given in the remarks at the end of this chapter. However, if the neutrino beam is described by plane waves which do not have an energy-momentum spread, there will not occur such effects in this treatment. This is the first hint on the incompleteness of the plane wave treatment, which, in the end, leads to the necessity of using wave packets instead.

Nevertheless, the probability (3.4) looks quite simple and it should be easy to calculate some results for real experiments. But after a closer look, one sees that it lacks in two points. First, in real experiments no one measures the time between the production and detection of the neutrino. And in most cases, e. g. solar neutrinos, it is rather impossible to know the production time. Therefore, one has to convert the time dependence into a distance dependence, since this is usually known to a much higher accuracy. Second, the exponential in (3.4) contains the energy and momentum of the different mass neutrinos. These are impossible to know, because the neutrinos cannot be measured directly. In particular, the only observed particles are the ones that participate in the detection process, or the production process if it is observed. Those particles are the corresponding leptons or nuclei to the neutrinos and have only one specific energy and momentum. Thus, the energy and momentum of the mass neutrinos have to be rewritten in terms of the measured momenta of the actually detected particles. These easy looking tasks give rise to a vast number of papers written by different authors who all claim different ways to be the only right ones.

3.2 Time to Space Conversion

The rewriting of the oscillation probability (3.4) in an only space dependent way is not unique and in fact there are basically four different possibilities mentioned in the literature. They are usually called:

1. *equal-energy assumption*,
2. *equal-momentum assumption*,
3. *equal-velocity assumption*,
4. *energy-momentum conservation*.

In the following the main points in these assumptions will be summarized while the discussion of the different cases is postponed to the next section. For simplicity the derivations will be done in only one spatial dimension. This is not a crude approximation since the neutrinos travel a macroscopic distance, which means a deviation from the one dimensional case would cause a separation of the mass neutrinos and thus no oscillation would be measurable. A further simplification will be the restriction to the argument of the exponential in (3.4), which is the only interesting quantity here. In particular it will be denoted as

$$\phi := (p_i - p_j)L - (E_i - E_j)T, \quad (3.5)$$

and shall get an index E , p , v and c in the four different cases, respectively.

Equal-Energy Assumption This assumption is the easiest one. It was first mentioned by LIPKIN in 1995 and then repeated and defended in several following papers [Lip95, GL97, Sto98, Lip99, Lip02, Lip06]. Equal energy means, all mass states are supposed to have the same energy, that is, $E_i = E_j = E$. Then the time dependence of (3.5) simply vanishes and the resulting argument can be written as

$$\phi_E = (p_i - p_j)L = \frac{p_i^2 - p_j^2}{p_i + p_j}L = \frac{m_j^2 - m_i^2}{p_i + p_j}L, \quad (3.6)$$

where the last step requires the use of the dispersion relation (3.3). In order to get a relation in terms of the measured momentum, we can define an average momentum

$$\bar{p} := \frac{1}{N_m} \sum_i p_i. \quad (3.7)$$

Since the difference of the momenta p_i must be smaller than the momentum uncertainty of the flavour neutrino—as already mentioned in section 2.1—they have to be nearly equal and we can approximate them by the average momentum. Thus, the argument becomes

$$\phi_E \approx \frac{m_j^2 - m_i^2}{2\bar{p}}L. \quad (3.8)$$

If the neutrinos are considered to be relativistic (i. e., $p_i \gg m_i$)—as they in fact are in practice—the momenta can be expanded in terms of the mass

$$p_i = \sqrt{E^2 - m_i^2} = p_0 - \frac{m_i^2}{2p_0} + \mathcal{O}\left(\frac{m_i^4}{p_0^3}\right), \quad (3.9)$$

where $p_0 = E$ is the momentum and energy of a massless neutrino. Using this expansion to rewrite \bar{p} in terms of p_0 changes the argument to

$$\phi_E \approx \frac{m_j^2 - m_i^2}{2p_0}L. \quad (3.10)$$

The terms proportional to m_i^2 in (3.9) are neglected, which can be done if

$$\frac{m_j^2 - m_i^2}{2p_0} \frac{m_i^2}{4p_0^2} \ll \frac{m_j^2 - m_i^2}{2p_0}. \quad (3.11)$$

The validity of this estimation can be seen from experimental details, since in usual experiments only neutrinos with an energy higher than about 100 keV can be detected, while the mass is found to be smaller than about one eV (see [Giu04b] footnote 3 and references therein). Thus, $m_i^2/p_0^2 \lesssim 10^{-10}$, which is obviously in good agreement with (3.11).

Equal-Momentum Assumption This is the oldest and most common assumption used in nearly every publication on neutrino oscillations. Equal momentum means $p_i = p_j = p$. Thus, this time the space dependent term in (3.5) vanishes and the argument can be written as

$$\phi_p = -(E_i - E_j)T = -\frac{E_i^2 - E_j^2}{E_i + E_j}T = \frac{m_j^2 - m_i^2}{E_i + E_j}T. \quad (3.12)$$

In the last step we again used the dispersion relation (3.3). With the same argumentation as in the *equal-energy assumption* we can introduce an average energy

$$\bar{E} := \frac{1}{N_m} \sum_i E_i \quad (3.13)$$

and rewrite the argument as

$$\phi_p \approx \frac{m_j^2 - m_i^2}{2\bar{E}} T. \quad (3.14)$$

Now, in the same manner as in the previous assumption, the relativistic limit can be considered. Here, the expansion of the energy in terms of the mass yields

$$E_i = \sqrt{p^2 + m_i^2} = p_0 + \frac{m_i^2}{2p_0} + \mathcal{O}\left(\frac{m_i^4}{p_0^3}\right), \quad (3.15)$$

where $p_0 = p$ is the momentum and energy of a massless neutrino. Here, p_0 rather than E_0 is used in order to have a consequent notation in the different assumptions. This expansion allows us to rewrite the average energy and thus the argument to

$$\phi_p \approx \frac{m_j^2 - m_i^2}{2p_0} T. \quad (3.16)$$

The terms proportional to m_i^2 in (3.15) are again neglected. This can be validated by the same argument as in the *equal-energy assumption*. The last step is the conversion of the temporal dependence into a spatial dependence. This can be done by using the classical velocity $v_i = p_0/E_i$, which is also the group velocity for a wave packet with mean momentum p_0 . However, in principle, no such velocity is defined in the case of plane waves, which is again a point against the usage of plane waves and for a description in terms of wave packets. Expanding the velocity in terms of the mass in the relativistic limit yields

$$\frac{L}{T} = v_i = \frac{p_0}{E_i} = 1 - \frac{m_i^2}{2p_0^2} + \mathcal{O}\left(\frac{m_i^4}{p_0^4}\right). \quad (3.17)$$

We can again neglect the second term since it is smaller than $\sim 10^{-10}$ if the same arguments as in the equal-energy case are used. This allows us to approximate $T \approx L$ and the final form of the argument is

$$\phi_p \approx \frac{m_j^2 - m_i^2}{2p_0} L, \quad (3.18)$$

which is the same as the one in the *equal-energy assumption*.

Equal-Velocity Assumption The *equal-velocity assumption* deals—as suggested by the name—with the velocities of the mass neutrinos, in particular the group velocities, which is in the same way insufficient as in the equal-momentum assumption. These velocities are assumed to be equal for all neutrinos:

$$v_i = \frac{p_i}{E_i} = v = \frac{L}{T} \quad \forall i. \quad (3.19)$$

The first mention of this assumption was independently made by TAKEUCHI et al. [TTTTY99] and DE LEO et al. [DLDR00]. The argumentation here is the one of TAKEUCHI et al.

We can use (3.19) to rewrite the general argument (3.5). In order to do so we should note that (3.19) implies $E_i T = p_i L / v^2$. Then the argument changes to

$$\begin{aligned}\phi_v &= (p_i - p_j) \left(1 - \frac{1}{v^2}\right) L \\ &= (p_j - p_i) \frac{L}{v^2 \gamma^2},\end{aligned}\tag{3.20}$$

where we introduced the usual relativistic γ -factor $\gamma = 1/\sqrt{1-v^2}$. The factor also gives a relation between the momentum and mass: $p_i = \gamma v m_i$. Inserting this in (3.20) after the momentum part is extended, yields

$$\begin{aligned}\phi_v &= \frac{p_j^2 - p_i^2}{p_i + p_j} \frac{L}{v^2 \gamma^2} \\ &= \frac{m_j^2 - m_i^2}{p_i + p_j} L,\end{aligned}\tag{3.21}$$

which is the same as in the *equal-energy assumption*. Hence, we get the same result if we first introduce an average momentum and then go to the relativistic limit:

$$\phi_v \approx \frac{m_j^2 - m_i^2}{2\bar{p}} L \stackrel{p_i \ll m_i}{\approx} \frac{m_j^2 - m_i^2}{2p_0} L.\tag{3.22}$$

Energy-Momentum Conservation In this approach, a rigorous treatment of the energy and momentum conservation in the production process of the neutrino is performed. This was first done by WINTER in 1981 [Win81] and then used by several other authors [Gol96, GK01, Giu01, Giu06] to show that neither the *equal-energy assumption* nor the *equal-momentum assumption* are satisfying.

If the production process of the neutrino is a two-body decay, within the rest-frame of the decaying particle, the energy-momentum conservation relation can be written as

$$E_I = \sqrt{p_i^2 + M^2} + \sqrt{p_i^2 + m_i^2}.\tag{3.23}$$

Here, E_I is the energy of the decaying particle, p_i is the momentum of the neutrino and the recoiling particle and m_i^2 and M^2 are the neutrino and recoiling particle mass, respectively. Of course, the same considerations can be done for more complicated processes. We just have to insert the energy of the entire initial state on the left-hand side and the entire mass and the sum of the momenta of all recoiling particles on the right-hand side, while working in the center of mass system of the neutrino and the recoiling particles. This is the reason why the left-hand side is called E_I rather than just the mass of the decaying particle.

From (3.23) we can find the value of the momentum p_i :

$$\begin{aligned}p_i &= \sqrt{p_0^2 - \frac{2(E_I^2 + M^2)m_i^2 - m_i^4}{4E_I^2}} \\ &= p_0 - \frac{E_I^2 + M^2}{2E_I^2} \frac{m_i^2}{2p_0} + \mathcal{O}\left(\frac{m_i^4}{p_0^3}\right),\end{aligned}\tag{3.24}$$

where again $p_0 = \frac{E_I^2 - M^2}{2E_I}$ is the momentum of a massless neutrino. The higher order terms in this expansion can be neglected even in the case of relatively low neutrino energies. For example, a neutrino with mass 1 eV and momentum $p_0 \sim 100$ eV, which is not even observable [Giu04b], would give a correction of order 10^{-6} , four orders less than the first correction term.

Using the dispersion relation (3.3) yields the energy of the neutrino

$$\begin{aligned} E_i &= \sqrt{p_i^2 + m_i^2} = \sqrt{p_0^2 + \frac{2(E_I^2 - M^2)m_i^2 + m_i^4}{4E_I^2}} \\ &= p_0 + \frac{E_I^2 - M^2}{2E_I^2} \frac{m_i^2}{2p_0} + \mathcal{O}\left(\frac{m_i^4}{p_0^3}\right). \end{aligned} \quad (3.25)$$

In the notation of [GK01], the factors $E_I^2 + M^2/2E_I^2$ and $E_I^2 - M^2/2E_I^2$ are called ξ and $1 - \xi$, respectively. Since this simplifies the notation we will also use the factor ξ here. If we now insert (3.24) and (3.25) into the general argument (3.5) and neglect the higher order terms, we get the following argument for the *energy-momentum conservation* assumption:

$$\phi_c \approx \xi \frac{m_j^2 - m_i^2}{2p_0} L + (1 - \xi) \frac{m_j^2 - m_i^2}{2p_0} T. \quad (3.26)$$

In order to get a relation between the time and distance, we can again compute the group velocity (where the question of the existence of this velocity is the same as in the other assumptions):

$$\begin{aligned} v_i &= \frac{p_i}{E_i} = 1 - \xi \frac{m_i^2}{2p_0^2} - (1 - \xi) \frac{m_i^2}{2p_0^2} + \mathcal{O}\left(\frac{m_i^4}{p_0^4}\right) \\ &= 1 - \frac{m_i^2}{2p_0^2} + \mathcal{O}\left(\frac{m_i^4}{p_0^4}\right). \end{aligned} \quad (3.27)$$

Again the higher order terms can be neglected for relatively low neutrino energies. In particular, using the same example would give a correction of order 10^{-8} here.

Inserting (3.27) into (3.26) then yields

$$\begin{aligned} \phi_c &\approx \xi \frac{m_j^2 - m_i^2}{2p_0} L + (1 - \xi) \frac{m_j^2 - m_i^2}{2p_0} L \\ &= \frac{m_j^2 - m_i^2}{2p_0} L, \end{aligned} \quad (3.28)$$

where we also neglected the terms which come from the m_i^2 terms in the velocity, because as in the *equal-energy assumption* and the *equal-momentum assumption* the bound

$$\frac{m_j^2 - m_i^2}{2p_0} \frac{m_i^2}{2p_0^2} \ll \frac{m_j^2 - m_i^2}{2p_0} \quad (3.29)$$

holds for realistic neutrinos.

As can be seen from (3.28), the factor ξ does not show up in the final argument.

3.3 Discussion

There is an extensive discussion on the different options in the literature (see the citations in this section) without reaching a consensus on any of those assumptions. We will present parts

of the discussion in this section. In particular, the *equal-energy assumption* and the *energy-momentum conservation* are the most favoured ones. BEUTHE [Beu03] gave a comprehensive summary of the arguments used in the literature. In the following there will be some overlap with his presentation but we will present some additional arguments.

As can be seen from the last section, there is no difference in (3.10), (3.18), (3.22) and (3.28) after the relativistic limit is applied. That is, the argument of the phase is for all cases

$$\phi = \frac{m_j^2 - m_i^2}{2p_0} L,$$

or by introducing the, so called, oscillation length $L_{ji}^{osc} = 4\pi p_0 / (m_j^2 - m_i^2)$

$$\phi = 2\pi \frac{L}{L_{ji}^{osc}}.$$

This result is not surprising since the relativistic limit was applied by expanding the energy and momentum in terms of the mass and then neglecting all terms of second or higher order in the mass. Hence, only terms that describe massless neutrinos are considered. But in the massless case all energies and momenta are the same: $E_i = E_j = p_i = p_j = p_0$ and every distinction in the assumptions simply vanishes. This means, in order to discuss the validity of the assumptions one cannot use the results obtained in the relativistic limit.

A first look at the four assumptions shows that they are incompatible. For example, if the neutrinos with different masses had the same energy they could not have the same momentum and vice versa. In particular, this means they also could not have the same velocities in this cases. If energy-momentum conservation is assumed, (3.24) and (3.25) show that neither the energy nor the momentum can be equal. Nevertheless, the *equal-velocity assumption* could, in principle, be used in this case.

A closer look then shows that the plane wave treatment itself cannot be the last step. That is, assuming a production process for a neutrino where all particles are described by plane waves. Then the energies and momenta of the non-neutrino particles are fixed and energy-momentum conservation determines the exact values for the neutrino. This is a *missing mass experiment* [Lip95] and allows only the production of a mass eigenstate. Hence, no oscillations would occur. But, since oscillations are observed one has to improve the treatment. This will be, as usual, the description in terms of wave packets which we will do in the next chapter. However, the plane-wave description already leads to the oscillation phase, which will be the same in the wave-packet treatment. Therefore, we should first discuss the easier treatment of plane waves before going over to wave packets. Nevertheless, in order to do so, we have to assume some results from the wave packets. In particular, these will be the uncertainties for energy, momentum and position as well as the group velocities.

A rough estimation of the uncertainties yields—as already noted in section 2.1

$$\sigma_E > |E_2 - E_1| \quad \text{and} \quad \sigma_p > |p_2 - p_1|,$$

to be necessary in order to allow oscillations.

In the following, we will discuss the arguments for each of the assumptions separately.

- The *equal-energy assumption* is the by far most discussed assumption. There are basically three different arguments given for using equal energies for the mass neutrinos.

First of all, in [Lip95, Lip99] the uncertainties in the production process are considered and it is claimed that the energy uncertainty can be neglected compared to the momentum

uncertainty. The argument goes as follows: Assuming the initial particle, which decays to the neutrino, to be at rest gives a momentum uncertainty from the localisation of the particle, which, in turn, gives an energy uncertainty $\sigma_{E_I} \sim \sigma_{p_I}^2/m_I$. Since σ_{E_I} is of second order in σ_{p_I} , it is neglected. This is actually an over simplified view, because the produced neutrino is not at rest and thus the energy uncertainty is of first order in the momentum uncertainty: $\sigma_{E_\nu} \sim \sigma_{p_\nu} p_\nu/E_\nu \sim \sigma_{p_\nu}$. Moreover, even if $\sigma_{E_\nu} \ll \sigma_{p_\nu}$, the uncertainty does not have to be exactly zero.

The second argument is given in [GL97, Lip99], where a boundary condition for the neutrino wave function is assumed. This is, the probability of finding a neutrino with a wrong flavour at the production point vanishes for all times. The boundary condition then leads to a factorization of the flavour and time dependence and, by Fourier transform of the time, to a factorization of flavour and energy. Therefore, only mass neutrinos with the same energy can give rise to a flavour change. However, the authors did not give a real reason why the condition should be valid for all times. And moreover, in the wave packet picture the packets leave the region of production rather quickly which then leads to a vanishing probability and the condition becomes meaningless [Beu03].

As a third argument a fuzziness in time is assumed [GL97, Lip99]. This comes from the spatial spread of the wave packets which results in a non-zero overlap time with the source (or detector). Thus, one has to average the flavour-change probability over this fuzzy arrival time. If the source (or detector) is stationary [Sto98, Lip02] this leads to a wash out of all terms with different energies. This can already be seen in the plane wave picture, where the average can be written as (cf. [Beu03, Giu06])

$$\int dt_P e^{i(p_i-p_j)(x_D-x_P)-i(E_i-E_j)(t_D-t_P)} = 2\pi \delta(E_i - E_j) e^{i(p_i-p_j)(x_D-x_P)-i(E_i-E_j)t_D}.$$

This is of course only true if the integration range is infinity, which can be assumed in the case of stationarity. If this is not the case, the delta function becomes a narrow peak around $E_i - E_j$. However, real processes are not stationary on the microscopic scale, because the particle that decays into the neutrino has an inherent energy uncertainty due to its instability of the order $1/\tau$ the inverse life-time. Likewise, the same is true for the detection process.

A further argument, which is against the assumption of equal energy, is the question of Lorentz invariance. It can be shown that the Lorentz transformation, from one frame to another, of the energy and momentum leads to a non-vanishing difference in the energy even if it is zero in one particular frame [Giu01].

In conclusion, the arguments for the *equal-energy assumption* are not very convincing, but at least from the third argument it can be said that in the case of a nearly stationary process the interference terms of the wave packet components with equal energies will give the main contribution. However, this does not fix the energies to be exactly equal.

- The *equal-momentum assumption* is more or less the poor cousin in the discussion on which is the right assumption. On the one side it is used by almost all authors to come to a quick solution for the probability, but on the other side no one gives arguments for the correctness of this assumption. On the contrary, all given arguments predict that it is wrong. These are, in principle, all arguments that are used for the *equal-energy assumption*. For example, if the uncertainty of the momentum is larger than the energy uncertainty there is no reason to assume the momenta to be equal. Moreover, the equal energies in a nearly stationary process predict different momenta due to the different

masses. In addition, the question of Lorentz invariance provides the same difficulty as in the energy case. That is, the assumption of equal momenta is only valid in one particular frame [Giu01].

- The *energy-momentum conservation* assumption has the advantages of already containing the conservation laws of energy and momentum and on the other hand being the most general assumption in the sense that neither the energy nor the momentum are constrained. The missing of a constraint allows the usage of this assumption in any Lorentz frame. One just has to transform the factor ξ into the relevant frame. In fact, one can show that either $\xi > 1/2$ or $\xi < 1/2$ is valid in any system. In particular, this means that a system, where the energies are equal, cannot be transformed into a system, where the momenta are equal [Giu01], because equal energies correspond to $\xi = 1$ while equal momenta correspond to $\xi = 0$.

The only real criticism on this assumption is made in [Beu03]. Therein, BEUTHE claims that, for example, in the case of a pion decay at rest ($\pi \rightarrow \mu\nu$), the energy-momentum uncertainty of the pion is much larger than the difference in masses $m_j^2 - m_i^2$ and thus it should be meaningless to compute the values of the neutrino energy and momentum to order m^2 . However, this would also mean that it is meaningless to compute a oscillation probability since the interference terms are of order $m_j^2 - m_i^2$. A more detailed analysis of the uncertainties shows the following: From (3.24) and (3.25) the computation of the uncertainties of p_i and E_i , under the assumption of a pion uncertainty σ_{E_I} , yields

$$\sigma_{p_i} \sim \sigma_{E_i} \sim \frac{M^2 m_i^2 \sigma_{E_I}}{E_I^2 p_0 E_I}.$$

Since $M/E_I \sim 1$ for a pion decay it is obvious that the uncertainty for the neutrino can be neglected—as long as $\sigma_{E_I} \ll E_I$ —for the computation of the oscillation phase.

- The last assumption to be discussed is the *equal-velocity assumption*. As already said, it could only be used in the case of different energy and momenta. However, there is one argument that makes it very unlikely to use this assumption. That is, equal velocities means equal γ -factors and hence (cf. [OT00])

$$\frac{E_i}{E_j} = \frac{p_i}{p_j} = \frac{m_i}{m_j}.$$

Obviously, this can not be true for the equal energy and equal momentum case since then also the masses must be equal. But even in the case of different energies and momenta this relation cannot hold, because it is inherent Lorentz invariant and it was shown [Giu01] that one can always transform to a system where either the energies or the momenta are equal, which then, in turn, breaks the relation above. Therefore, the *equal-velocity assumption* can be ruled out.

