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Worldsheet Boundaries, Supersymmetry, and Quantum Geometry

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Kurzfassung

Gegenstand der vorliegenden Abhandlung ist die Beziehung zwischen Randbedingungen in zwei-dimensionalen, $\mathcal{N} = 2$ supersymmetrischen und konformen Quantenfeldtheorien auf der einen Seite, und D-branes, die als ausgedehnte geometrische Objekte in nicht-perturbativer Stringtheorie auftreten, auf der anderen. Das Hauptforschungsinteresse gilt den Eigenschaften von D-branes in nicht-klassischen oder stark gekrümmten Hintergründen. Es werden in dieser Arbeit Methoden der konformen Feldtheorie auf Weltflächen mit Rändern entwickelt und zum Studium der Quantengeometrie von D-branes angewendet.

Das erste Resultat, das in dieser Arbeit vorgestellt wird, ist ein Beitrag zu dem Problem, Randbedingungen in rationalen konformen Feldtheorien zu definieren. Solche Theorien sind auf geschlossenen Riemannschen Flächen durch eine chirale Symmetriealgebra und eine modular invariante Toruszustandssumme gekennzeichnet. Das Problem, Randbedingungen zu definieren, hängt von beidem ab—Randbedingungen müssen einerseits nicht die gesamte Symmetrie der geschlossenen Theorie erhalten, und dürfen andererseits nur Felder einschliessen, die in der Zustandssumme vorkommen. Nun sind die meisten modularen Invarianten in rationalen konformen Feldtheorien vom simple-current Typ, und können unter Umständen eine erweiterte chirale Symmetrie aufweisen. Das Problem, das hier behandelt wird, ist, alle Randbedingungen zu bestimmen, die die nicht erweiterte chirale Algebra erhalten, für eine beliebige modulare Invariante vom simple-current Typ.

Der Hauptteil der Arbeit handelt von Feldtheorien, die zusätzlich zu konformer Invarianz $\mathcal{N} = 2$ Supersymmetrie besitzen. Solche Theorien treten in der störungstheoretischen Definition auf der Weltfläche des Superstrings auf. Zusätzliche Einschränkungen sind notwendig, um Stabilität und ein Raumzeit-supersymmetrisches Spektrum zu erhalten. Für σ-Modelle ist das Kriterium, dass die Zielmannigfaltigkeit eine Calabi-Yau Mannigfaltigkeit ist. In einer abstrakten algebraischen Konstruktion können die notwendingen Projektionen ausgehend von einer beliebigen $\mathcal{N} = 2$ rationalen konformen Feldtheorie ausgeführt werden.

Eine mikroskopische Beschreibung von D-branes erfordert das Einführen von Weltflächen mit Rändern, und die Bedingungen für Supersymmetrie müssen neu analysiert werden. Im Besonderen erklärt die vorliegende Arbeit, wie die Projektionen, die im abstrakten Rahmen der algebraischen konformen Feldtheorie auftreten, in systematischer Weise behandelt werden können, und wie die wichtigsten Eigenschaften von D-branes in diesem Zugang kodiert sind.

Die allgemeine Theorie wird anschliessend in einer Anzahl von Beispie-

len illustriert, nämlich dem zwei-dimensionalen Torus, $\mathcal{N} = 2$ minimalen Modellen, Gepner-Modellen und $\mathcal{N} = 2$ Coset-Modellen (Kazama-Suzuki-Modellen).

Gepner-Modelle sind ausgezeichnete Beispiele, in denen rationale konforme Feldtheorien im inneren Teil einer Stringkompaktifizierung verwendet werden. Sie werden aus Tensorprodukten von $\mathcal{N} = 2$ minimalen Modellen aufgebaut. Das wichtigste neue technische Resultat zu Gepner-Modellen ist die Auflösung von simple-current-Fixpunkten, die in der Konstruktion von B-Typ Randbedingungen auftreten. In der physikalischen Interpretation führt dies zu einem interessanten neuen Mechanismus für die Erhöhung der Eichsymmetrie auf D-branes. Weiterhin wird in zwei Beispielen gezeigt, wie sich die Beziehung zwischen Gepner-Modellen und Calabi-Yau-Hyperflächen in gewichteten projektiven Räumen auf Randbedingungen und D-branes ausdehnen lässt.

$\mathcal{N} = 2$ Coset-Modelle, die die minimalen Modelle als Spezialfall enthalten, sind eine weitere Klasse von Beispielen von rationalen konformen Feldtheorien mit $\mathcal{N} = 2$ Supersymmetrie. Eine Untermenge von superkonformen Randbedingunen in diesen Modellen können über die Cardy-Konstruktion definiert werden. Über ihre Schnitteigenschaften erhalten die Randzustände eine geometrische Interpretation in der Homologie der Auflösung einer zugeordneten Singularität. Ausserdem zeigt sich, dass die strukturelle Verwandschaft zu Grassmannschen Mannigfaltigkeiten im offenen String bestehen bleibt.

Abstract

The subject matter of the present dissertation is the relation between, on one side, boundary conditions in two-dimensional, $\mathcal{N} = 2$ supersymmetric, and conformal, quantum field theories, and D-branes, which are extended geometric objects appearing in non-perturbative string theory, on the other side. The primary research interest are the properties of D-branes in non-classical or strongly curved backgrounds. In this work, techniques of conformal field theory on worldsheets with boundaries are developed and applied to study the quantum geometry of D-branes.

The first result presented in this thesis is a contribution to the problem of defining boundary conditions in rational conformal field theories. These theories are specified, on closed Riemann surfaces, by a chiral symmetry algebra and a modular invariant torus partition function. The problem of defining boundary conditions depends on both—boundary conditions may, on the one hand, be allowed to break part of the bulk symmetry, and must, on the other hand, only involve bulk fields that are present in the bulk partition function. Now most modular invariants in rational conformal field theories are of simple-current type, and they may or may not exhibit an enlarged chiral symmetry. The problem which is treated here is to determine all boundary conditions that preserve the unextended chiral symmetry algebra, for an arbitrary modular invariant of simple-current type.

The main part of the thesis deals with field theories that in addition to conformal invariance exhibit $\mathcal{N} = 2$ supersymmetry. Such theories appear in the perturbative definition on the worldsheet of the superstring. Additional restrictions are needed to achieve stability and a space-time supersymmetric spectrum. For σ-models, the criterion is that the target be a Calabi-Yau manifold. In an abstract algebraic construction, the necessary projections can be performed starting from any $\mathcal{N} = 2$ rational conformal field theory.

A microscopic description of D-branes requires the introduction of worldsheet boundaries, and the conditions for supersymmetry have to be reexamined. In particular, the present thesis explains how to deal in a systematic way with the projections that arise in the abstract setting of algebraic conformal field theory, and how the most important characteristics of D-branes are encoded in this approach.

The general theory is then illustrated in a number of examples, namely the two-dimensional torus, $\mathcal{N} = 2$ minimal models, Gepner models, and $\mathcal{N} = 2$ coset models (Kazama-Suzuki models).

Gepner models are examples in which rational conformal field theories are used for the internal part of a string compactification. They are built on tensor products of $\mathcal{N} = 2$ minimal models. The main new technical

result on Gepner models is the resolution of simple-current fixed points that appear in the construction of B-type boundary conditions. In the physical interpretation, this leads to an interesting new mechanism for enhancement of gauge symmetry on D-branes. Furthermore, it is shown in two examples how the connection between Gepner models and Calabi-Yau hypersurfaces in weighted projective spaces can be extended to include boundary conditions and D-branes.

$\mathcal{N} = 2$ coset models, which contain minimal models as a special case, are another class of examples of rational conformal field theories with $\mathcal{N} = 2$ supersymmetry. A subset of superconformal boundary conditions in these models can be obtained by Cardy's construction. Through their intersection properties, the boundary states receive a geometric interpretation in terms of the homology of the resolution of an associated singularity. Also, the structural resemblance to Grassmannian spaces is found to extend to the open string.

Chapter 1

Introduction

String theory intends to be a serious candidate for a next unification step in theoretical physics. At the present state of development, quantized, supersymmetric strings have convinced a significant part of the theoretical physics community of their aptitude to describe, in a unified manner, all elementary particles and their known fundamental interactions, including gravity.

In twentieth century high-energy physics, fundamental interactions were described by quantum field theories with gauge symmetries. At the currently most fundamental testable level, physicists rely on the Standard Model of particle physics, with gauge group $SU(3) \times SU(2) \times U(1)$. It accounts for the strong interaction and the electro-weak interaction. The matter content of the Standard Model are the well-known three lepton generations, three quark generations, and the as yet unsignificantly established Higgs field. It is generally expected that many *ad-hoc* features of the Standard Model can be explained from unification at very high energies. The energy scale of this Grand Unification typically is of the order 10^{16}GeV. This is much higher than energies accessible with today's accelerators, and just below the Planck scale of 10^{19}GeV, at which effects of quantum gravity are expected to become relevant.

Part of the appeal of string theory arises from its ability to account for gauge theories in a unified framework in which symmetries and gauge and matter fields have a common, geometric origin.

Besides unification, string theory offers the advantage of a better behaved perturbation theory for the computation of scattering amplitudes for physical processes. For String Theory, the low-energy limit of these scattering amplitudes—at today's accessible high energies—takes the role played by the classical limit for Quantum Mechanics, the low curvature limit for General Relativity, or the low velocity limit for Special Relativity. As in these historical examples, the existence of the limit, and the recovery of previously known results, is the major touchstone for the theory and together with very few (or even without?) non-trivial verifiable new predictions will suffice for a general acceptance of string theory as an embracing physical theory.

In addition to its role for high-energy physics, "string theory predicts gravity" (E. Witten), in the sense that the spectrum of the quantized string contains a spin two excitation, identifiable with the graviton. The classical equations of motion of General Relativity are recovered as the on-shell condition of string perturbation theory. At the same time, string theory makes a quantum theory of gravitation well-defined, at least at the perturbative level.

String theory also has interesting consequences for mathematics, more particularly for geometry. If string theory contains a quantum version of gravity, then, since gravity is fundamentally linked with the geometry of space-time, it must be that strings probe a quantum structure of space-time itself. In the mathematical part of the theory, the usual notions of classical geometry have to be abandoned, and must be replaced with new ones. In other words, in string theory, space-time and its geometry must become derived concepts, and cannot remain fundamental or *a priori*. This aspect of string theory, which, honestly, is largely undiscovered, is referred to as quantum geometry.

While many non-perturbative properties of quantized strings—the very definition of "non-perturbative string theory" included—are still out of sight at present, string theory provides a fascinating guessing ground for theoretical physicists searching for a satisfactory, unified theory of space, time, and matter.

1.1 String perturbation theory

The basic idea of string theory is that elementary particles—the "fundamental" constituents of matter—should not be pictured as pointlike objects, but rather as little strings—one-dimensional extended objects moving in space-time. The following briefly sketches the main steps from classical point-particles to quantized superstrings. For textbook treatments of string theory, see [1, 2].

■ The classical action for a relativistic point-particle moving in space-time, M, is essentially equal to the length of the worldline γ swept out by the moving particle,

$$S(g,\gamma) = m \int_\gamma |\dot{\gamma}(\tau)| \mathrm{d}\tau \,, \tag{1.1}$$

where τ is a coordinate along the worldline $\gamma \subset M$ of the particle, and m is its mass[1]. The action depends on the choice of a metric g on M, which is thought of as defining the "background" that acts on the particle.

▪ The classical description is good as long as $S(g, \gamma)$ is large compared to the fundamental quantum of action, Planck's constant h. This constant then appears in the quantum mechanical description of the particle's propagation in M, for example in the study of path-integrals of the form

$$\int \mathcal{D}\gamma \; \mathrm{e}^{2\pi \mathrm{i} S(g,\gamma)/h} \,. \tag{1.2}$$

▪ It is possible to let also other background fields, for example electromagnetic, act on the particle, simply by adding further terms to the action (1.1). However, the *re-actio* of the particle on the background is not described by (1.1). In fact, already the correct quantum mechanical description of interactions of relativistic particles requires the framework of quantum field theory. Quantum field theory and the Standard Model are extremely successful in describing interactions of fundamental particles, but also have important mathematical problems and well-known conceptual shortcomings. Most notably, the gravitational interaction—mediated by the space-time metric g—cannot be described in this way. "By a historical accident" in the late 1960's, high-energy physicists were led to try to overcome these shortcomings with strings.

▪ Strings moving in space-time sweep out a two-dimensional surface, the worldsheet Σ. The classical action is proportional to the area of Σ in space-time, again computed with the help of a background metric g. Choosing for Σ a parametrization (σ, τ) and a metric h, and denoting by $X : \Sigma \to M$ the embedding into space-time, the classical action is[2]

$$S(g, X) = \frac{1}{4\pi\alpha'} \int_\Sigma \sqrt{h}\, h^{\alpha\beta}\, g_{\mu\nu}(X)\, \partial_\alpha X^\mu\, \partial_\beta X^\nu \,\mathrm{d}\sigma\,\mathrm{d}\tau \,, \tag{1.3}$$

where α' is a fundamental constant with the dimension of a length squared. Notice that the action(1.3) defines a classical field theory, in which the coordinates $X(\sigma, \tau)$ are fields living on Σ. As before, the action may be supplemented with other terms to include more background (space-time) fields. In particular, one may add the coupling to a dilaton field, ϕ, which is of the form $\int_\Sigma \phi(X) R$, where R is the curvature of h.

▪ Armed with 100 years of experience with quantum theory, it is a simple

[1]Here and below, it is taken for granted that the understanding of the role of the speed of light in theoretical physics is complete and allows setting $c = 1$.

[2]To be precise, (1.3) is the generalization of the analog of (1.1) for a massless particle.

matter to quantize (1.3), for example using path integrals. However, one would like to quantize respecting all classical symmetries of (1.3). In particular, the classical action is conformally invariant, and the preservation of this symmetry at the quantum level puts severe constraints on the classical background. While the appearance of the equations of General Relativity (for g) might be rather surprising, it is an even more astonishing feature that the classical equations of motion in string theory constrain even the number of space-time dimensions, to 26 for the bosonic string and 10 for the superstring. Thus, the worldsheet theories of strings are conformally invariant, two-dimensional quantum field theories. It is mainly from this perspective that strings will be studied in this thesis.

▪ That being so simple, it is natural to ask how to describe the interactions of strings and the backreaction on the background. It turns out that interactions can be accommodated in a simple fashion by allowing strings to split and to join. More precisely, quantization of the action (1.3) on a cylinder, $\Sigma = S^1 \times \mathbb{R} \ni (\sigma, \tau)$ corresponds to a single non-interacting string. Considering the corresponding conformal field theory on worldsheets of more complicated topology than the cylinder amounts to including interactions of strings. In the perturbative prescription, the fundamental excitations of the string are interpreted as elementary particles and scattering amplitudes for physical processes describing their interactions are obtained from a sum over all worldsheet topologies. Higher topologies are suppressed by powers of $g_s = \exp(\phi)$, which therefore plays the role of the string coupling constant.

▪ Supersymmetry is a symmetry that relates bosonic and fermionic degrees of freedom of a quantum theory. The discovery of supersymmetry in the early 1970's was motivated in part by the tendency of supersymmetric theories to have milder divergences as compared to ordinary quantum field theories. The main phenomenological interest for supersymmetry lies in the fact that the supersymmetric version of the Standard Model predicts a convergence of the coupling constants of the electro-magnetic, the weak, and the strong force at a single unification scale around 10^{16}Gev. The late 1970's also witnessed a major interest in theories with local versions of supersymmetry, yielding supergravity theories. Indeed, the improved convergence properties of supersymmetric theories led to the hope that gravity could be consistently quantized after introduction of supersymmetry. Until today, local supersymmetry still is an important ingredient for models of quantum gravity, mainly as the low energy limit of superstring theory.

▪ Supersymmetry was incorporated into string theory from the beginning. The basic idea is to add fermionic degrees of freedom that propagate along the string, in such a way that after quantization, the worldsheet theory has superconformal invariance. In the development of string theory until around

1985 it was realized that supersymmetry allows the construction of truly consistent and phenomenologically interesting string models. It turned out that the presence of extended worldsheet supersymmetry, *i.e.*, from $\mathcal{N} = 1$ to $\mathcal{N} = 2$, allows performing the GSO projection that eliminates the tachyon from the string spectrum and guarantees space-time supersymmetry of the resulting spectrum. Also, it was shown that the uncompactified version of the superstring, which naturally lives in 10 space-time dimensions is a satisfactory quantum theory, in which anomalies cancel by the Green-Schwarz mechanism. Until 1985, then, the basic five types of perturbatively defined uncompactified string theories had been discovered: Type I with gauge group SO(32), type IIA and type IIB, and two heterotic theories with gauge group SO(32) and E8 $\times$ E8, respectively.

1.2 String compactification

As explained in the last section, strings require space-time to have a critical dimension, which is $26 = 25 + 1$ for the bosonic and $10 = 9 + 1$ for the supersymmetric string. Thus, if string theory describes the world of experience, which has dimension $4 = 3 + 1$, it is natural to ask how this may come about. A possible answer is that only 4 of the 10 (or 26) dimensions are actually extended to sizes larger than the Planck scale, and the remaining 6 dimensions are "curled up" and too small to be resolved. This means that the background of space-time fields, or vacuum of the quantized string, describes a manifold with 6 dimensions of typical size much smaller than the remaining 4. To ensure conformal invariance of the perturbative string theory, and space-time supersymmetry for a stable vacuum, the vacuum must be, to lowest order, a supersymmetric solution of the supergravity equations of motion. A careful analysis of these equations of motions and supersymmetry conditions reveals that the 6 compact dimensions must form a Calabi-Yau manifold.

A closely related, if seemingly different, approach to compactification is to directly specify a suitable conformal field theory for the internal part without any reference to a classical geometric space-time.

In either case, the choice of the internal part of the compactification has important consequences for the physics in the external part, which is the usual flat extended Minkowski space. As is familiar from Kaluza-Klein theory, the spectrum of fields of the internal conformal field theory determines the field content of the low-energy effective theory in the external dimensions.

1.3 Non-perturbative aspects, string dualities, and D-branes

Interacting string theory contains two parameters along which the theory is modified from the free case. On the one hand, there is the string coupling, g_s, that governs interactions among strings in space-time. The role played by g_s in the string loop expansion is similar to the role played by $\hbar$ in field theory. On the other hand, also the worldsheet theory, which is a two-dimensional conformal field theory, is not "free". "Interactions" on the worldsheet are controlled by the string tension, α', or more precisely by the scale of space-time curvature in units of $1/\alpha'$. The couplings in the low-energy effective theory depend on both g_s and α'. In a first attack, α' and g_s are treated as perturbative parameters, leading to σ-model- and string perturbation theory, respectively. But then there are also quantum corrections to the classical description that are non-perturbative in the two couplings. The last ten years of string theory have witnessed significant progress in understanding these non-perturbative effects.

Choosing a specific Calabi-Yau manifold as compactification space for the superstring solves the zeroth-order requirement for a string vacuum. Possible quantum corrections to this classical solution are tightly constrained by symmetries. For instance, the relevant symmetry algebra on the worldsheet is the $\mathcal{N} = 2$ super-Virasoro algebra [3], which combines two very powerful algebraic structures: conformal symmetry in two dimensions [4] and $\mathcal{N} = 2$ supersymmetry. It plays an important role in the problems treated in the present thesis.

On the worldsheet, non-perturbative effects are due to worldsheet instantons, [5, 6] which are topologically non-trivial embeddings of the worldsheet into the target Calabi-Yau space. Worldsheet instantons correct string scattering amplitudes—or effective couplings of space-time fields—beyond σ-model perturbation theory. In string theory, the low-energy space-time fields can be viewed as moving on the parameter space of the target manifold, and the couplings as data from additional geometric structure on this parameter space. Thus, one may interpret worldsheet instantons as effectively deforming the classical parameter space of the manifold into a "quantum moduli space". This quantum parameter space is the relevant object for low-energy physics. It can be regarded as a first glimpse into "quantum geometry", the modification of classical geometry described by string theory.

Exact results on the structure of the quantum moduli space of Calabi-Yau manifolds for type II and heterotic strings have been obtained by mirror symmetry [7]. A basic consequence of supersymmetry is that the moduli space is locally a direct product of two separate structures [8, 9]. From the point of view of the Calabi-Yau manifold, these are the complex structure moduli space and the Kähler moduli space. The corresponding structures from the space-time point of view are, for instance, the vector- and hypermultiplet moduli spaces of the low-energy effective theory of the type II string. Non-perturbative corrections by worldsheet instantons only affect the Kähler moduli space and leave the complex structure moduli space untouched. The proposal of mirror symmetry [10, 11] is that to every Calabi-Yau manifold Y, there is a mirror manifold Y^* such that the quantum Kähler moduli space of Y is isomorphic to the complex structure moduli space of Y^* and vice-versa. Thus, worldsheet instanton corrections of the Kähler moduli space of Y can be computed purely classically in the complex structure moduli space of Y^* [12, 13].

While non-perturbative quantum corrections on the string worldsheet are under control at least conceptually, and partly also computationally, this is not true for non-perturbative corrections in space-time. The main result of the "third superstring revolution" in 1995 was the access to certain non-perturbative effects through string dualities [14, 15]. Here, the word "duality" refers to a non-perturbative equivalence of physical theories that look rather different at the perturbative level. In other words, one and the same physical theory admits several distinct—dual—perturbative definitions, each valid in a different regime of parameters. Mirror symmetry is actually an example of a duality, albeit one that does not involve the string coupling, g_s. The current picture is that there is an extended "web of dualities" that involves and connects all five perturbative string theories. This picture has also led to the expectation that there is an underlying, even more fundamental theory called M [16], that reduces to the various string theories in certain regions of parameter space.

An important role in the context of string dualities is played by D-branes. The branes in question arose as certain solitonic solutions of the classical supergravity equations [17]. These solutions are extended in a certain number of space-time directions and include a non-trivial configuration of the Ramond-Ramond (RR) fields. It was then noticed that string solitons are the degrees of freedom that appear in non-perturbative sectors of and provide the link between, the various perturbatively defined string theories. The non-perturbative character of these objects is easily seen. On the one hand, the curvature of space-time diverges at the position of the brane, and perturbative string theory in the traditional sense breaks down. On the other hand, they contain RR field configurations, to which the elementary string does not couple.

It was realized by Polchinski [18] that in fact there is an object in string theory that does carry RR charge, namely the boundary condition for open strings [19]. Hence, branes should be described by letting open strings end on them. The fact that the open string boundary conditions are of Dirichlet type in the directions orthogonal to the brane led to the name D(irichlet)-brane. Upon quantization, the open strings represent the elementary degrees of freedom of the brane [20]. The open string picture also avoids the singularity at the position of the brane [21], and hence gives an interesting alternative approach to the exploration of space-time at small distances.

The role of D-branes for string theory is twofold. On the one hand, they are fundamental for string dualities as microscopic degrees of freedom. On the other hand, when appropriately combined with orientifolds [22, 23] they can be included in the background geometry, and thus multiply the freedom of choice of a perturbative vacuum. In both cases, open string sectors with appropriate boundary conditions appear in the description. This necessitates the study of conformal field theories on the string worldsheet also in the presence of boundaries.

1.4 Summary and outline

The subject of this research arises from a combination of various topics that were discussed above. The general goal is to develop techniques for finding boundary conditions for open strings in type II string theory, to carefully analyze the implications of supersymmetry, both on the worldsheet and in space-time, for the construction, and to determine the consequences for the modification of geometry described by strings and branes. In the following outline of the thesis, the references point partly to the places where this work is published, and partly to additional literature which is relied upon in the presentation of background material.

Chapter 2 contains a review of two-dimensional conformal field theory with boundaries [24, 25]. The ideas are developed along the guiding principle that there are two conceptual stages of conformal field theory (CFT), chiral CFT and full CFT. Chiral CFT is the stage at which the chiral symmetries of the theory are implemented. In fact, chiral CFTs can be completely reconstructed from the representation theory of algebraic objects known as vertex operator algebras. The step from chiral CFT to full CFT then is a projection problem, subject to the physical requirements of locality, modular invariance, and factorization. Without boundaries, the field content of a full CFT is encoded in the modular invariant torus partition function. Boundary conditions can be understood as parametrizing solutions to the projection problem if the worldsheet of the full CFT has boundaries. The second part of this chapter presents results on the problem of defining boundary conditions for rational conformal field theories with arbitrary modular invariant of simple-current type [26]. Simple currents are the invertible elements of the fusion ring of a rational CFT, and constitute a powerful combinatorial tool to analyze modular constraints on a CFT. In particular, most known modular invariants of rational CFT can be constructed with simple-current techniques. These modular invariants may also exhibit an enlarged chiral algebra, but the boundaries will not be required to preserve this larger algebra. Thus, the results presented in section 2.2 cover a rather general class of situations in rational CFT with boundaries.

Chapter 3 turns to a second important ingredient of worldsheet theories for perturbative string theory: $\mathcal{N} = 2$ supersymmetry. The first part of the chapter is a review of σ-models on Kähler and Calabi-Yau manifolds, which are used as the internal part of compactifications of the type II string, and of the definition of supersymmetric boundary conditions (D-branes) for these models [27, 28, 29]. The boundary conditions fall into two main classes. Those of A-type correspond to special Lagrangian submanifolds equipped with a flat U(1) connection, while those of B-type correspond to holomorphic objects such as stable holomorphic vector bundles. The second part of the chapter explains the algebraic approach to string compactification and boundary conditions therein. It is laid out in detail how the various projections can be taken into account in a systematic way in defining boundary conditions [30]. Furthermore, the space-time supersymmetries that are broken or preserved by the boundary conditions are identified, and many other characteristics of D-branes, such as their mass and Ramond-Ramond charge, are identified in the abstract setting. Furthermore, a general formula for the intersection index of two boundary states is derived [31].

Chapter 4 illustrates the general theory of chapters 2 and 3 in a large class of examples. The examples show that the combination of algebraic and geometric methods leads to interesting results about D-branes in the strong-curvature regime.

Section 4.1 contains material about boundary conditions for the two-dimensional torus. The main goal is to show that here the algebraic and geometric approaches lead to the same results. This also gives a good check on the formalism and the developing intuition.

Section 4.2 reviews boundary conditions in $\mathcal{N} = 2$ minimal models [32, 33, 29], and their relation to strings and branes near simple singularities of ADE type. The results about minimal models are also useful input for the next section.

Section 4.3 then deals with Gepner models, which are built on tensor products of $\mathcal{N} = 2$ minimal models. A-type boundary conditions in Gepner models can be defined following the prescriptions of section 3.2. The discussion is aimed in part at clarifying results that have appeared in the literature [34, 35, 36]. B-type boundaries are then constructed by making use of the self-mirror property of Gepner models. Namely, the Greene-Plesser construction allows to reduce the problem to the question of boundary conditions for simple-current modular invariants, for which the results of section 2.2 provide the clue. Fixed points under the projections are identified and resolved both for A- and B-type boundary conditions. In two examples [37], the B-type boundary conditions of the Gepner are then related to geometric objects on the associated Calabi-Yau hypersurfaces, following the work of [32] for the quintic. The two models under considerations have the structure of K3-fibrations over $\mathbb{P}^1$. The fixed points in B-type boundary conditions can be interpreted in physical terms. The stabilizers are realized only projectively, and this implies that the worldvolume theory exhibits an unusual enhancement of gauge symmetry [38], somewhat similarly to orbifolds with discrete torsion.

Finally, section 4.4 analyzes the properties of boundary states in $\mathcal{N} = 2$ coset models based on Grassmannians $\mathrm{Gr}(n, n+k)$ [31]. The underlying intersection geometry is given by the fusion ring $\mathrm{U}(n)$. This is isomorphic to the quantum cohomology ring of $\mathrm{Gr}(n, n+k+1)$, and thus can be encoded in a "boundary" superpotential whose critical points correspond to the boundary states. In this way, the intersection properties can be represented in terms of a soliton graph that forms a generalized, $\mathbb{Z}_{n+k+1}$ symmetric McKay quiver. Investigating the spectrum of bound states, it turns out that the states obtained from rational conformal field theory produce only a small subset of the possible quiver representations.

Chapter 5 contains conclusions.

Chapter 2

Boundary conditions in rational conformal field theory

Worldsheet theories of perturbative string theory are conformally invariant quantum field theories (CFTs). This chapter is devoted to such CFTs, allowing in particular the presence of boundaries, as necessary for a worldsheet description of D-branes. The first part of the chapter reviews certain aspects, mainly of algebraic nature, of conformal field theory in two dimensions. It is based on [24, 25]. The second part is more specifically concerned with boundary conditions for arbitrary simple-current modular invariants in rational conformal field theories. This part contains results of [26].

2.1 Conformal field theory with boundaries

2.1.1 From chiral CFT to full CFT

For the constructions described below, it is necessary to distinguish two conceptual levels of CFT, chiral conformal field theory (χCFT) and full conformal field theory (full CFT), and to understand the construction of a full CFT as a two-step process. The underlying physical idea is to "split a CFT into two chiral halves", and to recombine them afterwards by a projection.

Thus, χCFT considers only chiral (left- or right-moving) degrees of freedom at a time. The large amount of symmetry implied by conformal invariance in two dimensions [4] imposes strong constraints on and greatly simplifies the study of χCFT. To define a χCFT, it suffices to specify a chiral algebra, which includes at least the (super-)Virasoro algebra, and a set of irreducible representations of the chiral algebra, closed under fusion. The natural arena for the Euclidean version of χCFT are complex one-dimensional (super-)manifolds (complex curves). In a mathematical language, χCFT can be reduced to the representation theory of the chiral algebra, formalized in what are known as vertex operator algebras.

Full CFT, on the other hand, contains both left- and right-moving degrees of freedom. Typically, a full CFT is obtained from a χCFT by a projection operation, in a way that will be described further below. All remaining constraints on the theory, such as locality, modular invariance, factorization constraints, etc., are implemented in going from χCFT to full CFT. Accordingly, the definition of a full CFT based on the data of a given χCFT, amounts to finding a solution of these constraints. It is important to realize that a string background can be defined only after a full CFT has been constructed. For instance, modular invariance, required for integrating correlation functions over the moduli space of curves, is satisfied only by full CFT. But string theory also imposes constraints on the CFT that can be traced back to the chiral level, such as worldsheet supersymmetry. In addition, the choice of a particular D-brane background is equivalent to the assignment of Chan-Paton multiplicities to the various possible boundary conditions on worldsheet boundaries. Again, this is subject to certain conditions, such as supersymmetry and absence of anomalies in the space-time theory.

While full CFT is the starting point of (perturbative) string theory, χCFT also has numerous and beautiful applications in physics and has found sounding resonance in mathematics itself. An particularly nice example is the use of χCFT for the description of incompressible quantum Hall fluids (see [39], and [40] for recent work on this problem).

The starting point of the discussion in this section is the description of the "arenas" on which the CFTs are defined. Then a summary of χCFT will follow, and, finally, the projection to full CFT is treated.

