MEAN LIFETIME MEASUREMENTS IN LOW-STATISTICS EXPERIMENTS*

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Statistical methods, based on maximum likelihood function, suitable for mean lifetime measurements in low-statistic experiments are presented. Examples of two typical experimental methods — implantation and moving tape collector — are discussed.

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1. Introduction

Mean lifetimes (τ) are among the most basic properties of atomic nuclei, and are often the first property of a new isotopes to be measured experimentally. With the advances of methods of production of rare isotopes, we reach for more and more exotic nuclides. As a general rule, the further the isotope lies from the valley of stability the shorter is the half-life and the smaller is the production cross section. The newly discovered isotopes are often produced in small quantities, down to a couple observed atoms. This causes a need for a special statistical treatment of collected decay times and extraction of τ from a low statistics data.

2. Implantation experiments

In a typical implantation experiment, the time of implantation of individual radioactive ions, the detector setup, and a subsequent decay time are observed. The decays are correlated with implantations, and the half-life of isotope of interest may be measured. The decays are observed within a time window with lower threshold t_1 (originating in *e.g.* detector dead time after implantation) and an upper threshold t_2 (which is limited by observation window lenght, subsequent implantations, background signals *etc.*).

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A very popular method for low-statistic experiment was described by Schmidt [1]. The estimator for τ in this method is a mean of observed decay times ($\hat{\tau} = \bar{t_i}$). The requirements of this method, as described by the author, are full time range coverage (very small t_1 and very large t_2 compared to τ), and exclusion of contribution of other radioactive species. However, it was not stated quantitatively what values should be considered as "small" and "large". This will be discussed in more details in this section.

The first moment of a distribution of measured decay times is an unbiased estimator only when $t_1 = 0$ and $t_2 \to \infty$. However, a more realistic situation is when both t_1 and t_2 have some finite values. In such a case, the probability density function has a form

$$\rho(t,\tau) = \frac{1}{\tau}\frac{\exp(-t/\tau)}{\exp\left(-t_1/\tau\right) - \exp\left(-t_2/\tau\right)}\,. \tag{1}$$

The first moment in this case is a biased estimator of a mean lifetime

$$\bar{t} = \int\limits_{t_1}^{t_2} \frac{t\exp\left(-t/\tau\right)dt}{\exp\left(-t_1/\tau\right) - \exp\left(-t_2/\tau\right)} = \tau - \frac{t_1\exp\left(-t_1/\tau\right) - t_2\exp\left(-t_2/\tau\right)}{\exp\left(-t_1/\tau\right) - \exp\left(-t_2/\tau\right)}\,. \tag{2}$$

The difference between $\bar{t}$ and τ is plotted in Fig. 1. In order to achieve estimator biased by less than 5%, it is recommended that the t_2 value should be larger than $5 \times \tau$ and the t_1 shorter than $0.05 \times \tau$.

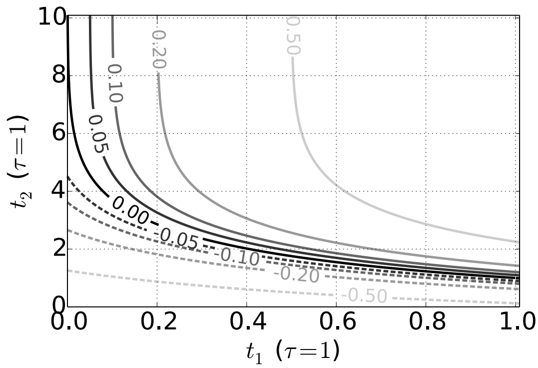

Fig. 1. Difference between mean lifetime (τ) and a estimator based on the first moment of measured decay time distribution ($\bar{t}$) in experiment with a finite measurement window (t_1, t_2). The values on axes are expressed in units of $\tau = 1$. Solid lines indicate positive value of bias ($\bar{t_i} > \tau$), while dashed — negative ($\bar{t_i} < \tau$).

Using the density function from Eq. (1), we can build the likelihood function [2, 3]. Additionally, we assume that in general situation each event may have a different observation window (t_1^i, t_2^i)

$$\ln \mathcal{L} = -n \ln \tau - \frac{1}{\tau} \sum_{i=0}^{n} t_i - \sum_{i=0}^{n} \ln \left(\exp\left(-t_1^i/\tau\right) - \exp\left(-t_2^i/\tau\right) \right) . \quad (3)$$

The maximum of this function may be found by solving an equation $\frac{\partial \mathcal{L}}{\partial \tau} = 0$ and the maximum likelihood estimator (MLE) for τ is [3]

$$\hat{\tau} = \bar{t_i} - \frac{1}{n} \sum_{i=1}^{n} \frac{t_1^i \exp\left(-t_1^i/\tau\right) - t_2^i \exp\left(-t_2^i/\tau\right)}{\exp\left(-t_1^i/\tau\right) - \exp\left(-t_2^i/\tau\right)} . \quad (4)$$

This equation cannot, in principle, be solved analytically, however it is easy to find its solution numerically with a simple recursion. The uncertainty in the MLE method is calculated by finding values of parameter τ, where $\ln L(\tau) - \ln L(\tau_{\max}) = -0.5$, where $\tau_{\max}$ is the solution of Eq. (4). Notice that if the t_1 and t_2 are identical for all events, the equation (4) can be further simplified.