In conclusion, the assumption of *energy-momentum conservation* seems to be the most satisfying one. On the one side it is the most general one that includes the conservation laws. On the other side, all the other assumptions lack in some point or are only approximately valid. However, it is possible that the *equal-energy assumption* or the *equal-momentum assumption* are valid in some particular Lorentz frames, while the time average yields the energies to be, at least, nearly equal.

3.4 Remarks on the Plane Wave Treatment

The Wash-Out of the Oscillations As already mentioned in 3.1 the energy-momentum spread of a real neutrino beam leads to a wash out of the oscillations if the time or distance between production and detection is large enough. This can be seen in the following way: The energy-momentum spread requires an averaging of the probability over the spread. Hence,

$$\tilde{\mathcal{P}}(\alpha \rightarrow \beta; \mathbf{L}, T) = \frac{1}{\sigma_p^6 \sigma_E^2} \int d^3 p_i d^3 p_j dE_i dE_j \mathcal{P}(\alpha \rightarrow \sigma; \mathbf{L}, T), \quad (3.30)$$

where $\tilde{\mathcal{P}}$ is the averaged probability. In (3.4) the only part that depends on the energy and momentum is the phase and the integration of this yields a term proportional to $(\sigma_p^6 \sigma_E^2 \mathbf{L}^2 T^2)^{-1}$ which vanishes for large \mathbf{L} and T . Thus, the result after the average is

$$\tilde{\mathcal{P}}(\alpha \rightarrow \beta; \mathbf{L}, T) = \frac{1}{(2\pi)^3} \sum_i |U_{\beta i}|^2 |U_{\alpha i}^*|^2,$$

which can be non-zero even in the case of $\alpha \neq \beta$.

The Factor of Two Problem Several authors showed that it is possible to get an oscillation phase that is two times the standard one [Lip95, SWS95a, SWS95b, WS96, DLDR00, DLNR04, DLNR03]. As it is pointed out in the replies to this problem [L⁺96, Kay97, BLSG99, GK01, Giu02b, OST03, Giu04b], the origin of the factor is the superposition of different mass states at different times (or positions). In particular, if different energies or momenta are assumed, the neutrinos will have different group velocities and thus arrive at different times $T_i = (E_i/p_i)L$ (or positions $L_i = (p_i/E_i)T$) at the detector. Then the phase difference is given by

$$\phi = (p_i - p_j)L - (E_i T_i - E_j T_j) = \frac{m_j^2}{p_j} L - \frac{m_i^2}{p_i} L$$

or

$$\phi = (p_i L_i - p_j L_j) - (E_i - E_j)T = \frac{m_j^2}{E_j} T - \frac{m_i^2}{E_i} T,$$

which in the case of equal momenta (or energies) yields an argument twice as large as the standard ones (3.10), (3.18), (3.22) and (3.28). This result is general and can be obtained in all assumption schemes—except the *equal-velocity assumption*, where the arrival times are all the same and the computation shown in section 3.2 then gives the standard result.

However, the important point to explain this wrong result was given by [GK01]. That is, the different waves must be coherent, which implies additional phases as we showed in section 2.4. If these phases are properly taken into account we obtain the standard result back.

In section 2.4 we also showed that the use of the coherence condition gives a result which is the same as if production and detection at the same space-time points is assumed. This means, if we ignore the possibility of different production and detection points we can, in turn, state that the interference must be calculated at the same space-time point. This argument was used by [L⁺96, Kay97, BLSG99, OST03, Giu04b].

Do Charged Leptons Oscillate? Closely related to the factor of two problem is the question whether the recoil particle, in for example the decay $W^\pm \rightarrow l_\sigma^\pm + \nu_\sigma$, can oscillate. The authors of [SSW95, SWS95a, SWS95b, WS96] claimed that this oscillations should appear for recoil leptons—or, for example Λ_s , in the case of kaon oscillation. As pointed out by [DMOS97, BLSG99] this effect actually bases on the assumption of superpositions at different times. Thus, the argumentation on the correctness of these oscillations is the same as in the *factor of two* case. The calculation particularly reduces to the standard case with no oscillations if the superposition is calculated at the same space-time point. Moreover, in [GK01, Giu04b] it is argued that oscillations in the probability of finding a charged particle are, in principle, impossible, because this would violate the conservation of the probability. However, the mixing matrix in (1.16) can be considered to act on the leptons rather than the neutrinos. Then the mass leptons are different from the flavour leptons and oscillations could occur. But since the masses of the leptons differ from around 500 keV to 2 GeV very high energies which exceed a few hundred TeV and some new physics in order to detect such a flavour superposition are needed to get this effect [Akh07].

The Intermediate Wave Packet Model

4.1 The Uncertainties of a Neutrino

In the previous chapter the description of the neutrinos was done in the plane-wave approximation, although the arguments in the discussion often relied on a wave-packet interpretation. This already shows the insufficiency of regarding neutrinos as plane waves. A more important reason why one has to use wave packets instead comes from the experimental conditions. In particular, neutrino-oscillation experiments are localised in space and the important measurement is the distance between production and detection. This is contrary to usual high-energy physics experiments, where one is more interested in the energies and momenta of the particles. Thus, the usual plane-wave approximation cannot hold for neutrinos. Moreover, as will be presented shortly, a too precise measurement of energy and momentum destroys the oscillation behaviour.

Even though the plane-wave and wave-packet approach seems to be quite different, there is an interesting feature that should be noted. In 1996, KIERS, NUSSINOV and WEISS [KNW96] showed that it is impossible to perform an experiment which can discriminate between an ensemble of wave packets with equal mean energy \bar{E} and energy spread σ_E and an ensemble of plane waves, whose definite energies are distributed around \bar{E} with the same spread $\Delta E = \sigma_E$.

However, before the wave-packet treatment will be described some comments on the involved uncertainties shall be made. Since this is done in detail in [KP93, GK07] only a summary of the important points will be given here.

The first who considered the effects of wave packets on neutrino oscillations was NUSSINOV in 1976 [Nus76] who used simple physical arguments in order to estimate a size of the packets. From the separation of the packets due to their different velocities, he could then predict a coherence length, which describes the maximal distance in which oscillation can occur. Twenty years later, he together with KIERS and WEISS [KNW96, KW98] showed, that the coherence of the different mass neutrinos could be restored, even if they are spatially separated, by taking into account the temporal resolution of the detection process. This was already noted when the interference at different space-time points was discussed in section 2.4.

In 1981 KAYSER [Kay81] studied the influence of the uncertainties on neutrino oscillations. In particular, he showed, that the energy or momentum uncertainties, σ_E or σ_p , for the neutrino become smaller than the energy or momentum difference of the corresponding mass neutrinos, $|E_i - E_j|$ or $|p_i - p_j|$, by for example more precise measurements on the initial and recoil particles of the production. Under this assumption, on the one hand one precisely knows which mass neutrino is produced and on the other hand the induced spatial uncertainty $\sigma_x \sim 1/\sigma_E$ or $\sigma_x \sim 1/\sigma_p$ becomes larger than the oscillation length, which leads to a wash out of the

oscillations.

Therefore, the requirements for oscillations to occur can be summarised (for relativistic neutrinos) as follows:

$$\frac{1}{\sigma_E} \sim \frac{1}{\sigma_p} \sim \sigma_x < L_{osc} \sim \frac{1}{|E_i - E_j|} \sim \frac{1}{|p_i - p_j|}. \quad (4.1)$$

In the following we will give an estimation of the wave-packet size for the general case and particularly for the muon decay (the more detailed description can be found in [GK07] p. 311ff). There are basically two different mechanisms influencing the size of the wave packet. They depend on the physical situation.

Natural Linewidth If the neutrino is produced in a decay of a free particle, the important influence on the wave-packet size comes from the mean life time of the decaying particle. That is, the emission of a coherent wave train will be interrupted due to the decay of the particle. Therefore, the maximum size of the wave packet—for a relativistic neutrino—can be estimated to

$$\sigma_x \sim \tau_X,$$

where τ_X is the mean life time of the decaying particle X . This is valid in the rest frame of the decaying particle. If it decays in flight, the relativistic time dilation changes the size of the wave packet to

$$\sigma_x \sim \gamma \tau_X,$$

where the usual γ -factor was included. This factor is given by $\gamma = E_X/m_X$, the ratio of the energy and mass of the decaying particle.

For the muon decay

$$\mu^+ \longrightarrow e^+ + \nu_e + \bar{\nu}_\mu$$

at rest we can estimate the size of the wave packet to

$$\sigma_x \sim \tau_\mu \sim 2.2 \cdot 10^{-6} \text{s} \sim 10^3 \text{m} \sim 10^{10} \text{eV}^{-1}.$$

The corresponding momentum uncertainty is then

$$\sigma_p \sim \frac{1}{\sigma_x} \sim 10^{-10} \text{eV}.$$

Collision Broadening If the neutrino is produced in a decay of a particle contained in a medium, there are more effects that influence the size of the wave packet. In particular, the emission of a coherent wave train will be interrupted if one of the involved particles collides with a particle of the medium. Thus, the important time scale is the average time between two collisions:

$$\tau \sim \min_X \left[\frac{\ell_X}{v_X} \right],$$

where ℓ_X is the mean free path of particle X and v_X its velocity. X can be any of the involved particles without the neutrinos, because they are relativistic and leave the region as soon as

they are produced. The velocities can be estimated for the initial particles by the mean thermal velocity and for the produced particles by the kinematics of the process. The mean free path is a bit more complicated. It is defined as

$$\ell_X = \frac{1}{\pi b^2 N},$$

where N is the particle number density of the medium and b the impact parameter for the Coulomb scattering with the electric field of the nuclei, which can be written as

$$b \approx \frac{4\pi\alpha Z}{T}.$$

Here, α is the fine-structure constant, T the kinetic energy and Z the atomic number of the nuclei in the medium. There is, however, an upper bound for the impact parameter. That is, it cannot exceed half the mean internuclear distance, which can be estimated by $1/N^{1/3}$. Therefore, ℓ cannot be smaller than $\sim 1/N^{1/3}$.

For a muon that decays in graphite at 300K. The values are $Z = 6$ and $N \approx 10^{23}\text{cm}^{-3}$. Then the internuclear distance is $1/N^{1/3} \approx 2 \cdot 10^{-8}\text{cm}$ and the kinetic energy is $T \approx 4 \cdot 10^{-2}\text{eV}$. From this we get a mean free path of $\ell_\mu \approx 5 \cdot 10^{-17}\text{cm}$, which is smaller than the internuclear distance. Therefore, the value for the mean free path is given by the internuclear distance. The velocity of the muon is $v_\mu = \sqrt{2T/m_\mu} \sim 2 \cdot 10^{-5}$. Thus, $\sigma_x \sim 10^{-3}\text{cm}$. The mean free path of the positron is the internuclear distance since it is annihilated as soon as it meets an electron. The velocity can be assumed to be ~ 1 since it is relativistic. Thus $\sigma_x \sim \ell_e \sim 2 \cdot 10^{-8}\text{cm}$. Which is smaller than the muon contribution. Therefore, for the neutrino we can estimated $\sigma_x \sim 2 \cdot 10^{-8}\text{cm}$ and $\sigma_p \sim 2 \cdot 10^3\text{eV}$.

There is an important difference between the uncertainties and the production and detection at different space-time points. That is, if all waves are produced at the same space-time point they will have a spatial uncertainty due to the localisation of the production process and the momentum uncertainty involved in this process. After they travelled to the detector, they will be detected at one specific space-time point. But since the detection process also involves some spatial and temporal uncertainties due to the localisation and finite detection time, this detection can in principle take place everywhere inside these uncertainties. Therefore, the detector has to be described as a wave packet and the probability is the overlap integral over the wave packet and the detector. On the other side, if production and detection at different space-time points are considered, this does not influence the shape of the wave packets neither at the detector nor at the source and only effects the phases. But the phases have to satisfy the coherence condition which means they can be regarded as produced at some point and detected at some other specific point

4.2 Gaussian Wave Packets

In this section we will review the calculation of the flavour changing probability for the case of Gaussian wave packets. This was first done in 1991 by GIUNTI, KIM and LEE [GKL91] (a summary of this paper can be found in [KP93]) and extended by GIUNTI and KIM in 1997 [GK98], where they took into account the spatial coherence width of the detection process. Finally, in 2002 GIUNTI [Giu02b] also included the temporal coherence width of the detection process. Moreover, a summary of the whole calculation in three spatial dimensions can be found in [GK07].

Since we already introduced the most general treatment in chapter 2, we can directly calculate the version of [Giu02b] just by fixing the wave functions $\psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P)$ for the propagating mass neutrinos and the functions $\psi_i^D(\mathbf{x}, \mathbf{x}_D, t, t_D)$ for the detection process. As the title of this section implies, we will describe the mass neutrinos as Gaussian wave packets. Thus, we can use the following momentum space wave function

$$\psi_i(\mathbf{p}) = \frac{1}{(2\pi\sigma_{p_P}^2)^{3/4}} \exp\left(-\frac{(\mathbf{p} - \bar{\mathbf{p}}_i)^2}{4\sigma_{p_P}^2}\right), \quad (4.2)$$

where σ_{p_P} is the momentum width of the packet coming from the uncertainty of the production process and $\bar{\mathbf{p}}_i$ is the mean momentum of the wave packet. The factor in front of the exponential normalises the packet to

$$\int d^3p |\psi_i(\mathbf{p})|^2 = 1. \quad (4.3)$$

Nevertheless, we will forget about the normalisation factors in the following calculation. But, however, we will come back to this question in the final result for the probability, which will then be normalised to (2.18).

The next step is to Fourier transform the momentum space function (4.2) into the position space. Therefore, we use (2.11) which yields

$$\begin{aligned} \psi_i(\mathbf{x}, \mathbf{x}_P, t, t_P) &= \int \frac{d^3p}{(2\pi)^{3/2}} \psi_i(\mathbf{p}) e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_P) - iE(\mathbf{p})(t - t_P)} \\ &= \frac{1}{(2\pi\sigma_{x_P}^2)^{3/4}} \exp\left[i\bar{\mathbf{p}}_i \cdot (\mathbf{x} - \mathbf{x}_P) - i\bar{E}_i(t - t_P)\right] \\ &\quad \cdot \exp\left(-\frac{[(\mathbf{x} - \mathbf{x}_P) - \bar{\mathbf{v}}_i(t - t_P)]^2}{4\sigma_{x_P}^2}\right) \end{aligned} \quad (4.4)$$

In order to get this analytical result, we expanded the energy $E_i(\mathbf{p})$ around the mean momentum $\bar{\mathbf{p}}_i$. Thus,

$$E(\mathbf{p}) = \bar{E}_i + (\mathbf{p} - \bar{\mathbf{p}}_i) \cdot \bar{\mathbf{v}}_i \quad \text{with} \quad \bar{E}_i = \sqrt{\bar{\mathbf{p}}_i^2 + m_i^2}, \quad \bar{\mathbf{v}}_i = \frac{\bar{\mathbf{p}}_i}{\bar{E}_i}, \quad (4.5)$$

where $\bar{\mathbf{v}}_i$ is the group velocity of the wave packet. It is important to note that the neglect of higher order terms in the expansion also means a neglect of the spreading of the wave packet. This can be done for relativistic neutrinos. The new width σ_{x_P} introduced in (4.4) is the spatial wave packet width, which is related to the momentum width by

$$\sigma_{x_P} \sigma_{p_P} = \frac{1}{2}. \quad (4.6)$$

The integral in (4.4) and most of the following integrals are Gaussian. Therefore, they can be calculated analytically.

Before we can actually calculate the amplitude (2.16), we have to find an explicit form for the wave function that describes the detection process. In order to get an analytical result we use a Gaussian wave packet which shall be localised around the detection point x_D with the spatial and temporal widths σ_{x_D} and σ_{t_D} , respectively:

$$\psi_i^D(\mathbf{x}, \mathbf{x}_D, t, t_D) \propto \exp\left[i\bar{\mathbf{p}}_i \cdot (\mathbf{x} - \mathbf{x}_D) - i\bar{E}_i(t - t_D)\right] \exp\left(-\frac{(\mathbf{x} - \mathbf{x}_D)^2}{4\sigma_{x_D}^2} - \frac{(t - t_D)^2}{4\sigma_{t_D}^2}\right). \quad (4.7)$$

The mean energy and momentum of the detection process are the same as those of the propagating mass neutrino. This is necessary, because the incoming mass neutrino excites the degree of freedom of the detection process which corresponds to its mean energy and momentum [Giu02b].

Now, we have everything together in order to calculate the amplitude (2.16). That is,

$$\begin{aligned}
\mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T) &\propto \sum_i U_{\beta i} U_{\alpha i}^* \int dt \int d\mathbf{x} \\
&\cdot \exp \left[-i\bar{\mathbf{p}}_i \cdot (\mathbf{x} - \mathbf{x}_D) + i\bar{E}_i(t - t_D) \right] \exp \left(-\frac{(\mathbf{x} - \mathbf{x}_D)^2}{4\sigma_{x_D}^2} - \frac{(t - t_D)^2}{4\sigma_{t_D}^2} \right) \\
&\cdot \exp \left[i\bar{\mathbf{p}}_i \cdot (\mathbf{x} - \mathbf{x}_P) - i\bar{E}_i(t - t_P) \right] \exp \left(-\frac{[(\mathbf{x} - \mathbf{x}_P) - \bar{\mathbf{v}}_i(t - t_P)]^2}{4\sigma_{x_P}^2} \right) \\
&\propto \sum_i U_{\beta i} U_{\alpha i}^* \exp \left(i\bar{\mathbf{p}}_i \cdot \mathbf{L} - i\bar{E}_i T \right) \exp \left(-\frac{(\mathbf{L} - \bar{\mathbf{v}}_i T)^2}{4\sigma_{x_i}^2} \right), \tag{4.8}
\end{aligned}$$

where for simplicity a new width σ_{x_i} was introduced. It is related to the other widths by

$$\sigma_{x_i}^2 := \sigma_{x_P}^2 + \sigma_{x_D}^2 + \bar{\mathbf{v}}_i^2 \sigma_{t_D}^2. \tag{4.9}$$

Result (4.8) shows that each mass neutrino is detected in a space-time region of width σ_{x_i} around the space-time point (\mathbf{L}, T) . The size of this region is determined by the width of the propagating wave packet and the spatial and temporal coherence widths of the detection process. The dominating contribution comes obviously from the largest of these widths. However, since we already assumed relativistic neutrinos in order to neglect the wave packet spreading, we can use this assumption again and approximate

$$\sigma_{x_i}^2 \approx \sigma_{x_P}^2 + \sigma_{x_D}^2 + \sigma_{t_D}^2 = \sigma_x^2. \tag{4.10}$$

From the amplitude (4.8) we can now derive the probability for the flavour change by taking the absolute square. Hence,

$$\begin{aligned}
\mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T) &\propto \sum_{i,j} U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \exp \left[i(\bar{\mathbf{p}}_i - \bar{\mathbf{p}}_j) \cdot \mathbf{L} - i(\bar{E}_i - \bar{E}_j)T \right] \\
&\cdot \exp \left(-\frac{(\mathbf{L} - \bar{\mathbf{v}}_i T)^2}{4\sigma_x^2} - \frac{(\mathbf{L} - \bar{\mathbf{v}}_j T)^2}{4\sigma_x^2} \right). \tag{4.11}
\end{aligned}$$

This probability contains the same problem as the one for the plane-wave treatment (3.4). That is, we need to know the time interval between the production and detection processes in order to really calculate $\mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}, T)$, but as we said in chapter 3.1 the time is not measured in usual neutrino oscillation experiments. Therefore, we have to dispose the time dependence. In the plane-wave treatment this was done by using different assumptions which then allow a conversion of the time into a distance dependence. However, for the wave-packet treatment we cannot use this assumptions, because they rely on well-defined energies and momenta for the mass neutrinos, which are not given for a wave packet that has inherent uncertainties. Therefore, we have to use another possibility, which will be the average over the time interval

T. Performing the corresponding integral, which is again Gaussian, yields

$$\begin{aligned}
 P(\alpha \rightarrow \beta; \mathbf{L}) &\propto \int dT P(\alpha \rightarrow \beta; \mathbf{L}, T) \\
 &\propto \sum_{i,j} U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \exp \left(i(\bar{\mathbf{p}}_i - \bar{\mathbf{p}}_j) \cdot \mathbf{L} - i(\bar{E}_i - \bar{E}_j) \frac{\bar{\mathbf{v}}_i + \bar{\mathbf{v}}_j}{\bar{v}_i^2 + \bar{v}_j^2} \cdot \mathbf{L} \right) \\
 &\quad \cdot \exp \left(-\frac{(\bar{E}_i - \bar{E}_j)^2 \sigma_x^2}{\bar{v}_i^2 + \bar{v}_j^2} - \frac{\mathbf{L}^2}{2\sigma_x^2} + \frac{(\bar{\mathbf{v}}_i \cdot \mathbf{L})^2 + (\bar{\mathbf{v}}_j \cdot \mathbf{L})^2}{2\sigma_x^2(\bar{v}_i^2 + \bar{v}_j^2)} - \frac{[(\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot \mathbf{L}]^2}{4\sigma_x^2(\bar{v}_i^2 + \bar{v}_j^2)} \right).
 \end{aligned} \tag{4.12}$$

In order to get a simplified expression we expand the mean momentum $\bar{\mathbf{p}}_i$ in powers of m_i^2 . This is the same as we have done in the *energy-momentum conservation* assumption in chapter 3.2, but now we will use the more general ansatz in three spatial dimensions. Thus,

$$\bar{\mathbf{p}}_i \approx \mathbf{p}_0 - \boldsymbol{\xi} \frac{m_i^2}{2p_0} \quad \text{with} \quad \frac{\boldsymbol{\xi}}{2p_0} = -\left. \frac{\partial \mathbf{p}_i}{\partial m_i^2} \right|_{m_i=0}, \tag{4.13}$$

where \mathbf{p}_0 is the vector and p_0 the absolute value of the momentum of a massless neutrino. While $\boldsymbol{\xi}$ is the three dimensional generalisation of the factor ξ introduced in chapter 3.2. The higher order terms can be neglected, because we will only consider relativistic neutrinos here. The same will be done in all following expression.