The worldsheet

A full CFT lives on a conformal manifold Σ which might be unoriented and can have boundaries. Topologically, such manifolds are classified by three quantities: the number $g \in \{0, 1, 2, \dots\}$ of handles, the number $b \in \{0, 1, 2, \dots\}$ of boundaries, and the number $c \in \{0, 1, 2\}$ of crosscaps. The Euler characteristic (that determines the order in the string loop expansion) of such a manifold is

$$\chi = 2 - 2g - b - c\,. \tag{2.1}$$

To every such manifold, there corresponds a double cover $\hat{\Sigma}$, which is an oriented manifold without boundaries. The defining property is that Σ can be obtained from $\hat{\Sigma}$ by dividing out an orientation reversing involution σ,

$$\Sigma = \hat{\Sigma}/\sigma\,. \tag{2.2}$$

The manifold $\hat{\Sigma}$ is characterized topologically by its Euler characteristic or its genus,

$$\hat{\chi} = 2 - 2\hat{g} = 2\chi\,. \tag{2.3}$$

The choice of a conformal structure on Σ induces a complex structure on $\hat{\Sigma}$, which hence naturally is a complex curve. The space of conformal (or complex) structures form the moduli space of Σ (or $\hat{\Sigma}$).

In the study of correlation functions in CFT, the conformal manifolds, or complex curves, appear with punctures, *i.e.*, insertion points of field operators. The moduli spaces of an n-punctured complex curve will generically be denoted by $\mathcal{M}(\hat{\Sigma}_n)$, the corresponding universal covering space (the Teichmüller space) by $\mathcal{T}(\hat{\Sigma}_n)$. For a corresponding conformal manifold, there is a distinction between bulk and boundary insertions. The moduli space with n bulk and m boundary insertions will be denoted by $\mathcal{M}(\Sigma_{n|m})$, the Teichmüller space by $\mathcal{T}(\Sigma_{n|m})$.

The central idea is the following:

> *Full CFT on a conformal manifold Σ is constructed from chiral CFT on the double $\hat{\Sigma}$ of Σ.*

Chiral CFT

A χCFT is a quantum field theory [1] on $\hat{\Sigma}$, respecting the given complex structure. Explicitly, this means that the fields of such a theory are holomorphic. Among all fields of a χCFT, the local ones are distinguished. These local chiral fields form an algebra of operator-valued distributions on the Hilbert space of physical states. This algebra is called the chiral algebra of the theory and is denoted by $\mathcal{A}$. Among the fields generating $\mathcal{A}$, there is the energy-momentum tensor T of the theory. The coefficients of the Laurent expansion of T in a chosen local complex coordinate z, L_n, then satisfy the commutation relations of the Virasoro algebra.

A unitary[2] χCFT with chiral algebra $\mathcal{A}$ can be reconstructed from the unitary representations of $\mathcal{A}$ [41]. Let λ label a unitary representation of $\mathcal{A}$, and let $\mathcal{H}_\lambda$ denote the corresponding representation space, which is a Hilbert

[1] of a rather special kind. It is not a local quantum field theory in the usual sense.

[2] Unitarity is assumed here for simplicity. It is a reasonable requirement when one is considering the internal part of a string compactification. However, in full string theory, non-unitary theories do appear.

space. To each such representation is assigned a nonnegative number, Δ_λ, called the conformal weight of the representation. It is defined as the minimal eigenvalue of the zero-mode operator, L_0, of the energy-momentum tensor,

$$\Delta_\lambda = \inf\left\{\langle v_\lambda, L_0 v_\lambda\rangle \mid v_\lambda \in \mathcal{H}_\lambda, \|v_\lambda\| = 1\right\} . \tag{2.4}$$

In a consistent theory there is always a unique irreducible vacuum representation, ω, characterized by the vanishing of the conformal weight, $\Delta_\omega = 0$.

Given two representations, λ and μ, one can define their fusion, namely a tensor product representation, $\lambda * \mu$, which is again a unitary representation of $\mathcal{A}$. A chiral algebra is called rational if the number of inequivalent, irreducible unitary representations is finite. Let I denote the set of such representations. For a rational chiral algebra, the tensor product of two representations can be decomposed into a direct sum of irreducible unitary representations. Thus, the set of unitary irreducible representations of a rational chiral algebra, furnished with the tensor product, has the structure of a commutative, associative ring. For λ_1, λ_2 and λ_3 in I, let $N_{\lambda_1,\lambda_2}^{\lambda_3}$ denote the multiplicity of λ_3 as a subrepresentation in the tensor product $\lambda_1 * \lambda_2$. The multiplicities $N_{\lambda_1,\lambda_2}^{\lambda_3}$ are the structure constants of the ring and are called fusion rules; for a rational chiral algebra, they are finite non-negative integers. The vacuum representation, ω, plays the role of the unit for the tensor product, *i.e.*, $\lambda * \omega = \omega * \lambda = \lambda$. For every irreducible representation λ there is a contragredient (or conjugate) representation λ^+ with the property that $\lambda * \lambda^+$ contains the vacuum representation ω exactly once as a subrepresentation.

Given a number n of irreducible unitary representations, $\lambda_1, \ldots, \lambda_n$, the linear space of conformal blocks is defined as the space of invariant tensors, *i.e.*, of invariant linear functionals, on the representation space of the tensor-product representation $\lambda_1 * \cdots * \lambda_n$. It actually turns out (see [42]) that in the definition of the tensor product representation $\lambda_1 * \cdots * \lambda_n$, one can introduce a dependence on n complex parameters $z_1, \ldots, z_n$, which can be considered as local coordinates of pairwise different points of the worldsheet $\hat{\Sigma}$. This is how the representation theory of $\mathcal{A}$ yields back the χCFT. For this reason, the space of conformal blocks,

$$V_{\hat{\Sigma}}(z_1, \lambda_1, \ldots, z_n, \lambda_n) , \tag{2.5}$$

depends on the complex parameters $z_1, \ldots, z_n$. Its dimension is given by

$$N_{\vec{\lambda}} := N_{\lambda_1,\ldots,\lambda_n} = \sum_{\mu_1,\ldots,\mu_{n-3}} N_{\lambda_1\lambda_2}^{\mu_1} N_{\mu_1\lambda_3}^{\mu_2} \cdots N_{\mu_{n-3}\lambda_{n-1}}^{\bar{\lambda}_n} , \tag{2.6}$$

and does not depend on the parameters $z_1, \ldots, z_n$. The more intuitive physical content of a conformal block is as a correlation function of the χCFT:

By state-field correspondence, to every vector $v_\lambda \in \mathcal{H}_\lambda$ is associated a field $\Phi_\lambda(v_\lambda, z)$, such that the conformal blocks (2.5), evaluated on $(v_1, \ldots, v_n) \in \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$ give precisely the vacuum expectation value of the product of the corresponding fields,

$$V_{\hat{\Sigma}}(z_1, \lambda_1, \ldots, z_n, \lambda_n)(v_1, \ldots, v_n) = \langle \Phi_{\lambda_1}(v_1, z_1) \cdots \Phi_{\lambda_n}(v_n, z_n) \rangle_{\hat{\Sigma}}\,, \qquad (2.7)$$

where $z_1, \ldots, z_n$ are local coordinates of pairwise distinct insertion points on Σ.

An important insight into the structure of χCFT is that the spaces of conformal blocks fit together into a vector bundle of rank given by $N_{\vec{\lambda}}$, over the moduli space of the n-punctured curve $\hat{\Sigma}_n$. This bundle is denoted by $\mathcal{V}(\hat{\Sigma}_n, \vec{\lambda})$. It is naturally equipped with a projectively flat connection, called Knizhnik-Zamolodchikov connection. In this language, the correlators of χCFT are just horizontal sections of $\mathcal{V}(\hat{\Sigma}_n, \vec{\lambda})$. Since this bundle is non-trivial, however, the horizontal sections are not global sections. The holonomy of the Knizhnik-Zamolodchikov connection defines a representation of the fundamental group of the moduli space of n-punctured curves (the braid group on n strands, if the curve is connected) on the spaces of conformal blocks.

To repeat, the data of a χCFT consists of a chiral algebra and a set of irreducible representations closed under fusion. The solution of a χCFT consists of the conformal blocks and the associated representation of the braid group (fusing and braiding matrices). The present discussion suffices for understanding the construction of a full CFT at the level of correlation functions. More precise representation-theoretic definitions of conformal blocks and references to mathematically rigorous studies of their properties can be found, *e.g.*, in [43, 24].

Full CFT

Assume given a conformal manifold Σ with boundary $\partial\Sigma$, and imagine a situation with a vector of insertion points in the bulk, $\vec{p} = (p_1, \ldots p_n)$, with $p_i \in \Sigma \setminus \partial\Sigma$ and a number of insertions on the boundary, $\vec{q} = (q_1, \ldots q_m)$ with $q_i \in \partial\Sigma$. By definition of the Schottky cover $\hat{\Sigma}$, every insertion point p_i has two preimages, $z_i \in \hat{\Sigma}$ and $\tilde{z}_i = \sigma(z_i)$ under the projection $\hat{\Sigma} \to \Sigma$, while every boundary point q_i has only a single preimage $w_i = \sigma(w_i)$. From the χCFT that is assumed given on $\hat{\Sigma}$, one may attach to each p_i two chiral labels: λ_i, associated with z_i, and $\tilde{\lambda}_i$, associated with $\tilde{z}_i$. On the other hand, every q_i carries but one chiral label, μ_i. This determines the labelling of fields of the full CFT, but only partially; what combinations $(\lambda_i, \tilde{\lambda}_i)$ of chiral labels are allowed for the bulk insertions, and what further labels are needed to distinguish boundary insertions, is part of the data to be determined.

At the level of the χCFT, the situation is then specified by data $\vec{\hat{z}} = (\vec{z}, \vec{\tilde{z}}, \vec{w})$ and $\vec{\hat{\lambda}} = (\vec{\lambda}, \vec{\tilde{\lambda}}, \vec{\mu})$. The χCFT then determines a bundle of conformal blocks over the moduli space of an $\hat{n}$-punctured curve with $\hat{n} = 2n + m$,

$$\mathcal{V}(\hat{\Sigma}_{\hat{n}}, \vec{\hat{\lambda}}) \,. \tag{2.8}$$

As explained above, the correlators of the χCFT are the local horizontal sections of this bundle of conformal blocks. To construct correlation functions for the full CFT on Σ, proceed as follows. Lift the bundle $\mathcal{V}(\hat{\Sigma}_{\hat{n}}, \vec{\hat{\lambda}})$ from $\mathcal{M}(\hat{\Sigma}_{\hat{n}})$ to $\mathcal{T}(\hat{\Sigma}_{\hat{n}})$, restrict this bundle to $\mathcal{T}(\Sigma_{n|m}) \subset \mathcal{T}(\hat{\Sigma}_{\hat{n}})$, and denote it by $\mathcal{W}(\Sigma_{n|m}, (\vec{\lambda}, \vec{\tilde{\lambda}}, \vec{\mu}))$. This bundle inherits a connection from the Knizhnik-Zamolodchikov connection on $\mathcal{V}$. The important point is that the bundle $\mathcal{W}$ might (and indeed should) admit horizontal sections that are globally defined over $\mathcal{M}(\Sigma_{n|m})$. This expresses the main requirement of locality of the correlation functions of the full CFT.

The intuitive reason that $\mathcal{W}$ admits global horizontal sections even if $\mathcal{V}$ does not is, for bulk insertions, that the monodromies in one chiral label λ are canceled by those in the other, $\tilde{\lambda}$. This clarifies that a judicious choice of allowed combinations $(\lambda, \tilde{\lambda})$ has to be made. For boundary insertions, the monodromies are lost basically because in Σ, boundary insertions cannot be moved around each other or around bulk insertions. The monodromies of the corresponding insertions in the cover $\hat{\Sigma}$ thus disappear.

The requirements of locality with respect to bulk insertions and moduli of the curves (modular invariance) are still not enough to single out a unique solution of the projection problem χCFT $\to$ full CFT. There are in general several possible choices of allowed pairings $(\lambda, \tilde{\lambda})$ that yield a consistent solution. There is even more freedom in the presence of boundaries, for roughly the same reasons as above. In string theory, it is not necessary to parametrize the set of solutions for every topology of Σ independently. This is because amplitudes are required to factorize. Factorization means that at the boundary of the moduli space of the two-dimensional manifold, where the manifold is singular, the amplitude can also be written in terms of amplitudes on a manifold with blown-up singularity. This allows reducing the problem to low topologies, namely to worldsheets with Euler characteristic $\chi \geq 0$.

The torus partition function

In the bulk (*i.e.*, without boundaries, and on orientable surfaces), the spectrum of the full CFT is determined from the modular invariant torus partition function. This is by definition the vacuum correlation function $Z(\tau)$ on the torus, where τ is the modular parameter of the torus. Written as

$$Z(\tau) = \sum_{\lambda,\tilde{\lambda}} Z_{\lambda\tilde{\lambda}} \bar{\chi}_\lambda(\bar{\tau})\, \chi_{\tilde{\lambda}}(\tau)\,, \tag{2.9}$$

the partition function encodes the spectrum of allowed bulk fields for any topology. Here, $\chi_\lambda(\tau)$ are the characters of the χCFT, and $Z_{\lambda\tilde{\lambda}}$ is a matrix with non-negative entries that commutes with the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on the characters, *i.e.*,

$$[Z,S] = 0 = [Z,T]\,, \tag{2.10}$$

where S and T are the modular S- and T-matrices, respectively. The condition (2.10) is the requirement of modular invariance on the torus.

Clearly, the expansion (2.9) depends on the choice of chiral algebra, which determines the range of λ and $\tilde{\lambda}$. When classifying modular invariants, it is expected that there is always a unique maximally extended chiral algebra, $\bar{\mathcal{A}}$, with respect to whose characters the entries of the matrix $Z_{\lambda\tilde{\lambda}}$ are either 0 or 1. In other words, working with the maximally extended chiral algebra, all bulk fields are uniquely labelled by the allowed combinations of chiral labels $(\lambda,\tilde{\lambda})$. However, it might be more convenient to neglect the extension of the chiral algebra to $\bar{\mathcal{A}}$ expressed in (2.9) and to work with a smaller algebra $\mathcal{A} \subset \bar{\mathcal{A}}$. It is then necessary to enlarge the range of chiral labels λ, because irreducible representations of $\bar{\mathcal{A}}$ can become reducible when restricted to $\mathcal{A}$, and also to allow for $Z_{\lambda\tilde{\lambda}} > 1$ to account for the fact that one and the same irreducible representations of $\mathcal{A}$ can be embedded in inequivalent ways into representations of $\bar{\mathcal{A}}$.

2.1.2 Boundary conditions

The presence of worldsheet boundaries requires new prescriptions for the projection from χCFT to full CFT. A natural proposal is based on the expectation that it should be possible to parameterize the set of solutions by attaching the label of a "boundary condition" to every boundary component. When referring to boundary components, also any boundary insertion is regarded as separating the boundary into different components. Any such boundary insertion is then interpreted as a boundary field Ψ, and receives, in addition to the chiral label μ, the two labels a, b of the adjacent boundary conditions. In particular, for $a \neq b$ a boundary field can be viewed as changing the boundary condition. As in the bulk, the spectrum of boundary fields is encoded in a partition function. In this case, it is the annulus with boundary conditions a and b on the two boundaries,

$$Z_{ab}(t) = \sum_{\mu} A^{\mu}_{ab} \chi_{\mu}(t) \,. \tag{2.11}$$

Here t is the modular parameter of the annulus, and $A^{\mu}_{ab} \in \mathbb{Z}_{\geq 0}$ are the annulus coefficients. An annulus coefficient greater than 1 indicates a degeneracy of boundary fields, *i.e.*, there is more than one way to transform boundary condition a into boundary condition b, using the same chiral representation μ. The boundary field Ψ then has an additional degeneracy label A, and the full labelling is therefore of the form $\Psi^{a\,A\,b}_{\mu}$.

On the other hand, boundary fields are not taking part in the characterization of the boundary conditions themselves. A boundary condition can therefore be regarded as a solution to the factorization constraints for surfaces Σ with a single boundary component and only bulk insertions. Moreover, factorization (*e.g.*, of the Möbius strip to a crosscap plus a disc) allows to restrict attention to the case where Σ is the disc and where there is a single bulk insertion.

Individual boundary conditions are thus determined by the properties of bulk correlators on a disc. At the chiral level, these correlators correspond to conformal blocks on the sphere, $\hat{\Sigma} = \mathbb{CP}^1$, with an even number of insertions. The moduli space of three or less points on a sphere is trivial, so non-trivial constraints arise for the first time from the four-point blocks. These blocks appear also in the familiar case of correlators of four bulk insertions on $\Sigma = S^2$, as well as for four boundary insertions on the disc. In both cases, factorization of the four point blocks is used to derive constraints for the operator product coefficients, and ultimately to solve for them. Here, two-point functions on the disc provide constraints for the boundary conditions.

The classifying algebra

To proceed, introduce the bulk-boundary operator product

$$\Phi_{\lambda\tilde{\lambda}}(z,\bar{z}) \sim \sum_{\nu\in I} \sum_{B} (1-|z|^2)^{-2\Delta_\lambda + \Delta_\nu} \mathcal{R}^{a}_{\lambda\tilde{\lambda}\nu} \Psi^{a\,B\,a}_{\nu}(\arg z) \qquad \text{for } |z| \to 1 \,. \tag{2.12}$$

This OPE expresses what happens when a bulk field $\Phi_{\lambda\tilde{\lambda}}$ approaches the boundary of the disc $|z| \leq 1$ with boundary condition a: It creates excitations on the boundary, described by the boundary operators $\Psi^{a\,B\,a}_{\nu}$.

Consider now the situation with two bulk insertions, $(\lambda_1, \tilde{\lambda}_1)$ and $(\lambda_2, \tilde{\lambda}_2)$ and one boundary insertion of the vacuum, ω. Two factorizations of this amplitude are possible. First using the bulk OPE to produce a single bulk field and then considering its bulk-boundary operator product, yields an expression which contains the reflection coefficient $\mathcal{R}^a$ once. The other factorization is to apply (2.12) to both bulk fields; then two reflection coefficients $\mathcal{R}^a$ appear. Comparison of the two factorizations yields an identity of the form

$$\mathcal{R}^a_{\lambda_1\tilde{\lambda}_1\omega}\,\mathcal{R}^a_{\lambda_2\tilde{\lambda}_2\omega} = \sum_{\lambda_3\in I} \tilde{\mathcal{N}}_{\lambda_1\lambda_2}^{\ \ \lambda_3}\,\mathcal{R}^a_{\lambda_3\tilde{\lambda}_3\omega}\,, \tag{2.13}$$

where $\tilde{\mathcal{N}}_{\lambda_1\lambda_2}^{\ \ \lambda_3}$ is some complicated combination of operator product coefficients and representation matrices for $\pi_1(\mathcal{M}_{4,0})$ acting on four-point blocks.

The structure encoded in (2.13) is that, for a fixed boundary condition, a, the quantities $\mathcal{R}^a$ form a one-dimensional representation of an associative algebra with structure constants $\tilde{\mathcal{N}}_{\lambda_1\lambda_2}^{\ \ \lambda_3}$. This algebra is called the classifying algebra [44]; it encodes a piece of structure that a consistent set of boundary conditions is expected to possess.

Ishibashi states and boundary states

The reflection coefficients $\mathcal{R}$ also determine the correlation functions of a single bulk field, $\Phi_{(\lambda,\tilde{\lambda})}$, at the center of a disc with boundary condition a. Other positions of the bulk field can be related to this case by using the action of the Möbius group $\mathrm{SL}(2,\mathbb{R})$ on the disc. In this situation, the reflection coefficients appear simply as the expansion coefficients of these one-point functions in terms of the relevant conformal blocks of the χCFT on the cover of the disc. Both these one-point functions and conformal blocks can be conveniently written in terms of "boundary states", which are introduced as follows.

On the chiral level, the correlator in question is given by two-point blocks on the sphere, $\mathbb{CP}^1$, with insertions at $z_\lambda = 0$ and $z_{\tilde{\lambda}} = \infty$. By definition, the two-point blocks are invariant functionals $\beta \in (\mathcal{H}_\lambda \otimes \mathcal{H}_{\tilde{\lambda}})^*$ with the property

$$\beta \circ (J^a_n \otimes \mathbf{1} + \mathbf{1} \otimes J^a_{-n}) = 0\,, \tag{2.14}$$

for every current $J^a(z) = \sum_n J^a_n z^{n-1}$ in the chiral algebra (assuming that the chiral algebra is a current algebra). By a version of Schur's lemma, non-trivial functionals β obeying (2.14) exist only if λ and $\tilde{\lambda}$ are conjugate representations of the chiral algebra. Even if these linear functionals are not in the Hilbert space dual of $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$, they are usually written as kets $\|\lambda\rangle\rangle$ and called Ishibashi states. In terms of those, formula (2.14) reads

$$(J^a_n \otimes \mathbf{1} + \mathbf{1} \otimes J^a_{-n})\,\|\lambda\rangle\rangle = 0\,. \tag{2.15}$$

It is sometimes possible to write down the Ishibashi state $\|\lambda\rangle\rangle$ explicitly; *e.g.*, for theories based on a free boson, it can be written as a generalized coherent state,

$$\|\lambda\rangle\rangle = \exp\Big(-\sum_{n=1}^{\infty} b_{-n}\otimes b_{-n}\Big)\, v_\lambda\,, \tag{2.16}$$

where v_λ is the highest weight state in the tensor product of Fock spaces, generated by the oscillator creation operators, b_{-n}. Such a realization is helpful for calculating one-point functions on a disc explicitly. It is, however, not necessary to know such an explicit realization to determine the spectrum of boundary fields. For this, it is sufficient to know how $\|\lambda\rangle\rangle$ behaves under factorization. The crucial information that allows to calculate concretely with boundary states is the following identity that relates two-point blocks and characters:

$$\chi_\lambda(2\tau) = \langle\langle\lambda\|\, \mathrm{e}^{2\pi\mathrm{i}\tau(L_0\otimes\mathbf{1}+\mathbf{1}\otimes L_0-c/12)}\, \|\lambda\rangle\rangle\,. \tag{2.17}$$

Returning to the level of full CFT, the information about one-point functions on the disc with boundary condition a is again encoded in a "boundary state" $|a\rangle\rangle$. Just like the Ishibashi states (2.15), this is a linear form on the space of bulk states. The correlator for the bulk field $\Phi(v\otimes\tilde{v};z=0)$ inserted in the center of the disc $|z|\le 1$ is given by the value of $|a\rangle\rangle$ on $v\otimes\tilde{v}$,

$$\langle\Phi(v\otimes\tilde{v};z=0)\rangle_a = \langle\langle a|v\otimes\tilde{v}\rangle\,. \tag{2.18}$$

The boundary state can then be written as a linear combination of Ishibashi states. With a suitable normalization of bulk fields, this expansion reads [45, 46, 25]

$$|a\rangle\rangle = \sum_{\lambda\in I}\frac{B_{\lambda,a}}{\sqrt{S_{\lambda,\omega}}}\,\|\lambda\rangle\rangle\,. \tag{2.19}$$

The Cardy coefficients $B_{\lambda,a}$ are, up to normalization, equal to the reflection coefficients $\mathcal{R}^a_{\lambda\lambda^+\omega}$. Eq. (2.17) is an example of the general idea that correlation functions in full CFT are special sections of the space of conformal blocks, *i.e.*, linear combinations of the basis elements, which are here given by the Ishibashi states.

2.1.3 Classification of boundary conditions in RCFT

The problem of classifying boundary conditions in conformal field theory depends on two data. The first is a choice of a bulk theory. As explained above, this is determined by the choice of a (maximally extended) chiral algebra $\bar{\mathcal{A}}$ to be used on the covers of all closed worldsheets and the choice of a modular invariant torus partition function.

The second ingredient in the classification problem is the amount of symmetry preserved by the boundary conditions, or in other words the choice of a chiral algebra $\mathcal{A}$ to be used on the covers of open worldsheets. If $\mathcal{A} \subsetneq \bar{\mathcal{A}}$, the boundary conditions are usually referred to as "symmetry breaking".

Concerning "symmetry breaking boundaries", two remarks are in order. First, the distinction between symmetry breaking and symmetry preserving is not an invariant one. Indeed, by viewing a bulk theory with chiral algebra $\bar{\mathcal{A}}$ and diagonal partition function as a theory with chiral algebra $\mathcal{A} \subset \bar{\mathcal{A}}$, but with a non-trivial partition function, "$\bar{\mathcal{A}}$ symmetry breaking boundary conditions" are mapped to "$\mathcal{A}$ symmetry preserving boundary conditions with non-trivial modular invariant".

The second comment is related to the way in which a bulk symmetry is broken. For every current $J(z)$ in the preserved algebra $\mathcal{A}$, the boundary states satisfy an equation of the type (2.15),

$$\left(J^a_n \otimes \mathbf{1} + \mathbf{1} \otimes J^a_{-n}\right) |a\rangle\rangle = 0 \,. \tag{2.20}$$

It sometimes happens that, for some currents in $\bar{\mathcal{A}}$, the boundary state satisfies a "twisted" version of this identity, namely,

$$\left(J^a_n \otimes \mathbf{1} + \mathbf{1} \otimes \Omega(J^a_{-n})\right) |a\rangle\rangle = 0 \,, \tag{2.21}$$

where Ω is an automorphism of the chiral algebra $\bar{\mathcal{A}}$. In this case, the boundary condition a is said to have "automorphism type Ω" [25] (or that there is a non-trivial "gluing condition", defined by the automorphism Ω [34]). However, not all $\bar{\mathcal{A}}$ symmetry breaking boundary conditions actually have a definite automorphism type, and boundary conditions which do, have rather special properties (for example in $\mathcal{N} = 2$ theories, see section 3). Also, it should be mentioned that boundary conditions with definite, but non-trivial automorphism type should not be considered as symmetry preserving. For the free boson, for instance, this point of view would lead to the conclusion that Dirichlet boundary conditions preserve the U(1) current that generates translations, which is obviously not the case.[3]

Since the work of Cardy [47], boundary conditions have been classified in the following situations of rational CFT.

1. The bulk partition function is given by charge conjugation, and the boundary conditions preserve the full bulk symmetry. This case is

[3]Of course, and this is one of the reasons why D-branes are so rich, Dirichlet boundary conditions preserve translational invariance of the T-dual circle.

known as the Cardy case, and is briefly described below. In the Cardy case, correlation functions for all worldsheet topologies can be described using techniques from topological field theory. Remarkable, in that framework, factorization properties and modular invariance under the relative modular group can be rigorously proven [48].

2. The bulk partition function is a simple-current modification of the charge conjugation invariant, and the boundary conditions preserve the full symmetry corresponding to the original partition function. The solution of this problem is the content of section 2.2 of the present thesis.

3. The bulk partition function is given by charge conjugation and the boundary conditions preserve an orbifold subalgebra of the original chiral algebra by an Abelian automorphism [49, 50]. This problem is related to the previous one by shifting the point of view from "symmetry breaking" to "non-trivial modular invariant", as explained above. The correspondence uses that simple-current extensions are inverse to orbifolding by an Abelian automorphism of finite order.

4. The bulk partition function is given by a modular invariant which is not of simple-current type, and the boundary conditions preserve the original chiral algebra. This problem was considered for Virasoro minimal models and SU(2) WZW models in [51]. These authors emphasize the role played by graphs in the classification of modular invariants and boundary conditions in rational CFT. See [52] for a recent review of these ideas. A different approach to going beyond simple currents, which emphasizes the symmetry breaking, was presented in [53].

In addition, solutions for a few other isolated cases are also known. Recent work includes [54, 55, 56].

Cardy's construction [47]

If the bulk partition function is given by the charge conjugation modular invariant, there is a primary bulk field $\Phi_{(\lambda,\lambda^+)}$ for every irreducible representation $\lambda \in I$ of the chiral algebra, and hence the Ishibashi states (giving a basis of the classifying algebra) are in one-to-one correspondence with the element λ of I. The structure constants of the classifying algebra, $\tilde{\mathcal{N}}_{\lambda_1\lambda_2}^{\ \ \lambda_3}$ can be computed explicitly. It is found that they are just the fusion rules. By the Verlinde formula,

$$\tilde{\mathcal{N}}_{\lambda_1\lambda_2}^{\ \ \lambda_3} = N_{\lambda_1\lambda_2}^{\lambda_3} = \sum_{a\in I} \frac{S_{\lambda_1,a}S_{\lambda_2,a}S^*_{\lambda_3,a}}{S_{\omega,a}}, \tag{2.22}$$

the one-dimensional representations of the fusion algebra are then labelled by the elements a of I and given by the (generalized) quantum dimensions,

$$R_a(\Phi_\lambda) = \frac{S_{\lambda,a}}{S_{\omega,a}}\,. \tag{2.23}$$

It follows in particular that also the boundary conditions are in one-to-one correspondence to the primary fields. In general, it is expected that the number of boundary conditions is equal to the number of Ishibashi states, and the matrix of Cardy coefficients is unitary. However, the seemingly natural correspondence between Ishibashi states and Cardy states does not generalize. Indeed, the difference in meaning of the labels for the boundary blocks (solution of the Ward identities) and of the labels for the boundary conditions (solution to the projection problem χCFT $\rightarrow$ full CFT on the disc) shows that asking for symmetry of the matrix $B_{\lambda,a}$ is not a natural requirement.

The explicit expression for the boundary states in the Cardy case is

$$|a\rangle\!\rangle = \sum_{\lambda\in I} \frac{S_{\lambda,a}}{\sqrt{S_{\lambda,\omega}}} \|\lambda\rangle\!\rangle\,, \tag{2.24}$$

and for the annulus coefficients,

$$\begin{aligned} A^{\mu}_{ab} &= \sum_{\lambda} \frac{B^*_{\lambda,a} B_{\lambda,b} S_{\lambda,\mu}}{S_{\lambda,\omega}} \\ &= \sum_{\lambda} \frac{S^*_{\lambda,a} S_{\lambda,b} S_{\lambda,\mu}}{S_{\lambda,\omega}} \\ &= N^{a}_{b\mu}\,. \end{aligned} \tag{2.25}$$

So, also the annulus coefficients are given by the fusion rules, and are manifestly non-negative integers.

2.2 Boundary conditions for simple-current modular invariants

Following the rather general theory presented in the last section, the present section deals with the concrete problem of defining possible boundary conditions for conformal field theories with rational chiral algebra, $\mathcal{A}$, and a torus modular invariant of simple-current type. The boundary conditions will be required to preserve the given chiral algebra, even if the partition function exhibits an extended chiral algebra.

The contents of this section have appeared in the paper [26]. The problem of constructing boundary conditions for the most general simple-current invariant appeared in the work on Gepner models. As will be expanded in section 4.3, B-type branes in Gepner models can be constructed precisely as boundary conditons in a certain modular invariant of simple-current type, which is not a pure extension, and the required methods did not exist in the literature. On the other hand, the problem also appears natural in the context of "open descendants" of CFTs defined on closed, oriented surfaces. This is how the problem was presented in [26], and how it will be introduced here.

Recall that for the construction of type I string vacua, one needs to include not only boundaries (*i.e.*, D-branes), but also crosscaps (*i.e.*, orientifolds) in the background. In the conformal field theory, such data is equivalent to the choice of a Klein bottle projection, and a collection of boundary conditions with certain Chan-Paton multiplicities. Such a system of CFTs is referred to as an "open descendant" [57].

