AXIOMATIC FIELD THEORY

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Major progress in axiomatic field theory in recent years has been achieved in three different areas:

(i) Algebraic approach. A general study of particle statistics (including parastatistics) and field commutation relations based on the assumption of commutativity of observables at spacelike distance has been successfully carried through by Doplicher, Haag, and Roberts[1,2] using an algebraic method. Their work was not reported on at this conference.

(ii) Constructive field theory. This is an attempt to construct a quantum field theory for a given interaction such as ϕ^4 and $\bar{\psi}\psi\phi$ in a mathematically satisfactory manner, thereby establishing the existence of nontrivial models satisfying the basic axioms of quantum field theory and enabling a further mathematical study of the physical properties of these models such as broken symmetry. Recent remarkable progress was reported in the mini-rapporteur talks of Jaffe and Wightman.

(iii) Properties of scattering amplitudes. There has been remarkable progress in the analysis of on-mass-shell n-point amplitudes by Bros, Epstein, and Glaser.[3] This subject along with some recent work on the Pomeranchuk theorem was discussed in the mini-rapporteur talk of Martin (#788, 312, 313, 314). In addition, a parametric dispersion representation, which contains only physical absorptive parts and follows from axiomatic analyticity for pion-pion scattering, was reported by Khuri (#787), and a connection between scaling, light-cone singularities and the asymptotic behavior of the Jost-Lehmann-Dyson spectral function was discussed by Vladimirov (#917) and Stichel.[4]

In addition to achievement in the above three areas, important progress in renormalization has been made by Epstein and Glaser.[5] This work was not reported on.

In the area of mathematical aspects of quantum field theory, E. Mihul reported her work on the Bargman-Hall-Wightman theorem and on the extended tube, and Swieca (#637) discussed the unitary implementability of special conformal transformations for free fields.

In the following, areas (ii) and (iii) listed above are discussed in somewhat more detail.

I. Constructive Field Theory

One considers a Hamiltonian

$$H = H_0 + H_I + H_c,$$

where H_0 is a free Hamiltonian, H_I is an interaction Hamiltonian, and H_c is an (infinite) counterterm. Typically, one starts out from a cutoff Hamiltonian, proves the self-adjointness and semiboundedness of the cutoff Hamiltonian, defines a Heisenberg field $\phi(\vec{x},t) = e^{itH}\phi(\vec{x})e^{-itH}$ (in a cutoff theory), proves a finite propagation property, goes to the limit of no cutoff, and proves the Wightman axioms one by one. For super-renormalizable interactions, i.e., for those interactions which produce essentially only a finite number of divergent graphs, such a program can be carried through by treating the divergences exactly, and remarkable progress has been achieved over the past several years, where outstanding contributions have been made by Glimm and Jaffe.

At the time of this conference, all the Wightman axioms including the existence of a mass gap between the vacuum and the rest of the energy momentum spectrum had been established for $P(\phi)_2$ theory (a theory with an interaction $H_I = \int :P(\phi(x)):dx$ where P is a polynomial which is bounded below such as $\lambda\phi^4 - \alpha\phi^2$, $\lambda > 0$, and the space-time dimension is 2) for small coupling constant. Jaffe predicted that the $(\text{Yukawa})_2$ theory will be in similar shape within 12 months because of recent results on the Euclidean formulation for Fermions.[6,7] (The Euclidean formulation in general will be discussed below.) Another recent breakthrough is a proof of the positivity of the Hamiltonian for $(\phi^4)_3$ theory. Rapid progress was also predicted for this interaction as well as the $(\text{Yukawa})_3$ theory.

One of the most recent technical developments, which has been of vital importance for the rapid progress in the past year and is responsible for the optimistic future predictions, is concerned with the so-called Euclidean method. In particular, relations among Euclidean field theory, classical statistical mechanics, and quantum field theory were emphasized by Wightman and will be explained below in some detail.

It has long been known that the vacuum expectation values of products of fields in quantum field theory (VEV) can be continued analytically to Schwinger points, i.e., points with pure imaginary times and real space coordinates. The VEV at Schwinger points is called a Schwinger function. Symanzik[8] has developed a Euclidean field theory which yields Schwinger functions of a Minkowski quantum field theory as the expectation values of commuting fields with respect to a positive measure. Decisive progress has been achieved in the past year by Nelson[9-11] who introduced the Markoff property to Euclidean field theory and showed that this property together with properties discussed by Symanzik permit the reconstruction of a Minkowski quantum field theory from a Euclidean field theory. Thus one can first construct the Euclidean field theory or Schwinger functions for a given interaction and then study the corresponding Minkowski quantum field theory. This is the so-called Euclidean method.