Using the dispersion relation $\bar{E}_i = \sqrt{\bar{\mathbf{p}}_i^2 + m_i^2}$ yields the expression

$$\bar{E}_i \approx p_0 + \left(1 - \frac{\mathbf{p}_0 \cdot \boldsymbol{\xi}}{p_0} \right) \frac{m_i^2}{2p_0} \tag{4.14}$$

for the mean energy of the i th mass neutrino. The last quantity that we will approximate is the mean velocity, which is given by the ratio of the mean momentum and energy. Hence,

$$\bar{\mathbf{v}}_i \approx \frac{\mathbf{p}_0}{p_0} - \left[\frac{\mathbf{p}_0}{p_0} \left(1 - \frac{\mathbf{p}_0 \cdot \boldsymbol{\xi}}{p_0} \right) + \boldsymbol{\xi} \right] \frac{m_i^2}{2p_0^2}. \tag{4.15}$$

Now we can insert these approximations into the probability (4.12) and neglect all appearing terms of higher than second order in m_i^2 . The resulting probability for relativistic neutrinos is

$$P(\alpha \rightarrow \beta; \mathbf{L}) = \sum_{i,j} U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \exp \left[-2\pi i \frac{L}{L_{ij}^{osc}} - 2\pi^2 \left(1 - \frac{\mathbf{L} \cdot \boldsymbol{\xi}}{L} \right)^2 \left(\frac{\sigma_x}{L_{ij}^{osc}} \right)^2 - \left(\frac{L}{L_{ij}^{coh}} \right)^2 \right] \tag{4.16}$$

with the oscillation and coherence lengths

$$L_{ij}^{osc} = \frac{4\pi p_0}{m_i^2 - m_j^2} \quad \text{and} \quad L_{ij}^{coh} = \frac{4\sqrt{2}p_0^2}{m_i^2 - m_j^2} \sigma_x. \tag{4.17}$$

Throughout the whole calculation we did not care about the factors in front of the exponential. This will be changed now. The normalisation condition (2.18) can easily be applied on (4.16). If we for example take the sum over α , we can use the unitarity of the PMNS-matrix and the fact that all terms in the exponential vanish if $i = j$ and get as a result just 1. This means, the probability is already normalised, which allows us to use the equal sign in (4.16) rather than the proportional sign.

In the following we will explain the physical meaning of the three exponential factors in the probability.

The phase factor

$$\exp \left[-2\pi i \frac{L}{L_{ij}^{osc}} \right] \quad (4.18)$$

This factor describes the actual oscillations. As can be seen from chapter 3, it is exactly the same as in the plane-wave treatment.

The localisation term

$$\exp \left[-2\pi^2 \left(1 - \frac{\mathbf{L} \cdot \boldsymbol{\xi}}{L} \right)^2 \left(\frac{\sigma_x}{L_{ij}^{osc}} \right)^2 \right] \quad (4.19)$$

This term suppresses the oscillations if $\sigma_x \gg L_{ji}^{osc}$. This means, in order to get oscillations the production and detection process must be localised in a region much smaller than the oscillation length. This is the reason why it is called localisation term. Since in usual experiments this condition is very well satisfied, we can ignore this term and only use the effective probability

$$P(\alpha \rightarrow \beta; \mathbf{L}) = \sum_{ij} U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \exp \left[-2\pi i \frac{L}{L_{ij}^{osc}} - \left(\frac{L}{L_{ij}^{coh}} \right)^2 \right]. \quad (4.20)$$

The coherence term

$$\exp \left[- \left(\frac{L}{L_{ij}^{coh}} \right)^2 \right] \quad (4.21)$$

This term suppresses the oscillations if the distance L becomes larger than the coherence length, that is $L \gg L_{ij}^{coh}$. As could be seen in the detailed calculation of the probability, the coherence term basically comes from the last term in (4.12), which becomes large if the velocity of the wave packets is different. Therefore, this term describes the suppression of the oscillations due to the separation of the wave packets for different mass neutrinos. If the packets are too much separated they cannot have an overlap with the detection process and thus they cannot be detected coherently. In this case the coherence term yields an effective flavour changing probability, which is simply the probability for an incoherent mixing

$$P(\alpha \rightarrow \beta; \mathbf{L}) = \sum_i |U_{\beta i}|^2 |U_{\alpha i}^*|^2. \quad (4.22)$$

If $L \ll L_{ij}^{coh}$, the coherence is satisfied and the effective probability reduces to the one already obtained in the plane-wave picture

$$P(\alpha \rightarrow \beta; \mathbf{L}) = \sum_i |U_{\beta i}|^2 |U_{\alpha i}^*|^2 + 2\text{Re} \left[\sum_{i < j} U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \exp \left(2\pi i \frac{L}{L_{ji}^{osc}} \right) \right]. \quad (4.23)$$

The External Wave Packet Model

During the last 20 years a number of models to describe neutrinos in QFT have been published. According to BEUTHE [Beu03, Beu02] all these models can be grouped into four different categories: the *external wave-packet models*, the *stationary boundary-conditions models*, the *source-propagator models* and the *Blasone-Vitiello models*. Since the last category of models uses a different approach than the other models, we will postpone its discussion to chapter 8. The *source-propagator models* proposed by SRIVASTAVA, WIDOM and SASSAROLI [SWS98], where the neutrino is described by its propagator, which is coupled to a source but not to a detector, leads to non-standard oscillation lengths or oscillations of the recoil particle. The reason is the same as in chapter 3.4 where the *factor of two problem* and the oscillation of recoil particles was discussed. Since we ruled out these result, we will not discuss the QFT version of it here. The *stationary boundary-conditions models* describe the neutrino as an internal particle produced in a certain process which then propagates and is detected in another certain process. The important boundary condition assumed in these models is the stationarity of the source and the detector. This condition leads to a unique value for the energy of the neutrinos. The best examples of this kind of models are given by GRIMUS and STOCKINGER [GS96] as well as CARDALL and CHUNG [CC99]. As was shown by BEUTHE [Beu03, Beu02], these models are actually special cases of the general *external wave-packet model*. Therefore, we will also not describe them here. The last category is the *external wave-packet model* whose first version for neutrinos was proposed by GIUNTI, KIM, LEE and LEE [GKLL93] and then a few years later generalised by GIUNTI, KIM and LEE [GKL98]. In this model they used Gaussian wave packets for the external particles and described the neutrino by its propagator. Similar models were used by KIERS and WEISS who replaced the external wave packets by quantum oscillators [KW98] and by CARDALL who pays more attention on the spin structure [Car00]. The most general version of this kind of models was given by BEUTHE [Beu03, Beu02]. In this chapter we will review his computation of the neutrino oscillation formula. The whole calculation involves quite a lot of steps, which are given in great detail in [Beu03]. Therefore, we only present the main steps and arguments here. To be consistent with the original computation we have to change some of our notations, but this will be explicitly noted when it is important.

5.1 The Jacob-Sachs Model

The first authors who used a model in line with the *external wave-packet model* were JACOB and SACHS. They used it to describe the propagation of unstable particles in QFT [JS61]. A few years later SACHS applied the model to Kaons in order to describe their oscillations

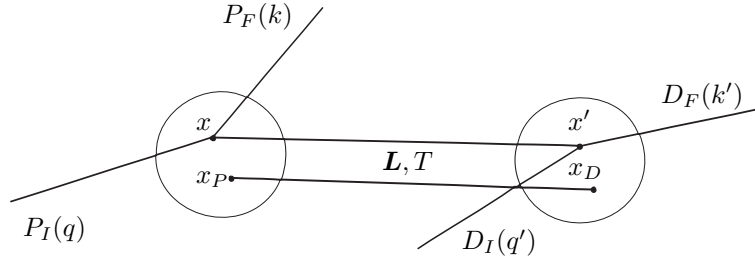


Figure 5.1: Propagation of the neutrino between the source and the detector centered around x_P and x_D , respectively (reproduced according to [Beu03]).

[Sac63]. In the following section we will describe this model and how it can be applied to the description of neutrino oscillations.

The Feynman Diagram The basic point in the *external wave-packet model* is that the neutrino is not considered as an observable particle, which is produced and/or detected, but the whole process starting from the particle which produces the neutrino up to the particles which are produced in the neutrino detection is considered as a quantum field theory process. This description avoids the problem that we cannot regard the neutrino as an asymptotically free particle. Of course, we have to assume that the external particles for the overall process can be considered as asymptotically free. In figure 5.1 we have depicted the situation. We consider the production process $P_I \rightarrow P_F + \nu$ and the detection process $\nu + D_I \rightarrow D_F$, where P and D symbolises all particles involved in these processes. The 4-momenta of the external particles are q, k, q' and k' , respectively. If we assume the production and detection to take place at $x_P = (t_P, \mathbf{x}_P)$ and $x_D = (t_D, \mathbf{x}_D)$ we can have the actual interaction points at $x = (t, \mathbf{x})$ and $x' = (t', \mathbf{x}')$, which lie somewhere inside the production and detection regions depicted as circles around x_P and x_D . These regions appear, because we assume the external particles to be described as wave packets. The neutrino is then an internal particle that propagates the macroscopically distance \mathbf{L} in the time T . Note that we can consider the diagram as a Feynman diagram if we remove the circles.

The Wave Packets Before we go on with this analogy we will briefly introduce the notation for the wave packets, which we will use in the following. The notation and normalisation follows roughly the one used in [PS95]. That is, the state vector which describes a particle with mass m is given as the wave packet

$$|\psi\rangle := \int [d^3k] \psi(\mathbf{k}) |\mathbf{k}\rangle,$$

where $[d^3q]$ is the integration measure given by

$$[d^3k] := \frac{d^3k}{(2\pi)^3 \sqrt{2E(\mathbf{k})}},$$

with $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$. Note that there is no time-dependence for the state vector $|\psi\rangle$. This follows from the fact that we work in the interaction picture, where the time-dependence of the momentum eigenvector $|\mathbf{k}\rangle$ is given by a phase $\exp(iE(\mathbf{p})t)$, while the momentum space wave function $\psi(\mathbf{k})$ evolves in time due to a phase factor $\exp(-iE(\mathbf{p})t)$. Thus, both dependencies

cancel each other. However, if we want to calculate the configuration space wave function by means of a Fourier transformation we have to take the time dependence into account.

Furthermore, we will denote a momentum wave function which is sharply peaked and symmetric around \mathbf{K} by $\psi(\mathbf{k}, \mathbf{K})$. The corresponding configuration space wave function is then centered around the spatial origin. By means of the space-time translation operator $\exp(i\mathbf{P}\cdot\mathbf{x}_0)$ we can shift this the maximum to \mathbf{x}_0 and time t_0 . The corresponding momentum space wave function is then denoted by

$$\Psi(\mathbf{k}, \mathbf{K}, \mathbf{x}_0, t_0) := \psi(\mathbf{k}, \mathbf{K})e^{iE(\mathbf{k})t_0 - i\mathbf{k}\mathbf{x}_0}.$$

The Fourier transformation of this wave function then gives the configuration space wave function

$$\tilde{\Psi}(\mathbf{x}, t, \mathbf{x}_0, t_0) = \int \frac{d^3k}{(2\pi)^3} \psi(\mathbf{k}, \mathbf{K}) e^{-iE(\mathbf{k})(t-t_0) + i\mathbf{k}(\mathbf{x}-\mathbf{x}_0)}. \quad (5.1)$$

For simplicity we will assume that the external particles P_I , P_F , D_I and D_F only consists of one particle whose corresponding state vectors are given as

$$\begin{aligned} |P_I\rangle &:= \int [d^3q] \Psi_{P_I}(\mathbf{q}, \mathbf{Q}, \mathbf{x}_P, t_P) |P_I(\mathbf{q})\rangle, \\ |P_F\rangle &:= \int [d^3k] \Psi_{P_F}(\mathbf{k}, \mathbf{K}, \mathbf{x}_P, t_P) |P_F(\mathbf{k})\rangle, \\ |D_I\rangle &:= \int [d^3q'] \Psi_{D_I}(\mathbf{q}', \mathbf{Q}', \mathbf{x}_D, t_D) |D_I(\mathbf{q}')\rangle, \\ |D_F\rangle &:= \int [d^3k'] \Psi_{D_F}(\mathbf{k}', \mathbf{K}', \mathbf{x}_D, t_D) |D_F(\mathbf{k}')\rangle, \end{aligned}$$

The Amplitude As we mentioned before, we will interpret the process shown in figure 5.1 as a Feynman diagram with an additional folding with the initial and final particles' wave function. In turn this means we can compute the amplitude for the whole process of neutrino production, propagation and detection by using the usual QFT rules for such diagrams. For simplicity we will consider the neutrino as a scalar particle and ignore the spin structure. This can be done because we can factorise the spin part from the rest of the amplitude. In order to “prove” this we have to distinguish between three different possibilities [Beu03]:

1. If the neutrinos are relativistic their chirality and helicity is the same as we already noted in chapter 1.2. Since the weak interactions involve only left-handed neutrinos we know that only this helicity eigenstates propagate. Thus, we can factorise the spin structure from the amplitude, which can then be evaluated as if the neutrinos were scalars, i. e.

$$\mathcal{A}^{\text{SPIN}} = H(s)\mathcal{A}^{\text{SCALAR}}$$

where $H(s)$ includes all spin relevant factors.

2. If the neutrinos are non-relativistic but have nearly degenerated masses, the spin structure for the different helicities is nearly the same and we can again take the spin part out of the sum over the mass eigenstates

$$\sum_i U_{\beta i} \mathcal{A}_i^{\text{SPIN}} U_{i\alpha} \sim \left(\sum_s H(s) \right) \sum_i U_{\beta i} \mathcal{A}_i^{\text{SCALAR}} U_{i\alpha}.$$

3. If the neutrinos are non-relativistic with very different masses we cannot factorise the amplitude. However, the case of very different masses is not relevant for NO because this situation corresponds to an incoherent mixing of the mass neutrinos and not a coherent one which is necessary for oscillations. However, even if an experiment can produce a coherent mixing for this case, the energy and momentum uncertainties would average out the oscillation effect quite rapidly.

Thus, in conclusion we can neglect the spin structure in all cases relevant for oscillations.

Now we can come back to the diagram in figure 5.1. Using the common Feynman rules yields the amplitude

$$\mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T) = \langle D_F, D_I | T \left[\exp \left(-i \int d^4x \mathcal{H}_I \right) \right] - \mathbf{1} | P_I, P_F \rangle, \quad (5.2)$$

where \mathcal{H}_I is the interaction Hamiltonian, T is the time-ordering operator and we included the $\mathbf{1}$ to avoid the calculation of the trivial process, because in that case no neutrino would propagate. For the explicit calculation of the amplitude we expand the exponential to second order in the coupling constant in \mathcal{H}_I , which yields

$$\mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T) = \int [d^3q] \Psi_{P_I} \int [d^3k] \Psi_{P_F}^* \int [d^3q'] \Psi_{D_I} \int [d^3k'] \Psi_{D_F}^* \mathcal{A}_{\text{PLANE}}(q, k, q', k') \quad (5.3)$$

where $\mathcal{A}_{\text{PLANE}}$ is the amplitude if all external particles were plane waves:

$$\mathcal{A}_{\text{PLANE}}(q, k, q', k') = \int d^4x \mathcal{M}_P(q, k) e^{-i(q-k)x} \int d^4x' \mathcal{M}_D(q', k') e^{-i(q'-k')x'} G(x' - x). \quad (5.4)$$

Here $\mathcal{M}_P(q, k)$ and $\mathcal{M}_D(q', k')$ are the amplitudes for the interactions at the production and detection vertex, respectively. The propagator for the neutrino is a bit more complicated. Since the neutrinos which interact at x and x' are flavour neutrinos we cannot just write down the propagator as in the case of a mass neutrino. However, by definition we have

$$G_{\beta\alpha}(x' - x) := \langle 0 | T(\nu_\beta(x') \nu_\alpha^*(x)) | 0 \rangle, \quad (5.5)$$

which is the time-ordered two-point function. If we now insert the mixing of the field operators

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i, \quad (5.6)$$

we obtain

$$\begin{aligned} G_{\beta\alpha}(x' - x) &= \sum_{ij} U_{\beta i} \langle 0 | T(\nu_i(x') \nu_j^*(x)) | 0 \rangle U_{j\alpha}^* \\ &= \sum_i U_{\beta i} G_{ii} U_{i\alpha}^*, \end{aligned} \quad (5.7)$$

with the diagonal propagator for the mass neutrinos

$$\begin{aligned} G_{ii}(x' - x) &= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x'-x)} \frac{i}{p^2 - m_i^2 + i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x'-x)} G_{ii}(p^2). \end{aligned} \quad (5.8)$$

We assume that the renormalisation has been carried out and thus m_i is the physical mass of the neutrino ν_i . After a shift in the variables x and x' we finally obtain the amplitude

$$\begin{aligned} \mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T) &= \sum_i U_{\beta i} \mathcal{A}_i(i \rightarrow i; \mathbf{L}, T) U_{i\alpha}^* \\ &= \int \frac{d^4 p}{(2\pi)^4} \psi(p^0, \mathbf{p}) G_{ii}(p^2) e^{-iE(\mathbf{p})T + i\mathbf{p}\mathbf{L}}, \end{aligned} \quad (5.9)$$

where $\psi(p^0, \mathbf{p})$ is the *overlap function* that describes the overlap of the wave packets both at the source and the detector. It is given by

$$\begin{aligned} \psi(p^0, \mathbf{p}) &= \int d^4 x e^{ipx} \int d^4 x' e^{-ipx'} \int [d^3 q] \psi_{P_i}(\mathbf{q}, \mathbf{Q}) e^{-iqx} \int [d^3 k] \psi_{P_F}^*(\mathbf{k}, \mathbf{K}) e^{ikx} \\ &\quad \cdot \int [d^3 q'] \psi_{D_i}(\mathbf{q}', \mathbf{Q}') e^{-iq'x'} \int [d^3 k'] \psi_{D_F}^*(\mathbf{k}', \mathbf{K}') e^{ik'x'} \mathcal{M}_P(q, k) \mathcal{M}_D(q', k'). \end{aligned} \quad (5.10)$$

Thus, the final amplitude $\mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T)$ is the sum of the amplitudes $\mathcal{A}_i(i \rightarrow i; \mathbf{L}, T)$, where the internal particle is a mass neutrino weighted by the mixing matrix elements $U_{\beta i} U_{i\alpha}^*$.

5.2 External Particles as Gaussian Wave Packets

The crucial step in the actual calculation of (5.9) are the integrations within the overlap function. In most cases it is not possible to do this analytically. One exception is the assumption of plane waves for the external particles. But this is spoiled by the same problems as in the quantum mechanical plane wave approximation. Another possibility is the assumption of Gaussian wave packets for the external particles. This is the approach done first in [GKLL93] and later in [Beu03]. As we already said we will not describe details here. The whole calculation can be found in [Beu03].

Gaussian Wave Packets A Gaussian wave packet in the normalisation of [Beu03] is given as

$$\psi_\chi(\mathbf{p}, \mathbf{p}_\chi) = \left(\frac{2\pi}{\sigma_{p_\chi}^2} \right)^{3/4} \exp \left(-\frac{(\mathbf{p} - \mathbf{p}_\chi)^2}{4\sigma_{p_\chi}^2} \right). \quad (5.11)$$

By using (5.1) we find the configuration space version of it if we expand the energy to second order around the mean momentum \mathbf{p}_χ

$$E(\mathbf{p}) \simeq E_\chi + \mathbf{v}_\chi \cdot (\mathbf{p} - \mathbf{p}_\chi) + \frac{1}{2E_\chi} \left((\mathbf{p} - \mathbf{p}_\chi)^2 - (\mathbf{v}_\chi \cdot (\mathbf{p} - \mathbf{p}_\chi))^2 \right) \quad (5.12)$$

with $E_\chi = \sqrt{\mathbf{p}_\chi^2 + m_\chi^2}$ and $\mathbf{v}_\chi = \mathbf{p}_\chi/E_\chi$. The Fourier transformation then yields

$$\tilde{\psi}_\chi(\mathbf{x}, t, \mathbf{p}_\chi) = \frac{(2\pi\sigma_{x_\chi}^2)^{-3/4}}{\sqrt{\det \Sigma}} \exp \left(-iE_\chi t + i\mathbf{p}_\chi \mathbf{x} - \frac{(\mathbf{x} - \mathbf{v}_\chi t) \Sigma^{-1} (\mathbf{x} - \mathbf{v}_\chi t)}{4\sigma_{x_\chi}^2} \right) \quad (5.13)$$

with

$$\sigma_{x_\chi} \sigma_{p_\chi} = \frac{1}{2} \quad \text{and} \quad \Sigma^{ij} = \delta^{ij} + (\delta^{ij} - v^i v^j) \frac{2it\sigma_{p_\chi}^2}{E_\chi}. \quad (5.14)$$

The matrix Σ is important because it determines the spreading of the wave packet. As can be shown this spreading is different for the directions transverse and along \mathbf{v}_χ . In particular, the transverse spreading begins at $t \simeq E_\chi/2\sigma_{p_\chi}^2$, whereas the longitudinal spreading begins at the later time $t \simeq E_\chi^3/2m_\chi^2\sigma_{p_\chi}^2$. This will become important later.

For the external wave packets we will neglect the spreading. Additionally, we assume that the factors $\mathcal{M}_P(q, k)$ and $\mathcal{M}(q', k')$ vary only slowly over the width of the wave packets. Therefore, we can approximate them by their value at the mean momentum of the wave packets and then take them out of the integrals in the overlap function. Moreover, we can take them out of the sum over the different mass eigenstates, because we suppose the neutrinos to be either relativistic or having nearly degenerate masses, which results in the same factor for all mass neutrinos.