The more basic data one wishes to determine is the set $\{m\}$ of Ishibashi labels, the set $\{a\}$ of boundary labels, and a matrix $B_{m,a}$ of boundary coefficients and a vector Γ_m of crosscap coefficients, which relate the Ishibashi states to boundary states and to the crosscap state, respectively. Quite generally, there is an Ishibashi label for each primary field λ that is paired with its conjugate, λ^+, in the torus partition function (defined by a modular invariant $Z_{\lambda\tilde{\lambda}}$). A difficulty arises when some of these terms in the torus partition function have a multiplicity larger than 1. The resulting degeneracy, which is precisely $Z_{\lambda\lambda^+}$, leads to Ishibashi labels being of the general form (λ, α), where λ labels an irreducible representation of $\mathcal{A}$, and α takes $Z_{\lambda\lambda^+}$ values.

These data must satisfy a large collection of "sewing constraints" [45, 58, 46]. Most of them are difficult to check explicitly because this would require detailed knowledge of fusing matrices, braiding matrices and OPE coefficients. There exists a set of simpler constraints, which are presumably a consequence of the sewing constraints, but are certainly necessary, namely the requirement of positivity and integrality of the partition functions. These partition functions correspond to the torus, annulus, Möbius strip and Klein bottle surface. Each partition function can be written as a linear combination of characters χ_μ with arguments that depend on the surface under consideration, and with coefficients that depend on the choice of boundary condition. Actually, characters of precisely which algebra should be used to expand the partition functions in, is part of the problem. A possible solution is to simply use characters of the given chiral algebra $\mathcal{A}$. Then, the coefficients are given by

$$\begin{aligned} A^{\mu}_{ab} &= \sum_{m=(\lambda,\alpha)} \frac{B^*_{m,a}\, B_{m,b}\, S_{\lambda,\mu}}{S_{\lambda,\omega}} \,, \\ M^{\mu}_{a} &= \sum_{m} \frac{\Gamma^*_m\, B_{m,a}\, P_{\lambda,\mu}}{S_{\lambda,\omega}} \,, \\ K^{\mu} &= \sum_{m} \frac{\Gamma^*_m\, \Gamma_m\, S_{\lambda,\mu}}{S_{\lambda,\omega}} \,, \end{aligned} \tag{2.26}$$

for annulus, Möbius strip, and Klein bottle, respectively. Here S is the usual modular transformation matrix of the RCFT, and $P = \sqrt{T}ST^2S\sqrt{T}$, as introduced in [59]. All quantities on the left of (2.26) must be integers. Furthermore, torus modular invariant $Z_{\lambda\tilde{\lambda}}$, annulus coefficients, and the combinations $\frac{1}{2}(Z_{\lambda\lambda}+K^{\lambda})$ and $\frac{1}{2}(A^{\mu}_{aa}+M^{\mu}_{a})$ (the closed and open string partition function coefficients) must be *non-negative* integers. And A^{ω}_{ab}, the boundary conjugation matrix (the label "ω" refers to the vacuum), must be a permutation of order 2.

In practice, these integrality conditions have turned out to be very restrictive. In principle, however, it must be checked that, indeed, the theory is well-defined, and one may need additional requirements for this. It should also be mentioned here that other, alternative and complementary approaches to the characterization of conformally invariant boundary conditions have been proposed in the literature. Each approach abstracts and generalizes a different aspect of the Cardy case, which is presumably the only case that is completely under control. For instance, given the relation between the classification of modular invariant partitions functions to graphs of various types, one can imagine generalizing this to the boundary problem, see [51] and references therein. Another, loosely related, idea is that there should be an analog of the fusion algebra for the boundary conditions (a "boundary fusion ring"), with structure constants given by the annulus coefficients, see [53].

2.2.1 Simple currents and modular invariants

The CFT under study in this section is given, in the bulk, by the modification of the charge conjugation modular invariant by a simple-current invariant. The modular invariant is thus of the form $(ZC)_{\lambda\mu} = Z_{\lambda\mu^+}$, where $Z_{\lambda\mu}$ is a modular invariant of simple-current type, and $C_{\lambda\mu} = \delta_{\mu,\lambda^+}$ is the charge conjugation matrix.

Simple currents [60] are the units (invertible elements) of the fusion ring of a rational conformal field theory. Recall that the fusion ring of a rational CFT is a commutative associative semi-simple ring with identity and basis (over $\mathbb{Z}$), and with a distinguished basis with respect to which all structure constants are positive integers (the fusion rules). Simple currents have a number of equivalent characterizations.

(i) The fusion product of the simple current J with any other field λ, yields just a single field, $\Phi_{\mathrm{J}} \star \Phi_\lambda = \Phi_{\mathrm{J}\lambda}$.

(ii) J and its conjugate satisfy $\Phi_{\mathrm{J}} \star \Phi_{\mathrm{J}^+} = \Phi_\omega$.

(iii) The quantum dimension of J is equal to 1.

Simple currents contain information about "accidental" additional symmetries of a conformal field theory, *i.e.*, symmetries not encoded in the chiral algebra [61]. Here is a summary of important properties of simple currents.

With respect to multiplication in the fusion ring, simple currents form a finite Abelian group, also called the "center" of the conformal field theory. Simple currents associate a conserved charge to every primary field of the conformal field theory, called the monodromy charge,

$$Q_{\mathrm{J}}(\lambda) = \Delta_{\mathrm{J}} + \Delta_\lambda - \Delta_{\mathrm{J}\lambda} \in \mathbb{Q}/\mathbb{Z}\,. \tag{2.27}$$

A consequence of the Verlinde formula is the following property of the modular S-matrix,

$$S_{\mathrm{J}\lambda,\mu} = \exp(2\pi\mathrm{i}Q_{\mathrm{J}}(\mu))\, S_{\lambda,\mu}\,. \tag{2.28}$$

This equation is central to all applications of simple currents. As an example, (2.28) implies that the monodromy charge can be written as [62]

$$Q_{\mathrm{J}}(\lambda) = \frac{r(\mathrm{ord(J)}-1)}{2\,\mathrm{ord(J)}}, \tag{2.29}$$

where ord(J) is the order of J, *i.e.*, the smallest integer with $\big(\Phi_{\mathrm{J}}\big)^{\mathrm{ord(J)}} = \Phi_\omega$, and where the monodromy parameter r is in $\mathbb{Z}/\big(2\,\mathrm{ord(J)}\big)$ or $\mathbb{Z}/\mathrm{ord(J)}$ if ord(J) is even or odd, respectively. In particular, $2\,\mathrm{ord(J)}\,\Delta_{\mathrm{J}} \in \mathbb{Z}$. Currents satisfying in addition $\mathrm{ord(J)}\,\Delta_{\mathrm{J}} \in \mathbb{Z}$ are "bosonic" and form the "effective center" of the CFT. They can be used for the construction of modular invariants.

Given a distinguished subgroup $\mathcal{G}$ of the group of all simple currents, it is useful to think of $\mathcal{G}$ as a group of symmetries acting on the primary fields of the conformal field theory. As a consequence, primary fields are organized in orbits of the $\mathcal{G}$-action given by the fusion product. Furthermore, it is possible to understand a modding out of the theory by $\mathcal{G}$. As always, this operation requires particular care if the action of $\mathcal{G}$ on the primary fields is not free.

Simple currents were first applied to the construction of modular invariant partition functions in [62]. A modular invariant,

$$Z(\tau) = \sum_{\lambda,\tilde{\lambda}\in I} Z_{\lambda\tilde{\lambda}}\, \bar{\chi}_\lambda(\bar{\tau})\, \chi_{\tilde{\lambda}}(\tau)\,, \tag{2.30}$$

is called a simple-current invariant if any two paired chiral labels are connected by the action of some simple current, *i.e.*,

$$Z_{\lambda\tilde{\lambda}} \neq 0 \implies \tilde{\lambda} = \mathrm{J}\lambda \quad \text{for some simple current J}\,. \tag{2.31}$$

The majority of all known modular invariants are of simple-current type, and in contrast to the other, exceptional modular invariants, simple-current invariants have been completely classified [63].

The prescription for constructing simple-current invariants is as follows: First choose a subgroup $\mathcal{G}$ of the effective center of the conformal field theory. The relative monodromies in $\mathcal{G}$ determine the symmetric part, $X + X^t$, of a pairing (bihomomorphism) $X : \mathcal{G} \times \mathcal{G} \to \mathbb{R} \bmod \mathbb{Z}$, by the prescription $(X+X^t)(\mathrm{J},\mathrm{K}) = Q_\mathrm{J}(\mathrm{K}) \bmod \mathbb{Z}$. One can then choose the antisymmetric part of X, fixing the ambiguity on the diagonal by the condition $X(\mathrm{J},\mathrm{J}) = \Delta_\mathrm{J}$. The requirement that X defined in this way be a homomorphism precludes the use of simple currents that do not satisfy $\mathrm{ord}(\mathrm{J})\Delta_\mathrm{J} \in \mathbb{Z}$. With this data, define a matrix $Z = Z(\mathcal{G}, X)$, where the matrix entry $Z_{\lambda\mu}$ is equal to the number of solutions, $\mathrm{J} \in \mathcal{G}$, to the equations

$$\begin{gathered} \mu = \mathrm{J}\lambda \\ Q_\mathrm{K}(\lambda) + X(\mathrm{K},\mathrm{J}) = 0 \bmod \mathbb{Z} \quad \forall \mathrm{K} \in \mathcal{G}\,. \end{gathered} \tag{2.32}$$

The results of [63] show that (2.32) indeed defines a modular invariant partition function and, furthermore, that any modular invariant of simple-current type is of this form $Z(\mathcal{G}, X)$. Using results from group theory, it is possible to show that, given $\mathcal{G}$, the modular invariants are classified by the cohomology group $H^2(\mathcal{G}, \mathbb{C}^*)$.

A somewhat finer characterization of simple-current invariants can be obtained by identifying the maximally extended chiral algebra encoded in the partition function. It is easy to see from (2.32) that the left (resp. right) moving chiral algebra is extended precisely by all simple currents in the left (resp. right) kernel of X.

2.2.2 Ishibashi and boundary labels

The simple-current modular invariant $Z(\mathcal{G}, X)$ specified by $\mathcal{G}$ and the alternating part of X is to be multiplied with the charge conjugation matrix. In general, Ishibashi states correspond to labels paired with their charge conjugate in the partition function, and, hence, they are here determined by the diagonal elements of $Z(\mathcal{G}, X)$, counting multiplicities. The only simple currents that can contribute by a solution to (2.32) are those that satisfy $\mathrm{J}\lambda = \lambda$. They form a group, the stabilizer $\mathcal{S}_\lambda$ of λ. If this group is non-trivial, multiplicities larger than 1 may occur, possibly leading to Ishibashi label degeneracies. For pure extensions (*i.e.*, $X = 0$), this was analyzed in [49, 50], and the conclusion is that the Ishibashi label degeneracy is actually equal to the fixed point degeneracy[4]. It is natural to extend this result to the general case, and to label the degeneracy by the simple currents that cause it. Hence the ansatz for the Ishibashi labels is

$$m = (\lambda, \mathrm{J}); \quad \mathrm{J} \in \mathcal{S}_\lambda \text{ with } Q_{\mathrm{K}}(\lambda) + X(\mathrm{K}, \mathrm{J}) = 0 \bmod \mathbb{Z} \text{ for all } \mathrm{K} \in \mathcal{G}\,. \tag{2.33}$$

This ansatz produces also the correct count for pure extension invariants, although the labelling chosen here is not the same as in [49, 50]. In those papers the dual basis—the characters ψ_α of $\mathcal{S}_\lambda$—was used for the degeneracy labels. This is not possible if the modular invariant involves also a non-trivial fusion rule automorphism, because the currents satisfying (2.33) do not form a group in that case. For pure extensions, the basis used here differs by a (discrete) Fourier transformation from the one in [49, 50].

A hint for the set of boundary labels can also be obtained from the results for pure extension invariants [49, 50], and the results for $\mathbb{Z}_2$ automorphism invariants [57, 64]. In those cases, the boundaries are in one-to-one correspondence with the complete set of $\mathcal{G}$ orbits (of arbitrary monodromy charge). As usual, fixed points lead to degeneracies. For pure $\mathbb{Z}_2$ automorphism invariants due to a half-integer spin simple current, the degeneracy was found to be given by the order of the stabilizer of the orbit, whereas for pure extensions it is the order of the untwisted stabilizer. The ansatz for the boundary labels in the present case will be a natural generalization of these two special cases.

The untwisted stabilizer in the case $X = 0$ is defined as follows. For every simple current J with fixed points there exists a "fixed point resolution matrix" S^{J}; these matrices can be used to express the unitary modular

[4] This result is non-trivial, because the degeneracy in the extended theory is in general not equal to the fixed point degeneracy, *i.e.*, the order of the stabilizer, but rather to the size of a subgroup, the untwisted stabilizer.

S-transformation matrix of the extended theory through quantities of the unextended theory. The matrices S^{J} are conjectured to be equal to the modular S-transformation matrices for the J-one-point conformal blocks on the torus, and are explicitly known for all WZW models [65, 66], their simple-current extensions [67] and also for coset conformal field theories. In these cases, the matrix S^{J} is obtained from the modular S-matrix of certain twisted affine Lie algebras, related to the current algebra of interest by folding of the Dynkin diagram. Therefore, S^{J} matrices have many properties in common with the unextended S-matrix. In particular, elements of the matrix S^{J} whose labels are related by the action of a simple current K obey an equation similar to (2.28), albeit with a twist,

$$S^{\mathrm{J}}_{\mathrm{K}\rho,\mu} = F_\rho(\mathrm{K},\mathrm{J})\, \mathrm{e}^{2\pi \mathrm{i} Q_{\mathrm{K}}(\mu)}\, S^{\mathrm{J}}_{\rho,\mu} \,. \tag{2.34}$$

The quantity F_ρ is called the simple-current twist, and the untwisted stabilizer $\mathcal{U}_\rho$ is the subgroup of $\mathcal{S}_\rho$ of currents that have twist 1 with respect to all currents in $\mathcal{S}_\rho$,

$$\mathcal{U}_\rho := \{\mathrm{J} \in \mathcal{S}_\rho ; F_\rho(\mathrm{K},\mathrm{J}) = 1 \ \text{ for all } \mathrm{K} \in \mathcal{S}_\rho\} \,. \tag{2.35}$$

It turns out that F_ρ is an alternating bihomomorphism on $\mathcal{S}_\rho$ (alternating simply means $F_\rho(\mathrm{J},\mathrm{J}) = 1$, which for a bihomomorphism implies $F_\rho(\mathrm{J},\mathrm{K}) = F_\rho(\mathrm{K},\mathrm{J})^{-1}$), and therefore the definition of $\mathcal{U}_\rho$ admits a cohomological interpretation [68].

In the present case, it is easy to see that, due to the presence of non-local currents in $\mathcal{G}$, F_ρ is not alternating any longer (*i.e.*, $F_\rho(\mathrm{J},\mathrm{J}) \neq 1$, in general). However, one can show that a modified twist, F^X_ρ, defined by

$$F^X_\rho(\mathrm{K},\mathrm{J}) := \mathrm{e}^{2\pi \mathrm{i} X(\mathrm{K},\mathrm{J})} F_\rho(\mathrm{K},\mathrm{J}) \,, \tag{2.36}$$

indeed is alternating, *i.e.*, obeys $F^X_\rho(\mathrm{J},\mathrm{J}) = 1$ for all $\mathrm{J} \in \mathcal{G}$. The untwisted stabilizer in the presence of X is then defined as before, replacing F_ρ with F^X_ρ, *i.e.*,

$$\mathcal{U}^X_\rho := \{\mathrm{J} \in \mathcal{S}_\rho | F^X_\rho(\mathrm{K},\mathrm{J}) = 1 \text{ for all } \mathrm{K} \in \mathcal{S}_\rho\} \,. \tag{2.37}$$

As mentioned, the twist F^X_ρ has a nice cohomological interpretation. More precisely, alternating bihomomorphisms of an Abelian group $\mathcal{G}$ are in one-to-one correspondence to cohomology classes $\mathcal{F}^X_\rho$ in $H^2(\mathcal{G},\mathrm{U}(1))$. In particular, the untwisted stabilizer provides a basis of the center of the twisted group algebra $\mathbb{C}_{\mathcal{F}^X_\rho}\mathcal{S}_\rho$. It is this characterization of the untwisted stabilizer—more appropriately called the central stabilizer because of its cohomological interpretation—that will be of particular interest in the application to Gepner models.

Given all these facts, there is only one natural ansatz for the labels of the boundaries in the general case: orbits of labels of the unextended theory with degeneracy given by the untwisted stabilizer,

$$a = [\rho, \psi_\rho] \,, \tag{2.38}$$

where ρ is the label of a representative of a $\mathcal{G}$-orbit, and ψ a character of $\mathcal{U}_\rho^X$. In (2.38), the bracket $[\cdot,\cdot]$ is defined as equivalence classes under the action of $\mathcal{G}$ on pairs (ρ, ψ_ρ) given by $\mathrm{K}(\rho, \psi_\rho) = (\mathrm{K}\rho, \mathrm{K}(\psi_\rho))$ with

$$\mathrm{K}(\psi_\rho)(\mathrm{J}) = F_\rho(\mathrm{K},\mathrm{J})^* \mathrm{e}^{-2\pi \mathrm{i} X(\mathrm{K},\mathrm{J})} \psi_\rho(\mathrm{J}) \tag{2.39}$$

for $\mathrm{J} \in \mathcal{U}_\rho^X$. Notice that the action of $\mathcal{S}_\rho$ is trivial.

2.2.3 The boundary coefficients

By definition, Ishibashi states are the conformal blocks for one-point correlation functions on the disc, *i.e.*, specific two-point blocks on the sphere. However, in the present case, it is more appropriate to view the Ishibashi state labelled by (λ, J) as a three-point block on the sphere, with insertions λ, λ^+, and J [5]. Moreover, already from [47] it is known that the relation between Ishibashi and boundary states essentially expresses the effect of a modular S-transformation. Together with the previous observation, it is then natural to expect that the fixed point resolution matrices S^{J} appear in the boundary coefficients.

The ansatz for the boundary coefficients is therefore

$$B_{(\lambda,\mathrm{J}),[\rho,\psi_\rho]} = \alpha(\mathrm{J})\, \tilde{S}_{(\lambda,\mathrm{J}),[\rho,\psi_\rho]} = \sqrt{\frac{|\mathcal{G}|}{\mathrm{s}_\rho \mathrm{u}_\rho}}\, \alpha(\mathrm{J})\, S^{\mathrm{J}}_{\lambda,\rho}\, \psi_\rho(\mathrm{J})^* \,, \tag{2.40}$$

where $\alpha(\mathrm{J})$ is a phase that will not be discussed here, but which must satisfy $\alpha(\omega) = 1$. Also, $\mathrm{s}_\rho := |\mathcal{S}_\rho|$, $\mathrm{u}_\rho := |\mathcal{U}_\rho^X|$. All previously studied cases are correctly reproduced by the remarkably simple formula (2.40).

The matrix B is well-defined on orbits $[\rho, \psi_\rho]$, as is apparent from

$$\begin{aligned} \mathrm{K}(\psi_\rho)(\mathrm{J})^* S^{\mathrm{J}}_{\lambda,\mathrm{K}\rho} &= [F_\rho(\mathrm{K},\mathrm{J})^* \mathrm{e}^{-2\pi \mathrm{i} X(\mathrm{K},\mathrm{J})} \psi_\rho(\mathrm{J})]^* \cdot \mathrm{e}^{2\pi \mathrm{i} Q_\mathrm{K}(\lambda)} F_\rho(\mathrm{K},\mathrm{J})^* S^{\mathrm{J}}_{\lambda,\rho} \\ &= \psi_\rho(\mathrm{J})^* \mathrm{e}^{2\pi \mathrm{i}(Q_\mathrm{K}(\lambda)+X(\mathrm{K},\mathrm{J}))} S^{\mathrm{J}}_{\lambda,\rho} \\ &= \psi_\rho(\mathrm{J})^* S^{\mathrm{J}}_{\lambda,\rho} \,, \end{aligned} \tag{2.41}$$

[5]This is actually the natural interpretation in the three-dimensional topological picture established in [69].

by using the simple-current property of S^{J}, the action of K on ψ_ρ defined in (2.39) and the fact that $\mathrm{J} \in \mathcal{S}^X$. Here, $\mathcal{S}^X$ is defined by the property in (2.33).

A non-trivial check of the formula (2.40) is completeness, *i.e.*, that the matrix of boundary coefficients is square and invertible. Indeed, $\tilde{S}$ is unitary with inverse

$$(\tilde{S}^{-1})^{[\rho,\psi_\rho],(\lambda,\mathrm{J})} = \tilde{S}^*_{(\lambda,\mathrm{J}),[\rho,\psi_\rho]}\,. \tag{2.42}$$

Before proving unitarity, note the following useful implication concerning the S^{J} matrices.

$$\mathrm{J} \in \mathcal{S}^X_\lambda \setminus \mathcal{U}^X_\rho \implies S^{\mathrm{J}}_{\lambda,\rho} = 0\,. \tag{2.43}$$

This follows from the fact that if $\mathrm{J} \in \mathcal{S}^X_\lambda \setminus \mathcal{U}^X_\rho$, there is a $\mathrm{K} \in \mathcal{S}_\rho$ with $F^X_\rho(\mathrm{K},\mathrm{J}) \neq 1$. But then,

$$\begin{aligned} S^{\mathrm{J}}_{\lambda,\rho} &= S^{\mathrm{J}}_{\lambda,\mathrm{K}\rho} \\ &= \mathrm{e}^{2\pi\mathrm{i}Q_{\mathrm{K}}(\lambda)} F_\rho(\mathrm{K},\lambda)^* S^{\mathrm{J}}_{\lambda,\rho} \\ &= F^X_\rho(\mathrm{K},\mathrm{J})^* S^{\mathrm{J}}_{\lambda,\rho}\,, \end{aligned} \tag{2.44}$$

where the last step uses $Q_{\mathrm{K}}(\lambda) = -X(\mathrm{K},\mathrm{J}) \mod \mathbb{Z}$. Hence $S^{\mathrm{J}}_{\lambda,\rho} = 0$.

Now turning first to the right-inverse property of (2.42), assume that $(\lambda, \mathrm{J}_\lambda)$ and (μ, J_μ) satisfy (2.33). Then

$$\begin{aligned} \sum_{[\rho,\psi_\rho]} \tilde{S}_{(\lambda,\mathrm{J}_\lambda),[\rho,\psi_\rho]} \tilde{S}^*_{(\mu,\mathrm{J}_\mu),[\rho,\psi_\rho]} &= |\mathcal{G}| \sum_{\substack{[\rho,\psi_\rho] \\ \mathcal{U}^X_\rho \ni \mathrm{J}_\lambda,\mathrm{J}_\mu}} \frac{1}{\mathrm{s}_\rho \mathrm{u}_\rho} \psi_\rho(\mathrm{J}_\lambda)^* \psi_\rho(\mathrm{J}_\mu) S^{\mathrm{J}_\lambda}_{\lambda,\rho} S^{\mathrm{J}_\mu}_{\mu,\rho}{}^* \\ &= \sum_\rho \sum_{\psi_\rho \in \mathcal{U}^{X*}_\rho} \frac{1}{\mathrm{u}_\rho} \psi_\rho(\mathrm{J}_\lambda)^* \psi_\rho(\mathrm{J}_\mu) S^{\mathrm{J}_\lambda}_{\lambda,\rho} S^{\mathrm{J}_\mu}_{\mu,\rho}{}^* \\ &= \sum_\rho \delta_{\mathrm{J}_\lambda,\mathrm{J}_\mu} S^{\mathrm{J}_\lambda}_{\lambda,\rho} S^{\mathrm{J}_\lambda}_{\mu,\rho}{}^* \\ &= \delta_{\lambda,\mu}\delta_{\mathrm{J}_\lambda,\mathrm{J}_\mu} = \delta_{(\lambda,\mathrm{J}_\lambda),(\mu,\mathrm{J}_\mu)} \end{aligned} \tag{2.45}$$

To prove that (2.42) is a left inverse, use a projector onto $Q_{\mathcal{G}}(\lambda) + X(\mathcal{G},\mathrm{J}) = 0$. It is given by

$$\delta^1(Q_{\mathcal{G}}(\lambda) + X(\mathcal{G},\mathrm{J})) = \frac{1}{|\mathcal{G}|} \sum_{\mathrm{K}\in\mathcal{G}} \mathrm{e}^{-2\pi\mathrm{i}(Q_{\mathrm{K}}(\lambda)+X(\mathrm{K},\mathrm{J}))}\,. \tag{2.46}$$

Now compute

$$\sum_{\substack{(\lambda,\mathrm{J})\\ \mathrm{J}\in\mathcal{S}_\lambda^X}} \tilde{S}_{(\lambda,\mathrm{J}),[\rho,\psi_\rho]}\tilde{S}^*_{(\lambda,\mathrm{J}),[\sigma,\psi_\sigma]}$$

$$= \frac{|\mathcal{G}|}{\sqrt{\mathrm{s}_\rho\mathrm{u}_\rho\mathrm{s}_\sigma\mathrm{u}_\sigma}} \sum_{\substack{(\lambda,\mathrm{J})\\ \mathrm{J}\in\mathcal{S}_\lambda^X\cap\mathcal{U}_\rho^X\cap\mathcal{U}_\sigma^X}} \psi_\rho(\mathrm{J})^*\psi_\sigma(\mathrm{J})S^{\mathrm{J}}_{\lambda,\rho}{S^{\mathrm{J}}_{\lambda,\sigma}}^*\,. \quad (2.47)$$

After inserting the projector (2.46), the constraint $\mathrm{J}\in\mathcal{S}_\lambda^X$ can be dropped, since otherwise, S^{J} is 0. Using unitarity of S^{J}, this yields

$$\begin{aligned}
&\sum_{\substack{(\lambda,\mathrm{J})\\ \mathrm{J}\in\mathcal{S}_\lambda^X}} \tilde{S}_{(\lambda,\mathrm{J}),[\rho,\psi_\rho]}\tilde{S}^*_{(\lambda,\mathrm{J}),[\sigma,\psi_\sigma]}\\
&= \frac{1}{\sqrt{\mathrm{s}_\rho\mathrm{u}_\rho\mathrm{s}_\sigma\mathrm{u}_\sigma}} \sum_{\mathrm{K}\in\mathcal{G}} \sum_{\mathrm{J}\in\mathcal{U}_\rho^X\cap\mathcal{U}_\sigma^X} \sum_{\lambda} \mathrm{e}^{-2\pi\mathrm{i}(Q_{\mathrm{K}}(\lambda)+X(\mathrm{K},\mathrm{J}))}\psi_\rho(\mathrm{J})^*\,\psi_\sigma(\mathrm{J})S^{\mathrm{J}}_{\lambda,\rho}{S^{\mathrm{J}}_{\lambda,\sigma}}^*\\
&= \frac{1}{\sqrt{\mathrm{s}_\rho\mathrm{u}_\rho\mathrm{s}_\sigma\mathrm{u}_\sigma}} \sum_{\mathrm{K}\in\mathcal{G}} \sum_{\mathrm{J}\in\mathcal{U}_\rho^X\cap\mathcal{U}_\sigma^X} \mathrm{e}^{-2\pi\mathrm{i}X(\mathrm{K},\mathrm{J})}F_\sigma(\mathrm{K},\mathrm{J})^*\,\psi_\rho(\mathrm{J})^*\,\psi_\sigma(\mathrm{J})\sum_\lambda S^{\mathrm{J}}_{\lambda,\rho}{S^{\mathrm{J}}_{\lambda,\mathrm{K}\sigma}}^*\\
&= \frac{1}{\mathrm{s}_\rho\mathrm{u}_\rho}\sum_{\mathrm{K}\in\mathcal{G}}\delta_{\rho,\mathrm{K}\sigma}\sum_{\mathrm{J}\in\mathcal{U}_\rho^X}\psi_\rho(\mathrm{J})^*\,F_\sigma(\mathrm{K},\mathrm{J})^*\,\mathrm{e}^{-2\pi\mathrm{i}X(\mathrm{K},\mathrm{J})}\psi_\sigma(\mathrm{J})\\
&= \frac{1}{\mathrm{s}_\rho\mathrm{u}_\rho}\sum_{\mathrm{K}\in\mathcal{G}}\delta_{\rho,\mathrm{K}\sigma}\sum_{\mathrm{J}\in\mathcal{U}_\rho^X}\psi_\rho(\mathrm{J})^*\,\mathrm{K}(\psi_\sigma)(\mathrm{J})\\
&= \frac{1}{\mathrm{s}_\rho}\sum_{\mathrm{K}\in\mathcal{G}}\delta_{\rho,\mathrm{K}\sigma}\delta_{\psi_\rho,K(\psi_\sigma)}\\
&= \delta_{[\rho,\psi_\rho],[\sigma,\psi_\sigma]}
\end{aligned} \quad (2.48)$$

The fact that the matrix of boundary coefficients is square implies rather non-trivial relations involving the number of orbits of various kinds and the orders of stabilizers. The finest such sumrule is

$$\#\{[\rho],\mathrm{J}\in\mathcal{U}_\rho^X\} = \#\{\lambda,\mathrm{J}\in\mathcal{S}_\lambda^X\}\,. \quad (2.49)$$

In the language of boundary conditions, this identity means that the number of boundary conditions arising with J in the untwisted stabilizer is equal to the number of diagonal terms (and hence, boundary blocks) in the partition function arising by the action of J. In particular, summing over J, the number of boundary blocks (or Ishibashi states) is equal to the number of boundary conditions.

2.2.4 Concluding remarks and open problems

The main results of this section are the ansätze for Ishibashi and boundary labels, and for the boundary coefficients. These are crucial ingredients in the application to Gepner models in section 4.3. The strongest indication that the ansätze are correct comes, besides naturalness, from the various integrality checks.

- First of all, it should be mentioned that there is also a natural ansatz for the vector of crosscap coefficients. They are given in [26], where also the phases α appearing in (2.40) are defined.
- The integrality of the annulus coefficients, as defined in eq. (2.26) has been verified by Huiszoon and Schellekens in a large class of examples (WZW models), and this is accepted to be a highly non-trivial check in this context.
- It has also been checked in a large class of examples that all other partition functions, Möbius strip and Klein bottle, and open and closed string partition functions satisfy the necessary integrality and positivity constraints.

For further support of the prescription, it would be helpful to justify more rigorously the ansatz for the Ishibashi states (2.33) from a representation theoretic point of view, similar to [49]. Furthermore, it would be nice to make contact with the pictures in topological field theory, along the lines of [48], and to interpret Ishibashi and boundary states in this language. This would not only provide a clean justification of the ansatz for the boundary coefficients. It could also help to prove that there is no inconsistency in situations of more complicated topology. Furthermore, it should help in developing further the connection between CFT in two dimensions and topological field theory in three dimensions, which exists in the bulk [70, 41], and is expected to hold in the same generality also in the presence of boundaries [53]. As an example, it should be possible to reexpress the annulus amplitudes in terms of "solitonic" characters of a certain extended chiral algebra, for which integrality is more apparent. Further work is in progress on these questions, and, hopefully, will be reported about elsewhere soon.

Chapter 3

Supersymmetry, worldsheet boundaries, and D-branes

Conformal quantum field theories in two dimensions, as described in the previous chapter, are important in various areas in physics. The application of interest in this thesis are worldsheet theories for strings. As indicated in the introductory chapter, a second important ingredient for the construction of consistent models in string theory is supersymmetry.

The aim in this chapter is to review the role played by supersymmetry in the construction of worldsheet theories for strings and to describe some of the new features that appear upon inclusion of worldsheet boundaries.