The Schwinger functions for a cutoff interaction are obtained by the Gell-Mann-Low formula, i.e., the infinite t limit of

$$Z_{t,\ell}^{-1} \int \phi(x_1)\cdots\phi(x_n)d\mu(\phi), \qquad Z_{t,\ell} = \int d\mu(\phi),$$

where $\phi(x)$ in this expression is a commuting Euclidean field.

$$d\mu(\phi) = \exp\left[-\int_0^t d\xi_0 \int_{-\ell/2}^{\ell/2} d\xi_1 V(\phi(\xi))\right] d\mu_0(\phi).$$

$V(\phi)$ is an interaction such as ϕ^4 and $d\mu_0(\phi)$ is the Gaussian measure for free fields. The model without a spatial cutoff is obtained in the limit $\ell \to \infty$. Thus one is interested in the limit of a state given by a measure $\exp[-\int_\Lambda H(\phi(\xi))d\xi]d\mu_0(\phi)$ as the rectangle Λ (in space-time) becomes infinite, which makes the correspondence with classical statistical mechanics more than mere analogy. Such a limit is used to obtain an equilibrium state in classical statistical mechanics. Since abstract characterizations of equilibrium states by a variational principle and by other equivalent conditions are known for classical statistical mechanics in an infinite volume, and since new techniques for predicting the existence of phase transitions are being developed, one can expect similar developments for quantum field theory. A study along these lines is contained in Refs. 12 and 13.

Apart from the connection with statistical mechanics, the power of the Euclidean method can be seen in Nelson's symmetry:

$$(\Phi_0, \exp[-tH_\ell]\Phi_0) = (\Phi_0, \exp[-\ell H_t]\Phi_0),$$

where Φ_0 is the free vacuum and H_ℓ denotes the cutoff full Hamiltonian, ℓ being the space cutoff. Such a symmetry between the (time) parameter t and the space cutoff ℓ is quite remarkable and is a simple consequence of the Feynman-Kac formula

$$(\Phi_0, \exp[-tH_\ell]\Phi_0) = Z_{t,\ell}$$

Guerra[14] was the first person to notice an important application of this symmetry, which started a fullscale use of the Euclidean method by many authors.[6,7,12,13,15-19] Some of the advantages of the Euclidean method are the availability of the Feynman-Kac formula, Euclidean symmetry, the symmetry of Green's function due to unrestricted commutativity of fields, and the availability of a perturbation expansion using the Feynman propagator $(k^2 + m^2)^{-1}$ which greatly simplifies earlier estimates.

We include in the list of references those quoted by Jaffe and Wightman in connection with $P(\phi)_2$ theory,[19-21] (ϕ^4) theory,[22-24] and (Yukawa)$_2$ theory.[25-28]

II. Analyticity of n-Point Amplitudes on Mass-Shell

Analyticity of the 2 particle-2 particle amplitude $A + B \to C + D$ (for particles satisfying stability conditions) in a neighborhood of the physical region except for the energy cut was established some time ago by Bros, Epstein, and Glaser on an axiomatic basis, and the analyticity domain was improved by Martin on the basis of unitarity and positivity.[29] The recent result of Bros, Epstein, and Glaser[3] is concerned with n-point amplitudes on mass shell and shows that the physical amplitude is the sum of a finite number of boundary values of analytic functions. (The fact that an n-point amplitude is not always a boundary value of a single analytic function in the neighborhood of some Landau singularities had been recognized earlier in perturbation theory.[30-32])