The different integrals in the overlap function can now be calculated. Finally we arrive at

$$\psi(p^0, \mathbf{p}) \sim \psi_P(p^0, \mathbf{p})\psi_D^*(p^0, \mathbf{p}) \quad (5.15)$$

with

$$\psi_{P/D}(p^0, \mathbf{p}) \sim \exp \left[-\frac{(\mathbf{p} - \mathbf{p}_{P/D})^2}{4\sigma_{p_{P/D}}^2} - \frac{(p^0 - E_{P/D} - (\mathbf{p} - \mathbf{p}_{P/D}) \cdot \mathbf{v}_{P/D})^2}{4\sigma_{E_{P/D}}^2} \right] \quad (5.16)$$

The new energy and momentum variables are

$$\begin{aligned} \mathbf{p}_P &:= \mathbf{Q} - \mathbf{K}, & E_P &:= E_{P_I} - E_{P_F}, \\ \mathbf{p}_D &:= \mathbf{K}' - \mathbf{Q}', & E_D &:= E_{D_F} - E_{D_I}. \end{aligned}$$

The explicit form of the uncertainties and velocities shall not be of interest to us at this point. However, they can be interpreted as the momentum and energy uncertainties as well as the velocities of the source and detector, respectively. Since the propagation distance is macroscopically for neutrino oscillations, the production and detection processes have to satisfy global energy-momentum conservation laws. Therefore, we can impose the conditions

$$\mathbf{p}_P = \mathbf{p}_D = \mathbf{p}_0 \quad \text{and} \quad E_P = E_D = E_0. \quad (5.17)$$

5.3 Three Different Amplitudes

The Pole Integration In the last section we found an expression for the overlap function. Thus, the last step in the computation of the amplitude is the momentum integral (5.9). We will do this integration in two steps, first we integrate over p^0 to get rid of the poles in the propagator and then we do the \mathbf{p} integration.

The integration over p^0 involves a careful choice of the integration contour, because the overlap function diverges at infinity in the complex plane. Since we will not go into detail, we just mention that we can perform the integration by using the *Jacob-Sachs theorem* (cf. [Beu03] appendix A). This theorem states that the integral has for $T \rightarrow \infty$ the asymptotical behaviour

$$\int dp^0 \psi(p^0, \mathbf{p}) G(p^2) e^{-ip^0 T} \rightarrow \frac{\pi}{E(\mathbf{p})} \psi(E(\mathbf{p}), \mathbf{p}) e^{-iE(\mathbf{p})T}. \quad (5.18)$$

Thus, the remaining part of the amplitude is given by

$$\mathcal{A}_i \sim \int \frac{d^3 p}{E_i(\mathbf{p})} \psi(E(\mathbf{p}), \mathbf{p}) e^{-iE(\mathbf{p})T + i\mathbf{p}\mathbf{L}}, \quad (5.19)$$

which can be seen as an amplitude in the intermediate wave packet model. The overlap function would then be the wave function for the i th mass neutrino. Hence, we can interpret the process in figure 5.1 as the production of a neutrino as a superposition of wave packets one for each mass neutrino, which are then detected. Nevertheless, there is an important difference to this model. That is, the overlap function contains the information on the source as well as the detector which spoils the causality. This point was discussed in detail by GIUNTI [Giu02a] who showed by using a different ansatz that the produced wave packets are only functions of $\psi_P(p^0, \mathbf{p})$. The phase factor then comes from the propagation and the detection process finally leads to the remaining function $\psi_D^*(p^0, \mathbf{p})$.

The 3-Momentum Integrations Unfortunately, the last integration over the 3-momentum cannot be done analytically. But according to the form of the integrand we have two possibilities for an approximation:

1. *Laplace's method* [BO78]: Since the overlap function is of the form $I = \exp(-af(\mathbf{p}))$ where a is a large parameter, we can expand the integrand around the minimum of $f(\mathbf{p})$ to second order and then perform a Gaussian integration.
2. *Method of stationary phase* [BO78]: Since the phase contains the large parameters T and L the phase oscillates rapidly. This allows us to expand the integrand around the stationary point of the phase to second order and then perform a Gaussian integration.

Since we cannot use both approximations at the same time, we have to decide which is more useful. *Laplace's method* is preferable if the phase varies slowly over the overlap function, whereas the *method of stationary phase* is applicable if the phase varies rapidly over the overlap function. It can be shown [Beu03] that this difference is connected to the spreading of the overlap function. If we can neglect the spreading we prefer *Laplace's method*. In the cases where we cannot neglect the spreading we will use the *method of stationary phase*. As we have seen before, the spreading depends on the direction which in the end leads us to divide the propagation range into three different parts:

1. The *no-dispersion regime*, where we can neglect the spreading in all directions and thus use Laplace's method.
2. The *transversal-dispersion regime*, where we can only neglect the spreading in the longitudinal direction. Therefore, we use Laplace's method for the longitudinal direction and the method of stationary phase for the transverse directions.
3. The *longitudinal-dispersion regime*, where we have to take into account the spreading in all directions and thus will use the method of stationary phase.

In the following we describe the oscillation in each regime separately.

Oscillations in the No-Dispersion Regime If the neutrino is not too far away from the source, we can assume that the phase in (5.9) varies slowly over the width of the overlap function. Therefore, we can use Laplace's method to get an approximation for the 3-momentum integral in (5.9). The term "not too far away" corresponds to a vanishing dispersion for the overlap function. It can be shown that the dispersion is negligible for distances $L \leq p_0/\sigma_p^2$, where σ_p is defined by

$$\frac{1}{\sigma_p^2} = \frac{1}{\sigma_{pP}^2} + \frac{1}{\sigma_{pD}^2}, \quad (5.20)$$

and is thus dominated by the smallest width among the production and detection widths [Beu03]. This threshold for the no-dispersion regime is rarely of the order of a macroscopic distance. And thus usual NO are not present in this regime. Nevertheless, we can compute the amplitude by finding the momentum \mathbf{p}_i which maximises the overlap function. The whole integrand can then be expanded to second order around this momentum, which allows us to perform a Gaussian integration. The final result for the amplitude has the form [Beu03]

$$\mathcal{A}_i \sim \exp\left(-iE_i T + i\mathbf{p}_i \mathbf{L} - f_i(\mathbf{p}_i) - F_i(T)\right), \quad (5.21)$$

where the factor $F_i(T)$ describes the space-time envelope of the propagating wave packet, whose explicit form does not interest us here. The factor $f_i(\mathbf{p}_i)$ represents the exponent in (5.16) taken at \mathbf{p}_i .

After having found the amplitude we can go on and compute the probability. In order to get a time-independent probability we average the absolute square of the amplitude over the propagation time T just as we have done it in the intermediate wave packet model. Thus, the probability is given by

$$\mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}) \sim \sum_{i,j} U_{\beta i} U_{i\alpha}^* U_{\beta j}^* U_{j\alpha} \int dT \mathcal{A}_i \mathcal{A}_j^*. \quad (5.22)$$

Inserting the amplitude (5.21) and performing the integration over T finally yields [Beu03]

$$\begin{aligned} \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}) \sim & \sum_{i,j} U_{\beta i} U_{i\alpha}^* U_{\beta j}^* U_{j\alpha} \exp\left(-\frac{(\tilde{\mathbf{v}}_0 \times \tilde{\mathbf{L}})^2}{2\sigma_x^2 \tilde{v}_0^2}\right) \\ & \cdot \exp\left(-2\pi i \frac{L}{L_{ij}^{osc}} - \frac{(m_i^2 - m_0^2)^2 + (m_j^2 - m_0^2)^2}{16\tilde{\sigma}_m^2 E_0^2} - 2\pi^2 \left(\frac{\tilde{\rho} \tilde{\sigma}_{x_{eff}}}{L_{ij}^{osc}}\right)^2\right) \end{aligned} \quad (5.23)$$

with

$$L_{ij}^{osc} = \frac{4\pi p_0}{m_i^2 - m_j^2}. \quad (5.24)$$

The mass m_0 corresponds to the momentum \mathbf{p}_0 and energy E_0 , while $\tilde{\sigma}_m$ can be interpreted as a mass uncertainty and $\tilde{\sigma}_{x_{eff}}$ as an effective spatial uncertainty build from the uncertainties of the production and detection region. The explicit form will not interest us. $\tilde{\rho}$ is a factor which describes the deviation of E_i and E_0 .

We will explain the different terms after the discussion of the two other regimes.

Oscillations in the Transverse-Dispersion Regime If the distance from the source does not satisfy the relation $L < p_0/\sigma_p^2$, we cannot use Laplace's method for all three spatial dimensions. However, if the distance satisfy the relation [Beu03]

$$L < L^{disp} = \frac{p_0 E_0^2}{2m_i^2 \sigma_{p_{eff}}^2}, \quad (5.25)$$

we can at least neglect the spreading in the direction of the momentum. $\sigma_{p_{eff}}$ is again an effective width given by the production and detection widths.

The integration over the 3-momentum in (5.9) involves now two steps. First we have to perform the integration over the transverse momenta by using the method of stationary phase.

That is we expand the integrand around the stationary point of the phase to second order and then perform a Gaussian integration. The second step is the integration over the longitudinal momentum using Laplace's method. The result is given by [Beu03]

$$\mathcal{A}_i \sim \frac{g(\boldsymbol{\ell})\sigma_{p_{eff}}}{T\sqrt{1+iT/T_i^{disp}}} \exp\left(-iE_i T + i\mathbf{p}_i \mathbf{L} - f_i(\mathbf{p}_i) - \frac{1}{1+iT/T_i^{disp}} \frac{(\mathbf{v}_i T - \mathbf{L})^2}{4\sigma_{x_{eff}}^2}\right) \quad (5.26)$$

with

$$g(\boldsymbol{\ell}) = \exp\left(-\frac{(\mathbf{p}_0 \times \boldsymbol{\ell})^2}{4\sigma_p^2}\right) \quad \text{and} \quad \boldsymbol{\ell} = \frac{\mathbf{L}}{L} \quad (5.27)$$

which restricts the propagation to a cone around the axis \mathbf{p}_0 with the angle $\arcsin(\sigma_p/p_0)$. The dispersion time T_i^{disp} is defined by $L_i^{disp} = v_0 T_i^{disp}$. Using (5.22) allows the computation of the corresponding probability [Beu03]

$$\begin{aligned} \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}) \sim & v_0 \sigma_{p_{eff}} \frac{g^2(\boldsymbol{\ell})}{L^2} \sum_{i,j} U_{\beta i} U_{i\alpha}^* U_{\beta j}^* U_{j\alpha} \\ & \cdot \exp\left(-2\pi i \frac{L}{L_{ij}^{osc}} - \frac{(m_i^2 - m_0^2)^2 + (m_j^2 - m_0^2)^2}{16\sigma_m^2 E_0^2} - 2\pi^2 \left(\frac{\rho\sigma_{x_{eff}}}{L_{ij}^{osc}}\right)^2 - \left(\frac{L}{L_{ij}^{coh}}\right)^2\right), \end{aligned} \quad (5.28)$$

where σ_m , $\sigma_{x_{eff}}$ and ρ have the same interpretation as in the no-dispersion regime but with a slightly different definition. The main difference to the result in the previous paragraph is the additional term which defines a coherence length L_{ij}^{coh} given by

$$L_{ij}^{coh} = \frac{1}{\sqrt{2\pi}} \frac{p_0}{\sigma_{p_{eff}}} L_{ij}^{osc}. \quad (5.29)$$

Oscillations in the Longitudinal-Dispersion Regime If the propagation distance is larger then the dispersion distance L^{disp} we cannot neglect the spreading in the longitudinal direction. Thus, we have to use the method of stationary phase for all integrals and expand the integrand in terms of the stationary momentum $\mathbf{p}_{cl,i}$. Performing the Gaussian integration yields [Beu03]

$$\mathcal{A}_i \sim \frac{g(\boldsymbol{\ell})\sigma_{p_{eff}}}{T\sqrt{1+iT/T_i^{disp}}} \exp\left(-im_i \sqrt{T^2 - L^2} - f_i(\mathbf{p}_i) + \sigma_{p_{eff}}^2 \frac{(f'_i(p_{cl,i}))^2}{1+iT/T_i^{disp}}\right), \quad (5.30)$$

where f' is the derivative of f . The remaining step is the computation of the probability according to (5.22). The final result reads

$$\begin{aligned} \mathcal{P}(\alpha \rightarrow \beta; \mathbf{L}) \sim & v_0 \sigma_{p_{eff}} \frac{g^2(\boldsymbol{\ell})}{L^2} \sum_{i,j} U_{\beta i} U_{i\alpha}^* U_{\beta j}^* U_{j\alpha} \\ & \cdot \exp\left(-2\pi i \frac{L}{L_{ij}^{osc}} - \frac{(m_i^2 - m_0^2)^2 + (m_j^2 - m_0^2)^2}{16\sigma_m^2 E_0^2} - 2\pi^2 \left(\frac{\rho\sigma_{x_{eff}}}{L_{ij}^{osc}}\right)^2 - \left(\frac{L}{L_{ij}^{coh}}\right)^2\right), \end{aligned} \quad (5.31)$$

which is exactly the same as in the transverse dispersion regime.

5.4 Analysis of the Probabilities

The probabilities in the different regimes are given by (5.23), (5.28) and (5.31). We will first discuss the probability for large distances, that is in the transverse and longitudinal dispersion regime and explain the different factors appearing in the result.

The geometrical factor

$$g(\ell) = \exp\left(-\frac{(\mathbf{p}_0 \times \ell)^2}{4\sigma_p^2}\right) \quad (5.32)$$

This factor yields a geometrical constraint between the momentum \mathbf{p}_0 and the direction of the observation $\ell = \mathbf{L}/L$. That is, the propagation is restricted to a cone with angle $\arcsin(\sigma_p/p_0)$ around the axis defined by \mathbf{p}_0 .

The phase factor

$$\exp\left[-2\pi i \frac{L}{L_{ij}^{osc}}\right] \quad (5.33)$$

This factor describes the oscillations. It is exactly the same as in all other approaches (cf. (4.18)).

The coherence term

$$\exp\left[-\left(\frac{L}{L_{ij}^{coh}}\right)^2\right] \quad (5.34)$$

This term implements a coherence length into the oscillations. That is, the oscillations vanish if $L \gg L_{ij}^{coh}$. There are two different physical explanations for this phenomena:

- If the coherence length is smaller than the dispersion length $L_{ij}^{coh} < L^{disp}$ then the coherence comes from the separation of the different wave packets due to their different group velocities.
- If the coherence length is larger than the dispersion length, then the decoherence takes place in the longitudinal dispersion regime and the wave packets spread out as quickly as they separate. Therefore, the decoherence is due to the spread of the wave packets, which at some point becomes comparable with the oscillation length and thus oscillations vanish.

Thus, even though the term has the same form in the intermediate wave-packet model (cf. (4.21)) we obtain a second explanation for the loss of the coherence, which is the dispersion of the wave packets. Since we neglected dispersion in the intermediate wave-packet model this is not a surprise.

The localisation terms

$$\exp\left[-\frac{(m_i^2 - m_0^2)^2 + (m_j^2 - m_0^2)^2}{16\sigma_m^2 E_0^2} - 2\pi^2 \left(\frac{\rho\sigma_{x_{eff}}}{L_{ij}^{osc}}\right)^2\right] \quad (5.35)$$

$$= \exp\left[-\frac{(m_i^2 - m_j^2)^2}{32\sigma_m^2 E_0^2} + \frac{(m_j^2 + m_j^2 - 2m_0^2)^2}{32\sigma_m^2 E_0^2} - 2\pi^2 \left(\frac{\rho\sigma_{x_{eff}}}{L_{ij}^{osc}}\right)^2\right] \quad (5.36)$$

The first term in this sum can be written as a term which suppresses the oscillations if $L_{ij}^{osc} \ll \sigma_x$. Thus, it demands the localisation of the source and detector to be small against the oscillation length. Such a term was also found in the intermediate wave packet model (cf. (4.19)). The third term gives a similar constraint. It can be shown that it implies that the oscillations vanish if $L_{ij}^{osc} \ll S_{P/D}$ where $S_{P/D}$ is the macroscopic size of the source or detector, respectively. The term in the middle will be described in the following.

The kinematical term

$$\exp \left[\frac{(m_j^2 + m_j^2 - 2m_0^2)^2}{32\sigma_m^2 E_0^2} \right] \quad (5.37)$$

Since m_0 is the mass of a particle with energy and momentum according to the energy-momentum conservation at the source and the detector, this term implies that the mass eigenstates have to be on-shell within the uncertainty σ_m .

Oscillations at short distance In the no-dispersion regime the probability looks slightly different. First of all, there is no coherence term in the probability. Which is more or less expected because at short distances the wave packets should not be driven apart too much and moreover there is no dispersion in this regime. Second there is no geometrical decrease $1/L^2$ which can be explained by the neglected dispersion. Finally the geometrical term does not constrain the propagation to a cone, but rather to a cylinder.

Chapter 6

Some Comments on Quantum Field Theory

In the last chapter we described the *external wave-packet model*, which already was a QFT model. Therein the neutrino was represented by its propagator and thus we did not need to worry about the description of the neutrino in terms of state vectors. However, the *Blasone-Vitiello model*, which we excluded in the last chapter, discusses the state vectors for flavour neutrinos in detail. Since we will encounter some usually not mentioned and not discussed properties of quantum field theories, we will describe them in a compact form in this chapter.

6.1 A One-Particle Theory

One possible starting point for a quantum field theory is the Lagrangian which describes the couplings between different fields. Therefrom, we can then find the equations of motion for the fields. Note that we will only discuss fermion fields in this chapter, because we do not need boson fields for the description of neutrinos. Nevertheless, we could rewrite this chapter without much effort in terms of bosons.

As an example we can consider the one-particle Dirac Lagrangian for a field $\psi(x)$ with mass m

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m)\psi, \quad (6.1)$$

which yields the following equation of motion

$$(i\cancel{\partial} - m)\psi = 0. \quad (6.2)$$

In order to solve this equation we can Fourier expand the field $\psi(x)$. Since we will later find a difference between the finite and the infinite volume case, we start with a finite volume V and take the limit $V \rightarrow \infty$ in the end. One important difference between the two cases is the countable number of degrees of freedom for the finite volume and an uncountable number for the infinite volume. Thus, we get

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}, r} (a_{\mathbf{p}}^r u_{\mathbf{p}}^r e^{-ipx} + b_{\mathbf{p}}^{r\dagger} v_{\mathbf{p}}^r e^{ipx}), \quad (6.3)$$

where r is the helicity and the spinors $u_{\mathbf{p}}^r$ and $v_{\mathbf{p}}^r$ are defined via

$$(\cancel{\not{p}} - m)u_{\mathbf{p}}^r = 0 \quad \text{and} \quad (\cancel{\not{p}} + m)v_{\mathbf{p}}^r = 0. \quad (6.4)$$

Note that the limit $V \rightarrow \infty$ can be performed by replacing

$$\frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \longrightarrow \frac{1}{(2\pi)^3} \int d^3p. \quad (6.5)$$

To actually get a quantised theory we impose the second quantisation and treat the field $\psi(x)$ as well as the Fourier coefficients $a_{\mathbf{p}}^r$ and $b_{\mathbf{p}}^{r\dagger}$ as operators defined in some Hilbert space. Therefore, we assume the following canonical anti-commutation relations (CCR)

$$\{\psi(x), \psi(y)\}_{x^0=y^0} = \{\psi^\dagger(x), \psi^\dagger(y)\}_{x^0=y^0} = 0, \quad (6.6a)$$

$$\{\psi(x), \psi^\dagger(y)\}_{x^0=y^0} = \delta(\mathbf{x} - \mathbf{y}), \quad (6.6b)$$

or equivalently

$$\{a_{\mathbf{p}}^r, a_{\mathbf{k}}^{s\dagger}\} = \{b_{\mathbf{p}}^r, b_{\mathbf{k}}^{s\dagger}\} = \delta_{\mathbf{p}\mathbf{k}}\delta_{rs} \quad \text{and all other zero.} \quad (6.7)$$

Using the creation and annihilation operators $a_{\mathbf{p}}^{r\dagger}$, $b_{\mathbf{p}}^{r\dagger}$ and $a_{\mathbf{p}}^r$, $b_{\mathbf{p}}^r$, respectively, allows us to define a Fock-space \mathcal{F} which we will then use as the Hilbert space for our theory. The basis for the Fock-space is the set of state vectors describing states with a finite number of particles. It can be shown that all other state vectors can be obtained by successively acting creation operators on the ground state, which is defined as the state vector annihilated by all operators $a_{\mathbf{p}}^r$ and $b_{\mathbf{p}}^r$

$$a_{\mathbf{p}}^r|0\rangle = b_{\mathbf{p}}^r|0\rangle = 0 \quad \forall \mathbf{p}, r. \quad (6.8)$$

This state vector is usually called the vacuum and interpreted as the zero-particle state.

An important assumption on the state vectors in the Fock space is that they describe asymptotically free particles. That is, the particles in this states do not interact with each other. Without this assumption we would not be able to interpret the vectors as states with a definite number of particles, because interactions in QFT mean the appearance of virtual particles which spoils the definition of a particle number. This also follows from the relation

$$[\mathbf{H}, a^\dagger] = E a^\dagger, \quad (6.9)$$

which is essential to interpret a^\dagger as a creation operator. But this relation is only valid if the Hamiltonian \mathbf{H} is the free Hamiltonian without interaction terms and thus is bilinear in a^\dagger . If the field $\psi(x)$ is an interacting field, we can still Fourier transform it with respect to the spatial coordinates, but the corresponding coefficients will then be time-dependent and do not satisfy (6.9). Thus, they cannot be interpreted as creation and annihilation operators. The only connection to the particle picture can then be obtained by the asymptotic condition. Of course, this is not important for the field $\psi(x)$, because there are no interactions. But it should be noted for later reference. It should also be noted that the Fock space is separable. That means the basis is countable [UMT82]. However, there are states which are not represented in the Fock space. In general these are all states which contain an infinite number of particles. If we build a Hilbert space \mathcal{H} which contains all possible states, with a finite number as well as an infinite number of particles, it can be shown that this space is uncountable and thus non-separable [UMT82]. The space \mathcal{F} can then be seen as a separable subspace of \mathcal{H} . In particular, there is an uncountable number of different separable subspaces, which can be grouped into equivalence classes [Bar63]. The different classes contain all subspaces that only differ in a finite number of particles. This shows that the Fock space is distinct to all other subspaces because it contains the unique zero-particle state.