Section 3.1 follows the geometric approach and characterizes symmetry preserving boundary conditions in a supersymmetric σ-model with target a Kähler manifold. This section closely follows the recent paper [29]. The geometric description will be appropriate to describe D-branes in the "large volume" region of moduli space, where worldsheet quantum corrections are suppressed. At a generic point in moduli space, the picture is that D-branes correspond to boundary conditions in the conformal field theory on the string worldsheet. Over most of moduli space, however, this conformal field theory is defined only implicitly as the fixed point of the renormalization group flow starting at the classical σ-model. Explicit calculations are restricted to the topological sector of the theory, in other words to the properties of the (quantum) moduli space itself. A simplification occurs again at special points in moduli space with enhanced symmetry, where the conformal field theory becomes rational. These rational points in moduli space allow to make contact with the algebraic construction of CFTs described in the previous chapter.

Section 3.2 takes up the algebraic approach to string compactification, reviews certain aspects of space-time supersymmetry and describes the construction of (a class of) supersymmetric boundary conditions in rational $\mathcal{N} = 2$ CFTs. This will prepare the ground for the analysis of explicit examples in chapter 4. This section is based on [30, 31].

3.1 Boundary conditions in Kähler and Calabi-Yau σ-models

It has been appreciated for a long time that there is an intimate connection between supersymmetry of a field theory and the special geometry of the classical target or field space. One well-known example in string theory is that $\mathcal{N} = 2$ supersymmetry of a two-dimensional σ-model requires the target space to be a Kähler manifold, *i.e.*, a manifold endowed with both a complex and a symplectic structure, which must be compatible in the sense that the symplectic form is a positive closed $(1,1)$ form with respect to the given complex structure. [1]

Upon quantization of the theory, it becomes more difficult to talk about the classical geometry of the target space. Supersymmetry, if it survives quantization, is then the only remnant of the special geometry of the target space. Thus, the amount of (worldsheet) supersymmetry of a quantum theory is the quantum equivalent of different structures of classical geometry, see, *e.g.*, ref. [73].

While the geometric conditions mentioned so far are common to all supersymmetric quantum theories, a further restriction on the target is special to string theory. Namely, strings require the worldsheet theory to be conformally invariant [74, 75, 76], and to allow for the GSO projection that eliminates the tachyon from the string spectrum and guarantees supersymmetry of the space-time theory. In the simplest cases, conformal invariance translates into the condition that the metric of the target be Ricci flat, while space-time supersymmetry requires the existence of a covariantly constant spinor, and leads to a restriction on the holonomy group of the compactification manifold. In mathematics, these two conditions are known to be equivalent on a Kähler manifold, and are called the Calabi-Yau property [77, 78]. Also, from the worldsheet point of view, $\mathcal{N} = 2$ supersymmetry intimately links conformal invariance and space-time supersymmetry [79, 80, 81].

Obviously, the various symmetry conditions and their mutual relationships have to be analyzed again after inclusion of worldsheet boundaries. Similarly to before, the first step is to characterize supersymmetric boundary conditions in terms of classical geometry. In a second step, one would like

[1] See the texts [71, 72] for references on complex and Calabi-Yau geometry.

to understand what further conditions are imposed by conformal invariance, space-time supersymmetry, and stability.

3.1.1 The σ-model[2]

In the compactification of the type II string, space-time is split into an external part, which is extended, for instance four-dimensional Minkowski space, and an internal part, which is typically given the form of a compact Calabi-Yau manifold. Restricting only to the internal part leads to the study of a supersymmetric non-linear σ-model. In the present subsection, the target, Y, is only assumed to be a Kähler manifold, the restriction to Calabi-Yau will be explained below. The field content of such a σ-model is as follows. [3]

- The worldsheet bosons are maps $X : \Sigma \to Y$ from the worldsheet to the target. Picking local coordinates, one can think of X as n complex bosons X^i, $i = 1, \ldots, n$, where n is the number of complex dimensions of Y.
- The worldsheet fermions $\psi_\pm$ are sections of the bundle $S^\pm(\Sigma) \otimes X^* T^{(1,0)} Y$, where $S^\pm(\Sigma)$ are spinor bundles on Σ, and $X^* T^{(1,0)} Y$ is the pullback of the holomorphic tangent bundle of Y.

In terms of these component fields, and in conformal gauge, the σ-model has the following action.

$$S = \int_\Sigma \mathrm{d}^2 z \Big[\frac{1}{2} g(\partial_+ X, \partial_- X) + \frac{\mathrm{i}}{2} g(\bar{\psi}_-, \overleftrightarrow{D}_+ \psi_-) + \frac{\mathrm{i}}{2} g(\bar{\psi}_+, \overleftrightarrow{D}_- \psi_+) \\ + g(\psi_+, R(\psi_-, \bar{\psi}_+) \bar{\psi}_-) \Big] , \quad (3.1)$$

where g is the Kähler metric and R the Riemann-tensor of Y. Furthermore, $\partial_\pm$ are ordinary derivatives on Σ, while $D_\pm = \partial_\pm + X^* \omega$ are covariant derivatives obtained from the pullback of the Levi-Civita connection, ω, in the tangent bundle of Y.

Kählerity of Y implies that the following global worldsheet supersymmetry transformations are symmetries of the theory,

$$\begin{aligned} \delta X &= \epsilon_+ \psi_- - \epsilon_- \psi_+ - \bar{\epsilon}_+ \bar{\psi}_- + \bar{\epsilon}_- \bar{\psi}_+ \\ \delta \psi_+ &= \mathrm{i} \bar{\epsilon}_- \partial_+ X^h + \epsilon_+ \, \omega_{\psi_+}(\psi_-) \\ \delta \bar{\psi}_+ &= -\mathrm{i} \epsilon_- \partial_+ X^{\bar{h}} - \bar{\epsilon}_+ \, \omega_{\bar{\psi}_+}(\bar{\psi}_-) \\ \delta \psi_- &= -\mathrm{i} \bar{\epsilon}_+ \partial_- X^h - \epsilon_- \, \omega_{\psi_-}(\psi_+) \\ \delta \bar{\psi}_- &= \mathrm{i} \epsilon_+ \partial_- X^{\bar{h}} + \bar{\epsilon}_- \, \omega_{\bar{\psi}_-}(\bar{\psi}_+) \, . \end{aligned} \quad (3.2)$$

[2]The following two subsections closely follow [29].

[3]A thorough mathematical presentation of supersymmetric σ-models can be found in [82]).

where $\partial_\pm X^h$ and $\partial_\pm X^{\bar{h}}$ denote holomorphic and antiholomorphic components of the tangent vectors $\partial_\pm X$, respectively. More precisely, under the transformations (3.2), the action varies by a boundary term,

$$\delta S = \frac{1}{2}\int_{\partial\Sigma}\Big[-\epsilon_+ g(\partial_- X, \psi_-) + \bar{\epsilon}_+ g(\partial_- X, \bar{\psi}_-) \\ - \epsilon_- g(\partial_+ X, \psi_+) + \bar{\epsilon}_- g(\partial_+ X, \bar{\psi}_+)\Big]\,. \quad (3.3)$$

Without boundaries, there are then four conserved supercurrents, $G_+ = g(\partial_+ X, \psi_+)$, $\bar{G}_+ = g(\partial_+ X, \bar{\psi}_+)$, $G_- = g(\partial_- X, \psi_-)$, and $\bar{G}_- = g(\partial_- X, \bar{\psi}_-)$. In the presence of boundaries, however, at most one half of these symmetries will be preserved.

3.1.2 Boundary conditions

Quite generally, the definition of a classical field theory in the presence of boundaries requires the specification of boundary conditions. In the Lagrangian framework, it is useful to distinguish boundary conditions imposed on the variation of fields in the variational principle from boundary conditions satisfied by the fields in the equations of motion. The first kind of boundary conditions can be viewed as external constraints imposed on the system, and the second kind depend on the first through consistency of the equations of motion. More precisely, the variation of the action functional under arbitrary variations of the fields contains boundary terms that depend on the variations and values of the fields at the boundary. Requiring vanishing of the boundary terms (independently of bulk terms in the variation) then determines boundary conditions in the equations of motion, which depend on the boundary conditions imposed in the variational principle. Furthermore, consistency of the constraints (boundary conditions on the allowed variations) with the equations of motion typically imply "secondary constraints" on the fields.

As an example, consider a free bosonic field in two dimensions with action $S = \int_\Sigma \mathrm{d}^2 z\, \partial^\mu X \partial_\mu X$. The variation of S under arbitrary variation of X is $\delta S = \int_\Sigma \big(-\partial^\mu \partial_\mu X\big)\delta X + \int_{\partial\Sigma} \partial_n X \delta X$, where $\partial_n X$ is the normal derivative at the boundary, $\partial\Sigma$. If δX is unconstrained, the boundary conditions for the equations of motion $-\partial^\mu\partial_\mu X = 0$ are of Neumann type, $\partial_n X|_{\partial\Sigma} = 0$. If on the other hand one imposes the constraint $X|_{\partial\Sigma} = \text{const.}$ (this implies the restriction $\delta X|_{\partial\Sigma} = 0$), the variational equation does not imply any boundary condition, but consistency with the equations of motion requires $\partial_t X|_{\partial\Sigma} = 0$, where ∂_t is the tangential derivative at the boundary. These boundary conditions are of Dirichlet type.

A typical feature of boundaries is that symmetries of the theory are broken by the boundary conditions. In the Lagrangian framework, continuous symmetries imply conserved currents by Noether's theorem, and boundary conditions are symmetry preserving under the following requirements.

- On the one hand, the transformation corresponding to an unbroken symmetry has to be consistent with the boundary conditions. In other words, the allowed variations have to contain the infinitesimal symmetry transformation at the boundary.
- On the other hand, the boundary conditions have to be such that the action is still invariant under the symmetry transformation, after imposing the boundary conditions. In other words, the normal component of the Noether current has to vanish at the boundary.

In the free boson example, it is easy to see that the U(1) symmetry of S, which is infinitesimally generated by $\delta X = \epsilon$ is preserved by Neumann and broken by Dirichlet boundary conditions. On the other hand, introducing a boundary breaks translational invariance on the worldsheet in both cases. As a consequence, only conformal transformations that preserve the boundary are symmetries of the theory.

Consider now the Kähler σ-model with action (3.1) on a two-dimensional worldsheet Σ with boundary $\partial\Sigma$. As before, denote the normal derivative at the boundary by ∂_n, and the tangent derivative by ∂_t. Assume that the boundary conditions are geometric, *i.e.*, there is a submanifold $\Gamma \subset Y$ such that the boundary is mapped to Γ, $X(\partial\Sigma) \subset \Gamma$. The boundary conditions on the bosonic fields are then

$$\delta X \parallel \Gamma\,; \qquad \partial_n X \perp \Gamma\,. \tag{3.4}$$

For the purpose of string theory, one is interested in boundary conditions that break half of the supersymmetries of eq. (3.2), and preserve the other half. There are essentially two ways to achieve this [27, 29], called A- and B-type supersymmetry, respectively.

- A-type supersymmetry is the diagonal combination of supersymmetries generated by $\epsilon_+ = \bar\epsilon_- = \epsilon_A$ and $\bar\epsilon_+ = \epsilon_- = \bar\epsilon_A$. Equation (3.3) then shows that the preserved supercurrents are $G^A = G_+ + \bar G_-$, and $\bar G^A = \bar G_+ + G_-$.
- B-type supersymmetry, on the other hand, is generated by $\epsilon_+ = -\epsilon_- = \epsilon_B$ and $\bar\epsilon_+ = -\bar\epsilon_- = \bar\epsilon_B$. The preserved supercurrents are $G^B = G_+ + G_-$ and $\bar G^B = \bar G_+ + \bar G_-$.

To understand the geometry of A-type boundary conditions, notice that in this case,

$$\delta X = \epsilon_A(\psi_- + \bar{\psi}_+) - \bar{\epsilon}_A(\bar{\psi}_- + \psi_+) \tag{3.5}$$

$$\delta\psi_+ = \epsilon_A\big(\mathrm{i}\partial_+ X^h + \omega_{\psi_+}(\psi_-)\big) \tag{3.6}$$

$$\delta\psi_- = \bar{\epsilon}_A\big(-\mathrm{i}\partial_- X^h + \omega_{\psi_-}(\psi_+)\big)\,. \tag{3.7}$$

In view of (3.4), the fermions then have to satisfy the boundary conditions $\psi_- + \bar{\psi}_+ \parallel \Gamma$ and $\bar{\psi}_- + \psi_+ \parallel \Gamma$. The condition that G_A be preserved becomes (recall $\partial_+ = \partial_t + \partial_n$, $\partial_- = \partial_t - \partial_n$),

$$g(\partial_t X, \psi_+ - \bar{\psi}_-) + g(\partial_n X, \psi_+ + \bar{\psi}_-) = 0\,. \tag{3.8}$$

Since $\partial_n X \perp \Gamma$, $\partial_t X \parallel \Gamma$, the second term vanishes, while the first implies $\psi_+ - \bar{\psi}_- \perp \Gamma$. If $\mathcal{J}$ denotes the complex structure of Y, one has that $\mathcal{J}(\psi_+ + \bar{\psi}_-) = \mathrm{i}(\psi_+ - \bar{\psi}_-)$. It follows that $\mathcal{J}$ maps vectors tangent to Γ to vectors orthogonal to Γ and vice-versa. In other words, Γ is a Lagrangian submanifold of Y [28, 27, 29].

Very similarly, one can show that in the case of B-type supersymmetry, the fermions have to satisfy $\psi_- + \psi_+ \parallel \Gamma$, $\bar{\psi}_- + \bar{\psi}_+ \parallel \Gamma$, $\psi_+ - \psi_+ \perp \Gamma$, and $\bar{\psi}_- - \bar{\psi}_+ \perp \Gamma$. This means that Γ has to be a holomorphic submanifold of X.

This characterization of boundary conditions—A-type as Lagrangian and B-type as holomorphic submanifolds—was in the simplest situation. More general cases include, for example, the addition of a B-field term, $\int_\Sigma X^*B$, to the action (3.1), where B is a closed two-form on Y. Furthermore, if the Kähler manifold Y is non-compact, it admits non-trivial holomorphic functions, and one can add a superpotential term. Last not least, one can couple the boundary to a target space gauge field A, by introducing a term of the form $\int_{\partial\Sigma} X^*A$. It turns out [28] that A-type supersymmetry is preserved if the gauge field is flat, while B-type supersymmetry requires the gauge field to be holomorphic. For a more precise description of the possible boundary conditions in the various cases, see ref. [29].

3.1.3 Quantum corrections

The foregoing analysis was purely at the level of classical field theory, and quantum effects will modify this description. As is familiar from the situation in the bulk, one may distinguish perturbative and non-perturbative quantum corrections on the worldsheet.

To start with, notice that the classical σ-model action (3.1) also has the symmetries of conformal invariance, and invariance under global R-rotations. R-rotations correspond to automorphisms of the $\mathcal{N} = 2$ supersymmetry algebra and act on the fermions as $\psi_\pm \to e^{2\pi i \alpha_V}\psi_\pm$ for vector, and $\psi_\pm \to e^{\pm 2\pi i \alpha_A}\psi_\pm$, for axial R-rotations, respectively.

Upon quantization of the σ-model, however, conformal invariance is broken, unless the beta-function vanishes. It turns out that to lowest order in σ-model perturbation theory, the beta-function for the target metric is proportional to the Ricci tensor corresponding to this metric. Thus, conformal invariance requires that the target admit a Ricci flat metric. The target must be a Calabi-Yau manifold.[4] The R-symmetries involve rotations of fermions and are also subject to quantum effects. Namely, the anomaly of the axial R-symmetry is proportional to the index of the Dirac operator for the worldsheet fermions. This index is equal to $2c_1(Y)$, where $c_1(Y)$ is the first Chern class of the tangent bundle of Y. Thus, according to Yau's theorem [78], the two conditions of conformal invariance and anomaly cancellation on the worldsheet are equivalent.

Besides these perturbative corrections to classical considerations, there is another kind of quantum effect on the worldsheet, namely non-perturbative instanton corrections [5, 6]. For a σ-model with Calabi-Yau target, instantons correspond, after Wick rotation to a Euclidean worldsheet, to holomorphic maps of the worldsheet Σ into Y. Worldsheet instantons modify the geometry of the moduli space of the Calabi-Yau manifold, and thereby also the masses and couplings in the low-energy effective field theory.

Recall that the moduli space of Ricci flat Kähler metrics on a Calabi-Yau manifold is locally the product of complex structure moduli space and Kähler structure moduli space. In physics terms, this factorization corresponds to a decoupling of vector- and hypermultiplet moduli of the $\mathcal{N} = 2$ supergravity theory, which is the effective description of compactified type II string theory at low energies [8, 9, 83].

Since, by definition, contributions of an instanton $\Phi : \Sigma \to Y$ must be weighted with $\exp^{-\int_\Sigma \Phi^*\omega/\alpha'}$, where ω is (the pullback of) the complexified Kähler form, they can only correct the Kähler moduli space. On the other hand, the complex structure moduli space is unaffected by worldsheet instantons, and can be computed classically. These two facts—decoupling and absence of non-perturbative corrections on the complex structure side—make mirror symmetry such a powerful tool. By exchanging Kähler moduli of Y with complex structure moduli of a mirror manifold, Y^*, mirror symmetry allows to map an a priori complicated calculation of worldsheet instanton

[4] Although there are higher order contributions to the beta-function even when the target is Ricci flat, the general criterion that requires Y to be Calabi-Yau is valid to all orders in perturbation theory.

corrections of the Kähler moduli space of Y to a classical computation in the complex structure moduli space of the mirror manifold, Y^* [13]. The mathematical interest of mirror symmetry computations arises from the relation to counting of holomorphic curves and Gromov-Witten invariants, see [84] for further references.

To summarize this brief review, it can be seen that the requirements of string theory impose certain constraints on the classical geometric target that defines the σ-model. Furthermore, some of the non-perturbative quantum corrections of string theory on Calabi-Yau manifolds are accessible using mirror symmetry.

3.1.4 D-branes

Similarly as in the bulk, it should be asked how the classical analysis of section 3.1.2 is modified by quantum effects. As before, one expects to first determine stronger geometric conditions that will ensure existence of a consistent, *i.e.*, conformally invariant and stable, quantum theory on the worldsheet. Note that these geometric conditions for the boundary sector can, or rather, must, depend on the moduli of the bulk theory. In a second step, one might then try to explicitly evaluate the non-perturbative quantum corrections.

These problems have attracted a lot of attention in recent years, see, *e.g.*, [85, 32, 86, 87, 88, 89, 90, 91] and it is not intended to review here the significant progress that has been made. The goal of this subsection is merely to summarize those aspects of the geometric characterization of D-branes on Calabi-Yau manifolds that will be needed in the examples, in particular, in section 4.3. This includes the topological classification of D-branes, basic stability (supersymmetry) criteria, the effective coupling to RR gauge fields, and the BPS central charge. For the rest of this section, the complex dimension of Y will be assumed to be $n = 3$.

Special Lagrangian submanifolds

As reviewed above, boundary conditions of A-type with respect to the $\mathcal{N} = 2$ worldsheet supersymmetry correspond geometrically to Lagrangian submanifolds, with a flat U(1) connection. Such submanifolds are topologically characterized by a homology class in $H_3(Y, \mathbb{Z})$.

It is shown in [92, 27] that the extra geometric condition ensuring spacetime supersymmetry is

$$\mathrm{Im}\, e^{i\gamma}\, \Omega|_{\Gamma} = 0\,, \tag{3.9}$$

where Γ is the cycle wrapped by the brane, and Ω is the unique holomorphic top-form of the Calabi-Yau. The angle γ is arbitrary. The alternative characterization is that the cycle wrapped by the brane has minimal volume in its homology class, or that $e^{i\gamma}\Omega|_\Gamma$ is proportional to the volume form induced by the Kähler metric. The condition (3.9) is called the "special Lagrangian condition", the corresponding branes in string theory are called A-type branes.

Physically, one can measure the homology class of an A-type brane as a charge under RR gauge fields. Recall that in the low-energy effective theory in the flat part of space-time, the RR gauge fields in hypermultiplets arise from dimensional reduction of RR form fields in 10 dimensions along the various cycles in the compactification manifold. In particular, for A-type branes that are point-like in the flat part of space-time, the relevant fields are the RR $p+1$-forms (with p odd) of the type IIB string. The natural space of RR gauge fields therefore is $H_3(Y)$. The coupling of an A-type brane (topologically also an element of $H_3(Y)$) to these RR gauge fields can then be shown to simply equal the natural symplectic pairing in $H_3(Y)$ (see, *e.g.*, [93]). Equivalently, the RR charge $Q^{(\mathrm{RR})}(\Gamma)$ of an A-type branes can be thought of as lying in the dual space, *i.e.*,

$$Q^{(\mathrm{RR})}(\Gamma) \in H^3(Y, \mathbb{Z}) . \tag{3.10}$$

More explicitly, given a basis of 3-cycles $\{\gamma^i, i = 1, \dots, h^3(Y)\}$ in $H_3(Y)$, one may expand $\Gamma = Q_i^{(\mathrm{RR})}\gamma^i$, and call $Q_i^{(\mathrm{RR})}$ the RR charges of the A-type brane.

Of central importance for later applications is the $\mathcal{N} = 2$ central charge of the brane. By definition, the central charge is the coupling of the brane to the central element of the $\mathcal{N} = 2$ supersymmetry algebra in flat space-time. This central element arises from the operator generating left-right symmetric spectral flow, which, for the type IIB string compactified to 4 dimensions, corresponds to the holomorphic three form $\Omega \in H^3(Y)$ on the Calabi-Yau three-fold. Thinking of the charge as an element of $H^3(Y)$, one should then really evaluate $Q^{(\mathrm{RR})}$ on the three cycle Poincaré dual to Ω. Equivalently, the central charge $Z(\Gamma)$ of an A-type brane wrapped on Γ is given by

$$Z(\Gamma) = \int_\Gamma \Omega = e^{-i\gamma} \int_\Gamma e^{i\gamma}\,\Omega . \tag{3.11}$$

Since Γ is special Lagrangian, (3.9), this expression shows in particular that the absolute value of $Z(\Gamma)$ is the volume of the cycle, which in physics terms is the mass of the brane, as viewed from flat space-time [2]. Thus, the mass of the brane is completely fixed by its RR charge, as befits a BPS state. Also, γ is equal to the phase of the central charge.

Stable coherent sheaves

For "B-type" branes, $\mathcal{N}=2$ worldsheet supersymmetry requires boundary conditions on holomorphic submanifolds, and a coupling to a connection in a holomorphic vector bundle. More generally, one considers coherent sheaves [93] or complexes thereof [94] as the natural geometric objects corresponding to B-type branes. The natural receptacle for the topological classification of D-branes is currently believed [95, 96] to be K-theory. But for the present purposes, it will be enough to approximate K-theory classes by cohomology classes. The topological class of a B-type branes $\mathcal{V}$, is then given by the Chern character, $\mathrm{ch}(\mathcal{V})$, and lies in the diagonal cohomology, $\oplus H^{i,i}(Y) \ni \mathrm{ch}(\mathcal{V})$.

Space-time supersymmetry for B-type boundary conditions is governed by Kähler moduli, and in distinction to the situation for A-type branes, depends on quantum corrections for the bulk. The precise geometric characterization of B-type branes throughout moduli space is subject of intensive current research, and will not be discussed in detail here. See [88, 94] for recent work. At large volume, the situation is somewhat better understood. The basic criterion [93] in the simplest case of a vector bundle (a brane wrapping the whole Calabi-Yau) is the existence of a hermitian Yang-Mills connection. By the Donaldson-Uhlenbeck-Yau theorem [97, 98], this is equivalent to mathematical definitions of stability of holomorphic vector bundles on Y. The general situation is more involved, and it is not easy to state the analog of eq. (3.9) in general.

Consider now the relation between the topological class of a B-type brane, $\mathcal{V}$, and its physical RR charges. Here, the relevant gauge fields arise from dimensional reduction of RR $p+1$-form fields, with p even, of the type IIA string. It is shown in [99, 93] that the charge describing the correct coupling to these RR gauge fields is given by the generalized Mukai vector

$$Q^{(\mathrm{RR})}(\mathcal{V}) = \mathrm{ch}(\mathcal{V})\,\sqrt{\hat{A}(Y)}\;\in\;\oplus_i H^{i,i}(Y)\,, \tag{3.12}$$

where $\mathrm{ch}(\mathcal{V}) = \mathrm{tr}(\mathrm{e}^F)$ is the Chern character, and $\hat{A}(Y)$ a topological invariant of Y.

As for A-type branes, the central charge of a B-type brane can be computed from its RR charge. Again, it is nothing but the coupling to the central element of the $\mathcal{N}=2$ space-time supersymmetry algebra, this time for the type IIA string. At large volume, one can write the central charge as [100],

$$Z(\mathcal{V}) \;=\int \mathrm{e}^{-K}\,Q^{(\mathrm{RR})}(\mathcal{V}) = \int \mathrm{e}^{-K}\,\mathrm{ch}(\mathcal{V})\,\sqrt{\hat{A}(X)}\,, \tag{3.13}$$

where K is the complexified Kähler class of the Calabi-Yau manifold. But this formula is subject to quantum corrections on the worldsheet. The best way to determine the central charge at a generic point in moduli space is to use mirror symmetry to map the charges of B-type branes on Y, eq. (3.12), to the charges of A-type branes on Y^*, eq. (3.10), and then compute Z using eq. (3.11).

Intersection indices

One more piece of information that will be useful in comparing results from algebraic conformal field theory with geometry in chapter 4 is the intersection index of branes. This intersection index is a topological invariant associated with two D-branes and can be thought of as a pairing between the respective topological charges. From the worldsheet point of view, the intersection index is the Witten index in the space of open strings stretching between the two branes.

For A-type branes, the natural pairing is the symplectic intersection form on $H_3(Y, \mathbb{Z})$. In particular, picking a symplectic basis of cycles, in which the intersection matrix is of the canonical form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, one may expand $Q^{(\mathrm{RR})}(\Gamma) = (Q_i^{(\mathrm{RR})}, \tilde{Q}_j^{(\mathrm{RR})})_{i,j=1,\ldots,h^3/2}$ and write for $\Gamma, \Gamma' \in H_3(Y)$,

$$\langle Q^{(\mathrm{RR})}(\Gamma), Q^{(\mathrm{RR})}(\Gamma')\rangle = \Gamma \cap \Gamma' = \sum_{i=1}^{h^3/2} Q_i^{(\mathrm{RR})}(\Gamma)\tilde{Q}_i^{(\mathrm{RR})}(\Gamma') - \tilde{Q}_i^{(\mathrm{RR})}(\Gamma)Q_i^{(\mathrm{RR})}(\Gamma') \,. \tag{3.14}$$

For B-type branes $\mathcal{V}, \mathcal{V}'$, with charges $Q^{(\mathrm{RR})}(\mathcal{V}), Q^{(\mathrm{RR})}(\mathcal{V}') \in H^{\mathrm{diag}}(Y)$, the analogous expression is

$$\langle Q^{(\mathrm{RR})}(\mathcal{V}), Q^{(\mathrm{RR})}(\mathcal{V}')\rangle = \int_Y Q^{(\mathrm{RR})}(\mathcal{V}^*)Q^{(\mathrm{RR})}(\mathcal{V}') = \int_Y \mathrm{ch}(\mathcal{V}^* \otimes \mathcal{V}')\hat{A}(Y) \,. \tag{3.15}$$

Using the index theorem, one can relate this expression to the index of the Dirac operator coupled to $\mathcal{V}^* \otimes \mathcal{V}'$, which directly shows its relation to the Witten index in the open string sector between the corresponding branes.

3.2 $\mathcal{N} = 2$ superconformal field theory with boundaries

The goal of this section is to analyze the interplay between the projections that arise in algebraic string compactifications and the construction of boundary conditions for such theories. The discussion centers around the internal part of the compactification, and is the algebraic counterpart of the classical geometric characterization of D-branes given in the previous section.

3.2.1 Algebraic construction of type II string compactifications

In the compactification of the type II string from 10 to D dimensions (D even), the role of the internal manifold Y is to provide an implicit definition of a conformal field theory. Since Y is a Calabi-Yau manifold, one expects that the renormalization group flow starting at the classical σ-model on Y defines a unique conformal field theory with $\mathcal{N} = 2$ worldsheet supersymmetry. In other words, at the conformal fixed point the modes of the energy momentum tensor, supersymmetry currents, and U(1) currents generate two copies of the $\mathcal{N} = 2$ super-Virasoro algebra, which is explicitly,[5]

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \\
\{G_r^\pm, G_s^\mp\} &= 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{3}\Big(r^2 - \frac{1}{4}\Big)\delta_{r,-s} \\
\{G_r^\pm, G_s^\pm\} &= 0 \\
[L_n, G_r^\pm] &= \Big(\frac{n}{2} - r\Big)G_{n+r}^\pm \\
[J_n, J_m] &= \frac{c}{3} n \delta_{n,-m} \\
[L_n, J_m] &= -mJ_{n+m} \\
[J_n, G_r^\pm] &= \pm G_{n+r}^\pm \,.
\end{aligned} \tag{3.16}$$

Abstracting from geometry, one may also ask for other, more explicit definitions of conformal field theories with the right properties required for string compactification. The class of conformal field theories that are used in such "algebraic" compactifications typically include "free" theories (*e.g.*, lattice CFTs and orbifolds thereof), or they are rational CFTs, which gives a good handle at explicit calculations. To be usable as a building block for string theory, the CFT must satisfy a number of conditions, the most basic one being the correct Virasoro central charge, $c = 3\hat{c} = 15 - 3D/2$. Here, $\hat{c}$ is numerically equal to the number of compactified complex spatial dimensions. Furthermore, the CFT must in particular have $\mathcal{N} = 2$ supersymmetry on the worldsheet and the associated U(1) charges must be integer [79, 80, 81]. It turns out that in many cases, these conditions can be imposed step by step, and a careful implementation of each step guarantees the ultimate success of the procedure.

[5]The reader should be warned about the following change of conventions. The worldsheet has been Wick-rotated to Euclidean signature. Hence, "right-moving" (anti-holomorphic) quantities on the worldsheet are distinguished from "left-moving" (holomorphic) by a bar. This is as in chapter 2 and different from section 3.1,where a $\pm$ subscript is used. Here, the $\pm$ subscript distinguishes the two worldsheet supercurrents.

The first step in the construction is the adjustment of the central charge. To this end, choose a certain number r of rational $\mathcal{N}=2$ superconformal field theories, $\mathcal{C}^{(i)}$, with central charge c_i, $i=1,\ldots,r$, such that $\sum_i c_i = 15-3D/2$. In fact, r might also be equal to 1, but ordinary constructions (*e.g.*, free bosons, coset models, orbifolds, etc., without using tensor products), very rarely produce such models.

To be able to make contact with chapter 2, the individual $\mathcal{N}=2$ building blocks will be described as bosonic theories with Virasoro central charge c_i and $\mathcal{N}=2$ structure appearing as simple-current symmetries.[6] In particular, each factor of the tensor product contributes two distinguished simple currents, denoted by v_i and s_i for $i=1,\ldots,r$. The primary field v_i has conformal weight $\Delta_{v_i}=3/2$, is a simple current of order 2 and contains at the lowest Virasoro degree two fields representing the worldsheet supercurrents, $G^{\pm}(z)$ of the superconformal theory. It is referred to as the vector current. The second simple current, s_i, has conformal weight $\Delta_{s_i}=c_i/24$, and a model dependent order. In supersymmetric language, the module corresponding to s_i contains at the lowest degree the unique Ramond ground state of maximal U(1) charge $q_i(s_i)=c_i/6$. The action of s_i by the fusion product is equivalent in supersymmetric language to spectral flow by half a unit, and the monodromy charge with respect to s_i satisfies

$$Q_{s_i}(\lambda) = \frac{q_i(\lambda)}{2} \mod \mathbb{Z}\,, \tag{3.17}$$

where $q_i(\lambda)$ is the i-th U(1) charge of the field λ in the tensor product.