The most spectacular progress has been achieved for the 5-point amplitude: $4 + 5 \to 1 + 2 + 3$. It is proved in this case that above a certain incident center-of-mass energy (4.8 times the common mass in the equal mass case, the calculation of 4.8 being due to Martin), the amplitude at any physical point is the boundary value of a single analytic function, holomorphic in a "local tube" in all (5 complex) variables. To describe the result, let E_1, E_2, E_3, be the center-of-mass energies of the final particles 1, 2, 3 and S_{12}, S_{23}, S_{31} be the (squared) two-particle subenergies of the final particles:

$$S_{12} = (E_1 + E_2 + E_3)^2 - 2E_3(E_1 + E_2 + E_3) + m_3^2, \text{ etc.}$$

In addition, two angular variables θ and ϕ, are necessary to describe completely the scattering process. For fixed E_1, E_2, E_3, analyticity in the angular variables θ and ϕ has been known[33,34] for some time. (In the quoted literature, an integration of the cross sections over all possible E_1, E_2, E_3 at a given total incident energy is done to obtain the analyticity of the production amplitude from the size of the ellipse of analyticity of the elastic amplitude $4 + 5 \to 4 + 5$. However, Omnes and Martin noticed that analyticity of the amplitude in the angular variables as a distribution in E_1, E_2, E_3 is obtained by integrating over an arbitrarily small cell in E_1, E_2, E_3 and using the Schwarz inequality.) Bros, Glaser, and Epstein obtained analyticity in all variables E_1, E_2, E_3, θ, ϕ in a region which is described by the following inequalities in the neighborhood of physical points:

$$\mathrm{Im}S_{kl} > 0,$$

$$\left\{\left(E_j^2 - m_j^2\right)^{1/2} - \epsilon E_j\right\}\mathrm{Im}S_{kl} + \left\{\left(E_j^2 - m_j^2\right)^{1/2} + \epsilon\left(E_j - m_j^2(E_1 + E_2 + E_3)^{-1}\right)\right\}\mathrm{Im}(S_{12} + S_{23} + S_{31}) < 0,$$

where (j, k, l) is any cyclic permutation of $(1, 2, 3)$ and $\epsilon = \pm 1$. Here E_i is the real part of the particle energy. As all E_1, E_2, E_3 tend to infinity, the second set of constraints on the relative magnitude of $\mathrm{Im}S_{12}$, $\mathrm{Im}S_{23}$, $\mathrm{Im}S_{31}$ becomes weaker and weaker.

III. Generalizations of the Pomeranchuk Theorem

The problem is to compare differential cross sections (and integrated cross sections) for line reversed processes

$$A + B \to C + D \text{ and } \bar{C} + B \to \bar{A} + D.$$

At fixed momentum transfer, Cornille and Martin proved that if the phases of amplitudes for both processes (defined by continuity assuming no physical region zeros) grow separately less fast than log s, then

$$\lim\left\{\frac{d\sigma}{dt}(s, t)_{AB\to CD}\Big/\frac{d\sigma}{dt}(s, t)_{\bar{C}B\to\bar{A}D}\right\} = 1$$

if in addition, this limit exists. (The case of a slowly varying momentum transfer can also be treated, but with great complications.) If the phases are bounded by (const) log s (a fact which can be proved from first principles for elastic scattering if the amplitude has no zeros at physical points), then Cornille and Martin prove that the limit is finite.

In the case of elastic scattering ($A = C$, $B = D$), it is shown that (1) widths of the diffraction peaks are asymptotically equal if the width of one amplitude has a nonzero limit and (2) if one amplitude exhibits persistent shrinking (i.e., monotonously tends to zero) and $(d\sigma/dt)(t = 0) \geq$ const. $(d\sigma/dt)(s, t)$, then the ratio of widths tends to 1. Here the width $\Delta(s)$ of an amplitude $F(s, t)$ is defined by $F(s, \Delta(s)) = \alpha F(s, 0)$ where α is any fixed constant such as $1/2$.

IV. Parametric Dispersion Representations

Starting from an analyticity domain in the two Mandelstam variables, Auberson and Khuri (#787) derive a parametric dispersion representation for equal mass elastic scattering amplitudes, which is symmetric in the three Mandelstam variables and contains only physical absorptive parts.

V. Scaling, Light-Cone Singularities and Asymptotic Behavior of the Jost-Lehmann-Dyson Spectral Function

The fact that scaling follows from certain light-cone singularities has been discussed by many authors. Vladimirov presented some work done in collaboration with N. N. Bogolubov, A. N. Tavkhelidze (#917) showing that a certain asymptotic behavior of the Jost-Lehmann-Dyson spectral function implies scaling and light-cone singularities. Stichel presented his work[4] showing that scaling implies a certain asymptotic behavior of the Jost-Lehmann-Dyson spectral function.

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