6.2 An Example

In order to describe the problems that arise from the discussion on the Fock space we will give an example for a theory with interacting particles. This can be found in [Bar63] as well as in a slightly different form in [UMT82]. For this example we consider two fields, $\psi_1(x)$ and $\psi_2(x)$, which obey the free Dirac equation (6.2). For simplicity the field $\psi_2(x)$ shall be massless $m_2 = 0$. Additionally, both fields should obey the CCR (6.6). If both fields are independent of each other we can build Fock spaces \mathcal{F}_1 and \mathcal{F}_2 for both of them. The generic space for a description of both fields is then the direct product

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2. \quad (6.10)$$

For neutrinos we need linear combinations of the field, for example $\nu_e = \cos\theta\nu_1 + \sin\theta\nu_2$, which can be obtained by taking an appropriate boundary condition. For simplicity, we will use a simpler case here and impose the condition that both fields are equal at time $x^0 = 0$

$$\psi_1(\mathbf{x}, 0) = \psi_2(\mathbf{x}, 0). \quad (6.11)$$

Because we assumed that both fields are representations of the CCR we are tempted to find a unitary operator which connects both fields

$$\psi_1(x) = U^{-1}\psi_2(x)U, \quad (6.12)$$

or in terms of the creation and annihilation operators

$$a_{\mathbf{p},1}^r = U^{-1}a_{\mathbf{p},2}^rU \quad \text{and} \quad b_{\mathbf{p},1}^r = U^{-1}b_{\mathbf{p},2}^rU. \quad (6.13)$$

The reason why we seek for unitary equivalence is the so-called *von Neumann theorem* [vN31], which is valid in quantum mechanical systems with a finite or countable number of degrees of freedom. It says simply spoken:

An irreducible set of operators satisfying the CCR is determined uniquely up to unitary equivalence.

For our example we can find the transformation (6.13) explicitly by using the Fourier expansion (6.3) and insert it in (6.11). $a_{\mathbf{p},1}^r$ and $b_{-\mathbf{p},1}^r$ can be isolated by taking an appropriate scalar product of the u and v spinors and using the orthogonality of u and v . This yields

$$a_{\mathbf{p},1}^r = \alpha_{\mathbf{p}} a_{\mathbf{p},2}^r + \beta_{\mathbf{p}} b_{-\mathbf{p},2}^{r\dagger}, \quad (6.14a)$$

$$b_{-\mathbf{p},1}^r = \alpha_{\mathbf{p}} b_{-\mathbf{p},2}^r - \beta_{\mathbf{p}} a_{\mathbf{p},2}^{r\dagger}, \quad (6.14b)$$

where $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{p}}$ are scalar products of the u and v spinors in $\psi_1(x)$ and $\psi_2(x)$. The explicit form of these scalars does not interest us here. Nevertheless, we should note that we can parametrise them by an angle $\theta_{\mathbf{p}}$ with $\alpha_{\mathbf{p}} = \cos\theta_{\mathbf{p}}$ and $\beta_{\mathbf{p}} = \sin\theta_{\mathbf{p}}$. The operator U in (6.13) is then given by

$$U = \exp \left[\sum_{\mathbf{p},r} \theta_{\mathbf{p}} \left(a_{\mathbf{p},2}^r b_{-\mathbf{p},2}^r + a_{\mathbf{p},2}^{r\dagger} b_{-\mathbf{p},2}^{r\dagger} \right) \right]. \quad (6.15)$$

To see that this operator really generates the mixing (6.14) we make use of the relation

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots \quad (6.16)$$

and explicitly compute the appearing commutators. A transformation of the form (6.14) is usually called a *Bogolubov transformation* which is well-known in thermal field theory [UMT82] and in the BCS-theory of superconducting.

The transformation of the creation and annihilation operators by U also yields a connection between the two Fock spaces \mathcal{F}_1 and \mathcal{F}_2 . If $a_{\mathbf{p},1}^r$ acts on an arbitrary state vector $|\rangle_1 \in \mathcal{F}_1$ we can rewrite this to

$$a_{\mathbf{p},1}^r |\rangle_1 = U^{-1} a_{\mathbf{p},2}^r U |\rangle_1 = U^{-1} a_{\mathbf{p},2}^r |\rangle_2 \quad \text{with} \quad |\rangle_2 := U |\rangle_1 \in \mathcal{F}_2 \quad (6.17)$$

For the particular case of the ground states $|0\rangle_1$ and $|0\rangle_2$ we then find

$$|0\rangle_1 = U^{-1} |0\rangle_2 = \prod_{\mathbf{p},r} \left\{ \alpha_{\mathbf{p}} - \beta_{\mathbf{p}} a_{\mathbf{p},2}^{r\dagger} b_{-\mathbf{p},2}^{r\dagger} \right\} |0\rangle_2. \quad (6.18)$$

As we can see the ground state $|0\rangle_1$ of \mathcal{F}_1 is a superposition of excitations of the ground state $|0\rangle_2$ of \mathcal{F}_2 . Each pair of excitations has total momentum zero and a helicity of ± 2 . This spoils our interpretation of $|0\rangle_1$ as the zero-particle state in \mathcal{F}_1 . Therefore, we should not call \mathcal{F}_1 as Fock space anymore. In turn the same is true for \mathcal{F}_2 , because we could invert (6.17) and get $|0\rangle_2$ as a superposition of excitations of $|0\rangle_1$. In conclusion, the ground state of $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, which is $|0\rangle_1 \otimes |0\rangle_2$, is not a zero-particle state but describes a condensate of infinitely many particles. Thus, due to the dependency which we imposed in (6.11) we encounter the problem of how to interpret the state vectors in \mathcal{F} . Nevertheless, the spaces \mathcal{F}_1 , \mathcal{F}_2 as well as \mathcal{F} are well defined and we will use them to find out more about the operator U . For example we can calculate the overlap of the two ground states, which is likewise the matrix element of U taken between the ground state of \mathcal{F}_1

$$\mathcal{O} = {}_1\langle 0|0\rangle_2 = {}_1\langle 0|U|0\rangle_1. \quad (6.19)$$

By using (6.17) we find

$$\mathcal{O} = \prod_{\mathbf{p},r} \alpha_{\mathbf{p}} = \prod_{\mathbf{p},r} \exp(\ln \alpha_{\mathbf{p}}) = \exp\left(2 \sum_{\mathbf{p}} \ln \alpha_{\mathbf{p}}\right). \quad (6.20)$$

Up to now we have not said anything about the finite volume and its consequences. From the infinite volume behaviour of the sum (6.5) we can write the overlap as

$$\mathcal{O} = \exp\left(\frac{2V}{(2\pi)^3} \int d^3p \ln \alpha_{\mathbf{p}}\right). \quad (6.21)$$

Since $\alpha_{\mathbf{p}} < 1$ (remember $\alpha_{\mathbf{p}} = \cos \theta_{\mathbf{p}}$) the exponent is negative and the overlap vanishes for $V \rightarrow \infty$. The same can be done for all state vectors and we find that all matrix elements of U taken between state vectors of one space, \mathcal{F}_1 or \mathcal{F}_2 , vanishes. An operator with this kind of behaviour is called an *improper unitary operator* in contrast to the usual *proper unitary operators*.

Eq. (6.21) also shows that in the infinite volume limit the spaces \mathcal{F}_1 and \mathcal{F}_2 are completely disjoint. Or, in other words, they are in different equivalence classes if we use the language of section 6.1. Such spaces are called *unitarily inequivalent* as a short form for *improperly unitarily equivalent*, which means their equivalence is due to an improper unitary operator.

The situation for the spaces in this example can be depicted as in figure 6.1.

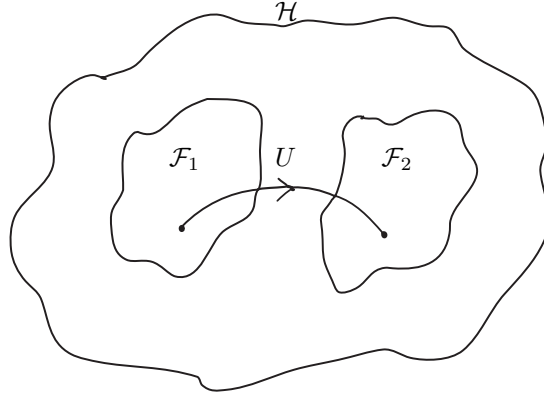


Figure 6.1: The two “Fock spaces” \mathcal{F}_1 and \mathcal{F}_2 are subspaces of the overall Hilbert space \mathcal{H} . These subspaces are connected by the improper operator U which maps a vector from \mathcal{F}_1 to \mathcal{F}_2 , while the creation and annihilation operators map within the respective subspaces.

6.3 Haag's Theorem

The reason why the operator becomes improper in the infinite volume limit is the breakdown of the *von Neumann theorem* which was only valid for a finite or countable number of degrees of freedom. But in the infinite volume limit the number of degrees of freedom becomes uncountable and thus two different representations of the CCR are, in general no longer unitarily equivalent. In turn this implies an uncountable number of different representations of the CCR because we have an uncountable number of equivalence classes.

However, the Fock space is unique in this number of representations, because it is the only representation which describes a definite number of asymptotically free particles.

The reason for all these problems we encountered so far is connected in *Haag's theorem* [Haa55, BLT75]. It says simply spoken:

- *If two fields are unitarily equivalent, then both are free if one is free.*
- *Only if the ground states $|0\rangle_1$ and $|0\rangle_2$ are equal, their corresponding “Fock spaces” are unitarily equivalent.*

Neutrinos and Haag's Theorem As emphasised before, the flavour-neutrino fields arise in the combinations

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i, \quad (6.22)$$

where the ν_i are the fields with definite masses m_1 and m_2 , respectively. Therefore, the linear combinations act in the Hilbert space $\mathcal{F}_1 \otimes \mathcal{F}_2$ which has the ground state $|0\rangle_1 \otimes |0\rangle_2$. But as we have seen this is not a zero-particle state and consequently $\mathcal{F}_1 \otimes \mathcal{F}_2$ is not a Fock space. Thus, we have no particle interpretation for excitations of $|0\rangle_1 \otimes |0\rangle_2$ due to ν_α . In other words, although ν_α and ν_i obey the same CCR they belong to different subspaces of \mathcal{H} . The same behaviour is manifest in the standard quantum mechanical approaches where we define the state vectors which should describe the flavour neutrinos as

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle. \quad (6.23)$$

That means particularly we define the Fock spaces \mathcal{F}_α and \mathcal{F}_i to be unitarily equivalent. Applying Haag's theorem then yields that both spaces describe free particles because we assume that at least the mass neutrinos can be regarded as asymptotically free particles. Thus, the flavour neutrinos should be asymptotically free. But this would not allow them to oscillate. In other words a particle which can oscillate into another particle cannot be considered as a free particle since it has at least interactions with the particle it oscillates in. Therefore, this kind of definition cannot be the right one and we expect that at some point problems will arise. We will show an example in the next chapter.

As we have seen, the problems for flavour neutrinos come from the implicitly assumed unitarily equivalence of the "Fock spaces" for flavour and mass neutrinos. Hence, one possible way out is to give up this equivalence. That is, we have to give up (6.23) and find a new relation which describes a unitarily inequivalence. This is the approach of BLASONE and VITIELLO, which we will describe in chapter 8.

However, even in this approach we will encounter a problem with the interpretation. If we assume the flavour and mass spaces to be unitarily inequivalent, only one of them can be the Fock space which includes the state vectors with a definite number of particles. Naturally, this would be the mass space, because we can assume the mass neutrinos to be asymptotically free. But then we already know that the flavour states are a condensate of mass neutrinos which means we cannot interpret a flavour state as a definite number of flavour neutrinos. Nevertheless, the states are well-defined as a condensate of infinitely many asymptotically free mass neutrinos. In chapter 8 we will show how NO can be calculated with this state vectors.

Chapter 7

The State Vectors for Flavour Neutrinos

As we have seen in the last chapter, the definition of the flavour neutrino state vectors as a superposition of mass neutrino state vectors in the form

$$|\nu_\alpha\rangle := \sum_i U_{\alpha i}^* |\nu_i\rangle \quad (7.1)$$

leads to problems in the interpretation. In order to show that these problems are not only of academical nature, but also lead to wrong results we will show an example in this chapter. This example was first pointed out by GIUNTI, KIM and LEE in 1992 [GKL92] without resort to the results in the last chapter. The second part of this chapter describes the attempt of GIUNTI et al. to avoid the problems arising from the so-called *weak states* (7.1). In particular, this is the introduction of so-called *weak-process states* as an alternative to the weak states.

7.1 Problems due to Weak States

GIUNTI, KIM and LEE showed that if the state vectors (7.1) are used, one could get non-vanishing transition amplitudes for forbidden processes [GKL92]. We will now shortly summarise their argumentation but make use of a notation which is more convinient for the next section.

If we consider the decay process

$$P_I \longrightarrow P_F + \ell_\beta^+ + \nu_\alpha \quad (7.2)$$

where P_I is the decaying particle and P_F represents all other final particles, while ℓ_β^+ is the corresponding lepton to the neutrino ν_α , the amplitude for this process should only be non-zero for $\beta = \alpha$, because the flavour of the lepton *defines* the flavour of the neutrino. However, let us use the general version and derive the amplitude

$$\mathcal{A}_{P_I \rightarrow P_F + \ell_\beta^+ + \nu_\alpha} = \langle \nu_\alpha, \ell_\beta^+, P_F | S | P_I \rangle, \quad (7.3)$$

where S is the S-matrix. Using the definition of the flavour neutrino state vectors (7.1) the amplitude can be written as

$$\begin{aligned} \mathcal{A}_{P_I \rightarrow P_F + \ell_\beta^+ + \nu_\alpha} &= \sum_i U_{\alpha i} \langle \nu_i, \ell_\beta^+, P_F | S | P_I \rangle \\ &= \sum_i U_{\alpha i} \mathcal{A}_{\beta i}, \end{aligned} \quad (7.4)$$

where we introduced the obvious abbreviation $\mathcal{A}_{\beta i}$.

In order to understand this relation better we can insert the explicit form of the S-matrix

$$S = \mathbf{1} - i \int d^4x H_{\text{CC}}(x), \quad (7.5)$$

where we only considered the first order contribution of the effective low-energy charged-current Hamiltonian

$$\begin{aligned} H_{\text{CC}} &= \frac{G_F}{\sqrt{2}} \sum_{\alpha=e,\mu,\tau} \bar{\nu}_\alpha(x) \gamma^\mu (1 - \gamma^5) \ell_\alpha(x) h_\mu^{P_I \rightarrow P_F}(x) + \text{h.c.} \\ &= \frac{G_F}{\sqrt{2}} \sum_{\alpha=e,\mu,\tau} \sum_i U_{\alpha i}^* \bar{\nu}_i(x) \gamma^\mu (1 - \gamma^5) \ell_\alpha(x) h_\mu^{P_I \rightarrow P_F}(x) + \text{h.c.}, \end{aligned} \quad (7.6)$$

where $h_\mu^{P_I \rightarrow P_F}$ is the current for the transition $P_I \rightarrow P_F$. In the second step we also included the mixing of the field operators (1.18).

Inserting (7.5) and (7.6) into (7.4) then yields

$$\mathcal{A}_{P_I \rightarrow P_F + \ell_\beta^+ + \nu_\alpha} = \sum_i U_{\alpha i} U_{\beta i}^* \mathcal{M}_{\beta i} \quad (7.7)$$

with the matrix element

$$\mathcal{M}_{\beta i} = -i \frac{G_F}{\sqrt{2}} \int d^4x \langle \nu_i, \ell_\beta^+, P_F | \bar{\nu}_i(x) \gamma^\mu (1 - \gamma^5) \ell_\beta(x) h_\mu^{P_I \rightarrow P_F}(x) | P_I \rangle. \quad (7.8)$$

This result is obviously not proportional to $\delta_{\alpha\beta}$, because the matrix element $\mathcal{M}_{\beta i}$ depends on the different masses m_i of the neutrinos and can therefore not be taken out of the sum. Thus, the amplitude for forbidden transitions can be non-zero.

Another important point is the probability for the process (7.2) which is the absolute square of (7.7)

$$\mathcal{P}_{P_I \rightarrow P_F + \ell_\beta^+ + \nu_\alpha} = \left| \sum_i U_{\alpha i} U_{\beta i}^* \mathcal{M}_{\beta i} \right|^2. \quad (7.9)$$

Since not the flavour but the mass neutrinos are the physical particles which propagate in space-time with definite kinematical properties, the probability should be the incoherent sum of the different matrix elements for the different decay channels weighted by the mixing matrix elements $|U_{\alpha i}|^2$ [Shr80, Shr81a, Shr81b]. Hence, the expected probability is

$$\mathcal{P}_{P_I \rightarrow P_F + \ell_\beta^+ + \nu_\alpha} = \sum_i |U_{\alpha i}|^2 |\mathcal{M}_{\beta i}|^2, \quad (7.10)$$

which is clearly different from (7.9).

However, if the process (7.2) is not sensitive to the different neutrino masses, which is for example the case for relativistic neutrinos, the matrix element can be approximated as

$$\mathcal{M}_{\beta i} \simeq \mathcal{M}_\beta. \quad (7.11)$$

This allows us to take the matrix elements out of the sums in (7.7), (7.9) and (7.10). Additionally, we can use the unitarity of the PMNS-matrix and get for (7.7)

$$\mathcal{A}_{P_I \rightarrow P_F + \ell_\beta^+ + \nu_\alpha} \simeq \mathcal{M}_\beta \delta_{\beta\alpha} \quad (7.12)$$

and (7.9)

$$\mathcal{P}_{P_I \rightarrow P_F + \ell_\beta^+ + \nu_\alpha} \simeq |\mathcal{M}_{\beta\alpha}|^2, \quad (7.13)$$

which is identical to (7.10) in this approximation.

This means, the weak states (7.1) can, strictly spoken, only be used if the production process is not sensitive to the different neutrino masses. In particular, this is not the case if some of the neutrinos are non-relativistic.

7.2 Weak-Process States

In order to avoid the problems that arise from the weak states (7.1), Giunti et al. [GKL92, Giu02a, Giu04a, Giu04b, Giu06] introduced so-called *weak-process states*. These states are defined in a way that they contain the information on the production process

$$P_I \longrightarrow P_F + \ell_\alpha^+ + \nu_\alpha. \quad (7.14)$$

The notation here is the same as in (7.2), but this time we assume $\beta = \alpha$. We should also note that the whole following computation can easily be adopted to the second possible production process, which is the scattering of a lepton and some initial particle P_I

$$\ell_\alpha^- + P_I \longrightarrow P_F + \nu_\alpha.$$

In terms of the state vectors, which describe the involved particles in the production process, we can get the general final state $|f\rangle$ by applying the S-matrix on the initial state. Hence

$$|f\rangle = S|P_I\rangle, \quad (7.15)$$

where the final state contains all possible decay channels for the particle P_I and P_I itself:

$$|f\rangle = |\nu_\alpha, \ell_\alpha^+, P_F\rangle + \dots = \sum_i \mathcal{A}_{\alpha i}^P |\nu_i, \ell_\alpha^+, P_F\rangle + \dots \quad (7.16)$$

Here, we singled out the important decay channel (7.14) and additionally took into account the coherent superposition of the mass neutrino state vectors. Since all different decay channels are represented by orthogonal state vectors and the vectors $|\nu_i, \ell_\alpha^+, P_F\rangle$ are orthogonal and normalised, we find the coefficients $\mathcal{A}_{\alpha i}^P$ to be the amplitudes

$$\mathcal{A}_{\alpha i}^P = \langle \nu_i, \ell_\alpha^+, P_F | S | P_I \rangle. \quad (7.17)$$

From (7.16) we can now identify the state vector for the flavour neutrino:

$$|\nu_\alpha^P\rangle = \frac{1}{\sqrt{\sum_i |\mathcal{A}_{\alpha i}^P|^2}} \sum_i \mathcal{A}_{\alpha i}^P |\nu_i\rangle, \quad (7.18)$$

where the first factor is due to the normalisation. Thus, in conclusion, the state vector which describes flavour neutrinos is a coherent superposition of the mass neutrino state vectors weighted by the amplitude for the production of the respective mass neutrino. These states are also called *production flavour neutrino states* [Giu06].

For the computation of neutrino oscillations we also need a description for the detected neutrino. This can be done in an analogous way as for the production. Let us consider the detection process

$$\nu_\alpha + D_I \longrightarrow D_F + \ell_\alpha^- . \quad (7.19)$$

Again the computation can easily be rewritten to describe the second possible detection process

$$\nu_\alpha + \ell_\alpha^+ + D_I \longrightarrow D_F .$$

Analogous to (7.16) the final state for the scattering $\nu_\alpha + D_I$ is given by

$$|f\rangle = S|\nu_\alpha, D_I\rangle = |D_F, \ell_\alpha^-\rangle + \dots \quad (7.20)$$

Using the unitarity of the S-matrix and decomposing the initial state vector in a coherent superposition of mass neutrino state vectors yields

$$\sum_i \mathcal{A}_{\alpha i}^D |\nu_i, D_I\rangle = S^\dagger |D_F, \ell_\alpha^-\rangle + \dots , \quad (7.21)$$

where $\mathcal{A}_{\alpha i}^D$ is given by

$$\mathcal{A}_{\alpha i}^D = \langle \nu_i, D_I | S^\dagger | D_F, \ell_\alpha^- \rangle , \quad (7.22)$$

which is the complex conjugate of the amplitude for the detection of a ν_i in the process (7.19).