Naively, one would now like to take the tensor product,

$$\mathcal{C}^{\text{ten. prod.}} = \bigtimes_{i=1}^{r} \mathcal{C}^{(i)}\,, \tag{3.18}$$

as the internal CFT of the compactification. However, the ordinary tensor product of the CFT factors is adapted to the bosonic language, and is not equivalent to their tensor product as superconformal theories. Indeed, the tensor product (3.18) is not $\mathcal{N}=2$ supersymmetric, essentially because the contributions of the individual factors to the putative worldsheet supersymmetry current,

$$\mathbf{1}\otimes\cdots\otimes G^{\pm}(z)\otimes\cdots\otimes\mathbf{1}\,, \tag{3.19}$$

[6] A summary of simple currents was given in section 2.2. For background material on the $\mathcal{N}=2$ superconformal algebra, see [10]; for more information about simple currents in such theories, see, for example, [101].

with $G^{\pm}$ in the i-th factor, do not lie in the same irreducible module of the bosonic tensor product, and it would not make sense to add them. In the other sectors of the theory, the problem reappears in the existence of fields that are NS in some factors of the tensor product, and R in others, *i.e.*, that the fermions are not "aligned". Fermion alignment can be achieved by a projection operation and is most easily implemented at the level of χCFT as a simple-current extension. Explicitly, the extension with the simple-current group generated by

$$\mathrm{w}_i = v_1 v_i \,, \tag{3.20}$$

for $i = 2, \ldots, r$, guarantees that the summands (3.19) all lie in the same irreducible representation of the extended algebra, namely in the equivalence class labelled by $v = [(v_1, \omega_2, \ldots \omega_r)]$. [7] Furthermore, since the monodromy charge with respect to vector currents distinguishes NS sector ($Q_{v_i} = 0$) from R sector ($Q_{v_i} = 1/2$), the projection, $Q_{\mathrm{w}_i} = 0 \bmod \mathbb{Z}$ ensures fermion alignment.

The general properties of simple-current extensions automatically guarantee that the resulting theory is a consistent CFT, in particular if the concern is modular invariance. The tensor product (3.18), extended by the alignment currents (3.20), will be referred to as $\mathcal{C}^{\mathrm{wsusy}}$, with chiral algebra $\mathcal{A}^{\mathrm{wsusy}}$.

Having obtained an $\mathcal{N} = 2$ superconformal theory with central charge $c = 15 - 3D/2$, the next condition on the CFT is that the total U(1) charge of all NS fields be integer. This condition is equivalent to the physical requirement that the string vacuum obtained after tensoring with external space-time, and performing the GSO projection, is stable (absence of tachyons) and exhibits space-time supersymmetry.

Since the total U(1) charge, q, is measured by the spectral flow operator, *i.e.*, the simple current

$$s = (s_1, s_2, \ldots, s_r) \,, \tag{3.21}$$

it is natural to understand also this second projection as a simple-current extension. More precisely, the extension is by the cyclic group generated by

$$\mathrm{u} = s^2\, v^{(D-2)/2} \,. \tag{3.22}$$

Indeed, the monodromy charge of a primary field λ of the theory $\mathcal{C}^{\mathrm{wsusy}}$ with respect to u is equal to

$$Q_{\mathrm{u}}(\lambda) = \begin{cases} q(\lambda) & \text{if } \lambda \text{ is in the NS sector} \\ q(\lambda) + \frac{D-2}{2} & \text{if } \lambda \text{ is in the R sector}\,, \end{cases} \tag{3.23}$$

[7] Recall that ω_i denotes the vacuum in the i-th factor of the tensor product.

and hence extension by u guarantees integrality of the U(1) charge in the NS sector. The factor $v^{((D-2)/2}$ in (3.22) and the shift by $(D-2)/2$ in the R sector in (3.23) is the usual dependence on odd or even complex dimensional compactification space (recall that $\hat{c} = c/3 = n = 5 - D/2$).

The tensor product $\mathcal{C}^{\text{ten. prod.}}$ (3.18), extended by the simple-current group

$$\mathcal{G}_{\text{ext}} := \langle \mathrm{w}_i, \mathrm{u} \rangle \,, \tag{3.24}$$

generated by the w_i and the current u is an $\mathcal{N} = 2$ superconformal theory with integer U(1) charge in the NS sector, and can constitute the internal sector of a string compactification. It will be denoted by $\mathcal{C}^{\text{inner}}$, with corresponding maximally extended chiral algebra $\mathcal{A}^{\text{inner}}$. In principle now, the construction of boundary conditions in this extended tensor product is a well-posed problem. However, for the physical interpretation of various ingredients of the construction, in particular to determine the correct amount of symmetry to be preserved by the boundary conditions, it is necessary to take a brief look at the remaining steps, involving the external space-time, towards a consistent string vacuum.

At the level of CFT the flat dimensions are described by the tensor product of D free bosons and D free fermions. This tensor product $\mathcal{C}_D^{\text{st,bos}} \times \mathcal{C}_{D/2}^{\text{st,ferm}}$ has $\mathcal{N} = 2$ supersymmetry (subscripts here stand for the Virasoro central charge). Gauging of the $\mathcal{N} = 1$ superconformal symmetry on the worldsheet can be performed by introducing a system of ghosts, $\mathcal{C}_{-26}^{\text{gh}}$, for the stress-energy tensor and a system of superghosts, $\mathcal{C}_{11}^{\text{sgh}}$ for the $\mathcal{N} = 1$ supercurrent. Consider then the tensor product

$$\mathcal{C}_D^{\text{st,bos}} \times \mathcal{C}_{D/2}^{\text{st,ferm}} \times \mathcal{C}^{\text{inner}} \times \mathcal{C}_{-26}^{\text{gh}} \times \mathcal{C}_{11}^{\text{sgh}} \,. \tag{3.25}$$

Again, to retain $\mathcal{N} = 2$ worldsheet supersymmetry, the space-time and internal worldsheet fermions have to be aligned by a simple-current extension. The GSO projection that avoids space-time tachyons and ensures a space-time supersymmetric spectrum amounts to projecting onto odd-integer total U(1) charge in (3.25). Full string theory proceeds from this by introducing the BRST operator (nilpotency follows from the vanishing of the total central charge) and restricting to physical observables in the BRST cohomology.

It turns out that there is a convenient prescription that allows to express also the GSO projection in the language of bosonic CFT as a simple-current extension, namely the so-called bosonic string map. In essence, it amounts to replacing superghosts and space-time fermions in (3.25) by a bosonic CFT corresponding to the SO($D+6$) WZW model at level 1. For a review of the bosonic string map, see [102]. For the purposes of the next subsection, suffice it to note that in this completely bosonic language, the space-time supercharges are the zero modes of the chiral fields ("spinor fields"),

$$S(z) = s_{\text{ext}}(z)s_{\text{int}}(z), \tag{3.26}$$

where s_{ext} is a spinor primary field of $\mathrm{SO}(D+6)_1$, and s_{int} is the primary field obtained from (3.21) after extension by u.

3.2.2 Automorphism types of boundary conditions and space-time supersymmetry

As seen in the geometric description of D-branes on Calabi-Yau manifolds, there are essentially two types of possible boundary conditions for the symmetry currents of an $\mathcal{N} = 2$ field theory, commonly called A- and B-type boundary conditions. In the algebraic framework, the two possibilities reappear as different automorphism types of boundary conditions with respect to the $\mathcal{N} = 2$ algebra. By construction, however, the chiral algebra of $\mathcal{C}^{\text{inner}}$ is much larger than the $\mathcal{N} = 2$ algebra and boundary conditions can be further classified according to the way in which this extended symmetry is realized. In particular, the algebra $\mathcal{A}^{\text{inner}}$ contains the simple current u. Since $\mathrm{u} \propto s^2$, and s is related to the space-time supercharge, it is reasonable to expect a connection between the realization of u on the boundary and the space-time supersymmetry preserved by the corresponding brane. It is the goal of this subsection to explain this connection.

It is easy to see that the automorphism group of the $\mathcal{N} = 2$ algebra (3.16) is isomorphic to $\mathrm{U}(1) \times \mathbb{Z}_2$, where the U(1) stems from inner, and $\mathbb{Z}_2$ from outer automorphisms. *A priori*, there are therefore two families of boundary conditions. Considering the theory on the upper half plane, with boundary at $z = \bar{z}$, one distinguishes

- A-type boundary conditions:

$$\begin{aligned} T(z) &= \bar{T}(\bar{z}) \\ G^{\pm}(z) &= \mathrm{e}^{\pm \mathrm{i}\alpha_A}\, \bar{G}^{\mp}(\bar{z}) \qquad \text{at } z = \bar{z} \\ J(z) &= -\bar{J}(\bar{z}) \end{aligned} \tag{3.27}$$

- B-type boundary conditions:

$$\begin{aligned} T(z) &= \bar{T}(\bar{z}) \\ G^{\pm}(z) &= \mathrm{e}^{\pm \mathrm{i}\alpha_B}\, \bar{G}^{\pm}(\bar{z}) \qquad \text{at } z = \bar{z} \\ J(z) &= \bar{J}(\bar{z}) \end{aligned} \tag{3.28}$$

Now recall that superstring theory is based on local invariance under $\mathcal{N} = 1$ worldsheet supersymmetry. This supersymmetry can be realized in different ways. It is easy to see that for each choice of φ, the combination $G^{(\varphi)} = e^{i\varphi}G^+ + e^{-i\varphi}G^-$ generates an $\mathcal{N} = 1$ super-Virasoro algebra. If this symmetry is to be gauged, it must not be broken by the boundaries.[8] Not every choice of inner automorphism is compatible with every embedding of the $\mathcal{N} = 1$ into the $\mathcal{N} = 2$. More precisely, each choice of α_A for A-type boundary conditions is compatible with one and only one choice of φ, namely $\alpha_A = -2\varphi$. One the other hand, B-type boundary conditions are only compatible with $\varphi = 0$, but this independently of the choice of α_B. This rather subtle distinction and its implications do not seem to have been analyzed in the literature. In any case, the angles α_A and α_B in (3.27) and (3.28) may be shifted by a redefinition of $G^{\pm}$, and most convenient is to simply set them to zero. See also [29] for arguments that this is no loss of generality in the σ-model context.

Another choice of convention is whether one describes a given theory using the diagonal or the charge conjugation modular invariant partition function. This freedom, which is in fact the origin of mirror symmetry, can be confusing. This is particularly true when applying results from conformal field theory with boundaries, because the natural choice there is the charge conjugation modular invariant, while geometrically, the diagonal modular invariant appears to be more suggestive. In this thesis, A- respectively B-type boundary condition will refer to a geometric interpretation, while trivial respectively mirror automorphism type will mean the algebraic characterization. Table 3.1 is the dictionary between the two formulations.

modular invariant	automorphism type (algebraic)	boundary condition (geometric)
diagonal	trivial	B-type
	mirror	A-type
charge conjugation	trivial	A-type
	mirror	B-type

Table 3.1: Automorphism types of A- and B-type boundary conditions

In a supersymmetric language, and in full string theory, BPS boundary states are constructed as GSO-invariant combinations of boundary states in

[8]Furthermore, the boundary conditions on the ghosts must be such that the ghost number and the BRST current are preserved, but this will not be discussed any further here.

the CFT (3.25). Such boundary states possess, in analogy with worldsheet spin structure (R or NS sector) in the bulk, a $\mathbb{Z}_2$ valued quantum number $\eta = \pm 1$, see [103, 104]. Which specific combination of boundary states is GSO-invariant depends on the projection, *i.e.*, whether one is dealing with a type IIA or type IIB theory. Furthermore, a GSO-invariant boundary state reflects the spin field in a specific way, and only a special linear combination of space-time supercharges is preserved, see [2].

Turning first to the internal part, a boundary state, $|a\rangle\rangle$, preserving the $\mathcal{N} = 1$ subalgebra, with "boundary spin structure" η, and with a definite automorphism type with respect to the $\mathcal{N} = 2$ algebra satisfies either

$$\begin{aligned} \big(L_n \otimes \mathbf{1} - \mathbf{1} \otimes L_{-n}\big)|a\rangle\rangle &= 0 \\ \big(G^{\pm}_r \otimes \mathbf{1} + \mathrm{i}\eta \mathbf{1} \otimes G^{\pm}_{-r}\big)|a\rangle\rangle &= 0 \\ \big(J_n \otimes \mathbf{1} + \mathbf{1} \otimes J_{-n}\big)|a\rangle\rangle &= 0\,, \end{aligned} \tag{3.29}$$

or,

$$\begin{aligned} \big(L_n \otimes \mathbf{1} - \mathbf{1} \otimes L_{-n}\big)|a\rangle\rangle &= 0 \\ \big(G^{\pm}_r \otimes \mathbf{1} + \mathrm{i}\eta \mathbf{1} \otimes G^{\mp}_{-r}\big)|a\rangle\rangle &= 0 \\ \big(J_n \otimes \mathbf{1} - \mathbf{1} \otimes J_{-n}\big)|a\rangle\rangle &= 0\,. \end{aligned} \tag{3.30}$$

In the first case, the automorphism type is trivial, while in the second case, it is equal to the mirror automorphism. Which case corresponds to A- and which to B-type boundary conditions on the symmetry currents depends on the choice between the description with diagonal or charge conjugation modular invariant, see table 3.1.

Of particular interest is the condition on the U(1) current J. For visualization, it is helpful to bosonize this current,

$$J = \mathrm{i}\sqrt{\frac{c}{3}}\partial X\,. \tag{3.31}$$

Then, A-type boundary conditions are like "Dirichlet", while B-type boundary conditions are like "Neumann" for the boson X. It is well-known that in terms of X, one can write $s = \mathrm{e}^{\mathrm{i}\sqrt{c/12}\,X}$, and hence

$$\mathrm{u} = v^{(D-2)/2}\,\mathrm{e}^{\mathrm{i}\sqrt{c/3}\,X}\,. \tag{3.32}$$

The fact that the U(1) charge is integer, or, equivalently, that $\mathrm{u} = s^2$ (respectively, $\mathrm{u}^2 = s^4$) is in the chiral algebra, implies that the chiral algebra contains as a subalgebra the chiral algebra of a boson at rational radius squared, $R^2 = c/3 = \hat{c}$ (respectively, $R^2 = 2\hat{c}$). Of course, the torus partition function is not a direct product, and a circle on which "X is compactified" is merely an analogy.

The conditions on the boundary states can also be translated into a purely bosonic language. The major difference is that the condition on the world-sheet supercurrents in (3.29) and (3.30) is no longer a condition on fields in the chiral algebra, but rather on a simple-current field. It is still possible to distinguish $\eta = \pm 1$ by the way in which the simple current corresponding to worldsheet supersymmetry is reflected at the boundary, *i.e.*, $v(z) = \eta\, v(\bar{z})$.

On the other hand, since the simple current u is in the chiral algebra, the preserved space-time supersymmetry charge can be measured as an automorphism type with respect to u. Now recall that for a compact boson at rational radius squared, the automorphism type with respect to the extended symmetry restricts the position of the Dirichlet boundary conditions on the circle, respectively the value of the Wilson line [9]. By analogy, this leads to an intuitive interpretation of the automorphism type with respect to u.

Explicitly, one has for A-type boundary conditions that $\mathrm{u}(z) = \mathrm{e}^{2\mathrm{i}\gamma}\, \bar{\mathrm{u}}^{+}(\bar{z})$, *i.e.*,

$$\mathrm{e}^{\mathrm{i}\sqrt{c/3}\,X_{\mathrm{L}}(z)} = \eta^{(D-2)/2}\mathrm{e}^{2\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}\sqrt{c/3}\,X_{\mathrm{R}}(\bar{z})}\,. \tag{3.33}$$

In the intuitive picture, this fixes the Dirichlet boundary conditions $(X_{\mathrm{L}} + X_{\mathrm{R}})|_{z=\bar{z}} = 2\gamma/\sqrt{\hat{c}} \bmod 2\pi/\sqrt{\hat{c}}$. By considering also the spin field, the ambiguity can be reduced. Namely, restricting to $\eta = +1$, one has for the spin field,

$$\mathrm{e}^{\mathrm{i}\sqrt{c/12}\,X_{\mathrm{L}}(z)} = \pm\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}\sqrt{c/12}\,X_{\mathrm{R}}(\bar{z})}\,. \tag{3.34}$$

which determines $X_{\mathrm{L}} + X_{\mathrm{R}} = 2\gamma/\sqrt{\hat{c}} \bmod 4\pi/\sqrt{\hat{c}}$.

For B-type boundary conditions, the analogous equation is

$$\mathrm{e}^{\mathrm{i}\sqrt{c/12}X_{\mathrm{L}}(z)} = \pm\mathrm{e}^{\mathrm{i}\beta}\mathrm{e}^{\mathrm{i}\sqrt{c/12}X_{\mathrm{R}}(\bar{z})}\,, \tag{3.35}$$

where $\beta/\sqrt{\hat{c}} \bmod 2\pi/\sqrt{\hat{c}}$ corresponds to the value of a Wilson line.

To specify the full boundary state for a D-brane in type II string theory, one has to decide about the extension of the brane in flat space-time. Assume that there are $p+1$ Neumann and $D-p-1$ Dirichlet boundary conditions in D-dimensional space-time. At the boundary, the space-time spinor current,

[9] In the simplest case, where $R^2/2$ is integer, the extension is by the field $\mathrm{e}^{\mathrm{i}RX}$. Then the "automorphism type" $\mathrm{e}^{\mathrm{i}RX_{\mathrm{L}}(z)} = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}RX_{\mathrm{R}}(\bar{z})}$ determines the position of the Dirichlet boundary condition to be at $X_{\mathrm{L}}(z) + X_{\mathrm{R}}(\bar{z})|_{z=\bar{z}} = \alpha/R$.

s_{ext}, is then reflected into a spinor of the same or of the opposite chirality if $p+1$ is even or odd, respectively. Given the reflection of the internal spinor current s_{int}, (3.34) and (3.35), and the fact that the total spinor current has to be reflected into itself for type IIB and into its conjugate for type IIA, one obtains the well-known conditions on the parity of p and the boundary condition in the internal sector. This is summarized in table 3.2.

	internal boundary condition	
	A-type	B-type
type IIA	p odd	p even
type IIB	p even	p odd

Table 3.2: Allowed combinations of number of external Neumann conditions, $p+1$ and internal boundary conditions, A-type or B-type, for compactified type IIA and type IIB string theory.

Given the identification between the total spinor current and space-time supersymmetry, it is now clear that the angles γ and β appearing in eqs. (3.34) and (3.35) measure which combination of space-time supersymmetry charges is preserved by the boundary condition. In other words, the angles are equal to the phase of the central charges. This also shows that the sign in (3.34) is simply the distinction between brane and antibrane.

An explicit construction

Assume now that the internal part of the string compactification, $\mathcal{C}^{\text{inner}}$, has been constructed along the lines described in the previous subsection, where the various projections correspond to simple-current extensions in a bosonic CFT. Using results of [49, 50], it is straightforward to construct boundary conditions for $\mathcal{C}^{\text{inner}}$ that do not preserve the maximally extended algebra $\mathcal{A}^{\text{inner}}$. The data characterizing the symmetries of the boundary conditions can be read off as monodromy charges with respect to the various simple currents. However, the constructions described here will always lead to boundary conditions of A-type with respect to the $\mathcal{N}=2$ algebra.

For details, recall that in [49, 50], boundary conditions for a theory based on a chiral algebra $\mathfrak{A}$ were constructed that preserve only a subalgebra $\underline{\mathfrak{A}}$, which is obtained from $\mathfrak{A}$ as the fixed algebra under a finite Abelian group of automorphisms. By the correspondence between simple-current extensions and orbifolds by finite Abelian groups, this can also be considered as boundary conditions in an $\underline{\mathfrak{A}}$ theory with a specific modular invariant of extension type. The simple-current group, $\mathfrak{G}$, and the automorphism group $\mathfrak{G}^*$ are related by duality. From the perspective of the $\underline{\mathfrak{A}}$ theory, the results are as follows. Boundary conditions preserving $\underline{\mathfrak{A}}$ correspond to $\mathfrak{G}$ orbits of $\underline{\mathfrak{A}}$ primaries $[\bar{\lambda}, \psi]$ (ψ is a degeneracy label given by a character of the untwisted stabilizer). The automorphism type of the boundary condition is an element of the dual orbifold group $\mathfrak{G}^*$, and can be computed as the monodromy charge of $\bar{\lambda}$ with respect to a simple current in $\mathfrak{G}$; in formulas,

$$\mathrm{aut}_{[\bar{\lambda},\psi]}(\mathrm{J}) = Q_{\mathrm{J}}(\bar{\lambda})\,. \tag{3.36}$$

In particular, the symmetries preserved by the boundary condition are given by the subgroup of $\mathfrak{G}$ on which $Q_{.}(\bar{\lambda})$ is trivial.

In the situation under study, there is a sequence

$$\mathcal{A}^{\text{ten. prod.}} \prec \mathcal{A}^{\text{wsusy}} \prec \mathcal{A}^{\text{inner}} \tag{3.37}$$

of embeddings of chiral algebras, where $\mathcal{A}^{\text{ten. prod.}}$, $\mathcal{A}^{\text{wsusy}}$, and $\mathcal{A}^{\text{inner}}$ are the chiral algebras of $\mathcal{C}^{\text{ten. prod.}}$, $\mathcal{C}^{\text{wsusy}}$, and $\mathcal{C}^{\text{inner}}$, respectively. Applying the results of [49, 50], boundary conditions in $\mathcal{C}^{\text{inner}}$ that preserve $\mathcal{A}^{\text{wsusy}}$ are in one-to-one correspondence with orbits under $\mathcal{G}_{\text{ext}}$, eq. (3.24), of primary field labels from the tensor product $\mathcal{C}^{\text{ten. prod.}}$, subject to the restriction of zero monodromy charge with respect to all alignment currents w_i.[10] The monodromy charge with respect to u is not restricted. It takes values in $\mathbb{Z}_N/N$, where N is the order of u, and is related to the space-time supersymmetry that is preserved by the boundary state. Moreover, the $\mathbb{Z}_2$ label $\eta = \pm 1$, [103, 104] is measured by the monodromy charge with respect to v.

The supersymmetry data of a boundary condition so constructed is summarized in table 3.3.

3.2.3 RR charges and intersection index

In the previous subsection, it was shown how to obtain A-type boundary conditions in the internal part of an algebraically constructed string compactification. These boundary conditions were then characterized with respect to the supersymmetry they preserve.

In this subsection, it is shown what and how information about a supersymmetric boundary condition a, is encoded in the expansion of the corresponding boundary state in terms of boundary blocks (Ishibashi states),

$$|a\rangle\!\rangle = \sum_{\lambda} \frac{B_{\lambda a}}{\sqrt{S_{\lambda 0}}} \|\lambda\rangle\!\rangle\,. \tag{3.38}$$

[10] When the orbits are stabilized by a non-trivial subgroup of $\mathcal{G}_{\text{ext}}$, the complete labelling also includes a character ψ of the untwisted stabilizer.

datum	values in	computed as
automorphism type with respect to u	$\frac{\mathbb{Z}_N}{N}$	$\frac{\gamma}{\pi} = Q_{\mathrm{u}}(\lambda) \mod \mathbb{Z}$
preserved space-time supersymmetry (phase of central charge)	$[0, 2\pi]$	$\gamma = 2\pi Q_s(\lambda) \mod 2\pi$
brane/antibrane	± 1	$(-1)^{2Q_s(\lambda)-Q_{\mathrm{u}}(\lambda)}$
η	± 1	$\eta = (-1)^{2Q_v(\lambda)}$

Table 3.3: Supersymmetry data of an A-type boundary condition, labelled by $[\lambda, \psi]_{\mathcal{G}_{\mathrm{ext}}}$. Here, $\mathcal{G}_{\mathrm{ext}}$ is the extension group from $\mathcal{C}^{\text{ten. prod.}}$ to $\mathcal{C}^{\text{inner}}$, and λ is a primary field of $\mathcal{C}^{\text{ten. prod.}}$ with $Q_{\mathrm{w}_i}(\lambda) = 0$ for all alignment currents $\mathrm{w}_i \in \mathcal{G}_{\mathrm{ext}}$. N is the order of the simple current, u, in the theory $\mathcal{C}^{\text{wsusy}}$.

The boundary condition labelled by a is required to have a well-defined automorphism type (A- or B-type) with respect to the $\mathcal{N} = 2$ algebra, and its extension by u. Upon inclusion in a string theory, the associated boundary states will represent wrapped BPS D-branes. It will, however, not be assumed that a belongs to the class of boundary conditions constructed in the previous subsection.[11]

To begin with, it is rather easy to compute the couplings of the boundary state to bulk fields. Quite generally, such a coupling is computed, up to a normalization, by the one-point amplitude of a bulk vertex operator Φ_λ at the center of a disc with boundary condition a. Since the boundary state $|a\rangle\rangle$ simply encodes the information about all such one-point functions, this is suggestively written as an inner product, and can be evaluated with the help of (3.38),

$$\langle \Phi_\lambda(z=0) \rangle_a = \langle \Phi_\lambda || a \rangle\rangle = \frac{B_{\lambda a}}{\sqrt{S_{\lambda 0}}} . \tag{3.39}$$

For this coupling to be non-vanishing, it is necessary that the bulk sector λ contributes an Ishibashi.

The most basic couplings of branes one might be interested in are the mass or tension, m_a, of the brane, as well as the RR charges. The mass is easily computed from the coupling to the vacuum sector, here labelled by 0

[11]To be on the safe side, one might require that the expansion (3.38) is finite, or that the preserved subalgebra is rational. It will not be discussed here whether such a condition is necessary. See also [105] for some arguments in this direction.

[105],

$$m_a \propto B_{0a}\,. \tag{3.40}$$

Notice that m_a never vanishes (in fact, in should always be positive), since the boundary condition preserves the Virasoro-algebra, and hence the vacuum sector always appears in the expansion (3.38).

On the other hand, the coupling to a massless RR field, i, which is proportional to the i-th RR charge,

$$Q_i^{(\mathrm{RR})}(a) \propto B_{ia}/\sqrt{S_{i0}}\,, \tag{3.41}$$

is non-vanishing only if the corresponding Ramond ground state contributes an Ishibashi state of the right automorphism type (*e.g.*, A-type (B-type) boundary states can only couple to Ramond ground states corresponding to the horizontal (vertical) cohomology of the appropriate Calabi-Yau space).

In particular, the central charge of the D-brane is the coupling of the boundary state to the spectral flow operator s. This was argued in section 3.1 for A-type boundary conditions, where the spectral flow is by half a unit on the left, and minus half a unit on the right. [12] Given that the phase of the central charge is the monodromy charge with respect to s, viewed as a simple current (see table 3.3), and the fact that, for a BPS state, the absolute value of the central charge is equal to the mass, it must be true that

$$B_{sa} = e^{2\pi \mathrm{i} Q_s(a)} B_{0a}\,. \tag{3.42}$$

This is of course reminiscent of the simple-current relation (2.28) for the modular S-matrix. Thus, at least in this particular case of the spectral flow, the simple-current relation must generalize to the boundary coefficients even when these are not given by the S-matrix as in the Cardy case.

More interesting than the simplest couplings, one can compute the "intersection index" of boundary states, which is the analog of the geometric quantities discussed in section 3.1.4. In [106, 29], the intersection of two boundary states, $|a\rangle\rangle$ and $|b\rangle\rangle$ is defined as an overlap amplitude in the RR sector. By a modular transformation, this is equal to the Witten index in the open string Hilbert space on the annulus, with boundary conditions a and b on the two sides of the annulus, respectively,

$$\mathcal{I}_{ab} = \langle\langle a||b\rangle\rangle_{\mathrm{RR}} = \mathrm{tr}_{\mathcal{H}_{ab}}(-1)^F\,. \tag{3.43}$$

[12] This is in CFT conventions, compare table 3.1. For B-type boundary conditions, the relevant spectral flow is left-right symmetric and also by half a unit.

The goal now is to derive a more convenient expression for $\mathcal{I}_{ab}$, in view of the cumbersome expansion (3.38).

In accordance with the general conventions adopted in this thesis, the list of boundary blocks will contain all bosonic-primary fields separately. The boundary blocks are normalized as in

$$\langle\langle i \| q^{L_0+\overline{L_0}-\frac{c}{12}} \| j \rangle\rangle = \delta_{ij}\chi_i(\tau)\,. \tag{3.44}$$

Inserting (3.38) in (3.43), and recalling that the definition of the overlap amplitude in the RR sector contains a phase factor $\mathrm{e}^{-\pi \mathrm{i} q_L(i)}$, this leads to

$$\mathcal{I}_{ab} = \sum_i \frac{B^*_{ia}B_{ib}}{S_{i0}}\mathrm{e}^{-\pi\mathrm{i}q_L(i)}\chi_i(\tau)\ , \tag{3.45}$$

where i is summed over all Ishibashi states from the RR sector. $q_L(i)$ is the left-moving U(1) charge of the state i. Usually, q_L has integer eigenvalues, whence $\mathrm{e}^{-\pi\mathrm{i}q_L} = (-1)^{F_L}$. For the considerations in section 4.4, however, it is convenient to have a slightly more general expression.

The expression (3.45) is in fact independent of τ, and can be computed in the limit $\tau \to \mathrm{i}\infty$, where only Ramond ground states (Rgs) contribute. Thus,

$$\mathcal{I}_{ab} = \sum_{i \text{ Rgs}} \frac{B^*_{ia}B_{ib}}{S_{i0}}\mathrm{e}^{-\pi\mathrm{i}q_L(i)}\,. \tag{3.46}$$

It is natural to view this expression as the intersection number in the closed string sector, in analogy with the geometric versions. At least formally, the expression (3.46) is simply the inner product of the RR charge vectors $\big(Q_i^{(\mathrm{RR})}(a)\big)$ and $\big(Q_j^{(\mathrm{RR})}(b)\big)$ with metric given by $\mathrm{e}^{-\pi\mathrm{i}q_L(i)}\delta_{ij}$.