3. Moving tape collector experiments

In a typical experiment with a moving tape collector (MTC), the activity is collected during $(0, T_1)$ period of time, subsequently the beam is turned off and during (T_1, T_2) a decay of activity is observed. The irradiated spot is next removed from the detection set-up in order to avoid a background from daughter decays. The distribution of events in time is typically fitted with an exponential grow-in and decay functions to estimate the τ. However, the fitting procedure is limited to experiments with sufficient statistics. Moreover, the shorter the half-life, the larger becomes the uncertainty of this method, since the fit is most sensitive to the sloped parts of the curve.

An alternative method, based on MLE, was developed for experiment in which the ^{86}Ga decay was measured for the first time [4]. The probability density function has a form

$$\rho(t;\tau) = \begin{cases} 1/\mathcal{N}\left(1 - e^{-t/\tau} + b(t)\right) , & 0 < \quad t < T_1 , \\ 1/\mathcal{N}\left[\left(1 - e^{-T_1/\tau}\right) e^{-(t-T_1)/\tau} + b(t)\right] , & T_1 \leq t < T_2 , \end{cases} \quad (5)$$

where $b(t)$ is a background events distribution and $\mathcal{N}$ is a normalization factor

$$\mathcal{N} = T_1 + \tau \exp\left(-T_2/\tau\right) - \tau \exp\left(\left(T_1 - T_2\right)/\tau\right) . \quad (6)$$

For a set of n events of decay time t_i, a likelihood function is constructed as $\mathcal{L}(\tau) = \prod_{i=1}^{n} \rho(t_i;\tau)$.

Figure 2 (a) presents the result of Monte Carlo simulation of MTC experiment. The events were generated from distribution described by Eq. (5) with $\tau = 0.05$, $T_1 = 1$, $T_2 = 2$, and a flat background events distribution (signal-to-noise ratio 1:1). Using an MLE method, the τ parameter was retrieved from a sample of $n = 100$ events. This procedure was repeated $N = 10\,000$ times, and the distribution of reconstructed τ values is presented in Fig. 2 (c). The calculated mean result $\bar{\tau} = 0.110$ (dotted line) is in agreement with the input value of 0.1 (dashed line) proving the validity of the method also for low-statistics, and short lifetime cases. The asymmetry in the distribution of results has a reflection in the non-symetrical uncertainties obtained with this method (Fig. 2 (b)).

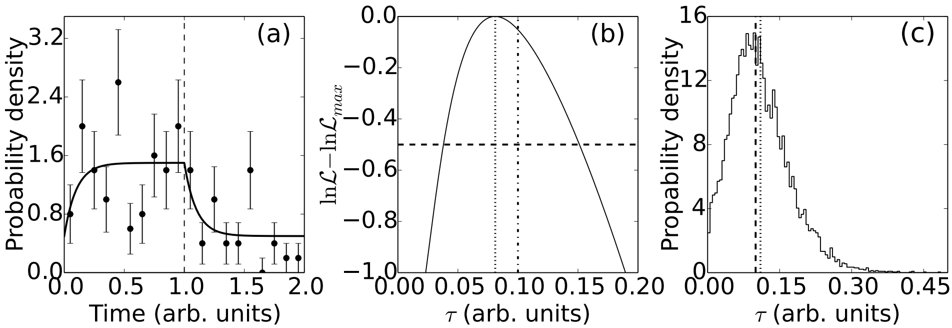

Fig. 2. Monte Carlo simulation of radioactive decay in a MTC-type experiment. (a) The events are generated from the density function (including random background) presented with a solid line, an example of generated distribution of 100 events is presented with points. (b) The events shown in (a) are analyzed with MLE method and a τ of 0.081 is found. (c) This procedure is repeated 10000 times to retrieve a distribution of results.

4. Summary

Examples of use of maximum likelihood method for life-time measurement in low-statistics experiments were presented. The MLE provides unbiased estimators also in complex, realistic situations where limited observation window or substantial background events influence is present.

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