Now we are able to single out the state vector for a detected neutrino which shall be normalised:

$$|\nu_\alpha^D\rangle = \frac{1}{\sqrt{\sum_i |\mathcal{A}_{\alpha i}^D|^2}} \sum_i \mathcal{A}_{\alpha i}^D |\nu_i\rangle \quad (7.23)$$

These state vectors are also called *detection flavour neutrino states*.

Although (7.18) and (7.23) have the same structure, they describe different things. On the one hand the production flavour neutrino state describes a neutrino produced in a charged current process which then propagates through space and time. On the other hand the detection flavour neutrino state does not describe a propagating neutrino but rather the component of a propagating neutrino which can generate a charged lepton through the charged current process (7.19) [Giu06].

In order to see the connection between the weak-process states and the weak states we use the explicit form of the S-matrix (7.5) and the effective low-energy charged-current Hamiltonian (7.6). This allows us to write the production and detection amplitudes $\mathcal{A}_{\alpha i}^P$ and $\mathcal{A}_{\alpha i}^D$, respectively, as

$$\mathcal{A}_{\alpha i}^P = U_{\alpha i}^* \mathcal{M}_{\alpha i}^P \quad \text{and} \quad \mathcal{A}_{\alpha i}^D = U_{\alpha i}^* \mathcal{M}_{\alpha i}^D , \quad (7.24)$$

with the matrix elements

$$\mathcal{M}_{\alpha i}^P = -i \frac{G_F}{\sqrt{2}} \int d^4x \langle \nu_i, \ell_\alpha^+, P_F | \bar{\nu}_i(x) \gamma^\mu (1 - \gamma^5) \ell_\alpha(x) h_\mu^{P_I \rightarrow P_F}(x) | P_I \rangle , \quad (7.25a)$$

and

$$\mathcal{M}_{\alpha i}^D = -i \frac{G_F}{\sqrt{2}} \int d^4x \langle \nu_i, D_I | \bar{\nu}_i(x) \gamma^\mu (1 - \gamma^5) \ell_\alpha(x) h_\mu^{D_I \rightarrow D_F}(x) | D_F, \ell_\alpha^- \rangle . \quad (7.25b)$$

The currents $h_\mu^{P_I \rightarrow P_F}(x)$ and $h_\mu^{D_I \rightarrow D_F}(x)$ describe the transitions $P_I \rightarrow P_F$ and $D_I \rightarrow D_F$, respectively.

Inserting (7.24) into the states (7.18) and (7.23) yields

$$|\nu_\alpha^P\rangle = \sum_i \frac{\mathcal{M}_{\alpha i}^P}{\sqrt{\sum_j |U_{\alpha j}|^2 |\mathcal{M}_{\alpha j}^P|^2}} U_{\alpha i}^* |\nu_i\rangle, \quad (7.26a)$$

and

$$|\nu_\alpha^D\rangle = \sum_i \frac{\mathcal{M}_{\alpha i}^D}{\sqrt{\sum_j |U_{\alpha j}|^2 |\mathcal{M}_{\alpha j}^D|^2}} U_{\alpha i}^* |\nu_i\rangle. \quad (7.26b)$$

These states have the same structure as the weak states (7.1) except for a factor that depends on the production or the detection process.

If the experiments are not sensitive to the different neutrino masses, the matrix elements can be approximated to be independent of i :

$$\mathcal{M}_{\alpha i}^P \simeq \mathcal{M}_\alpha^P \quad \text{and} \quad \mathcal{M}_{\alpha i}^D \simeq \mathcal{M}_\alpha^D. \quad (7.27)$$

In this case the vectors (7.26) reduce to the weak states up to an irrelevant phase. Thus, the weak states are approximations of the production and detection flavour neutrino states if the experiment is not sensitive to the different neutrino masses.

In the last section we said that the expected decay rate for the process (7.14) is an incoherent sum over the different decay channels which correspond to the different mass neutrinos. On the other side the computation of neutrino oscillations uses explicitly the coherent superposition of the mass neutrino state vectors. Thus, we should show that these two arguments can consistently be described with the weak-process states.

Decay Rates The amplitude for the process $P_I \rightarrow P_F + \ell_\alpha^+ + \nu_\alpha$ is

$$\mathcal{A}_{P_I \rightarrow P_F + \ell_\alpha^+ + \nu_\alpha} = \langle \nu_\alpha, \ell_\alpha^+, P_F | S | P_I \rangle. \quad (7.28)$$

This can be rewritten by using the production flavour neutrino state (7.18):

$$\mathcal{A}_{P_I \rightarrow P_F + \ell_\alpha^+ + \nu_\alpha} = \frac{1}{\sqrt{\sum_j |\mathcal{A}_{\alpha j}^P|^2}} \sum_i \mathcal{A}_{\alpha i}^{P*} \langle \nu_i, \ell_\alpha^+, P_F | S | P_I \rangle = \sqrt{\sum_i |\mathcal{A}_{\alpha i}^P|^2}. \quad (7.29)$$

Therefore, the probability is just the incoherent sum

$$\mathcal{P}_{P_I \rightarrow P_F + \ell_\alpha^+ + \nu_\alpha} = \sum_i |\mathcal{A}_{\alpha i}^P|^2. \quad (7.30)$$

If we additionally use the decomposition of the production amplitudes into the mixing matrix and the matrix element (7.24), the probability becomes of exactly the form which we expected in (7.10):

$$\mathcal{P}_{P_I \rightarrow P_F + \ell_\alpha^+ + \nu_\alpha} = \sum_i |U_{\alpha i}|^2 |\mathcal{M}_{\alpha i}^P|^2. \quad (7.31)$$

Neutrino Oscillations In order to describe neutrino oscillations with the above defined state vectors, we first rearrange them in a way that they fit into our notation of chapter 2, that is

$$|N_\alpha^P(t)\rangle = \sum_i \frac{\mathcal{M}_{\alpha i}^P}{\sqrt{\sum_j |U_{\alpha j}|^2 |\mathcal{M}_{\alpha j}^P|^2}} U_{\alpha i}^* \int d^3x e^{i\mathbf{p}_i \cdot (\mathbf{x} - \mathbf{x}_P) - iE_i(t-t_P)} |\mathbf{x}\rangle \otimes |\nu_i\rangle, \quad (7.32a)$$

and

$$|N_\beta^D\rangle = \sum_i \frac{\mathcal{M}_{\beta i}^D}{\sqrt{\sum_j |U_{\beta j}|^2 |\mathcal{M}_{\beta j}^D|^2}} U_{\beta i}^* \delta(t - t_D) |\mathbf{x}_D\rangle \otimes |\nu_i\rangle. \quad (7.32b)$$

Since the neutrinos are seen as plane waves in the treatment so far, we used the assumption of a point-like detection process. Now we can insert the modified vectors in the general oscillation amplitude (2.12) and get as result

$$\mathcal{A}(\alpha \rightarrow \beta; \mathbf{L}, T) = \sum_i \frac{\mathcal{M}_{\alpha i}^P}{\sqrt{\sum_j |U_{\alpha j}|^2 |\mathcal{M}_{\alpha j}^P|^2}} \frac{\mathcal{M}_{\beta i}^{D*}}{\sqrt{\sum_j |U_{\beta j}|^2 |\mathcal{M}_{\beta j}^D|^2}} U_{\alpha i}^* U_{\beta i} e^{-iE_i T + i\mathbf{p}_i \cdot \mathbf{L}} \quad (7.33)$$

which is exactly the same amplitude as in the standard case for plane waves, except for the factors in front which depend on the production and detection process. But, however, neutrino oscillation experiments are not sensitive to the different neutrino masses and we can use the approximation (7.27). In this case the factors in front become unity and the amplitude reduces to the standard one for which the probability was already given in (3.4).

In conclusion, the *weak-process states* give the same results as the *weak-states* in the case of experiments not sensitive to different neutrino masses. Moreover, their usage gives the expected result for the decay rate, which is not the case for the weak states. But however, they also describe a unitary equivalence and thus have the same principle problems as the weak states.

The Approach of Blasone and Vitiello

In this chapter we come back to the implications of Haag's theorem in chapter 6.3 and describe an approach where the mixing of the neutrinos is generated by an improper unitary operator. The main authors who pushed this approach forward are BLASONE and VITIELLO and their collaborators (see [BV95, ABIV95, BHV99, BV99, BJV01, BCV01, BCV02, BPT03, BMP05, BCTV05, BCJV06] and the comprehensive review [Cap04]). In the following we will abbreviate them by BV. However, there are also other authors who work in this field (e. g. [FHY99, FHY01, JM02, Giu05, LL06]). Note, that since we are not interested in boson mixing in this thesis, we only cite the papers concerning neutrino mixing and oscillations. Nevertheless, the whole treatment has been applied to bosons (see for example the review [Cap04]).

In the following sections we will summarise the BV-approach, but in order to avoid detours in the notation we will sometimes change or generalise the notation developed by BV.

8.1 The General Setup

Here we will only describe the simplest and most discussed version of the BV-approach. That is, the case of two flavour neutrinos which are superpositions of two mass neutrinos (for definiteness we call the flavours e and μ and the masses 1 and 2). Furthermore, we assume the neutrinos to be Dirac neutrinos. This restriction has the advantage, that we do not need to consider the influence of \mathcal{CP} -phases, which makes the approach considerably easier and allows us to focus on the important points.

However, the BV-approach can be used for three flavour neutrinos as well as for Majorana neutrinos (see e. g. [BCV02, Cap04]).

The Lagrangians The starting point for the following computations will be the Lagrangian

$$\mathcal{L}_{1,2} = (\bar{\nu}_1(x), \bar{\nu}_2(x)) \left[i\not{\partial} - \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \right] \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix}, \quad (8.1)$$

which describes two free fields with masses m_1 and m_2 . The field operators $\nu_1(x)$ and $\nu_2(x)$ shall satisfy the canonical equal-time anticommutation relations

$$\{\nu_i(x), \nu_j(y)\}_{x^0=y^0} = \{\nu_i^\dagger(x), \nu_j^\dagger(y)\}_{x^0=y^0} = 0, \quad (8.2a)$$

$$\{\nu_i(x), \nu_j^\dagger(y)\}_{x^0=y^0} = \delta_{ij} \delta(\mathbf{x} - \mathbf{y}). \quad (8.2b)$$

We now define the unitary transformed fields $\nu_e(x)$ and $\nu_\mu(x)$, which are called flavour eigenfields, by

$$\begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix} := U \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix}, \quad (8.3)$$

with U being the PMNS matrix for two flavours (cf. (1.20))

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (8.4)$$

These fields clearly satisfy the same canonical equal-time anticommutation relations as the mass fields

$$\{\nu_\alpha(x), \nu_\beta(y)\}_{x^0=y^0} = \{\nu_\alpha^\dagger(x), \nu_\beta^\dagger(y)\}_{x^0=y^0} = 0, \quad (8.5a)$$

$$\{\nu_\alpha(x), \nu_\beta^\dagger(y)\}_{x^0=y^0} = \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{y}). \quad (8.5b)$$

In order to find the Lagrangian for the flavour fields we introduce the mass matrix

$$\mathcal{M} := U \mathcal{M}_d U^T = \begin{pmatrix} m_{ee} & m_{e\mu} \\ m_{e\mu} & m_{\mu\mu} \end{pmatrix}, \quad (8.6)$$

where \mathcal{M}_d is the diagonal mass matrix for the mass fields. The different entries of the matrices are connected by

$$m_{ee} = m_1 \cos^2 \theta + m_2 \sin^2 \theta, \quad (8.7a)$$

$$m_{\mu\mu} = m_1 \sin^2 \theta + m_2 \cos^2 \theta, \quad (8.7b)$$

$$m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta. \quad (8.7c)$$

We can now finally write down the Lagrangian for the flavour fields as

$$\mathcal{L}_{e,\mu} = (\bar{\nu}_e(x), \bar{\nu}_\mu(x)) (i\cancel{\partial} - \mathcal{M}) \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix}. \quad (8.8)$$

Note that this Lagrangian clearly shows that the flavour fields are not free, because the mass matrix is non-diagonal they have interactions among each other.

The Creation and Annihilation Operators Since the Lagrangian (8.1) describes free neutrinos with definite masses, we can simply expand the field operators $\nu_i(x)$ in the usual way (see e. g. [PS95]) in terms of creation and annihilation operators:

$$\begin{aligned} \nu_i(x) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{p}, r} \left(a_{\mathbf{p}, i}^r u_{\mathbf{p}, i}^r e^{-iE_i t + i\mathbf{p} \cdot \mathbf{x}} + b_{\mathbf{p}, i}^{r\dagger} v_{\mathbf{p}, i}^r e^{iE_i t - i\mathbf{p} \cdot \mathbf{x}} \right) \\ &= \frac{1}{\sqrt{V}} \sum_{\mathbf{p}, r} \left(a_{\mathbf{p}, i}^r(t) u_{\mathbf{p}, i}^r + b_{-\mathbf{p}, i}^{r\dagger}(t) v_{-\mathbf{p}, i}^r \right) e^{i\mathbf{p} \cdot \mathbf{x}}, \end{aligned} \quad (8.9)$$

where we included the time dependence into the operators

$$a_{\mathbf{p}, i}^r(t) = a_{\mathbf{p}, i}^r e^{-iE_i t} \quad \text{and} \quad b_{-\mathbf{p}, i}^{r\dagger}(t) = b_{-\mathbf{p}, i}^{r\dagger} e^{iE_i t}. \quad (8.10)$$

The indices on the operators and spinors shall have the following meaning: the lower indices, here \mathbf{p} and i , denote the momentum and the neutrino species, while the upper index, here r denotes the helicity.

As in chapter 6 we will start with box normalisation in a finite space volume V . The limit $V \rightarrow \infty$ will then be taken in the end.

The spinors $u_{\mathbf{p},i}^r$ and $v_{-\mathbf{p},i}^r$ are defined as solutions of the equations

$$(\not{p} - m_i)u_{\mathbf{p},i}^r = 0 \quad \text{and} \quad (\not{p} + m_i)v_{-\mathbf{p},i}^r = 0 \quad (8.11)$$

and shall be normalised such that they obey the following orthogonality and completeness relations:

$$u_{\mathbf{p},i}^{r\dagger} u_{\mathbf{p},i}^s = v_{-\mathbf{p},i}^{r\dagger} v_{-\mathbf{p},i}^s = \delta_{rs}, \quad (8.12a)$$

$$u_{\mathbf{p},i}^{r\dagger} v_{-\mathbf{p},i}^s = v_{-\mathbf{p},i}^{r\dagger} u_{\mathbf{p},i}^s = 0, \quad (8.12b)$$

$$\sum_r \left(u_{\mathbf{p},i}^{r\dagger} u_{\mathbf{p},i}^r + v_{-\mathbf{p},i}^{r\dagger} v_{-\mathbf{p},i}^r \right) = \mathbf{1}. \quad (8.12c)$$

The explicit form for the spinors is the given by

$$u_{\mathbf{p},i}^r = \sqrt{\frac{E_i + m_i}{2E_i}} \begin{pmatrix} \xi^r \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_i + m_i} \xi^r \end{pmatrix} \quad \text{and} \quad v_{-\mathbf{p},i}^r = \sqrt{\frac{E_i + m_i}{2E_i}} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_i + m_i} \xi^r \\ \xi^r \end{pmatrix} \quad (8.13)$$

with

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (8.14)$$

The vector $\boldsymbol{\sigma}$ is the vector composed of the Pauli matrices σ^i .

Furthermore, the creation and annihilation operators satisfy the anticommutation relations for

$$\{a_{\mathbf{p},i}^r, a_{\mathbf{k},j}^{s\dagger}\} = \{b_{\mathbf{p},i}^r, b_{\mathbf{k},j}^{s\dagger}\} = \delta_{ij} \delta_{\mathbf{p}\mathbf{k}} \delta_{rs} \quad \text{and all other zero.} \quad (8.15)$$

Now we come to the field operators for the flavour neutrinos. Since they do not describe free particles—they have interactions among each other due to the non-diagonal mass matrix in (8.8)—the expansion in terms of creation and annihilation operators can only be made as

$$\nu_\alpha(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p},r} \left(a_{\mathbf{p},\alpha}^r(t) u_{\mathbf{p},\alpha}^r + b_{-\mathbf{p},\alpha}^{r\dagger}(t) v_{-\mathbf{p},\alpha}^r \right) e^{i\mathbf{p} \cdot \mathbf{x}}, \quad (8.16)$$

where the time-dependence of $a_{\mathbf{p},\alpha}^r(t)$ and $b_{-\mathbf{p},\alpha}^{r\dagger}(t)$ is a priori unknown, therefore we cannot simply postulate the anticommutation relations. However, they will follow from our computation in section 8.2. Note that we will call the $a_{\mathbf{p},\alpha}^r(t)$ and $b_{-\mathbf{p},\alpha}^{r\dagger}(t)$ operators for flavour fields creation and annihilation operators, although we already know from chapter chapter 6 that they can not be interpreted in a particle picture.

The choice of the spinors $u_{\mathbf{p},\alpha}^r$ and $v_{-\mathbf{p},\alpha}^r$ is a bit more complicated and involves a closer look on quantum field theoretical details. The original choice made by BV is

$$u_{\mathbf{p},\alpha}^{r(BV)} := u_{\mathbf{p},i}^r \quad \text{and} \quad v_{\mathbf{p},\alpha}^{r(BV)} := v_{\mathbf{p},i}^r \quad (8.17)$$

for the index pairs $(\alpha, i) = (e, 1)$ and $(\mu, 2)$. As it was pointed out in [FHY99, FHY01], this is not the most general case and in fact they showed that it is possible to use

$$v_{\mathbf{p},\alpha}^{r(FHY)} := u_{\mathbf{p},i}^r(\mu_i) \quad \text{and} \quad v_{\mathbf{p},\alpha}^{r(FHY)} := v_{\mathbf{p},i}^r(\mu_i) \quad (8.18)$$

which are defined by the Dirac equations

$$(\not{p} - \mu_i)u_{\mathbf{p},i}^r(\mu_i) = 0 \quad \text{and} \quad (\not{p} + \mu_i)v_{\mathbf{p},i}^r(\mu_i) = 0. \quad (8.19)$$

The μ_i are considered as arbitrary mass parameters. Moreover, in [Giu05] it was shown that these parameters can appear in measurable quantities, which lead GIUNTI to the conclusion that the BV-treatment is unphysical.

However, BV showed in [BV99, BCV01] that the parameters do not appear in the important quantities, namely the charges and the oscillation formula and furthermore they argued in [BCTV05] that the possibility of different mass parameters is intrinsic to the structure of quantum field theory. Here we will shortly summarise their argumentation in order to justify our choice of the spinors.

The main point is that every field operator—at least every Dirac operator—can be expanded in different spinor basis which correspond to different mass parameters. For example if we consider the field operator $\nu_1(x)$ for the mass neutrino ν_1 , we can expand it as

$$\nu_1(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p},r} \left(a_{\mathbf{p},1}^r(t) u_{\mathbf{p},1}^r + b_{-\mathbf{p},1}^{r\dagger}(t) v_{-\mathbf{p},1}^r \right) e^{i\mathbf{p}\cdot\mathbf{x}} \quad (8.20)$$

and as

$$\nu_1(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p},r} \left(a_{\mathbf{p},1}^r(\mu_1, t) u_{\mathbf{p},1}^r(\mu_1) + b_{-\mathbf{p},1}^{r\dagger}(\mu_1, t) v_{-\mathbf{p},1}^r(\mu_1) \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (8.21)$$

where the different creation and annihilation operators are connected via

$$a_{\mathbf{p},1}^r(\mu_1, t) := \sum_s \left(u_{\mathbf{p},1}^{r\dagger}(\mu_1) u_{\mathbf{p},1}^s a_{\mathbf{p},1}^s(t) + u_{\mathbf{p},1}^{r\dagger}(\mu_1) v_{-\mathbf{p},1}^s b_{-\mathbf{p},1}^{s\dagger}(t) \right), \quad (8.22a)$$

$$b_{-\mathbf{p},1}^{r\dagger}(\mu_1, t) := \sum_s \left(v_{-\mathbf{p},1}^{r\dagger}(\mu_1) u_{\mathbf{p},1}^s a_{\mathbf{p},1}^s(t) + v_{-\mathbf{p},1}^{r\dagger}(\mu_1) v_{-\mathbf{p},1}^s b_{-\mathbf{p},1}^{s\dagger}(t) \right). \quad (8.22b)$$

This is just the *Bogoliubov transformation* which appeared in chapter 6. The equivalence of these two expansion can simply be proven by inserting (8.22) into (8.20) and using the completeness relation (8.12).

Therefore, it is possible to expand a given field operator in different basis for different mass parameters and the only reasonable choice for this parameters is govern by the physical mass of the particle. In our case the only physical relevant masses are m_1 and m_2 , the masses of the mass neutrinos, which in turn corresponds to the choice of BV for the spinors.

In the following we will use the same assumption for the spinors.

The Group Structure of the Lagrangians For later reference we will also have a look on the group structure of the Lagrangians (8.8) and (8.1). Both Lagrangians are invariant under a global U(1) transformation, which changes the field operators by adding a phase

$$\begin{pmatrix} \nu_1'(x) \\ \nu_2'(x) \end{pmatrix} := e^{i\alpha_m} \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu_e'(x) \\ \nu_\mu'(x) \end{pmatrix} := e^{i\alpha_f} \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix}, \quad (8.23)$$

where α_m and α_f are real constants. If we apply Noether's theorem to this symmetry, we obtain the conserved charges

$$\begin{aligned} Q_m &= \int d^3x (\bar{\nu}_1(x), \bar{\nu}_2(x)) \gamma^0 \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} \\ &= \int d^3x \left(\nu_1^\dagger(x) \nu_1(x) + \nu_2^\dagger(x) \nu_2(x) \right), \end{aligned} \quad (8.24a)$$

$$Q_f = \int d^3x \left(\nu_e^\dagger(x) \nu_e(x) + \nu_\mu^\dagger(x) \nu_\mu(x) \right), \quad (8.24b)$$

which can be shown to be equal, $Q_m = Q_f = Q$, by inserting the mixing for the field operators (8.3).