Several properties of $\mathcal{I}$ can be read off from (3.46). For instance, it is obvious that the rank of $\mathcal{I}$ (viewed as a matrix with entries labelled by the boundary states) cannot exceed the dimension of the chiral ring (the number of Ramond ground states is equal to the dimension of the chiral ring). Therefore, the topological charges of the D-branes lie in a lattice of rank bounded by the dimension of the chiral ring. What is not immediate from (3.46), however, is the fact that this lattice, equipped with $\mathcal{I}$ as metric, is integral. Integrality is more apparent in the open string sector. To demonstrate this, make a modular transformation in (3.45) to obtain,

$$\mathcal{I}_{ab} = \sum_{i,m} \frac{B^*_{ia}B_{ib}S_{im}}{S_{i0}}\mathrm{e}^{-\pi\mathrm{i}q_L(i)}\chi_m(-1/\tau)\ , \tag{3.47}$$

where i runs over Ramond Ishibashis and m over all fields. The restriction on i is relaxed by using that

$$S_{i,vm} = \begin{cases} -S_{i,m} & i \text{ in Ramond sector} \\ S_{i,m} & i \text{ in Neveu-Schwarz sector} \end{cases}, \tag{3.48}$$

where vm denotes the worldsheet superpartner of m (v is the simple current corresponding to the worldsheet supercurrent). Furthermore, the U(1) charge is given by half the monodromy charge with respect to the simple current, s, implementing spectral flow by half a unit. Hence, $e^{-\pi i q_L(i)} S_{im} = S_{i,s^{-1}m}$, and

$$\mathcal{I}_{ab} = \frac{1}{2} \sum_{i,m} \frac{B_{ia}^* B_{ib} S_{i,s^{-1}m}}{S_{i0}} (\chi_m(-1/\tau) - \chi_{vm}(-1/\tau)) , \tag{3.49}$$

where now i runs over all fields. This expression is further reduced by using the well-known relation between the Cardy coefficients and the annulus coefficients, $A_{ab}^{s^{-1}m} = \sum_i B_{ia}^* B_{ib} S_{i,s^{-1}m}/S_{i0}$. The annulus coefficients are non-negative integers by the Cardy condition.

To obtain a manifestly integral expression, one can use a slightly different normalization convention for the construction of the true supersymmetric boundary states, as was done in [32, 29]. However, the better alternative seems to be that the factor 1/2 is removed in the last steps (GSO projection) of the construction of the BPS state, where two states with $\eta = \pm 1$ are superposed. Their respective contributions to the intersection index are the same. Then,

$$\mathcal{I}_{ab} = \sum_m A_{ab}^{s^{-1}m} (\chi_m(-1/\tau) - \chi_{vm}(-1/\tau)) . \tag{3.50}$$

Now, $\chi_m - \chi_{vm}$ is a supersymmetric character. It is equal to one (or -1) if m (or vm) corresponds to a Ramond ground state primary and zero otherwise. This yields the intersection number, written in the open string sector with the help of the annulus coefficients,

$$\mathcal{I}_{ab} = \sum_{m \text{ Rgs}} A_{ab}^{s^{-1}m} - A_{ab}^{vs^{-1}m} . \tag{3.51}$$

The intersection index is now written in a manifestly integer form. It follows that the lattice spanned by the boundary states with metric given by $\mathcal{I}$ is an integral lattice, of rank bounded by the dimension of the chiral ring.

Various other interesting properties of the intersection matrix can be derived from (3.51) in a completely model independent way. For instance, if the $\mathcal{N} = 2$ theory constitutes the internal sector of a string compactification, the relation $sv = c$ holds (because of the extension with $\mathcal{G}_{\text{ext}}$), where c is the conjugate of the spectral flow. Using conjugation properties of the annulus coefficients and of the chiral ring, one can then show that the intersection index is (anti)^n-symmetric, where n is the number of compact complex dimensions.

Chapter 4

Examples

The simplest examples of conformal field theories with $\mathcal{N} = 2$ supersymmetry are built on free fields, and correspond geometrically to tori. In such theories, it is expected that a comparison between the classical geometric and abstract algebraic approaches to D-branes is possible and reveals a perfect match. This is demonstrated in section 4.1, in which the two-dimensional torus T^2 is examined in some detail. The main motivation is to obtain a reasonable intuition for the less trivial constructions in subsequent sections.

The next class of examples are the minimal models. These are rational theories whose chiral algebra consists only of the (bosonic part of) the $\mathcal{N} = 2$ super-Virasoro algebra. This property allows the complete analysis of the models, including also a simple treatment of boundary conditions. The physical interest of minimal models is on one hand due to their role as the conformal fixed points of simple Landau-Ginzburg theories. They can thus serve as local mirror models for string propagation near certain singularities, and as a consequence appear in a BCFT description of parts of Seiberg-Witten theory. On the other hand, minimal models are the simplest building blocks for the algebraic construction of string compactifications along the lines of section 3.2. Minimal models and their boundary sectors are reviewed in section 4.2, following [33, 31].

According to the two roles of $\mathcal{N} = 2$ minimal models, there are then two directions to pursue. In section 4.3, which is based in part on [37, 30, 38], the minimal models are tensored together to form Gepner models, their boundary sectors are analyzed, and it is explained how to extract interesting geometric information about D-branes on Calabi-Yau manifolds in the stringy regime. In particular, the analysis of simple-current fixed points and their untwisted stabilizers provides insight into a new mechanism for enhancement of gauge symmetry on D-branes. The other route, namely generalizations to $\mathcal{N} = 2$ coset models and the connection to local singularities is taken up in section

4.4. This material has appeared in [31].

4.1 The torus

4.1.1 The bulk theory

For the purposes of string theory, the two-dimensional torus is equipped with a "background" metric, g_{ij}, and a "background" B-field B_{ij}. The σ-model is defined by the action

$$S = \int \mathrm{d}^2 z \left[g_{ij} \partial X^i \bar{\partial} X^j + \mathrm{i} B_{ij} \partial X^i \bar{\partial} X^j + \mathrm{i} g_{ij} \big(\psi_-^i \partial \psi_-^j + \psi_+^i \bar{\partial} \psi_+^j \big) \right] . \quad (4.1)$$

The quantization of this model is standard. The solutions to the classical equations of motion can be written in the familiar mode expansion, which are quantized as the well-known infinite collection of harmonic oscillators. The zero modes have to satisfy a quantization condition and are labelled by momentum and winding numbers. On the fermions $\psi_\pm^i$, either periodic (R) or antiperiodic (NS) boundary conditions can be imposed. After quantization, it will be useful to bosonize the fermions and express them as an $\mathrm{SO}(2) = \mathrm{U}(1)$ WZW model. This part is rather trivial, but extremely important for supersymmetry considerations. It will be dealt with a little later.

To discuss details for the bosonic part of the theory, denote the two real coordinates on the torus by $X \in [0,1]$ and $Y \in [0,1]$, and suppose that the lengths of the two cycles of the torus are R_1 and R_2, forming an angle α. The metric is

$$g = \begin{pmatrix} R_1^2 & R_1 R_2 \cos\alpha \\ R_1 R_2 \cos\alpha & R_2^2 \end{pmatrix} , \quad (4.2)$$

with inverse,

$$g^{-1} = \frac{1}{R_1^2 R_2^2 \sin^2\alpha} \begin{pmatrix} R_2^2 & -R_1 R_2 \cos\alpha \\ -R_1 R_2 \cos\alpha & R_2^2 \end{pmatrix} , \quad (4.3)$$

and the B-field

$$B = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} . \quad (4.4)$$

Geometrically, the torus is $T^2 = \mathbb{R}^2/\Gamma$, where $\Gamma = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$. This lattice Γ with metric defined by g^{-1} is the lattice containing the winding modes $n = (n_x, n_y) \in \Gamma$, while the momentum modes naturally lie in the dual lattice $\Gamma^* = \mathbb{Z} \times \mathbb{Z} \ni m = (m_x, m_y)$, with metric defined by g.

It is fairly obvious that the (bosonic part of the) conformal field theory obtained from the torus σ-model is the theory for the product of two U(1) current algebras. The left- and right-moving U(1) charges, $q_{L,R} \in \Gamma_{\mathbb{C}}$ are constrained to be of the form,

$$
\begin{aligned}
q_L &= n + Bm + gm \\
q_R &= n + Bm - gm\,.
\end{aligned}
\tag{4.5}
$$

The modular invariant torus partition for this lattice CFT is

$$
Z(\tau,\bar\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{n\in\Gamma, m\in\Gamma^*} \mathrm{e}^{2\pi\mathrm{i}\bar\tau\left(\frac14 g^{-1}(q_L,q_L)-1/12\right)} \mathrm{e}^{2\pi\mathrm{i}\tau\left(\frac14 g^{-1}(q_R,q_R)-1/12\right)}\,, \tag{4.6}
$$

where $\frac14 g^{-1}(q_{L,R},q_{L,R})$ are left- and right-moving conformal weights and $1/12$ is the zero-point energy $c/24$. The factor $|\eta|^2$ is from the oscillators.

In components, the charges read

$$
\begin{aligned}
q_{L,x} &= n_x + Bm_y + \left(R_1^2 m_x + R_1R_2\cos\alpha\, m_y\right) \\
q_{R,x} &= n_x + Bm_y - \left(R_1^2 m_x + R_1R_2\cos\alpha\, m_y\right) \\
q_{L,y} &= n_y - Bm_x + \left(R_1R_2\cos\alpha\, m_x + R_2^2 m_y\right) \\
q_{R,y} &= n_y - Bm_x - \left(R_1R_2\cos\alpha\, m_x + R_2^2 m_y\right).
\end{aligned}
\tag{4.7}
$$

The natural supersymmetric language uses complex coordinates, Z and $\bar Z$. They are defined by[1]

$$
\mathrm{d}Z = \mathrm{d}X + \tau \mathrm{d}Y \tag{4.8}
$$

where $\tau = \tau_1 + \mathrm{i}\tau_2$ is the modular parameter of the torus. In terms of the previous variables, one has $\tau = \mathrm{e}^{\mathrm{i}\alpha}R_2/R_1$. The metric becomes

$$
\mathrm{d}^2 s = \frac{V}{\tau_2}|\mathrm{d}Z|^2\,, \tag{4.9}
$$

where $V = R_1R_2\sin\alpha$ is the volume of the torus. The components of the B-field are $b = B_{z\bar z} = -B_{\bar z z} = -B/2\mathrm{i}\tau_2$. In the complex plane, one then has $T^2 = \mathbb{C}/\Gamma$ with winding lattice $\Gamma = \mathbb{Z} + \tau\mathbb{Z} \ni n_z$ and momentum lattice $\Gamma^* = -\frac{1}{\mathrm{i}\tau_2}(\tau\mathbb{Z} - \mathbb{Z}) \ni m^z$. The corresponding conformal field theory is the theory of one complex boson, with left and right moving charges

$$
\begin{aligned}
q_{L,z} &= n_z + b\,m^{\bar z} + \frac{V}{2\tau_2} m^{\bar z} \\
q_{R,z} &= n_z + b\,m^{\bar z} - \frac{V}{2\tau_2} m^{\bar z}\,.
\end{aligned}
\tag{4.10}
$$

[1]The objects implied by the symbols Z and τ here have nothing to do with the corresponding ones in (4.6).

Yet another equivalent way of writing the charges is obtained by combining not only the complex structure parameters τ_1 and τ_2, but also the volume and the B-field into one complex Kähler parameter,

$$\begin{aligned} \tau &= \tau_1 + \mathrm{i}\tau_2 \\ \rho &= \rho_1 + \mathrm{i}\rho_2 = B + \mathrm{i}V\,. \end{aligned} \tag{4.11}$$

The charges are then

$$\begin{aligned} q_{L,z} &= -\frac{1}{2\mathrm{i}\tau_2}\big(\bar{\tau}n_x - n_y + \bar{\rho}m_x + \bar{\rho}\bar{\tau}m_y\big) \\ q_{R,z} &= -\frac{1}{2\mathrm{i}\tau_2}\big(\bar{\tau}n_x - n_y + \rho m_x + \rho\bar{\tau}m_y\big)\,. \end{aligned} \tag{4.12}$$

Mirror symmetry

It is in this last form, (4.12), that the dualities of the torus are most obvious. The conformal weight of the left moving representations is computed as

$$\begin{aligned} \Delta_L &= \frac{1}{4}\frac{4\tau_2}{V}|q_L|^2 \\ &= \frac{1}{4V\tau_2}\big|\bar{\tau}n_x - n_y + \bar{\rho}m_x + \bar{\rho}\bar{\tau}m_y\big|^2\,, \end{aligned} \tag{4.13}$$

and the spectrum is invariant under an $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ action on (ρ, τ). Furthermore, the spectrum is invariant in a most obvious way under "mirror symmetry", namely the exchange of τ and ρ. Since this operation is also equivalent to the exchange of n_x and m_x, it is easily identified as T-duality along the X-direction of the torus.

4.1.2 Supersymmetry

As mentioned above, the fermionic part of the torus σ-model is equivalent to the SO(2) WZW model at level 1. This in turn is the same as the theory of a free real boson compactified on a circle of radius 2. In fact, this boson is nothing but the U(1) part of the chiral algebra present in any $\mathcal{N} = 2$ compactification, as discussed in section 3.2. Consequently, it will be denoted by X. The vacuum, spinor, vector and conjugate spinor of $\mathrm{SO}(2)_1$ correspond, respectively, to the irreducible representations of U(1) with charge $0, 1/2, 1, -1/2$. The corresponding primary fields are written as vertex operators $1, \mathrm{e}^{\mathrm{i}X/2}, \mathrm{e}^{\mathrm{i}X}, \mathrm{e}^{-\mathrm{i}X/2}$, and have conformal weight 0, 1/8, 1/2, and 1/8, respectively. To make more explicitly contact with chapter 3, note that in this bosonic language, the holomorphic worldsheet supercurrents are nothing but

$$\begin{aligned} G^+(z) &= \partial Z(z)\, e^{iX(z)} & G^-(z) &= \partial \bar{Z}(z)\, e^{-iX(z)} \\ \bar{G}^+(\bar{z}) &= \bar{\partial} Z(\bar{z})\, e^{iX(\bar{z})} & \bar{G}^-(\bar{z}) &= \bar{\partial}\bar{Z}(\bar{z})\, e^{-iX(\bar{z})}\,, \end{aligned} \tag{4.14}$$

while the U(1) current of the $\mathcal{N} = 2$ algebra is

$$J(z) = i\partial X(z)\,. \tag{4.15}$$

Thus, the current denoted by u in section 3.2 ensuring integrality of the U(1) charge is

$$\mathrm{u} = s^2 v = e^{2iX}\,, \tag{4.16}$$

and is in the chiral algebra by construction.

4.1.3 A-type boundary conditions

The identification and construction of boundary conditions and D-branes for the torus model, whose bulk theory was reviewed in the last subsection, will be somewhat sketchy and maybe not as complete as one may wish for. However, recall that the main idea is to develop some intuition for the various ingredients that enter in more complicated examples, and not to produce an overkill for the torus. Various other aspects of the torus model are discussed in the literature [35, 107].

D-branes of A-type are special Lagrangian submanifolds. On the two-dimensional torus, with any Kähler form, any one-dimensional submanifold is Lagrangian. The special Lagrangian condition amounts to the one-dimensional submanifold being a straight line in the Z-plane, as shown in figure 4.1.

Denote the angle between the special Lagrangian cycle and the real axis in the Z plane by γ. It is then readily verified that the condition on worldsheet fields corresponding to Dirichlet boundary conditions on the cycle and to Neumann boundary conditions in the orthogonal direction is

$$e^{-i\gamma}\,\partial Z(z) = e^{i\gamma}\,\bar{\partial}\bar{Z}(\bar{z}) \qquad \text{at } z = \bar{z}\,. \tag{4.17}$$

Namely, in terms of the real and imaginary part of Z, the conditions are[2]

$$\begin{aligned} \cos\gamma\,\partial_n \operatorname{Re} Z + \sin\gamma\,\partial_n \operatorname{Im} Z &= 0 \\ \cos\gamma\,\partial_t \operatorname{Im} Z - \sin\gamma\,\partial_t \operatorname{Re} Z &= 0 \qquad \text{at } z = \bar{z}\,. \end{aligned} \tag{4.18}$$

[2]As in chapter 3, ∂_n and ∂_t stand for derivatives in the direction normal and tangent to the worldsheet boundary respectively.

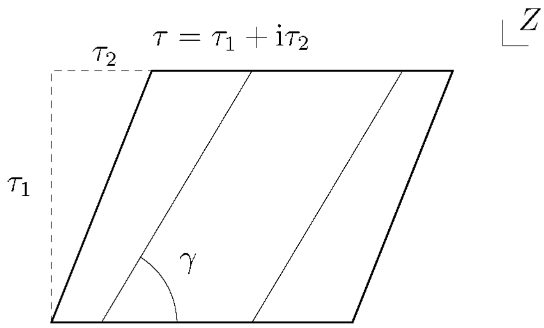

Figure 4.1: A special Lagrangian submanifold of T^2

Elementary geometric considerations show that not any choice of γ corresponds to a closed submanifold of T^2. Rather, γ has to be such that the cycle closes after winding a finite number of times around each cycle of the torus. Thus, there have to be integers n_1, n_2 such that

$$\tan\gamma = \frac{n_2\,\tau_2}{n_1 + n_2\,\tau_1}\,; \tag{4.19}$$

in other words, $(\tau_2 - \tau_1\,\tan\gamma)/\tan\gamma$ has to be a rational number. This is the geometric quantization condition on γ.

To rediscover the same condition from conformal field theory with boundaries, notice that the geometric condition (4.17) is nothing but the expression for an automorphism type of a boundary condition with respect to the chiral $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry of the lattice CFT describing the complex boson Z. To proceed, one has to identify the left-right combinations of representations in the partition function (4.6) that can contribute Ishibashi states for a given automorphism type. In terms of charges, the condition is

$$-\mathrm{e}^{-\mathrm{i}\gamma}q_{L,z} + \mathrm{e}^{\mathrm{i}\gamma}q_{R,\bar{z}} = 0\,, \tag{4.20}$$

with $q_{L,z}$ and $q_{R,z}$ as in (4.12). It is straightforward to show that the condition reduces to

$$\begin{aligned} m_y(\cos\gamma\,\tau_2 + \sin\gamma\,\tau_1) + m_x\sin\gamma &= 0 \\ n_x(\cos\gamma\,\tau_2 + \sin\gamma\,\tau_1) - n_y\sin\gamma &= 0\,. \end{aligned} \tag{4.21}$$

In other words, one obtains the condition that

$$\frac{\cos\gamma\,\tau_2 + \sin\gamma\,\tau_1}{\sin\gamma} = \cot\gamma\,\tau_2 + \tau_1 = \frac{n_1}{n_2} \tag{4.22}$$

has to be rational number, which is the same as the geometric quantization condition. Also, contributing Ishibashi states arise from

$$\begin{aligned} \frac{m_y}{m_x} &= -\frac{n_2}{n_1}\,, \\ \frac{n_x}{n_y} &= \frac{n_2}{n_1}\,. \end{aligned} \tag{4.23}$$

Rather obviously now, the classifying algebra for boundary conditions with automorphism type given by (4.17) is equal to the group algebra of $\mathbb{Z} \times \mathbb{Z}$, admitting irreducible representations labelled by two numbers $a, b \in \mathbb{R}$ mod $2\pi\mathbb{Z}$. It is clear that one may identify a geometrically with the intercept of the special Lagrangian cycle with the real axis and b with the value of a U(1) Wilson line along the one-dimensional world-volume of the brane.

In summary, A-type branes or boundary states on T^2 are classified topologically by two numbers n_1 and n_2, and have two continuously adjustable parameters a and b. The branes couple to closed string winding and momentum modes satisfying the first and the second equation of (4.23), respectively.

Supersymmetry

To check that the boundary conditions (4.17) are indeed consistent with supersymmetry, notice that by equation (4.14) and the usual A-type boundary condition on the supercurrents, $G^+(z) = \bar{G}^-(\bar{z})$, one obtains the condition

$$e^{i\gamma} e^{iX(z)} = e^{-i\gamma} e^{-iX(\bar{z})} \qquad \text{at } z = \bar{z}. \tag{4.24}$$

This is indeed consistent with Dirichlet boundary conditions

$$\partial X(z) = -\bar{\partial} X(\bar{z}) \qquad \text{at } z = \bar{z}, \tag{4.25}$$

on the U(1) current, and in fact shows that the Dirichlet boundary condition has to be chosen at one of the positions, γ, $\gamma + \pi$. The ambiguity corresponds to a choice of orientation of the D-brane, and as in chapter 3, is resolved by considering the condition on the spinor,

$$e^{iX(z)/2} = e^{-i\gamma} e^{-iX(\bar{z})/2}\,, \tag{4.26}$$

which ultimately decides about conserved and broken supersymmetry charges in the remaining eight non-compact space-time directions. It is interesting to note that only rational positions of the Dirichlet condition on X are allowed and that the position, γ, completely specifies the topological sector of the D-brane. In particular, for example, the mass of the D-brane is a completely discontinuous function of γ. In contrast, the geometric position, a, and Wilson line, b, are continuous parameters or moduli, of the special Lagrangian cycle.

As a final observation, notice that of course the choice of origin for the position γ is ambiguous. This can be traced back to the one-parameter freedom in A-type boundary conditions on the supercurrents (see chapter 3). Any two conventions for the origin $\gamma = 0$ are always related by a redefinition $X \mapsto X + \delta$ or, in supersymmetric language, to a chiral R-rotation on the fermions [29].

4.1.4 B-type boundary conditions

Given mirror symmetry as the exchange $\rho \leftrightarrow \tau$, it is apparent what changes in the formulae are necessary to go from A-type to B-type boundary conditions. Geometrically, B-type branes correspond to holomorphic vector bundles. The boundary of the worldsheet is coupled to a gauge field, and the boundary conditions also depend on the background B-field. B-type boundary conditions are mixed Dirichlet-Neumann boundary conditions, and are as such slightly less pictorial than A-type boundary conditions.

It will be argued below that B-type boundary conditions are given by the automorphism type

$$e^{-i\beta}\,\partial Z(z) = e^{i\beta}\,\bar\partial Z(\bar z) \qquad \text{at } z = \bar z\,, \tag{4.27}$$

in place of (4.17) on the complex boson Z.

Similarly to before, the combination of charges contributing Ishibashi states is

$$\begin{aligned} m_y(-\sin\beta\,\rho_2 + \cos\beta\,\rho_1) + n_x\cos\beta &= 0 \\ m_x(-\sin\beta\,\rho_2 + \cos\beta\,\rho_1) - n_y\cos\beta &= 0\,. \end{aligned} \tag{4.28}$$

Under the assumption that β satisfies the quantization condition that

$$\frac{\sin\beta\,\rho_2 - \cos\beta\,\rho_1}{\cos\beta} = \tan\beta\,\rho_2 - \rho_1 = \frac{c_1}{r} \tag{4.29}$$

be a rational number, there is again a $\mathbb{Z}\times\mathbb{Z}$ variety of Ishibashi states. This rational number will soon be identified with the slope of the holomorphic bundle, c_1/r. As a check, notice that all-Dirichlet boundary conditions, $\partial Z = -\bar\partial Z$, correspond to $\cos\beta = 0$, or $r = 0$, $c_1 = 1$. On the other extreme, assume that the B-field vanishes. Then all-Neumann boundary conditions, $\partial Z = \bar\partial Z$, correspond to $\sin\beta = 0$, or $r = 1$, $c_1 = 0$.

Summarizing B-type boundary conditions, they are classified topologically by two integers c_1 and r, and also have two continuous parameters whose geometric interpretation should be that of Wilson lines, respectively of the position of the D0 brane in the case of all-Dirichlet boundary conditions.

To obtain the geometric interpretation of the boundary condition (4.27), it is necessary to couple the worldsheet boundary to a gauge field. Namely, the path integral has to contain a Wilson line factor

$$\mathrm{tr}\big(P\exp\oint A\big)\,. \tag{4.30}$$

For the two-torus, the geometric boundary conditions derived with the coupling (4.30) are

$$\begin{aligned}\partial_n\,\mathrm{Re}\,Z+\frac{B+F}{V}\,\partial_t\,\mathrm{Im}\,Z&=0\\ \partial_n\,\mathrm{Im}\,Z-\frac{B+F}{V}\,\partial_t\,\mathrm{Re}\,Z&=0\,,\end{aligned} \tag{4.31}$$

which upon identifying $\tan\beta=(B+F)/V$ is the same as (4.27),

$$\begin{aligned}\cos\beta\,\partial_n\,\mathrm{Re}\,Z+\sin\beta\,\partial_t\,\mathrm{Im}\,Z&=0\\ \cos\beta\,\partial_n\,\mathrm{Im}\,Z-\sin\beta\,\partial_t\,\mathrm{Re}\,Z&=0\,.\end{aligned} \tag{4.32}$$

In other words, the quantization condition (4.29) is

$$\frac{B+F}{V}\,\rho_2-\rho_1=F=\frac{c_1}{r}\,. \tag{4.33}$$

4.2 From $\mathcal{N}=2$ minimal models to ADE singularities

4.2.1 Introduction

Similarly to the situation for the Virasoro algebra, $\mathcal{N}=2$ minimal models come in a discrete series, labelled by a positive integer k, with central charge $c=3k/(k+2)$. Viewed as bosonic CFTs, minimal models can be obtained from the coset construction as

$$\frac{\mathrm{SU}(2)_k\times\mathrm{SO}(2)_1}{\mathrm{U}(1)}\,, \tag{4.34}$$

simple current	minimal model labels	order	conformal weight mod$\mathbb{Z}$
v	$(0,0,2)$	2	$3/2$
s	$(0,1,1)$	$\begin{cases} 4h & k \text{ odd} \\ 2h & k \text{ even} \end{cases}$	$k/8h$
p	$(0,2,0)$	h	$-1/h$
f	$(k,0,0)$	2	$k/4$

Table 4.1: The most important simple current of a single $\mathcal{N} = 2$ minimal model, at level $k = h - 2$. Note that $s^2v = p$ and $f = p^{h/2}v$.

where the $\mathrm{SO}(2)_1 \equiv \mathrm{U}(1)_4$ comes from the fermions and the "level" of the U(1) in the denominator (related to the radius for the boson in the usual way) is determined from the embedding to be $2h = 2k + 4$. Accordingly, the primary fields of an $\mathcal{N} = 2$ minimal model are labelled by (l, m, s), with $l = 0, 1, \ldots, k$; $m \in \mathbb{Z}_{2h}$, $s \in \mathbb{Z}_4$, subject to the most usual restriction $l+m+s$ even, and field identification $(l, m, s) \equiv (k-l, m+h, s+2)$. Minimal models are the simplest examples of $\mathcal{N} = 2$ coset models, to be considered in more detail in section 4.4.

For later use, it will be convenient to have an overview of the simple currents in $\mathcal{N} = 2$ minimal models. These simple currents are summarized in table 4.1. Notice that when k is odd, the center, *i.e.*, the group of all simple currents, of the model is generated by s and is isomorphic to $\mathbb{Z}_{4h}$, whereas for even k the center is generated by v and s and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2h}$.

The simple current f is special because it is the only one with fixed points; it leaves fixed all fields of the form $(k/2, m, s)$. The fixed point S-matrix has entries

$$S^f_{(k/2,m,s),(k/2,m',s')} = \mathrm{e}^{-2\pi\mathrm{i}\,3k/16} \cdot \frac{1}{\sqrt{2h}}\,\mathrm{e}^{2\pi\mathrm{i}mm'/2h} \cdot \frac{1}{2}\,\mathrm{e}^{-2\pi\mathrm{i}ss'/4}\,, \tag{4.35}$$

as can be verified using eq. (6.1) of ref. [66].

4.2.2 Superconformal boundary conditions in $\mathcal{N} = 2$ minimal models

Boundary conditions in $\mathcal{N} = 2$ minimal models that yield BPS states upon inclusion in a string model must have a definite automorphism type with respect to the $\mathcal{N} = 2$ algebra. A-type boundary conditions for the charge conjugation modular invariant are completely under control by the Cardy construction. B-type boundary conditions are slightly more difficult to obtain. They can be constructed either by computing the orbifold of the minimal model by the mirror (*i.e.*, charge conjugation) automorphism of the $\mathcal{N}=2$ algebra, *e.g.*, along the lines of [108]. Or one can use the Greene-Plesser construction of mirror pairs to obtain B-type boundary conditions as A-type in the mirror, using results from section 2.2. This second possibility will be taken up in the context of Gepner models, the present section being restricted to the Cardy case.

To fix notation, A-type boundary conditions are labelled by the same set as the primary fields, (L,M,S) with $0\le L\le k$, $M\in\mathbb{Z}_{2h}$, $S\in\mathbb{Z}_4$ with $L+M+S$ even and identification $(L,M,S)\equiv(k-L,M+h,S+2)$. The explicit expression for the boundary coefficients is obtained by combining the modular S-matrices from the individual factors in (4.34). Thus, the boundary states are expressed in terms of the boundary blocks as

$$|(L,M,S)\rangle\!\rangle=\sum_{(l,m,s)}\frac{1}{\sqrt{h}}\frac{\sin\frac{\pi(l+1)(L+1)}{h}\,\mathrm{e}^{2\pi\mathrm{i}mM/2h}\,\mathrm{e}^{-2\pi\mathrm{i}sS/4}}{\sqrt{\sin\frac{\pi(L+1)}{h}}}\,\|(l,m,s)\rangle\!\rangle\,. \tag{4.36}$$

The meaning of the label S is clear in view of table 3.3. Namely, S mod 2 gives the monodromy charge of the boundary condition with respect to the simple current $v=(0,0,2)$. Thus, upon inclusion of the minimal model in a string compactification and fermion alignment (which implies alignment of S with the corresponding label from the remaining factors of the CFT), S will be identified with the $\mathbb{Z}_2$ label η in table 3.3. Also note that the label M mod h measures the reflection of the simple current $p=(0,2,0)=s^2$ at the boundary (compare the first row in table 3.3). The corresponding phase is given by $\mathrm{e}^{2\pi\mathrm{i}M/h}$. Finally, the contribution to the phase of the central charge is measured by $s=(0,1,1)$, and equal to $\mathrm{e}^{2\pi\mathrm{i}M/2h}\,\mathrm{e}^{-2\pi\mathrm{i}S/4}$. Thus, in particular, $S\to S+2$ is equivalent to exchanging brane with anti-brane.

In view of this, it will generally be sufficient to consider only one half (or even one quarter) of the (bosonic) boundary states, conventionally chosen to have $S=0,2$ (or simply $S=0$).

Intersection index

To compute the "intersection index" of A-type boundary states in $\mathcal{N}=2$ minimal models, it suffices to substitute equation (4.36) or the explicitly known fusion coefficients into either of the general formulas (3.46) or (3.51). To perform the computation, recall that the labels of the Ramond ground states are of the form $(l, l+1, 1)$ and that the corresponding chiral primaries are labelled by $(l, l, 0)$, where $l = 0, 1, \ldots, k$. The result [32, 100, 29] is as follows. For fixed labels L and L', the intersection of boundary states (L, M, S) and (L', M', S') can be viewed as a matrix in M and M'. Since the supersymmetry labels S and S' can be set to 0 or 2, the parities of M and M' in $\mathbb{Z}_{2h}$ are also fixed. Thus the intersection matrix is of size $h \times h$. However, a slightly more flexible notation is appropriate. Introduce, for any integer n, the $n \times n$ dimensional matrix

$$g_n := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix} . \tag{4.37}$$

The intersection matrix between the L and L' sector then is

$$\mathcal{I}_{LL'} = \sum_{l=0}^{k} N_{L'l}^{L} \Big(\big(g_{2h}\big)^{2l} - \big(g_{2h}\big)^{2l+2} \Big) , \tag{4.38}$$

where $N_{L'l}^{L}$ is an $\mathrm{SU}(2)_k$ fusion coefficient, and where it is implicitly understood that only the entries corresponding to the allowed parity of M and M' really have a meaning. The other rows and columns can be deleted upon desire. For example, for $L = L' = 0$, one may rewrite

$$\mathcal{I}_{00} = 1 - g_h . \tag{4.39}$$

An interesting observation [32] is that the states with $L \neq 0$ can be viewed, as far as their topological properties are concerned, as bound states of the "elementary states" with $L = 0$. Namely, one can show [100] that

$$\mathcal{I}_{LL'} = t_L^t \, \mathcal{I}_{00} \, t_{L'} , \tag{4.40}$$

where

$$t_L = \sum_{l=-L}^{L} \big(g_{2h}\big)^{l} , \tag{4.41}$$

and t_L^t is the transpose of this matrix. The matrix $(t_L)_M^{M'}$, restricted to the allowed parity of M, gives the expansion coefficients of the RR charges of higher L states in terms of those of states with $L' = 0$, $M' = 0, 2, \ldots, 2h-2$. Indeed, one may verify that the RR charges computed from the Cardy coefficients, (4.36) satisfy the property implicit in (4.41).