A second interesting transformation is the SU(2) mixing

$$\begin{pmatrix} \nu_1'(x) \\ \nu_2'(x) \end{pmatrix} := e^{i\alpha_m^j \tau^j} \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu_e'(x) \\ \nu_\mu'(x) \end{pmatrix} := e^{i\alpha_f^j \tau^j} \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix}, \quad (8.25)$$

with $\tau^j := \sigma^j/2$ and σ^j being the usual Pauli matrices, while the α_m^j and α_f^j are real constants. Note, that we used the sum convention for simplicity.

The Lagrangians (8.8) and (8.1) are, in general, not invariant under this transformation, which can be seen by calculating the change of the Lagrangians induced by (8.25)

$$\delta \mathcal{L}_{1,2} = i\alpha_m^j (\bar{\nu}_1(x), \bar{\nu}_2(x)) [\tau^j, \mathcal{M}_d] \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix}, \quad (8.26a)$$

$$\delta \mathcal{L}_{e,\mu} = i\alpha_f^j (\bar{\nu}_e(x), \bar{\nu}_\mu(x)) [\tau^j, \mathcal{M}] \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix}. \quad (8.26b)$$

Only the commutator $[\tau^3, \mathcal{M}_d]$ vanishes, which implies a conserved charge for this case. Nevertheless, we can use Noether's theorem to compute all charges, which will be in general time-dependent

$$Q_{m,1}(t) = \frac{1}{2} \int d^3x \left(\nu_1^\dagger(x) \nu_2(x) + \nu_2^\dagger(x) \nu_1(x) \right), \quad (8.27a)$$

$$Q_{m,2}(t) = -\frac{i}{2} \int d^3x \left(\nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right), \quad (8.27b)$$

$$Q_{m,3} = \frac{1}{2} \int d^3x \left(\nu_1^\dagger(x) \nu_1(x) - \nu_2^\dagger(x) \nu_2(x) \right), \quad (8.27c)$$

$$Q_{f,1}(t) = \frac{1}{2} \int d^3x \left(\nu_e^\dagger(x) \nu_\mu(x) + \nu_\mu^\dagger(x) \nu_e(x) \right), \quad (8.27d)$$

$$Q_{f,2}(t) = -\frac{i}{2} \int d^3x \left(\nu_e^\dagger(x) \nu_\mu(x) - \nu_\mu^\dagger(x) \nu_e(x) \right), \quad (8.27e)$$

$$Q_{f,3}(t) = \frac{1}{2} \int d^3x \left(\nu_e^\dagger(x) \nu_e(x) - \nu_\mu^\dagger(x) \nu_\mu(x) \right). \quad (8.27f)$$

The situation here is somehow similar to the situation in the GSW-model (see chapter 1), where the U(1) symmetry would be the weak-hypercharge symmetry and the SU(2) the weak-isospin symmetry. Therefore, we will define in analogy to the electric charge in the GSW-model

the following charges

$$Q_1 := \frac{1}{2}Q + Q_{m,3} = \int d^3 \nu_1^\dagger(x) \nu_1(x), \quad (8.28a)$$

$$Q_2 := \frac{1}{2}Q - Q_{m,3} = \int d^3 \nu_2^\dagger(x) \nu_2(x), \quad (8.28b)$$

$$Q_e(t) := \frac{1}{2}Q + Q_{f,3}(t) = \int d^3 \nu_e^\dagger(x) \nu_e(x), \quad (8.28c)$$

$$Q_\mu(t) := \frac{1}{2}Q - Q_{f,3}(t) = \int d^3 \nu_\mu^\dagger(x) \nu_\mu(x). \quad (8.28d)$$

The charge Q can be interpreted as the total lepton number of a system, while the charges Q_1 , Q_2 , $Q_e(t)$ and $Q_\mu(t)$ are the lepton number for each neutrino species separately. The time-dependence of the flavour charges $Q_e(t)$ and $Q_\mu(t)$ already show that something like oscillations must occur, because the sum $Q_e(t) + Q_\mu(t) = Q$ is conserved and thus a decreasing charge for the electron neutrino can only result in an increasing charge for the muon neutrino.

8.2 An Improper Generator for the Mixing

The main point in the work of BV is the definition of an operator which generates the mixing (8.3) between the field operators for the mass and the flavour neutrinos, respectively. This operator is usually called $G(\theta, t)$ and corresponds to the operator U in chapter 6. Its explicit form is given by

$$G(\theta, t) := \exp \left[\theta \int d^3x \left(\nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right]. \quad (8.29)$$

As can be seen from (8.27) the argument of the exponential is proportional to the second SU(2) charge for the mass fields:

$$G(\theta, t) = \exp \left[-2i\theta Q_{m,2}(t) \right]. \quad (8.30)$$

The mixing between the mass and flavour field operators in terms of this operator can then be written as

$$\begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix} = G^{-1}(\theta, t) \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} G(\theta, t). \quad (8.31)$$

In order to see that this really generates the mixing (8.3), we use the relation

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots,$$

where the commutators are given by

$$\begin{aligned} [2i\theta Q_{m,2}(t), \nu_1(x)] &= \theta \nu_2(x), \\ [2i\theta Q_{m,2}(t), \nu_2(x)] &= -\theta \nu_1(x). \end{aligned}$$

Thus, we get by recombining the infinite sum

$$\begin{aligned} \nu_e(x) &= \cos \theta \nu_1(x) + \sin \theta \nu_2(x), \\ \nu_\mu(x) &= -\sin \theta \nu_1(x) + \cos \theta \nu_2(x), \end{aligned}$$

which is just the mixing (8.3).

After having defined the mixing of the field operators in terms of $G(\theta, t)$, we can go a step further and try to find the relation between the creation and annihilation operators of the mass and flavour neutrinos. This relation is called the *dynamical map* [UMT82]. In order to find this map we insert the expansions for the field operators (8.9) and (8.16) into the mixing relation (8.31) and get

$$\begin{pmatrix} a_{\mathbf{p},e}^r(t) \\ a_{\mathbf{p},\mu}^r(t) \\ b_{-\mathbf{p},e}^{r\dagger}(t) \\ b_{-\mathbf{p},\mu}^{r\dagger}(t) \end{pmatrix} = G^{-1}(\theta, t) \begin{pmatrix} a_{\mathbf{p},1}^r(t) \\ a_{\mathbf{p},2}^r(t) \\ b_{-\mathbf{p},1}^{r\dagger}(t) \\ b_{-\mathbf{p},2}^{r\dagger}(t) \end{pmatrix} G(\theta, t). \quad (8.32)$$

For the calculation of this transformation we use the same relation as above. In order to calculate the commutators we have to express the charge in terms of the creation and annihilation operators. We get this expression by inserting the expansion (8.9) into the charge (8.27), which yields

$$\begin{aligned} 2i\theta Q_{m,2}(t) = \theta \sum_{\mathbf{p}} \sum_{r,s} & \left(a_{\mathbf{p},1}^{r\dagger}(t) a_{\mathbf{p},2}^s(t) u_{\mathbf{p},1}^{r\dagger} u_{\mathbf{p},2}^s + b_{-\mathbf{p},1}^r(t) a_{\mathbf{p},2}^s(t) v_{-\mathbf{p},1}^{r\dagger} u_{\mathbf{p},2}^s \right. \\ & a_{\mathbf{p},1}^{r\dagger}(t) b_{-\mathbf{p},2}^{s\dagger}(t) u_{\mathbf{p},1}^{r\dagger} v_{-\mathbf{p},2}^s + b_{-\mathbf{p},1}^r(t) b_{-\mathbf{p},2}^{s\dagger}(t) v_{-\mathbf{p},1}^{r\dagger} v_{-\mathbf{p},2}^s \\ & a_{\mathbf{p},2}^{r\dagger}(t) a_{\mathbf{p},1}^s(t) u_{\mathbf{p},2}^{r\dagger} u_{\mathbf{p},1}^s + b_{-\mathbf{p},2}^r(t) a_{\mathbf{p},1}^s(t) v_{-\mathbf{p},2}^{r\dagger} u_{\mathbf{p},1}^s \\ & \left. a_{\mathbf{p},2}^{r\dagger}(t) b_{-\mathbf{p},1}^{s\dagger}(t) u_{\mathbf{p},2}^{r\dagger} v_{-\mathbf{p},1}^s + b_{-\mathbf{p},2}^r(t) b_{-\mathbf{p},1}^{s\dagger}(t) v_{-\mathbf{p},2}^{r\dagger} v_{-\mathbf{p},1}^s \right). \end{aligned} \quad (8.33)$$

The commutators are then given by

$$[2i\theta Q_{m,2}(t), a_{\mathbf{p},1}^r(t)] = -\theta \sum_s \left(u_{\mathbf{p},1}^{r\dagger} u_{\mathbf{p},2}^s a_{\mathbf{p},2}^s(t) + u_{\mathbf{p},1}^{r\dagger} v_{-\mathbf{p},2}^s b_{-\mathbf{p},2}^{s\dagger}(t) \right), \quad (8.34a)$$

$$[2i\theta Q_{m,2}(t), a_{\mathbf{p},2}^r(t)] = \theta \sum_s \left(u_{\mathbf{p},2}^{r\dagger} u_{\mathbf{p},1}^s a_{\mathbf{p},1}^s(t) + u_{\mathbf{p},2}^{r\dagger} v_{-\mathbf{p},1}^s b_{-\mathbf{p},1}^{s\dagger}(t) \right), \quad (8.34b)$$

$$[2i\theta Q_{m,2}(t), b_{-\mathbf{p},1}^r(t)] = -\theta \sum_s \left(v_{-\mathbf{p},1}^{r\dagger} u_{\mathbf{p},2}^s a_{\mathbf{p},2}^s(t) + v_{-\mathbf{p},1}^{r\dagger} v_{-\mathbf{p},2}^s b_{-\mathbf{p},2}^{s\dagger}(t) \right), \quad (8.34c)$$

$$[2i\theta Q_{m,2}(t), b_{-\mathbf{p},2}^r(t)] = \theta \sum_s \left(v_{-\mathbf{p},2}^{r\dagger} u_{\mathbf{p},1}^s a_{\mathbf{p},1}^s(t) + v_{-\mathbf{p},2}^{r\dagger} v_{-\mathbf{p},1}^s b_{-\mathbf{p},1}^{s\dagger}(t) \right). \quad (8.34d)$$

If we recombine the terms, we get as a final result for the dynamical map

$$\begin{pmatrix} a_{\mathbf{p},e}^r(t) \\ a_{\mathbf{p},\mu}^r(t) \\ b_{-\mathbf{p},e}^{r\dagger}(t) \\ b_{-\mathbf{p},\mu}^{r\dagger}(t) \end{pmatrix} = \sum_s \begin{pmatrix} \cos \theta \delta_{rs} & \sin \theta u_{\mathbf{p},1}^{r\dagger} u_{\mathbf{p},2}^s & 0 & \sin \theta u_{\mathbf{p},1}^{r\dagger} v_{-\mathbf{p},2}^s \\ -\sin \theta u_{\mathbf{p},2}^{r\dagger} u_{\mathbf{p},1}^s & \cos \theta \delta_{rs} & -\sin \theta u_{\mathbf{p},2}^{r\dagger} v_{-\mathbf{p},1}^s & 0 \\ 0 & \sin \theta v_{-\mathbf{p},1}^{r\dagger} u_{\mathbf{p},2}^s & \cos \theta \delta_{rs} & \sin \theta v_{-\mathbf{p},1}^{r\dagger} v_{-\mathbf{p},2}^s \\ -\sin \theta v_{-\mathbf{p},2}^{r\dagger} u_{\mathbf{p},1}^s & 0 & -\sin \theta v_{-\mathbf{p},2}^{r\dagger} v_{-\mathbf{p},1}^s & \cos \theta \delta_{rs} \end{pmatrix} \begin{pmatrix} a_{\mathbf{p},1}^r(t) \\ a_{\mathbf{p},2}^r(t) \\ b_{-\mathbf{p},1}^{r\dagger}(t) \\ b_{-\mathbf{p},2}^{r\dagger}(t) \end{pmatrix}. \quad (8.35)$$

This relation also gives us the explicit time-dependence of the flavour operators, because we already know the time-dependence of the mass operators from the definition (8.10). If we choose as a particular reference frame the one wherein $\mathbf{p} = (0, 0, |\mathbf{p}|) = (0, 0, p)$, we can further simplify the matrix in the relation (8.35). From the explicit form of the spinors (8.13) we see that only the spinor products with $r = s$ remain. Furthermore, we can introduce the

abbreviations U_p and V_p which correspond to the coefficient α_p and β_p in chapter 6. They are defined as

$$\begin{aligned} U_p &:= u_{\mathbf{p},2}^{r\dagger} u_{\mathbf{p},1}^r = v_{-\mathbf{p},1}^{r\dagger} v_{-\mathbf{p},2}^r \\ &= \sqrt{\frac{E_1 + m_1}{2E_1}} \sqrt{\frac{E_2 + m_2}{2E_2}} \left(1 + \frac{p^2}{(E_1 + m_1)(E_2 + m_2)} \right), \end{aligned} \quad (8.36a)$$

$$\begin{aligned} (-1)^r V_p &:= u_{\mathbf{p},1}^{r\dagger} v_{-\mathbf{p},2}^r = -u_{\mathbf{p},2}^{r\dagger} v_{-\mathbf{p},1}^r \\ &= (-1)^r \sqrt{\frac{E_1 + m_1}{2E_1}} \sqrt{\frac{E_2 + m_2}{2E_2}} \left(\frac{p}{E_2 + m_2} - \frac{p}{E_1 + m_1} \right) \end{aligned} \quad (8.36b)$$

and satisfy the relation

$$U_p^2 + V_p^2 = 1. \quad (8.37)$$

Both, U_p and V_p have a minimum or maximum at $p = \sqrt{m_1 m_2}$. For $p \gg \sqrt{m_1 m_2}$, that is in particular for relativistic neutrinos, $U_p \rightarrow 1$ and $V_p \rightarrow 0$. The same happens for the cases where $m_1 = m_2$, which means no mixing is present.

Relation (8.35) can now be written as

$$\begin{pmatrix} a_{\mathbf{p},e}^r(t) \\ a_{\mathbf{p},\mu}^r(t) \\ b_{-\mathbf{p},e}^{r\dagger}(t) \\ b_{-\mathbf{p},\mu}^{r\dagger}(t) \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta U_p & 0 & (-1)^r \sin \theta V_p \\ -\sin \theta U_p & \cos \theta & (-1)^r \sin \theta V_p & 0 \\ 0 & -(-1)^r \sin \theta V_p & \cos \theta & \sin \theta U_p \\ -(-1)^r \sin \theta V_p & 0 & -\sin \theta U_p & \cos \theta \end{pmatrix} \begin{pmatrix} a_{\mathbf{p},1}^r(t) \\ a_{\mathbf{p},2}^r(t) \\ b_{-\mathbf{p},1}^{r\dagger}(t) \\ b_{-\mathbf{p},2}^{r\dagger}(t) \end{pmatrix}. \quad (8.38)$$

Using the anticommutation relations for the mass operators (8.2) and the orthogonality and completeness relations for the spinors (8.12), we can show by a straightforward calculation, that the flavour operators satisfy the usual equal-time anticommutation relations

$$\{a_{\mathbf{p},\alpha}^r(t), a_{\mathbf{p},\beta}^{s\dagger}(t)\} = \{b_{-\mathbf{p},\alpha}^r(t), b_{-\mathbf{p},\beta}^{s\dagger}(t)\} = \delta_{\alpha\beta} \delta_{\mathbf{p}\mathbf{k}} \delta_{rs} \quad \text{and all other zero.} \quad (8.39)$$

Furthermore, we can calculate unequal-time anticommutators from (8.38). The same could be done for the mass operators, but there are only the two non-vanishing anticommutators in (8.2) that get a time-dependence and nothing new appears, while the flavour operators give new non-vanishing anticommutators:

$$\{a_{\mathbf{p},e}^r(t), a_{\mathbf{k},e}^{s\dagger}(0)\} = \cos^2 \theta e^{-iE_1 t} + \sin^2 \theta \left(U_p^2 e^{-iE_2 t} + V_p^2 e^{iE_2 t} \right), \quad (8.40a)$$

$$\{a_{\mathbf{p},e}^r(t), a_{\mathbf{k},\mu}^{s\dagger}(0)\} = \sin \theta \cos \theta U_p \left(e^{-iE_2 t} - e^{-iE_1 t} \right), \quad (8.40b)$$

$$\{a_{\mathbf{p},e}^r(t), b_{\mathbf{k},e}^s(0)\} = \sin^2 \theta U_p V_p \left(e^{iE_2 t} - e^{-iE_2 t} \right), \quad (8.40c)$$

$$\{a_{\mathbf{p},e}^r(t), b_{\mathbf{k},\mu}^s(0)\} = \sin \theta \cos \theta V_p \left(e^{iE_2 t} - e^{-iE_1 t} \right), \quad (8.40d)$$

$$\{a_{\mathbf{p},\mu}^r(t), a_{\mathbf{k},\mu}^{s\dagger}(0)\} = \cos^2 \theta e^{-iE_2 t} + \sin^2 \theta \left(U_p^2 e^{-iE_1 t} + V_p^2 e^{iE_1 t} \right), \quad (8.40e)$$

$$\{a_{\mathbf{p},\mu}^r(t), b_{\mathbf{k},e}^s(0)\} = \sin \theta \cos \theta V_p \left(e^{iE_1 t} - e^{-iE_2 t} \right), \quad (8.40f)$$

$$\{a_{\mathbf{p},\mu}^r(t), b_{\mathbf{k},\mu}^s(0)\} = \sin^2 \theta U_p V_p \left(e^{-iE_1 t} - e^{iE_1 t} \right), \quad (8.40g)$$

$$\{b_{-\mathbf{p},e}^{r\dagger}(t), b_{\mathbf{k},e}^s(0)\} = \cos^2 \theta e^{iE_1 t} + \sin^2 \theta \left(U_p^2 e^{iE_2 t} + V_p^2 e^{-iE_2 t} \right), \quad (8.40h)$$

$$\{b_{-\mathbf{p},e}^{r\dagger}(t), b_{\mathbf{k},\mu}^s(0)\} = \sin \theta \cos \theta U_p \left(e^{iE_2 t} - e^{iE_1 t} \right), \quad (8.40i)$$

$$\{b_{-\mathbf{p},\mu}^{r\dagger}(t), b_{\mathbf{k},\mu}^s(0)\} = \cos^2 \theta e^{iE_2 t} + \sin^2 \theta \left(U_p^2 e^{iE_1 t} + V_p^2 e^{-iE_1 t} \right). \quad (8.40j)$$

If we consider the case of relativistic neutrinos, which corresponds to the large- p limit of U_p and V_p , we see that for (8.38) and (8.40) the case of the standard mixing is recovered.

In conclusion, the mixing of neutrinos is similar to our example in chapter 6. In both cases we have two fields ψ_1, ψ_2 and ν_1, ν_2 , which become connected by imposing boundary conditions. In chapter 6 this condition was $\psi_1(\mathbf{x}, 0) = \psi_2(\mathbf{x}, 0)$. For the neutrinos we have two conditions $\nu_e(x) = \cos \theta \nu_1(x) + \sin \theta \nu_2(x)$ and $\nu_\mu(x) = -\sin \theta \nu_1(x) + \cos \theta \nu_2(x)$. Therefore, we assume that similar things happen to the ‘‘Fock spaces’’ and the operator $G(\theta, t)$. In particular, we assume that the ‘‘Fock spaces’’ are disjoint and $G(\theta, t)$ is an improper unitary operator.

8.3 A Fock Space for Flavour Neutrinos

In the previous section we found the relation between the creation and annihilation operators for the mass and flavour neutrinos, respectively. Since the ‘‘Fock space’’ is build upon the ground state by acting creation operators on it, we can use the mixing relation to construct the ‘‘Fock space’’ for the flavour neutrinos.

We will start with the Fock space $\mathcal{H}_{1,2}$ for the mass neutrinos. The vacuum is defined as the state which is annihilated by all annihilation operators

$$a_{\mathbf{p},i}^r(t)|0\rangle_{1,2} = b_{\mathbf{p},i}^r(t)|0\rangle_{1,2} = 0 \quad \forall \mathbf{p}, r, i, t. \quad (8.41)$$

From this vacuum state vector the whole Fock space can be constructed by successive applying of creation operators on the vacuum. For example the one neutrino state is described by

$$|\nu_i(t)\rangle_{1,2} := a_{\mathbf{p},i}^{r\dagger}(t)|0\rangle_{1,2}. \quad (8.42)$$

In order to find the ‘‘Fock space’’ for the flavour neutrinos we have a look at the vacuum expectation value for the mass neutrino annihilation operator:

$${}_{1,2}\langle 0|a_{\mathbf{p},i}^r(t)|0\rangle_{1,2} = 0. \quad (8.43)$$

Since we know the connection to the flavour neutrino annihilation operator (8.32), we can identify the flavour neutrino vacuum state vector

$$0 = {}_{1,2}\langle 0|G(\theta, t)a_{\mathbf{p},i}^r(t)G^{-1}(\theta, t)|0\rangle_{1,2} := {}_{e,\mu}\langle 0(\theta, t)|a_{\mathbf{p},\alpha}^r|0(\theta, t)\rangle_{e,\mu} \quad (8.44)$$

with

$$|0(\theta, t)\rangle_{e,\mu} := G^{-1}(\theta, t)|0\rangle_{1,2}. \quad (8.45)$$

This is the corresponding relation to (6.13) in chapter 6. The ‘‘Fock space’’ $\mathcal{H}_{e,\mu}$ for the flavour neutrinos shall then be the space build up by applying flavour creation operators on this vacuum. For example the ‘‘one flavour neutrino’’ state is

$$\begin{aligned} |\nu_\alpha(\theta, t)\rangle_{e,\mu} &:= a_{\mathbf{p},\alpha}^{r\dagger}|0(\theta, t)\rangle_{e,\mu} \\ &= G^{-1}(\theta, t)|\nu_i(t)\rangle_{1,2} \end{aligned} \quad (8.46)$$

and we can regard the operator $G^{-1}(\theta, t)$ as the bijective map:

$$G^{-1}(\theta, t) : \mathcal{H}_{1,2} \rightarrow \mathcal{H}_{e,\mu} \quad (8.47)$$

which connects the two "Fock spaces".