4.2.3 $\mathcal{N} = 2$ minimal models and simple singularities; the boundary sector

Superconformal boundary states in $\mathcal{N} = 2$ minimal models have an interesting application in the context of non-perturbative solutions of supersymmetric Yang-Mills theory. This connection was pointed out in [33] and further investigated in [109, 110, 111]. It will be reviewed here as an example of the geometric connections of $\mathcal{N} = 2$ BCFT and also as a motivation for the investigations in section 4.4.

The connection between $\mathcal{N} = 2$ minimal models in two dimensions and strongly coupled $\mathcal{N} = 2$ super Yang-Mills theory in four dimensions brings together several circles of ideas. First, there is the well-known reconstruction of the Seiberg-Witten solution of $\mathcal{N} = 2$ gauge theories from type II string compactification on Calabi-Yau threefolds [112]. Decoupling of gravity amounts to restricting to a neighborhood of an appropriate isolated singularity on the threefold. In the simplest cases, the local (non-compact) geometry is described by a fibration of a two-dimensional ALE space over a $\mathbb{P}^1$ base. This ALE space, in turn, can be viewed as the resolution by a chain of $\mathbb{P}^1$'s of a simple singularity of ADE type. The strong coupling spectrum of the 4d gauge theory is represented by D-branes wrapping the various compact cycles in the ALE geometry.

A second ingredient then is the description of the simple singularities of ADE type in terms of Landau-Ginzburg CFTs with a certain superpotential. At the Gepner or orbifold point, which corresponds to strong coupling, the superpotential is of the form $W = x^N + 1/z^N$ for the series of A_{N-1} singularities. The first term in this superpotential can be viewed as describing the compact part of the ALE geometry, while the second term subsumes the non-compact dynamics, which is non-universal, but decoupled.

This is the point where $\mathcal{N} = 2$ minimal models enter. Namely, it is well-known that a minimal model at level k with diagonal modular invariant (also called minimal models of type A_{k+1} [3]) describes the conformal fixed point of a Landau-Ginzburg theory with superpotential $W = x^{k+2}$. Boundary states in $\mathcal{N} = 2$ minimal models should therefore correspond to D-branes in the corresponding Landau-Ginzburg model. That this is indeed the case, and that geometric considerations can be used, for example, to compute (some of) the Cardy coefficients (4.36), was shown in [29]. This identification of minimal boundary states in a Landau-Ginzburg theory will also play a role in the context of Gepner models in section 4.3.

As a result of all this, it is possible to obtain the strong coupling spectrum

[3]For the minimal models of D and E type, see the comments below.

of $\mathcal{N} = 2$ SYM with gauge group SU(N) in terms of the spectrum of boundary states in an $\mathcal{N} = 2$ minimal model at level $k = N - 2$. The proposal is substantiated by the following results; for more details, see [33].

The Landau-Ginzburg theory corresponding to the non-compact ALE space of type A_{N-1} is described by a Gepner type model starting with a coset $\mathrm{SU}(2)_{N-2}/\mathrm{U}(1)$, which is the compact minimal model, and a non-compact coset $\mathrm{SL}(2)_{N+2}/\mathrm{U}(1)$. Although not everything is known about the general definition of the latter model, one can determine the basic properties of the boundary sector by simple analogy with unitary minimal models. Essentially, the $\mathrm{SL}(2)_{N+2}/\mathrm{U}(1)$ can be viewed as a non-unitary coset $\mathrm{SU}(2)/\mathrm{U}(1)$ at negative level $-N-2$. Therefore, if a single minimal model at level k has basic intersection form $1 - g_h$, the appropriately projected tensor product $\mathrm{SU}(2)_{N-2}/\mathrm{U}(1) \times \mathrm{SL}(2)_{N+2}/\mathrm{U}(1)$ has intersection[4]

$$\mathcal{I}_{00} = (1 - g_N)(1 - g_N^{-1}) = 2 - g_N - g_N^{-1}\,. \tag{4.42}$$

This matrix is easily seen to equal the extended Cartan matrix of the SU(N) gauge group. The corresponding boundary states should therefore be interpreted in terms of D-branes wrapped around the compact 2-cycles of the ALE space, which also intersect precisely with this pattern. The basic $L = 0$ states here correspond to a set of simple roots (plus the highest root) of SU(N).

What is more, one can consider also states with $L \neq 0$ in the minimal model and their charges and intersection, see eq. (4.40). Surprisingly, it turns out [33] that the collection of all these states can be mapped one-to-one to the set of roots of SU(N)! After fibration of the ALE space, these are exactly the BPS states in the Yang-Mills theory that are expected to be stable at the origin of the moduli space. Notice that *a priori*, the intersection matrix (4.42), which can be computed either from CFT or from geometry, merely determines the lattice of BPS charges that are consistent with charge quantization at any point in moduli space. However, the occupation of this charge lattice with stable BPS states depends on the region of moduli space one is considering, and there may be lines of marginal stability on which some of the states decay. In particular, the easily understood truncation in rational CFT to a finite number of supersymmetric boundary states is translated to the quantum truncation of BPS states to a finite number at strong coupling in the Yang-Mills.

To conclude this section, it should be mentioned that the analysis goes through in very much the same way also for the minimal models with modular

[4] The effect of the projection on the intersection matrix is explained in section 4.3

invariants of D and E-type [109]. The boundary states in those models can be determined using results of [51], and again one finds agreement both between intersection form of boundary states and the corresponding extended Dynkin matrix, and between the finite number of boundary states and the finite number of roots of the simple Lie algebra.

4.3 From Gepner models to Calabi-Yau hypersurfaces

4.3.1 Introduction

After the analysis of the simplest non-trivial example of $\mathcal{N} = 2$ superconformal field theories and their open string sectors in the previous section, namely the minimal models, a natural next class of example are the Gepner models. Gepner models [113] are examples of string compactifications where the CFT describing the compact part of space-time is constructed algebraically, and the corresponding CFT is rational. The general strategy was described in section 3.2. The original Gepner construction uses tensor products of minimal models in the internal sector, other possibilities are built on more general $\mathcal{N} = 2$ coset models, yielding the Kazama-Suzuki models.

The importance of Gepner models for the development of string theory in the late 1980's is largely due to the connection to geometric string compactifications on Calabi-Yau manifolds. More precisely, it is known that a Gepner model is the exact solution of a σ-model on a Calabi-Yau manifold at a special, so-called Gepner, point in moduli space. This is particularly interesting because at the Gepner point, the curvature of the classical target space is large and σ-model perturbation theory is not reliable. Therefore, the description by an exactly solvable, algebraically constructed, CFT offers the possibility of exploring σ-models at large worldsheet coupling. Historically, the exploration of Gepner models and the associated geometrical string models led to the discovery of mirror symmetry [10, 12, 114, 13], with exciting physical and mathematical applications, see [7].

After the third superstring revolution and the advent of D-branes, it could be expected that the analysis of open string sectors in Gepner models for type II strings would provide the basis for similarly interesting developments, concerning not solely the manifold and its Kähler and complex structure moduli space, but also submanifolds of the Calabi-Yau, vector bundles over them, etc., and the associated quantum geometry. On the physics side, D-branes on Calabi-Yau manifolds have important applications for testing string dualities, as well as for the construction of realistic $\mathcal{N} = 1$ string vacua.

A first step towards such goals is the description of boundary conditions in Gepner models, and the identification of corresponding geometric objects. Given the non-geometric nature of the Gepner, this identification will only concern a certain subset of topological properties, like RR charges and the intersection of branes. These can be transported over bulk moduli space from the Gepner point all the way to the large volume limit, where classical geometry applies again.

In the past few years, there have been several approaches to the study of open string sectors in Gepner models and the comparison with geometric objects at large volume. The first geometric characterization of D-branes in Calabi-Yau manifolds was given in [27], building in part on earlier work in [92]. A first ansatz for boundary conditions in Gepner models [34] was inspired by Cardy's construction, generalized so as to account for the special projections that arise in the Gepner construction. These boundary states were analyzed regarding preserved space-time supersymmetry in [35] and compared to a Landau-Ginzburg description in [36]. A parallel development gave rise to a geometrical interpretation of the boundary states in terms of D-branes wrapped on the Calabi-Yau manifolds corresponding to the Gepner models. This was developed, and applied to the quintic, in [32], and to several other models in [100, 37, 115]. The stability of D-branes upon transport in moduli space was studied in [88, 116, 94]. Meanwhile, an independent approach to D-branes in strongly curved Calabi-Yau manifolds has been undertaken in [117, 29], using the connection to Landau-Ginzburg models and linear σ-models. This has led to an independent check of the identification of Gepner model boundary states and bundles in [118, 119, 120, 121].

It is the goal of this section to describe in detail the construction of boundary conditions in Gepner models and the comparison with geometric objects at large volume along the lines of [32, 29]. Of particular interest will be the careful implementation of the appropriate projections, and the resolution of the arising fixed points. For A-type states (associated with real submanifolds) the algebraic problems with fixed points were first pointed out in [36], analyzed in an example in [122], and solved in [30]. For B-type states, associated to holomorphic geometry, it was noticed in [100, 37] that owing to the presence of fixed points, for some of the B-type states constructed in [34] the open string vacuum is non-unique. In [38], the resolution of the fixed points along with a geometric interpretation in terms of enhanced gauge symmetries and singular bundles was achieved.

The plan of this section is as follows. The internal part of a Gepner model is obtained along the lines described in section 3.2, starting from the tensor product of $\mathcal{N} = 2$ minimal models. In particular therefore, the construction of (a subclass of) A-type boundary conditions is a special case of the general methods described there. In subsection 4.3.2 the detailed implementation of this prescription is presented along with the computation of the intersection matrix.

Subsection 4.3.3 is devoted to B-type boundary conditions. On general grounds, there is no reason to expect the construction of B-type boundary conditions to be particularly simple[5]. However, Gepner models enjoy the pleasant property that the mirror model can be obtained according to the Greene-Plesser construction [12] as an orbifold. This allows the construction of B-type boundary conditions as A-type in the mirror model, given a technique to construct boundary conditions in the Greene-Plesser orbifold. In algebraic language, the self-mirror property of the Gepner model amounts to the fact that the charge conjugation invariant is a simple-current modification of the diagonal modular invariant (and vice versa). Thus, the general methods of section 2.2 can be applied.

Subsection 4.3.4 then describes the main steps leading to an identification between the lattices of RR charges in the BCFT and in the geometric description, and presents the results for an explicit example.

It is worthwhile pointing out that the constructions described in the following subsections do by no means exhaust the possible supersymmetric boundary conditions in Gepner models—let alone the non-supersymmetric ones. The reputation of Gepner models of being exactly solvable examples of string compactifications has to be significantly tempered in the open string sector. Indeed, while in the closed string sector spectra and all couplings can be computed exactly, at least in principle, this crucially depends on rationality with respect to a largely extended chiral symmetry. As soon as the CFT is perturbed by a truly marginal operator most of the chiral symmetry is broken, generically only leaving the $\mathcal{N} = 2$ superconformal symmetry. Additional tools are then needed to obtain any quantitative information at all. Something similar happens in the construction of D-branes. The existing general, and powerful, methods for constructing open string boundary conditions in rational CFT require that also the preserved subalgebra be rational. But the boundary conditions so obtained will only be a tiny subset of all possible $\mathcal{N} = 2$ superconformal ones. One can, for example, imagine perturbing the open string background by D-brane moduli, keeping the bulk

[5]Indeed, from an abstract point of view, B-type boundary conditions for Gepner models are characterized by both an increase and a breaking of the chiral symmetry. In general, there might be obstructions to combine these two operations for boundary conditions. See the appendix of [30].

moduli fixed, and thereby generating more general boundary conditions. Information about those is incomplete. The set of boundary conditions that preserve most of the chiral symmetries of the Gepner model is still rather non-trivial. One would expect that the corresponding points in the moduli spaces of D-branes are special —maybe singular—points, analogously to the situation with bulk moduli.

4.3.2 A-type boundary conditions

The bulk theory

Recall from section 3.2 that starting from any tensor product of $\mathcal{N} = 2$ superconformal field theories, $\mathcal{C}^{\text{ten. prod.}}$, with appropriate total central charge, one can obtain the internal sector of a string compactification, $\mathcal{C}^{\text{inner}}$, by a sequence of simple-current extensions. Here, $\mathcal{C}^{\text{ten. prod.}}$ is explicitly chosen to be the tensor product of r $\mathcal{N} = 2$ minimal models,

$$\mathcal{C}^{\text{ten. prod.}} = \mathcal{C}_{k_1} \otimes \mathcal{C}_{k_2} \otimes \cdots \otimes \mathcal{C}_{k_r} \,, \tag{4.43}$$

with total central charge $\sum_i c_i = \sum_i 3k_i/h_i = 15 - 3D/2$. Borrowing notation from section 4.2, primary fields and simple currents in the tensor product will simply be denoted by appending a subscript i to the labels of the fields in a single minimal model.

The relevant simple-current group that extends $\mathcal{C}^{\text{ten. prod.}}$ to $\mathcal{C}^{\text{inner}}$ is then generated by $r-1$ order-2 currents $\mathrm{w}_i = v_1 v_i$ and by another current,[6]

$$\mathrm{u} = v_1^{n+r} \prod p_i \tag{4.44}$$

of order,[7]

$$H := \text{l.c.m.}(h_i)_{i=1,\ldots,r} \,. \tag{4.45}$$

Thus the extension group $\mathcal{G}_{\text{ext}}$ is of the form

$$\mathcal{G}_{\text{ext}} := \langle \mathrm{w}_i, u \rangle \;\cong\; (\mathbb{Z}_2)^{r-1} \times \mathbb{Z}_H \,. \tag{4.46}$$

[6]This definition of u is equal to the one of section 3.2 modulo identification by alignment currents. The present definition is more convenient combinatorially, but the distinction is of no importance, since alignment currents are always preserved at the boundary.

[7]Notice that $n + r$ odd implies that at least one k_i is even. This follows by a simple argument from the basic central charge condition $\sum_i 1 - 2/h_i = n$, which is equivalent to

$$H(r-n) = 2 \sum \frac{H}{h_i} \,.$$

Primary fields in $\mathcal{C}^{\text{ten. prod.}}$, eq. (4.43) are labelled by collections $(\lambda,\mu) := (l_1, l_2, \ldots, l_r, m_1, \ldots, m_r, s_1, \ldots, s_r)$. The extension by $\mathcal{G}_{\text{ext}}$ imposes well-known restrictions and identifications on these field labels (λ,μ). A label (λ,μ) appears in the extension only if

$$Q_{\mathrm{w}_i}(\lambda,\mu) = \frac{s_1}{2} + \frac{s_i}{2} = 0 \mod \mathbb{Z}\ , \text{ for all } i = 2, \ldots, r, \text{ and} \tag{4.47}$$

$$Q_{\mathrm{u}}(\lambda,\mu) = (n+r)\,\frac{s_1}{2} - \sum_{i=1}^{r} \frac{m_i}{h_i} = 0 \mod \mathbb{Z}\,. \tag{4.48}$$

Furthermore, two labels (λ,μ) and (λ',μ') are considered equivalent in the extension if there are $\epsilon_i \in \{0,1\}$ and $\alpha \in \{0,\ldots,H-1\}$ such that

$$\begin{aligned}(\lambda',\mu') &= \mathrm{u}^{\alpha}\Big(\prod_{i=2}^{r} \mathrm{w}_i^{\epsilon_i}\Big)(\lambda,\mu)\\ &= \big(\lambda, m_1 + 2\alpha, m_2 + 2\alpha, \ldots, m_2 + 2\alpha,\\ &\qquad s_1 + 2{\textstyle\sum_i}\epsilon_i + 2(n+r)\alpha, s_2 + 2\epsilon_2, \ldots, s_r + 2\epsilon_r\big)\,.\end{aligned} \tag{4.49}$$

The modular invariant partition function of the Gepner model is given by a sum over equivalence classes, weighted with the order of the stabilizer,

$$Z = \sum_{[(\lambda,\mu)]} |\mathcal{S}_\lambda|\, |\chi_{[(\lambda,\mu)]}|^2\,, \tag{4.50}$$

where $\mathcal{S}_\lambda$ is the stabilizer in the extension group $\mathcal{G}_{\text{ext}}$ of a given (λ,μ), and will be computed below. Furthermore,

$$\chi_{[(\lambda,\mu)]} = \sum_{\mathrm{J}\in\mathcal{G}_{\text{ext}}/\mathcal{S}_\lambda} \chi_{\mathrm{J}(\lambda,\mu)} \tag{4.51}$$

is the extended character corresponding to an equivalence class of labels. The action of $\mathrm{J} \in \mathcal{G}_{\text{ext}}$ on (λ,μ) is as given in (4.49).

Notice that for obtaining the modular invariant partition function, the simple-current language is completely equivalent to Gepner's method of "β-vector" [113]. However, the subtleties associated with fixed points are best captured by simple-current techniques. The fixed point combinatorics is the same for bulk fields as for A-type boundary conditions, so it is convenient to first also label the latter.

Labelling of boundary conditions

According to the results of section 3.2, boundary conditions in $\mathcal{C}^{\text{inner}}$ that preserve the chiral symmetry $\mathcal{A}^{\text{wsusy}}$ are labelled by $\mathcal{G}_{\text{ext}}$ orbits of primary fields from $\mathcal{C}^{\text{ten. prod.}}$ with vanishing monodromy charge with respect to the w_i. Thus, the starting point for the labelling of the boundary conditions are again the labels of the tensor product of minimal models,

$$(\Lambda, \mathtt{M}) := (L_1, \dots, L_r, M_1, \dots, M_r, S_1, \dots, S_r)\,, \tag{4.52}$$

with $L_i + M_i + S_i$ even. The basic goal now, and in all subsequent discussions of labelling issues, is to use the relevant selections and identifications to reduce the set of labels to a "standard" one.

First, for A-type boundary conditions in $\mathcal{C}^{\text{ten. prod.}}$, field identification in individual minimal models allows arranging $L_i \le k_i/2$ as well as $S_i \in \{0, 1\}$ for those i with $L_i = k_i/2$.

Next, the condition that $\mathcal{A}^{\text{wsusy}}$ be preserved imposes the selection rule $S_1 = S_i \bmod 2$ for all i. Identification by the w_i,

$$(\Lambda, \mathtt{M}) \equiv (\Lambda, M_1, \dots, M_r, S_1 + 2, \dots, S_i + 2, \dots, S_r)\,, \tag{4.53}$$

allows replacing all S_i's by a single S, say $S = S_1 \in \{0, \pm 1, 2\}$.

There is no selection rule for the M_i labels. The quantity

$$-\sum_i \frac{M_i}{h_i} + (n+r)\,\frac{S}{2} \mod \mathbb{Z}\,, \tag{4.54}$$

defined mod $\mathbb{Z}$, is the automorphism type of the boundary condition with respect to the current u, and unrestricted.

The explicit implementation of the identification of labels implied by the current u,

$$(\Lambda, \mathtt{M}) \equiv (\Lambda, M_1 + 2, \dots, M_r + 2, S_1 + 2(n+r), S_2, \dots, S_r)\,, \tag{4.55}$$

is a little more difficult. It involves questions of the divisibility of the heights h_i, which in general does not have a simple structure. In special cases, for instance when $\mathrm{l.c.m.}(h_i) = h_j$ for some j, the corresponding label M_j can be set to zero using this identification. However, from an abstract point of view, it is rather simple to obtain an overview over the labels after imposing identification by u. This is described in appendix 4.A.

The significance of these labels for the supersymmetry of the boundary conditions follows from table 3.3. The label $S_0 \equiv S \mod 2$ measures the $\mathbb{Z}_2$ quantum number $\eta = (-1)^{S_0}$. The automorphism type with respect to u, equal to $\gamma/\pi \bmod \mathbb{Z}$, determines the angle γ, but only up to the $\mathbb{Z}_2$ brane/antibrane ambiguity. This ambiguity may be fixed by computing the "automorphism type" with respect to the total spectral flow operator. Explicitly,

$$\frac{\gamma}{2\pi} = Q_s(\Lambda, \mathtt{M}) = -\sum_i \frac{M_i}{2h_i} + \frac{S}{4} + (r-1)\frac{S_0}{4} \mod \mathbb{Z}\,. \tag{4.56}$$

This shows in particular that changing S by 2 (with M_i's fixed) exchanges brane with antibrane. *Nota bene:* This is not saying that S alone allows a distinction between branes and antibranes. Among others, the fact that u changes S for $n+r$ odd, see (4.55), clearly prohibits this. Somewhat arbitrarily, one may call a state with $0 \le \gamma < \pi$ a brane and a state with $\pi \le \gamma < 2\pi$ an antibrane.

Fixed points

To find out about fixed points, both for bulk fields and for A-type boundary conditions, it suffices to determine which combinations of f_i's appear in $\mathcal{G}_{\text{ext}}$ (recall from table 4.1 that $f_i = (k_i, 0, 0)$ is the only simple current with fixed points in a single minimal model). This is simple, and the result is as follows.

- When all levels k_i are odd, there are no fixed points.
- Assume on the contrary that at least one level is even. Then $f_i = p_i^{h_i/2} v_i$ for all even k_i. Thus, for

$$F := \mathrm{u}^{\alpha} \prod_{i=2}^{r} \mathrm{w}_i^{\epsilon_i} \tag{4.57}$$

to have fixed points, *i.e.*, to be equal to $\prod_{i \in I_F} f_i$, for some $I_F \subset \{1, \dots, r\}$, it is necessary and sufficient that

$$\begin{aligned} h_i \text{ divides } \alpha \text{ and } \epsilon_i = 0 \quad & \text{for } i \notin I_F\,, \\ \frac{1}{2} h_i \mid \alpha \text{ and } h_i \nmid \alpha \text{ and } \epsilon_i = 1 \quad & \text{for } i \in I_F \setminus \{1\}\,, \\ \frac{1}{2} h_1 \mid \alpha \text{ and } h_1 \nmid \alpha \text{ and } {\textstyle\sum_i} \epsilon_i + (n+r)\,\alpha = 1 \bmod 2 \quad & \text{for } i = 1\,, \end{aligned} \tag{4.58}$$

where, without loss of generality, $1 \in I_F$ is assumed. As a minimal condition, $\alpha = \big(\text{l.c.m.}(h_i)\big)/2 = H/2$. To proceed, denote for any positive integer m, the power of 2 contained in m by $\sigma(m)$. [Examples: $\sigma(8) = 3, \sigma(3) = 0, \sigma(24) = 3$.] Furthermore, introduce $\Sigma := \max\{\sigma(h_i)\} = \sigma(H) = \sigma(\alpha) + 1$. Then, (4.58) is equivalent to $I_F = \{i, \sigma(h_i) = \Sigma\}$, and to the condition

$$|I_F| + (n+r)\, 2^{\Sigma-1} = 0 \bmod 2\,. \tag{4.59}$$

It turns out that this last condition is always trivial. Namely, from the anomaly cancellation condition, $n = \sum_i \frac{k_i}{k_i+2}$, one deduces (see footnote 7 on page 78),

$$\sum_{i=1}^{r} \frac{H}{h_i} = (r-n)\,\frac{H}{2}\,,$$
$$\sum_{i\in I_F} \frac{H}{h_i} = (r-n)\;2^{\Sigma-1} \mod 2\,, \tag{4.60}$$

where the last step relies on $\sigma(H/2) = \Sigma - 1$ and on $H/h_i = 0 \bmod 2$ if $i \notin I_F$. Eq. (4.60) implies (4.59).

As a result, there is only a single simple current in $\mathcal{G}_{\text{ext}}$ with fixed points, namely $F = \prod_{i,\sigma(h_i)=\Sigma} f_i$. It fixes all fields (λ, μ) respectively all boundary conditions $(\Lambda, \mathtt{M})$ with l_i, respectively L_i, equal to $k_i/2$ for all $i \in I_F = \{i, \sigma(h_i) = \Sigma\}$.

Intersection index

It is straightforward, but not very illuminating, to write down the Cardy coefficients for all A-type states constructed above; see [30] for explicit expressions. However, the intersection index of boundary states can be computed without much trouble. It suffices to know the general behavior of fusion coefficients under simple-current extensions [65].

Neglecting fixed points, the fusion coefficients of the extension by a group $\mathfrak{G}$ of local simple currents are given as combinations of fusion coefficients of the unextended theory as

$$^{\mathfrak{G}}N_{[\lambda],[\mu]}^{[\nu]} = \sum_{\mathrm{K}\in\mathfrak{G}} N_{\mathrm{K}\lambda,\mu}^{\nu}\,. \tag{4.61}$$

Here, $[\lambda]$, $[\mu]$, and $[\nu]$ are orbits of primaries of the unextended theory that are allowed in the extended theory, *i.e.*, have vanishing monodromy charge with respect to all currents in $\mathfrak{G}$. For boundaries, one is also interested in orbits that do not label any primary in the extended theory. Still, there is a similar formula for the annulus coefficients (see [50], in particular for fixed point issues). It is convenient to write this formula in terms of matrices of annulus and fusion coefficients,

$$^{\mathfrak{G}}A^{[\lambda]} \otimes \mathbf{1}_{|\mathfrak{G}|} = \sum_{\mathrm{K}\in\mathfrak{G}} N_{\mathrm{K}\lambda}\,.$$

The factor $\mathbf{1}_{|\mathfrak{G}|}$ accounts for the fact that the size of the matrices for the extended theory is smaller by a factor of $\mathfrak{G}$. Using the fusion algebra, this is also equal to

$$ {}^{\mathfrak{G}}\!A^{[\lambda]} \otimes \mathbf{1}_{|\mathfrak{G}|} = \sum_{\mathrm{K}\in\mathfrak{G}} N_{\mathrm{K}}\, N_{\lambda} \,. \tag{4.62} $$

This is easily applied to A-type boundary conditions in Gepner models after recalling the expression for the intersection matrix in terms of annulus coefficients, (3.51). As for a single minimal model, one may reduce the set of boundary labels to $S = 0$. The fusion matrix of the simple current $\prod_i p_i$ in the tensor product is simply $g_1 \otimes g_2 \cdots \otimes g_r$, where $g_i := g_{h_i}$. That u differs from this by a factor of v^{n+r} simply means that there is an extra minus sign in the action of the corresponding fusion matrix. Therefore, the intersection matrix of the Λ and Λ' boundary sectors is

$$ \mathcal{I}^{\mathrm{A}}_{\Lambda\Lambda'} \otimes \mathbf{1}_H = \Big[\sum_{j=0}^{H-1} \big((-1)^{n+r} g_1 \otimes \cdots \otimes g_r\big)^j\Big] \bigotimes_{i=1}^{r} \mathcal{I}_{L_i\, L'_i} \,, \tag{4.63} $$

where $\mathcal{I}_{L_i\, L'_i}$ is given by equation (4.38). In particular, for $\Lambda = \Lambda' = 0$, one finds,

$$ \mathcal{I}^{\mathrm{A}}_{00} \otimes \mathbf{1}_H = \big[1 + (-1)^{n+r} \otimes_i g_i + \big(\otimes_i g_i\big)^2 + \cdots\big] \;\otimes_{i=1}^{r} (1 - g_i) \,. \tag{4.64} $$

The factor of $\mathbf{1}_H$ can be made explicit on the right hand side of these equations by substituting the relation

$$ \otimes_i g_i = (-1)^{n+r} \,, \tag{4.65} $$

which is effectively imposed by the multiplication with the square bracket [32, 37].

For boundary states with $\Lambda \neq 0$, the property (4.40) still holds in the (projected) tensor product. A slight complication arises with non-trivial stabilizers. From the expression for the intersection matrix in the closed sector, (3.46), it easily follows that the intersection of boundary conditions with non-trivial stabilizer is of the general form

$$ \mathcal{I}_{(\Lambda,\psi),(\Lambda',\psi')} = \frac{1}{|\mathcal{S}_\Lambda|\,|\mathcal{S}'_\Lambda|} \Big(\mathcal{I}^0_{\Lambda,\Lambda'} + \psi\psi'\,\tilde{\mathcal{I}}\Big) \,, \tag{4.66} $$

where $\mathcal{I}^0$ is the result obtained from naive application of (4.63), and $\tilde{\mathcal{I}}$ depends on the existence of Ramond ground states that arise from the resolution of fixed points in the bulk. More precisely, $\tilde{\mathcal{I}} = 0$ if there are no such fixed point Ramond ground states, and ± 1 if there are. The details depend on the model under considerations, and no general rule has been obtained so far.

4.3.3 B-type boundary conditions

General strategy

While for A-type boundary conditions, the general theory of section 3.2 applies, for B-type boundary condition the Greene-Plesser construction, which is special to Gepner models, enters crucially. The general idea is to view B-type boundary conditions as A-type in the mirror model, and to describe the mirror model as a Greene-Plesser orbifold. This is explained in the present subsection.

Looking back at table 3.1, B-type boundary conditions are of automorphism type "mirror" when viewed from the charge conjugation modular invariant, but of trivial automorphism type when viewed from the diagonal modular invariant. By viewing the diagonal modular invariant as a simple-current modification (Greene-Plesser orbifold) of the charge conjugation modular invariant, the construction of boundary conditions requires the generalizations of the Cardy construction to simple-current modular invariants. Since the chiral symmetries before and after charge conjugation are the same, the discussion of section 3.2.2 regarding supersymmetry properties of the boundary conditions apply integrally.

The Greene-Plesser construction with simple currents

To apply the results of section 2.2, it is necessary to first reformulate the Greene-Plesser construction [12] of the mirror model in simple-current language. Incidentally, this will also resolve certain confusion that seems to persist in the literature about the Greene-Plesser construction, as applied in CFT.

In the CFT construction of mirror models, charge conjugation is applied to one chiral half of the model. Obviously, it is not essential from which side of the mirror one starts, *i.e.*, from the diagonal or from the charge conjugation modular invariant. However, the choice must be specified in order to avoid confusion. In the context of boundary conditions, it is most natural to start from the charge conjugation invariant, so that "A-type boundary conditions are the Cardy case". Then the mirror model is the diagonal modular invariant, up to a slight subtlety to be discussed below. On the other hand, most of the literature about the Greene-Plesser construction considers only the bulk theory and thus starts from the diagonal invariant. This convention will also be adopted here.