From (8.45) we can explicitly compute the form of the flavour vacuum [BV95]

$$\begin{aligned} |0(\theta, t)\rangle_{e,\mu} = \prod_{\mathbf{p},r} & \left[(1 - \sin^2 \theta V_p^2) - (-1)^r \sin \theta \cos \theta V_p \left(a_{\mathbf{p},1}^{r\dagger}(t) b_{-\mathbf{p},2}^{r\dagger}(t) + a_{\mathbf{p},2}^{r\dagger}(t) b_{-\mathbf{p},1}^{r\dagger}(t) \right) \right. \\ & + (-1)^r \sin^2 \theta U_p V_p \left(a_{\mathbf{p},1}^{r\dagger}(t) b_{-\mathbf{p},1}^{r\dagger}(t) - a_{\mathbf{p},2}^{r\dagger}(t) b_{-\mathbf{p},2}^{r\dagger}(t) \right) \\ & \left. + \sin^2 \theta V_p^2 a_{\mathbf{p},1}^{r\dagger}(t) b_{-\mathbf{p},2}^{r\dagger}(t) a_{\mathbf{p},2}^{r\dagger}(t) b_{-\mathbf{p},1}^{r\dagger}(t) \right] |0\rangle_{1,2}. \end{aligned} \quad (8.48)$$

This shows that the flavour vacuum is a condensate of mass neutrinos. By using (8.48) we can compute the condensation density, which is the number of mass neutrinos with momentum \mathbf{p} condensated in the flavour vacuum:

$${}_{e,\mu} \langle 0(\theta, t) | a_{\mathbf{p},i}^{r\dagger}(t) a_{\mathbf{p},i}^r(t) | 0(\theta, t) \rangle_{e,\mu} = {}_{e,\mu} \langle 0(\theta, t) | b_{-\mathbf{p},i}^{r\dagger}(t) b_{-\mathbf{p},i}^r(t) | 0(\theta, t) \rangle_{e,\mu} = \sin^2 \theta V_p^2. \quad (8.49)$$

Thus, we encountered the same problem as in chapter 6. The "Fock space" for the flavour neutrinos is no real "Fock space" because, the state vectors in it cannot be interpreted as states with a definite particle number. However, the vectors are well-defined by the map (8.47).

As a next step we will show that the two spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ are disjoint in the infinite volume limit. Therefore, we compute the overlap of the two ground states

$$\begin{aligned} \mathcal{O} &= {}_{1,2} \langle 0 | 0(\theta, t) \rangle_{e,\mu} = {}_{1,2} \langle 0 | G^{-1}(\theta, t) | 0 \rangle_{1,2} \\ &= \prod_{\mathbf{p},r} (1 - \sin^2 \theta V_p^2) = \prod_{\mathbf{p}} \exp \left(2 \ln (1 - \sin^2 \theta V_p^2) \right) \\ &= \exp \left(\sum_{\mathbf{p}} 2 \ln (1 - \sin^2 \theta V_p^2) \right). \end{aligned} \quad (8.50)$$

Using relation (6.5) for the behaviour of the sum in the infinite volume we obtain

$$\mathcal{O} = \exp \left(\frac{2V}{(2\pi)^3} \int d^3 p \ln (1 - \sin^2 \theta V_p^2) \right). \quad (8.51)$$

Since the argument of the logarithm is always smaller than 1, the exponent is negative and the overlap vanishes for $V \rightarrow \infty$ just as it was the case in chapter 6. The same calculation can be done for all state vectors of the different spaces with the same result. This also shows that the matrix elements of $G(\theta, t)$ vanish if taken between vectors of the space $\mathcal{H}_{1,2}$ or $\mathcal{H}_{e,\mu}$. Thus, $G(\theta, t)$ is an improper unitary operator.

In conclusion, we have shown in this chapter so far that it is possible to consider the mixing between flavour and mass neutrinos as generated by an improper unitary operator $G(\theta, t)$. This implies a unitary inequivalence of the mass and flavour "Fock spaces". Haag's theorem then allows us to consider the mass neutrinos as asymptotically free particles, while at the same time the flavour neutrinos have interactions among each other. But this also spoils the interpretation of a flavour neutrino state vector as a state with a definite number of flavour neutrinos. In particular, these states are condensates of mass neutrinos. Nevertheless, the

state vectors in $\mathcal{H}_{e,\mu}$ are well-defined and we can compute an oscillation probability as we will show in the next section. However, due to the problems with the interpretation of these states we cannot interpret the probability as a particle with flavour α which propagate and in the meanwhile oscillates into a flavour β . But we can consider the change in the expectation value of the flavour charge, defined in (8.28), in some particular state. This is possible for all kind of states even if they describe a condensate. As we will show in section 8.5 the state vectors in $\mathcal{H}_{e,\mu}$ are just the eigenvectors of the flavour charges and thus they are the natural choice to describe the "flavour neutrinos".

8.4 Neutrino Oscillations

As we said in the last section we will interpret the oscillations between different flavours as the change of the expectation value of the flavour charges taken in some particular state. Therefore, we have to express the flavour charge operators (8.28) in terms of the creation and annihilation operators. As usual we will work with normal ordered operators defined by

$$:: Q_\alpha(t) ::= Q_\alpha(t) - {}_{e,\mu} \langle 0(\theta, 0) | Q_\alpha(t) | 0(\theta, 0) \rangle_{e,\mu}. \quad (8.52)$$

We used the symbol $:: \dots ::$ in order to note that the normal ordering is done with respect to the flavour vacuum. In contrast, $: \dots :$ will denote the normal ordering with respect to the mass vacuum. Note that we will work in the *Heisenberg picture*, where the time-dependence is transferred to the operators. Since the flavour state vectors are different for each time, we have to choose a reference time—in our case $t = 0$.

Using the expansion (8.16) and the definition of the flavour charge (8.28) we find

$$:: Q_\alpha(t) ::= \sum_{\mathbf{p}, r} \left(a_{\mathbf{p}, \alpha}^{r\dagger}(t) a_{\mathbf{p}, \alpha}^r(t) - b_{-\mathbf{p}, \alpha}^{r\dagger}(t) b_{-\mathbf{p}, \alpha}^r(t) \right). \quad (8.53)$$

In order to find the probability amplitudes for the oscillation $\nu_e \rightarrow \nu_e$ and $\nu_e \rightarrow \nu_\mu$ we compute the following expectation values

$$\mathcal{P}(e \rightarrow e; t) = {}_{e,\mu} \langle \nu_e(\theta, 0) | :: Q_e(t) :: | \nu_e(\theta, 0) \rangle_{e,\mu}, \quad (8.54a)$$

$$\mathcal{P}(e \rightarrow \mu; t) = {}_{e,\mu} \langle \nu_e(\theta, 0) | :: Q_\mu(t) :: | \nu_e(\theta, 0) \rangle_{e,\mu}. \quad (8.54b)$$

The state vector $|\nu_e(\theta, 0)\rangle_{e,\mu}$ is defined according to (8.46). To actually calculate the probabilities we insert the flavour charges (8.53) and use the relation

$${}_{e,\mu} \langle 0(\theta, 0) | :: Q_\alpha(t) :: | 0(\theta, 0) \rangle_{e,\mu} = 0 \quad (8.55)$$

which follows from (8.52). The result can be written in terms of anticommutators to unequal times

$$\mathcal{P}(e \rightarrow e; t) = \left| \{a_{\mathbf{p}, e}^r(t), a_{\mathbf{p}, e}^{r\dagger}(0)\} \right|^2 + \left| \{b_{-\mathbf{p}, e}^{r\dagger}(t), a_{\mathbf{p}, e}^{r\dagger}(0)\} \right|^2, \quad (8.56a)$$

$$\mathcal{P}(e \rightarrow \mu; t) = \left| \{a_{\mathbf{p}, \mu}^r(t), a_{\mathbf{p}, e}^{r\dagger}(0)\} \right|^2 + \left| \{b_{-\mathbf{p}, \mu}^{r\dagger}(t), a_{\mathbf{p}, e}^{r\dagger}(0)\} \right|^2. \quad (8.56b)$$

Since we already know the form of these anticommutators, as they are given in (8.40), we can compute the final result

$$\mathcal{P}(e \rightarrow e; t) = 1 - \sin^2(2\theta) U_p^2 \sin^2 \left(\frac{E_2 - E_1}{2} t \right) - \sin^2(2\theta) V_p^2 \sin^2 \left(\frac{E_2 + E_1}{2} t \right), \quad (8.57a)$$

$$\mathcal{P}(e \rightarrow \mu; t) = \sin^2(2\theta) U_p^2 \sin^2 \left(\frac{E_2 - E_1}{2} t \right) + \sin^2(2\theta) V_p^2 \sin^2 \left(\frac{E_2 + E_1}{2} t \right). \quad (8.57b)$$

These two probabilities are the main results of the Blasone-Vitiello approach. They obviously differ from the results in the other approaches in two ways. First the probabilities depend on the momentum due to the dependency of U_p and V_p and second there is an additional term depending on the sum of the energies. Note that these results only depend on time and not on the spatial distance. However, this was only assumed for simplicity. We could get a spatial dependency if we use the flavour charge density $J(\mathbf{x}, t)$ with

$$Q_\alpha(t) = \int d^3x J_\alpha(\mathbf{x}, t) \quad (8.58)$$

instead of the charges. The probabilities are then given by

$$\mathcal{P}(\alpha \rightarrow \beta; L) = \int_0^T \int_{\Omega} \int_{e,\mu} \langle \nu_\alpha(\theta, 0) | J_\beta(\mathbf{x}, t)^i | \nu_\alpha(\theta, 0) \rangle_{e,\mu} dS^i, \quad (8.59)$$

which is the flux of neutrinos through the surface of the detector averaged over the time. This has been done in [BPT03]. Furthermore, we should note that in the case of relativistic neutrinos, that is $p \gg \sqrt{m_1 m_2}$, the standard formula is recovered due to $U_p \rightarrow 1$ and $V_p \rightarrow 0$ in this case.

8.5 Reasons in Favour of the BV-Approach

In this section we will discuss the important question whether the BV-approach is really more convenient than the standard approaches. This question can be reformulated to the one whether the state vectors

$$|\nu_\alpha(\theta, t)\rangle_{e,\mu} = G^{-1}(\theta, t) |\nu_i(t)\rangle_{1,2} \quad (8.60)$$

really describe the behaviour of flavour neutrinos better than the state vectors

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle. \quad (8.61)$$

We will discuss this question on the basis of two different attempts. First the flavour charges of the different state vectors and second the calculation of the process $W^+ \rightarrow e^+ + \nu_\mu$.

Flavour Charge Eigenstates In (8.28) we defined the flavour charges on the basis of the symmetry properties of the Lagrangian. Thus, we have not used any information concerning the state vectors. This allows us to test the different definitions (8.60) and (8.61) by computing their behaviour under a transformation with $Q_\alpha(t)$. This was done in [BCTV05, BCJV06].

If we use the different normal orderings defined in section 8.4 we can get the following relations for the charges

$$:: Q_e(t) :: = G^{-1}(\theta, t) : Q_1 : G(\theta, t), \quad (8.62a)$$

$$:: Q_\mu(t) :: = G^{-1}(\theta, t) : Q_2 : G(\theta, t) \quad (8.62b)$$

On the other hand, if we explicitly insert the mixing for the field operators into the definition of the flavour charges, we get

$$Q_e(t) = \cos^2 \theta Q_1 + \sin^2 \theta Q_2 + \sin \theta \cos \theta \int d^3x (\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1), \quad (8.63a)$$

$$Q_\mu(t) = \sin^2 \theta Q_1 + \cos^2 \theta Q_2 - \sin \theta \cos \theta \int d^3x (\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1). \quad (8.63b)$$

It can be shown by writing the charges in terms of creation and annihilation operators, that the mass neutrino state vectors are eigenvectors of the charges Q_1 and Q_2

$$: Q_1 : |\nu_1\rangle_{1,2} = |\nu_1\rangle_{1,2}, \quad : Q_2 : |\nu_1\rangle_{1,2} = 0, \quad (8.64)$$

$$: Q_2 : |\nu_2\rangle_{1,2} = |\nu_2\rangle_{1,2}, \quad : Q_1 : |\nu_2\rangle_{1,2} = 0. \quad (8.65)$$

If we use all these relations we can easily show that the state vectors (8.60) are eigenvectors of $Q_e(t)$ and $Q_\mu(t)$ at the reference time $t = 0$

$$\begin{aligned} :: Q_e(0) :: |\nu_e(\theta, 0)\rangle_{e,\mu} &= G^{-1}(\theta, 0) : Q_1(t) : |\nu_1\rangle_{1,2} \\ &= |\nu_e(\theta, 0)\rangle_{e,\mu} \end{aligned} \quad (8.66)$$

and similar for the following relations

$$: Q_e(0) :: |\nu_\mu(\theta, 0)\rangle_{e,\mu} = 0, \quad \text{and} \quad : Q_\mu(0) :: |\nu_e(\theta, 0)\rangle_{e,\mu} = 0, \quad (8.67a)$$

$$: Q_\mu(0) :: |\nu_\mu(\theta, 0)\rangle_{e,\mu} = |\nu_\mu(\theta, 0)\rangle_{e,\mu} \quad (8.67b)$$

If we try the same for the state vectors (8.61), we find that they are obviously not eigenvectors of $Q_e(t)$ and $Q_\mu(t)$

$$\begin{aligned} : Q_e(t) : |\nu_e\rangle &= : Q_e(t) : (\cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle) \\ &= \cos^3\theta|\nu_1\rangle + \sin^3\theta|\nu_2\rangle + \sin\theta\cos\theta \int d^3x : (\nu_1^\dagger\nu_2 + \nu_2^\dagger\nu_1) : (\cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle) \\ &\neq |\nu_e\rangle. \end{aligned} \quad (8.68)$$

The same holds for the other state vectors. Thus, we found that only the state vectors defined via the improper mixing (8.60) describe states with the expected flavour charge.

Amplitude for the Fundamental Vertex Now we will consider the process $W^+ \rightarrow e^+ + \nu_\mu$ whose amplitude should vanish. This is similar to the considerations in chapter 7 but however, we will encounter some new insight in the problem. The first who computed this process for the BV state vectors were [LL06] who found that the amplitude is non-vanishing. Later BV [BCJV06] showed that the actual problem is to consider the state vectors as asymptotically free and take a time integration in the amplitude from $-\infty$ to ∞ . As we pointed out in chapter 6 the neutrino states cannot be considered as asymptotically free because they have interactions among each other. Thus, the infinite times in the amplitude are meaningless. However, we expect that the problems due to the oscillations will not be present immediately and thus we can regard our neutrinos to be free for times shortly after the production. Thus, in the following we will calculate the amplitude for small times around the interaction time.

If we consider the amplitude

$$\mathcal{A}(W^+ \rightarrow e^+ + \nu_\mu) = \langle \nu_\mu, e^+ | S | W^+ \rangle \quad (8.69)$$

and expand the S-matrix to first order in the charged current Hamiltonian the amplitude will certainly contain a factor

$$\mathcal{A}_\nu = \int_{x_{in}^0}^{x_{out}^0} dx^0 \langle \nu_\mu | \bar{\nu}_e(x) | 0 \rangle, \quad (8.70)$$

which is the only interesting term for us.

We will start with the state vectors in the BV-approach. Then \mathcal{A}_ν reads

$$\begin{aligned}\mathcal{A}_\nu &= \int_{x_{in}^0}^{x_{out}^0} dx^0 \langle \nu_\mu(\theta, 0) | \bar{\nu}_e(x) | 0(\theta, 0) \rangle_{e,\mu} \\ &= \int_{x_{in}^0}^{x_{out}^0} dx^0 \sum_{\mathbf{k},s} \left(u_{\mathbf{k},e}^{s\dagger} \{ a_{\mathbf{p},\mu}^r(0), a_{\mathbf{k},e}^{s\dagger}(x^0) \} + v_{-\mathbf{k},e}^{s\dagger} \{ a_{\mathbf{p},\mu}^r(0), b_{-\mathbf{k},e}^s(x^0) \} \right)\end{aligned}\quad (8.71)$$

where we used the expansion (8.16) for the field operator. The explicit form of the anticommutators is given in (8.40) which yields

$$\mathcal{A}_\nu = \int_{x_{in}^0}^{x_{out}^0} dx^0 \sin \theta \cos \theta \left(u_{\mathbf{k},e}^{s\dagger} (e^{iE_2 t} - e^{iE_1 t}) + v_{-\mathbf{k},e}^{s\dagger} (e^{-iE_1 t} - e^{iE_2 t}) \right)\quad (8.72)$$

We now assume that the initial and final times x_{in}^0 and x_{out}^0 , respectively, are small compared to $1/E_2$ and $1/E_1$. Note that these times are much larger than the time-scale for the actual interaction which is of the order $1/G_F^{1/2}$. Then we see that the exponentials can be approximated by the zero order term and the whole term \mathcal{A}_ν vanishes. Thus we obtain the expected result.

In terms of the state vectors (8.61) the factor reads

$$\mathcal{A}_\nu = \int_{x_{in}^0}^{x_{out}^0} dx^0 \langle \nu_\mu | \bar{\nu}_e(x) | 0 \rangle\quad (8.73)$$

$$= \int_{x_{in}^0}^{x_{out}^0} dx^0 \sin \theta \cos \theta \left(u_{\mathbf{p},2}^{r\dagger} e^{iE_2 t} - u_{\mathbf{p},1}^{r\dagger} e^{iE_1 t} \right),\quad (8.74)$$

where we first wrote the factor in terms of the mass field operators and state vectors and then used the Fourier expansion for the field operators. If we now use the same assumption for the initial and final times as before we get

$$\mathcal{A}_\nu = \sin \theta \cos \theta \left(u_{\mathbf{p},2}^{r\dagger} - u_{\mathbf{p},1}^{r\dagger} \right) \neq 0,\quad (8.75)$$

which is not the expected result.

Thus, we have seen that if we consider the amplitude for times in a small range around the interaction time, only the BV-state vectors give the expected result of a vanishing amplitude. For this small times we can neglect the oscillation behaviour of the neutrinos and can consider them to be free. Note that we could do the same calculation for the weak-process states. The result would again be a non-vanishing amplitude.

In conclusion, this shows that the state vectors defined by the improper transformation (8.60) are preferable for the description of the flavour neutrino behaviour.

Summary and Outlook

In this thesis we have summarised the different theoretical approaches for the description of neutrino oscillations. As we have seen the quantum-mechanical plane-wave approach produces a number of problems. The main problem is the unclear notion of the state vector for a “particle” having no definite mass. Therefore, it should not be used for an accurate theoretical description of NO, even though it is the most used approach in the literature. The more advanced intermediate wave-packet model solves some of these problems but nevertheless it suffers from the same insufficiencies as all quantum-mechanical approaches. In particular, that are the Bargman superselection rule, which forbids the superposition of different mass states for non-relativistic particles, and the ill-defined flavour state vectors which do not reproduce the expected flavour neutrino behaviour. One possible way out is the external wave-packet model which bases on QFT and does not describe the neutrinos by state vectors but by their propagator. This reflects in an interesting way the unobservability of the flavour neutrino. In this model the oscillation probability is mainly the same as in the intermediate wave packet model. The differences are mostly due to QFT effects in the description of the source and detector. Another important point, which is not considered in the quantum mechanical approaches is Haag’s theorem. The main point of this theorem within the present circumstances is that if mass and flavour neutrinos are connected by a unitary transformation they are both free particles, because we consider the mass neutrinos as asymptotically free, which can explicitly be seen in the BV-approach, where the Lagrangian for the mass fields does not contain any interaction term. In contrast, the flavour neutrinos have interactions among each other leading to mixing transitions and can thus not be considered to be free. These problems, although known since the beginning of QFT, were not considered in the theory of neutrinos for a long time. Even today they are mostly ignored, which is partly justified by the fact that all approaches presented in this thesis give the same results if the neutrinos are considered to be relativistic. But from a theoretical viewpoint these approaches are unsatisfactory. Therefore, the BV-approach can be considered as a breakthrough for the theoretical description of neutrinos, because the state vectors in this approach really describe the behaviour of flavour neutrinos as we would expect it. However, the approach gives rise to a new problem, which concerns the interpretation of a flavour neutrino. As we have seen, the state vectors for flavour neutrinos can be considered as describing a condensate of infinitely many mass neutrinos and thus we do not have a definite particle number. Nevertheless, we can work with these vectors if we only consider quantities which do not resort to a particle interpretation. For example, we defined the oscillation probability as the expectation value of the flavour charge, which is not bothered by the infinite number of particles in the state.

As we said, for relativistic neutrinos all approaches yield the same result. Thus, in order to prove or disprove the different approaches empirically we have to measure oscillation effects

for non-relativistic neutrinos. This means we need a new kind of detection possibilities since today's experiments are only sensible to relativistic neutrinos. Another possibility could be the precise measurements made for the B-mesons and Kaons. Even though these are boson mixing effects most of the approaches and in particular the BV-approach can also be applied to boson mixing.

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