The mirror of a single $\mathcal{N}=2$ minimal model

The main idea of the Greene-Plesser construction is already apparent at the level of a single minimal model, so this will be considered as a warm-up example. Start from the diagonal modular invariant,

$$Z = \sum_{\substack{(l,m,s)\\ l+m+s\in 2\mathbb{Z},\ s=0,1}} \chi_{(l,m,s)}\,\overline{\chi}_{(l,m,s)}\,, \tag{4.67}$$

and consider first its modification by the simple-current group generated by p ("mod out the phase symmetry"). Since this group is cyclic, the pairing X is fixed by the conformal weight of p to be $X(p,p) = -1/h$ (see subsection 2.2.1). Then according to the general rules, the combination $(l,m,s)\,\overline{(l',m',s')}$ occurs in the new partition function if and only if

$$\begin{aligned} (l',m',s') = p^{\alpha}(l,m,s) \quad &\text{for some } \alpha, \text{ and} \\ Q_p(l,m,s) + X(p,p^{\alpha}) = 0\,. & \end{aligned} \tag{4.68}$$

Since $X(p,p^{\alpha}) = \alpha X(p,p) = -\alpha/h$ and $Q_p(l,m,s) = -m/h$, there is a unique solution $\alpha = -m$. Thus inserting $p^{\alpha}(l,m,s) = (l, m+2\alpha, s)$, it follows $l' = l$, $m' = -m$, $s' = s$, and so the modified partition function reads

$$Z^{(p)} = \sum_{\substack{(l,m,s)\\ l+m+s\in 2\mathbb{Z},\ s=0,1}} \chi_{(l,m,s)}\,\overline{\chi}_{(l,-m,s)}\,. \tag{4.69}$$

The next step is to include the simple current v in the simple-current group. Again $X(v,v) = 1/2$ is fixed, and vanishing "discrete torsion" is chosen, $X(p,v) = 0$. Then in addition to the condition (4.68),

$$\begin{aligned} (l',m',s') = v^{\epsilon}(l,m,s) \quad &\text{for } \epsilon \in \{0,1\}, \text{ and} \\ Q_v(l,m,s) + X(v,v^{\epsilon}) = 0\,. & \end{aligned} \tag{4.70}$$

Inserting the identities $v(l,m,s) = (l,m,s+2)$, $Q_v(l,m,s) = s/2$ and $X(v,v^{\epsilon}) = \epsilon/2$, the unique solution is seen to be $\epsilon = s$, so that

$$Z^{(p,v)} = \sum_{\substack{(l,m,s)\\ l+m+s\in 2\mathbb{Z},\ s=0,1}} \chi_{(l,m,s)}\,\overline{\chi}_{(l,-m,-s)}\,. \tag{4.71}$$

This is nothing but the charge conjugation modular invariant of the minimal model, and thus indeed the "mirror model" of the $\mathcal{N}=2$ minimal model with diagonal modular invariant.

Tensor products and their extensions

Now consider tensor products of r minimal models, of levels $k_1, k_2, \ldots, k_r$. As before, primary fields are labelled by collections (λ, μ) with $\lambda = (l_1, \ldots, l_r)$ and $\mu = (m_1, \ldots, m_r, s_1, \ldots, s_r)$, subject to the usual restrictions. Simple currents in the tensor product receive an additional index i according to the factor model to which they belong: v_i, s_i, p_i, f_i. The currents v_i and p_i generate a subgroup $\mathcal{G}_{\mathrm{ph}}$ of the center of the tensor product. Elements $\Pi \in \mathcal{G}_{\mathrm{ph}}$ are denoted by $\Pi = (\alpha_1, \ldots, \alpha_r, \epsilon_1, \ldots, \epsilon_r)$, standing for $\prod_i p_i^{\alpha_i} v_i^{\epsilon_i}$ with $\alpha_i \in \{0, 1, \ldots, h_i - 1\}$, $\epsilon_i \in \{0, 1\}$.

As an abstract group,

$$\mathcal{G}_{\mathrm{ph}} \cong (\mathbb{Z}_2)^r \times \prod_i \mathbb{Z}_{h_i}\,, \tag{4.72}$$

and $\mathcal{G}_{\mathrm{ph}}$ acts on the set of primary fields. Notice that the primary fields and group elements have similar labelling, but that the action and group composition laws are different. To avoid the confusion that this causes, the group law in $\mathcal{G}_{\mathrm{ph}}$ will be noted multiplicatively and the action of $\mathcal{G}_{\mathrm{ph}}$ on primary fields additively. The monodromy charge of a primary field (λ, μ) with respect to $\Pi \in \mathcal{G}_{\mathrm{ph}}$ is

$$Q_\Pi(\lambda, \mu) = \sum_i \Big(-\frac{\alpha_i m_i}{h_i} + \frac{\epsilon_i s_i}{2} \Big)\,, \tag{4.73}$$

($m_i \in \{-h_i + 1, \ldots, h_i\}$, $s_i \in \{-1, 0, 1, 2\}$). Note that $Q_\Pi(\lambda, \mu)$ depends on m_i only mod h_i and on s_i only mod 2.

By repeating the arguments for the single model above, one sees that the mirror model is obtained from the diagonal modular invariant of the tensor product by "dividing out" (*i.e.*, forming the simple-current modular invariant for) $\mathcal{G}_{\mathrm{ph}}$. However, the theory of interest here is the tensor product extended by the group $\mathcal{G}_{\mathrm{ext}}$ (4.46), imposing fermion alignment and integral U(1) charge in the NS sector. Since all currents in $\mathcal{G}_{\mathrm{ext}}$ are mutually local, left-right combinations $(\lambda, \mu)\,\overline{(\lambda', \mu')}$ occur in the partition function if and only if

$$\begin{aligned} (\lambda', \mu') &= \mathrm{J}(\lambda, \mu) \quad \text{for some } \mathrm{J} \in \mathcal{G}_{\mathrm{ext}} \quad \text{and} \\ Q_{\mathrm{K}}(\lambda, \mu) &= 0 \qquad\qquad \text{for all } \mathrm{K} \in \mathcal{G}_{\mathrm{ext}}\,. \end{aligned} \tag{4.74}$$

The goal is now to define a group $\mathcal{G}_{\mathrm{mirr}}$ of simple currents and a pairing X on $\mathcal{G}_{\mathrm{mirr}}$, such that the associated modular invariant is the mirror model of the $\mathcal{G}_{\mathrm{ext}}$-extension of the tensor product. Precisely what is this mirror model? It turns out that the mirror is not simply the charge conjugation invariant in the (bosonic) CFT sense, but rather the charge conjugation invariant multiplied by the invariant corresponding to the simple current v^n. This invariant is non-trivial only in the Ramond sector, where it exchanges the two bosonic primaries belonging to the same superfield.

The origin of this subtlety is that the mirror operation involves an exchange of the two supercurrents $G^\pm \leftrightarrow G^\mp$. In the Ramond sector, this operation exchanges spinor and conjugate spinor representation of the zero modes $G_0^\pm$ precisely if the central charge is odd. This follows from the representation theory of the $\mathcal{N}=2$ algebra, (3.16). Since in the bosonic formulation, the two supercurrents belong to the same primary field, v, this exchange must be performed "by hand". Thus, the mirror model is obtained by inverting all U(1) charges, modulo an extra action by v in the R sector if n is odd. The exchange is related to the flip from type IIA to type IIB string theory, if n is odd.

The claim is that the desired simple-current group is

$$\mathcal{G}_{\text{mirr}} := \langle \mathrm{w}_i,\ i=2,3,\ldots,r;\ v_1^\epsilon \textstyle\prod_j p_j^{\alpha_j}\ ,\ \dfrac{\epsilon}{2} - \displaystyle\sum_{j=1}^{r} \frac{\alpha_j}{h_j} = 0 \,\rangle\,, \tag{4.75}$$

with $\epsilon = 0$ for $n+r$ even and $\epsilon = 0,1$ for $n+r$ odd. The bilinear pairing X must be chosen in such a way that $\mathcal{G}_{\text{ext}}$ is local with respect to all other currents and such that there is no extension beyond $\mathcal{G}_{\text{ext}}$. This is satisfied if and only if X is the restriction to $\mathcal{G}_{\text{mirr}}$ of the pairing

$$X(p_i,p_j) = -\frac{\delta_{ij}}{h_i}\,, \qquad X(v_i,v_j) = \frac{1}{2}\,, \qquad X(p_i,v_j) = 0 = X(v_i,p_j) \tag{4.76}$$

(for all $i,j = 1,\ldots,r$) on $\mathcal{G}_{\text{ph}}$. To see this, first notice that $\mathcal{G}_{\text{ext}}$ is always a subgroup of $\mathcal{G}_{\text{mirr}}$ and that $X(\Xi,\mathrm{J}) = 0$ for every $\mathrm{J} \in \mathcal{G}_{\text{ext}}$ and every $\Xi \in \mathcal{G}_{\text{mirr}}$. Also, X is symmetric, so that left and right chiral algebras are extended by the same simple-current group, which contains at least $\mathcal{G}_{\text{ext}}$. To prove that there is no extension beyond $\mathcal{G}_{\text{ext}}$, it suffices to show that if $X(\Xi,\Pi) = 0$ for all $\Xi \in \mathcal{G}_{\text{mirr}}$, then $\Pi \in \mathcal{G}_{\text{ext}}$. To this end, consider $\Pi = (\alpha_1,\ldots,\alpha_r,\epsilon_1,\ldots,\epsilon_r) \in \mathcal{G}_{\text{mirr}}$. As $H = \text{l.c.m.}(h_i)$, there is a $t = \prod p_i^{t_i}$ with $X(\mathrm{u},t) = -\sum t_i/h_i = 1/H$. Then t together with all $\Xi \in \mathcal{G}_{\text{mirr}}$ generate all phase symmetries p_1, p_2, $\ldots$, p_r (including v_1 for $n+r$ odd). It follows that the simple current

$$\tilde{\Pi} := \mathrm{u}^{-H\,X(t,\Pi)}\,\Pi \tag{4.77}$$

satisfies $X(p_i,\tilde{\Pi}) = 0$ for all i and must hence be of the form $\tilde{\Pi} = (0,\ldots,0,\tilde{\epsilon}_1,\ldots,\tilde{\epsilon}_r)$. Now Π is in the kernel of X if and only if $\tilde{\Pi}$ is. In particular, $\tilde{\Pi}$

must be self-local, *i.e.*, satisfy $X(\tilde{\Pi}, \tilde{\Pi}) = 0$. Thus $\sum_i \tilde{\epsilon}_i$ is even, and hence $\tilde{\Pi} \in \mathcal{G}_{\mathrm{ext}}$, as claimed.

To show that this choice of $(\mathcal{G}_{\mathrm{mirr}}, X)$ gives the mirror invariant specified above, it needs to be checked that every solution to the analog of (4.74) for $(\mathcal{G}_{\mathrm{mirr}}, X)$, *i.e.*, to the conditions,

$$\begin{aligned} (\lambda', \mu') = \Pi(\lambda, \mu) \quad & \text{for some } \Pi \in \mathcal{G}_{\mathrm{mirr}} \text{ and} \\ Q_\Xi(\lambda, \mu) + X(\Xi, \Pi) = 0 \quad & \text{for all } \Xi \in \mathcal{G}_{\mathrm{mirr}}\,, \end{aligned} \tag{4.78}$$

is of the form

$$(\lambda', \mu') = (v_1)^\sigma (\lambda, -\mu) \mod \mathcal{G}_{\mathrm{ext}}\,, \tag{4.79}$$

where $\sigma = 0$ in the NS sector and $\sigma = n$ in the R sector[8].

To proceed, identify the label $\mu = (m_1, \ldots, m_r, s_1, \ldots, s_r)$ in the obvious manner with an element of $\mathcal{G}_{\mathrm{ph}}$ (*i.e.*, take m_i modulo h_i and s_i modulo 2), and set $\nu := \mu\Pi$ (in notation appropriate to the group law, not to the action of $\mathcal{G}_{\mathrm{ph}}$). Ideally, the second line of (4.78) should be expressed with the help of X and ν, so that the desired result would follow from the fact that the kernel of X is precisely $\mathcal{G}_{\mathrm{ext}}$. However, the monodromy charge $Q_\Xi(\lambda, \mu)$ does not coincide exactly with the pairing $X(\Xi, \mu)$, and moreover, $\nu \notin \mathcal{G}_{\mathrm{mirr}}$ in general. For example, when $n + r$ is even, then all $\Pi = (\alpha_1, \ldots, \alpha_r, \epsilon_1, \ldots, \epsilon_r) \in \mathcal{G}_{\mathrm{mirr}}$ must obey $\sum \epsilon_i$ even, but μ in the R sector satisfies this condition only if r is even. However, using that $\sum \zeta_i$ is even for $\Xi = (\beta_1, \ldots, \beta_r, \zeta_1, \ldots, \zeta_r) \in \mathcal{G}_{\mathrm{mirr}}$, it follows that $Q_\Xi(\lambda, \mu) = X(\Xi, (v_1)^\sigma \mu)$, with σ as above. A similar analysis for $n + r$ odd reveals that in all cases[9] $(v_1)^\sigma \nu$ satisfies both

$$Q_\Xi(\lambda, \mu) + X(\Xi, \Pi) = X(\Xi, (v_1)^\sigma \nu) \quad \text{and} \quad (v_1)^\sigma \nu \in \mathcal{G}_{\mathrm{mirr}}\,. \tag{4.80}$$

The conclusion is that $X(\Xi, (v_1)^\sigma \nu) = 0$ for all $\Xi \in \mathcal{G}_{\mathrm{mirr}}$, and thus by repeating the argument above, $(v_1)^\sigma \nu \in \mathcal{G}_{\mathrm{ext}}$. This implies (4.79).

The arguments can easily be extended to prove that one obtains "complementary mirrors" [12] by dividing out appropriate subgroups of $\mathcal{G}_{\mathrm{mirr}}$.

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[8] The freedom in the choice of v allows to use v_1.

[9] The general argument is as follows. Firstly, the s_i's and ζ_i's contribution to $Q_\Xi(\lambda, \mu)$ is $\frac{\sum \zeta_i}{2}\, s$, and to $X(\Xi, \mu)$ it is $\frac{\sum \zeta_i}{2}\, rs$. Therefore, to $X(\Xi, (v_1)^\sigma \mu)$ they contribute $\frac{\sum \zeta_i}{2}\, s(n + r) = \frac{\sum \zeta_i}{2}\, s \mod \mathbb{Z}$. Secondly, the condition on μ is $Q_{\mathrm{u}}(\lambda, \mu) = -\sum \frac{m_i}{h_i} + (n + r)\,\frac{s}{2} = 0$, so that $(v_1)^\sigma \mu$ satisfies $X(\mathrm{u}, \lambda, (v_1)^\sigma \mu) = -\sum \frac{m_i}{h_i} + (n + r)\,\frac{s}{2} = 0$ which is the condition for $\mathcal{G}_{\mathrm{mirr}}$.

Ishibashi states

Having reviewed and clarified the construction of mirror models due to Greene and Plesser in the language of simple currents, the next step in constructing the boundary conditions is to establish a list of boundary blocks (Ishibashi states) that will contribute to the boundary states. The reader should be alerted that the following discussion obtains after switching back to the conventions in which is original model is described by the charge conjugation modular invariant.

For the problem at hand, the boundary blocks are obtained from all those primary fields that are paired with their charge conjugate in the mirror model, which by the present convention is (almost) the diagonal invariant. These fields are precisely those that are self-conjugate up to the action of $\mathcal{G}_{\text{ext}}$ and, for R sector fields when n is odd, of v_1. Thus, the boundary blocks are determined by looking for all solutions to the requirement

$$(v_1)^{\sigma}(\lambda,-\mu)=\prod_{i=2}^{r}(\mathrm{w}_i)^{\epsilon_i}\,\mathrm{u}^m\prod_{i=1}^{r}(\kappa_i)^{\epsilon_i'}(\lambda,\mu)\,, \tag{4.81}$$

where (λ,μ) must be an allowed field in the $\mathcal{G}_{\text{ext}}$-extension and κ_i denotes field identification in the i-th factor; also, all ϵ's are 0 or 1, and $m\in\{0,1,\dots,H-1\}$.

Obviously, ϵ_i' can be non-zero only if $l_i=k_i/2$. Let I be the set of those i for which $l_i=k_i/2$ **and** $\epsilon_i'=1$. By fermion alignment, $s_i=s \bmod 2$ for all i with $s\in\{0,\pm1,2\}$. The m_i must satisfy

$$\begin{aligned} -m_i &= m_i+2m && \bmod 2h_i && \text{for } i\notin I\,,\\ -m_i &= m_i+2m+h_i && \bmod 2h_i && \text{for } i\in I\,. \end{aligned} \tag{4.82}$$

Notice that $I\neq\emptyset$ requires $m+s\in 2\mathbb{Z}+1$, because of the selection rule in the corresponding minimal models: Modulo two, the second line in (4.82) implies $0=k_i/2+m_i+s=k_i/2+m+h_i/2+s=m+s+1$ for $i\in I$.

The condition on the s_i's reads

$$\begin{aligned} -s_i &= s_i+2\epsilon_i+2\epsilon_i' \bmod 4 && \text{for } i=2,3,\dots,r\,,\\ -s_1+2\sigma &= s_1+2m\,(n+r)+2\epsilon_1'+2\sum_i\epsilon_i \bmod 4 && \text{for } i=1\,. \end{aligned} \tag{4.83}$$

Since $\sigma=sn \bmod 2$, these two equations yield

$$2\sum_i s_i+2sn=2\sum_i\epsilon_i'+2m\,(n+r)\bmod 4\,, \tag{4.84}$$

and this finally gives

$$(n+r)(s+m)+|I| = 0 \mod 2\,. \tag{4.85}$$

As a check, notice that zero monodromy charge with respect to u requires

$$\begin{aligned} Q_{\mathrm{u}}(\lambda,\mu) &= \sum_i \frac{-m_i}{h_i} + \frac{s}{2}(n+r) = -m\,\frac{1}{2}(n+r) - \frac{|I|}{2} + \frac{s}{2}(n+r) \\ &= \frac{1}{2}\,(n+r)(s+m) + \frac{|I|}{2} = 0 \mod 1\,, \end{aligned} \tag{4.86}$$

and coincides with the condition on the s_i's, eq. (4.85).

Counting Ishibashis thus involves finding solutions of (4.85). When $n+r$ is even, $|I| \in 2\mathbb{Z}$ is a necessary and sufficient condition for having a solution to (4.85). For $n+r$ odd and $|I| = 0$, $s = 0, 1$ and m even and odd are allowed, but with their sum $s+m$ restricted to be even. For $n+r$ odd and $|I| \neq 0$, as already noted, $m+s$ must be odd, and this implies that $|I|$ must be odd.

Restricting to boundary conditions that preserve $\mathcal{A}^{\mathrm{wsusy}}$, these solutions have then to be counted modulo (field identification and) identification by alignment currents[10]. On the other hand, since the chiral symmetry generated by u need not be preserved, this current does not lead to identifications among Ishibashis.

Given the choice of m, s and I satisfying (4.85), eq. (4.82) determines m_i modulo h_i, for each i. Field identification in a single minimal model is fully used up by fixing, in the integers, $m_i = m$ for $i \notin I$ and $m_i = m + h_i/2$ for $i \in I$.

Identification by alignment currents is then used to replace all s_i's with a single s, say $s = s_1 \in \{0, \pm 1, 2\}$. Because spectral flow connects one-to-one s even with s odd, it suffices to consider only $s = 0, 2$.

The enumeration of Ishibashis now starts with a set $I \subset \{1, \ldots, r\}$ with k_i even for $i \in I$, and with the correct parity of $|I|$, depending on $n+r$ even or odd. Then one chooses an m with the correct parity. This in turn fixes the m_i's as described above, and hence restricts also the parity of allowed l_i's for $i \notin I$. For $i \in I$, of course, $l_i = k_i/2$.

The number of l_i's with given parity is $(k_i+1)/2$ for k_i odd. For k_i even, it is $k_i/2+1$ if m is even, and $k_i/2$ if m is odd.

[10]In the language of [50], this amounts to going to the corresponding ideal of the classifying algebra.

Putting things together yields the expressions

$$\begin{aligned} n+r \text{ even}: \quad & 2\times\frac{H}{2}\times\prod_{k_i \text{ odd}}\Big(\frac{k_i+1}{2}\Big)\times\Big[\prod_{k_i \text{ even}}\Big(\frac{k_i}{2}+1\Big)+\prod_{k_i \text{ even}}\frac{k_i}{2}\Big] \\ & +\sum_{I,\,|I| \text{ even}}\Big[2\times\frac{H}{2}\times\prod_{k_i \text{ odd}}\Big(\frac{k_i+1}{2}\Big)\times\prod_{k_i \text{ even},\, i\notin I}\frac{k_i}{2}\Big] \\ n+r \text{ odd}: \quad & 2\times\frac{H}{2}\times\prod_{k_i \text{ odd}}\Big(\frac{k_i+1}{2}\Big)\times\prod_{k_i \text{ even}}\Big(\frac{k_i}{2}+1\Big) \\ & +\sum_{I,\,|I| \text{ odd}}\Big[2\times\frac{H}{2}\times\prod_{k_i \text{ odd}}\Big(\frac{k_i+1}{2}\Big)\times\prod_{k_i \text{ even},\, i\notin I}\frac{k_i}{2}\Big]. \end{aligned} \tag{4.87}$$

for the total number of Ishibashi states with $s = 0, 2$. Expression (4.87) makes sense in all cases with fairly obvious conventions about products and sums. In particular, $|I| = 0$ is not included as even, here.

The boundary conditions

The starting point for describing the labelling of the boundary conditions is again the labels of the tensor product of minimal models,

$$(L_1, \dots, L_r, M_1, \dots, M_r, S_1 \dots, S_r) \tag{4.88}$$

with $L_i + M_i + S_i$ even. As usual, the S_i's are aligned, and can be replaced with a single S, say $S = S_1 \in \{0, \pm 1, 2\}$, and $S_i \in \{0, 1\}$ for $i = 2, \dots, r$. The supersymmetry data is determined exactly as in the case of A-type boundary conditions. Taken modulo 2, S determines η, and the quantity

$$-\sum_i \frac{M_i}{h_i} + (n+r)\,\frac{S}{2} \mod \mathbb{Z} \tag{4.89}$$

gives the automorphism type with respect to u. Given the role of η for space-time supersymmetry, inequivalent branes are counted by restricting to $S = 0, 2$. One can then compute the phase of the central charge,

$$\frac{\gamma}{2\pi} = \frac{M}{2H} := -\sum_i \frac{M_i}{2h_i} + \frac{S}{4} \mod \mathbb{Z}\,. \tag{4.90}$$

Before counting boundary conditions explicitly, a few preparatory remarks are in order. First notice that

$$M = -\sum_i w_i M_i + H\,\frac{S}{2} \mod 2H\,, \tag{4.91}$$

where $w_i := H/h_i$. Hence, because of minimal model selection rules, for fixed L_i and S, only a definite parity of M occurs.

Next, assume that $-\sum_i M_i/2h_i + S/4 = -\sum_i M_i'/h_i + S'/4$. Then, if $n+r$ is odd, $(M_1, \ldots, M_r, S)$ and $(M_1', \ldots, M_r', S')$ are related by an allowed phase symmetry in $\mathcal{G}_{\mathrm{mirr}}$, namely $\alpha_i = (M_i - M_i')/2$, $\epsilon = (S - S')/2$ (see eq. (4.75)), and hence label the same boundary condition. As a consequence, the single M label is sufficient to distinguish inequivalent labels, if $n + r$ is odd. On the other hand, if $n + r$ is even, two such labels are related by an allowed phase symmetry only if $S = S'$.

Furthermore, because of the definition of H as least common multiple, there exists a combination $(t_1, \ldots, t_r)$ such that $-\sum_i t_i/h_i = 1/H \bmod \mathbb{Z}$ (Compare the discussion of the Greene-Plesser construction). Shifting $(M_1, \ldots, M_r)$ by multiples of $(2t_1, \ldots, 2t_r)$ therefore produces all possible M's with a given parity.

Counting boundary conditions leading to inequivalent branes, *i.e.*, with S even, now proceeds as follows. First choose a collection $\Lambda \equiv (L_1, \ldots, L_r)$, and denote the subset of those i's with $L_i = k_i/2$ by I_Λ. Fermion alignment and field identification in minimal models is completely used up by letting $L_i \leq k_i/2$, setting $S_i = 0$ for $i = 2, \ldots, r$, and, if $I_\Lambda \neq \emptyset$, $S = S_1 = 0$.

If $n + r$ is even and $I_\Lambda = \emptyset$, one has M running over H values (even or odd numbers between 0 and $2H - 1$), and $S = 0, 2$ distinguishing branes and antibranes. In all other cases, M and S are not independent. A good choice is to only retain M, running over H values. Loosing S as an independent label entails in particular that a brane and its antibrane lie on the same "$\mathbb{Z}_H$-orbit" as soon as $n + r$ is odd or $I_\Lambda \neq \emptyset$.

Fixed points

As for A-type states, the presence of fixed points slightly spices up the combinatorics. Such fixed points occur when $L_i = k_i/2$ for some i. According to the results of section 2.2, the necessary data for dealing with fixed points are stabilizer $\mathcal{S}_a$, simple-current twist F_a^X and from there the untwisted stabilizer $\mathcal{U}_a \subset \mathcal{S}_a$ of a boundary label a. While for A-type states the stabilizer can only be trivial or equal to $\mathbb{Z}_2$, implying that the untwisted stabilizer coincides with the full stabilizer, for B-type the situation is more complicated, and more interesting.

Consider then a boundary label $\Lambda \equiv (L_1, \ldots, L_r)$.[11] It remains un-

[11]The labels M and S are irrelevant for this discussion, and will hence be suppressed.

changed when acting with those simple currents

$$\mathrm{F} = \prod_{i \in I_{\mathrm{F}}} f_i \,, \tag{4.92}$$

with $I_{\mathrm{F}} \subseteq I_\Lambda$ (recall that $f_i = (k_i, 0, 0) \equiv (0, h_i, 2)$). Which f_i's or combinations thereof are in $\mathcal{G}_{\mathrm{mirr}}$ depends on $n+r$ being even or odd. For $n+r$ odd, every f_i satisfies

$$f_i = p_i^{h_i/2} v_i \quad \text{and hence} \quad X(\mathrm{u}, f_i) = -\frac{h_i}{2h_i} + \frac{1}{2} = 0 \mod \mathbb{Z}\,, \tag{4.93}$$

so it is an allowed phase symmetry. Therefore,

$$|\mathcal{S}_\Lambda| = 2^{|I_\Lambda|}\,, \tag{4.94}$$

In contrast, for $n{+}r$ even, a single f_i does not satisfy the condition $X(\mathrm{u}, f_i) = 0 \mod \mathbb{Z}$ and hence is not allowed. But every pair $f_i f_j$ is allowed. This implies

$$|\mathcal{S}_\Lambda| = 2^{|I_\Lambda|-1}\,. \tag{4.95}$$

For the computation of the untwisted stabilizer, consider first the case $n{+}r$ even. To be specific, distinguish some $a_0 \in I_\Lambda$, denote the corresponding simple current by f_0, and let the stabilizer $\mathcal{S}_\Lambda$ be generated by $f_{0a} := f_0 f_a$, with $a \in I_\Lambda$. Note $\mathcal{S}_\Lambda \cong (\mathbb{Z}_2)^{|I_\Lambda|-1}$.

Assume now that Λ is of the form $(\ldots, k_{a_0}/2, \ldots, k_a/2, \ldots, k_b/2, \ldots)$ (not excluding $a = b$), and consider a second label of the form $\lambda = (\ldots, k_{a_0}/2, \ldots, k_a/2, \ldots, l_b, \ldots)$. Then the twisting of the simple-current relation for Λ is determined from[12]

$$S^{f_{0a}}_{\lambda, f_{0b}\Lambda} = \begin{cases} (-1)^{l_b} S^{f_{0a}}_{\lambda,\Lambda} & \text{for } a \neq b\,, \\ S^{f_{0a}}_{\lambda,\Lambda} & \text{for } a = b\,. \end{cases} \tag{4.96}$$

Thus,

$$F_\Lambda(f_{0b}, f_{0a}) = \begin{cases} (-1)^{k_{a_0}/2} & \text{for } a \neq b\,, \\ (-1)^{k_{a_0}/2 + k_a/2} & \text{for } a = b\,. \end{cases} \tag{4.97}$$

[12]The relevant pieces of the fixed point matrices come from the SU(2) part in eqs. (4.35) and (4.36), *i.e.*, for $a \neq b$,

$$\cdots\ \mathrm{e}^{-2\pi\mathrm{i}\, 3k_{a_0}/16} \cdot \mathrm{e}^{-2\pi\mathrm{i}\, 3k_a/16} \cdot \sin \frac{\pi(k_b/2+1)(l_b+1)}{h_b}\ \cdots$$

Since $f_a = p_a^{h_a/2} v_a$ and $X(p_a, p_b) = -\delta_{ab}/h_a$, one has

$$X(f_{0a}, f_{0b}) = \frac{h_{a_0}}{4} + \frac{h_a}{4}\,\delta_{ab}\,. \tag{4.98}$$

Putting the pieces together, the full simple-current twist is

$$F_\Lambda^X(f_{0a}, f_{0b}) \equiv F_\Lambda(f_{0a}, f_{0b})\,\mathrm{e}^{-2\pi\mathrm{i}X(f_{0a},f_{0b})} = (-1)^{1+\delta_{ab}}\,. \tag{4.99}$$

Computing the untwisted stabilizer is now an easy exercise. Consider some $\mathrm{F} = \prod_{b\in I_\mathrm{F}} f_{0b} \in \mathcal{S}_\Lambda$, with $I_\mathrm{F} \subseteq I_\Lambda \setminus \{a_0\}$. Then,

$$F_\Lambda^X(f_{0a}, \mathrm{F}) = \begin{cases} (-1)^{|I_F|} & \text{for } a \notin I_\mathrm{F}\,, \\ (-1)^{|I_F|-1} & \text{for } a \in I_\mathrm{F}\,. \end{cases} \tag{4.100}$$

For F to be in $\mathcal{U}_\Lambda$, $F_\Lambda^X(f_{0a}, \mathrm{F})$ must be equal to 1 for all a. This is only possible if $I_\mathrm{F} = \emptyset$, or if $I_\mathrm{F} = I_\Lambda \backslash \{a_0\}$ and $|I_\Lambda|$ is even. Thus,

$$\mathcal{U}_\Lambda = \begin{cases} \mathbb{Z}_2 & \text{for } |I_\Lambda| \text{ even}\,, \\ \{\mathrm{id}\} & \text{for } |I_\Lambda| \text{ odd}\,. \end{cases} \tag{4.101}$$

The combinatorics for the boundary states in the case $n + r$ odd can be mapped to the other case by appending a trivial factor (with $k = 0$) to the tensor product of minimal models. Put differently, the above derivation still holds by letting $\mathcal{S}_\Lambda$ be generated by $f_{0a} = f_a$, without distinguishing any particular $a \in I_\Lambda$. Simple-current twists and monodromy carry over *mutatis mutandis*, alone the final result is a little different: $\mathrm{F} \in \mathcal{U}_\Lambda$ if either $I_\mathrm{F} = \emptyset$ or $I_\mathrm{F} = I_\Lambda$, and $|I_\Lambda|$ odd. In this case then,

$$\mathcal{U}_\Lambda = \begin{cases} \mathbb{Z}_2 & \text{for } |I_\Lambda| \text{ odd}\,, \\ \{\mathrm{id}\} & \text{for } |I_\Lambda| \text{ even}\,. \end{cases} \tag{4.102}$$

The number of B-type boundary conditions

Adding together all the above combinatorics leads to the following explicit formula for the total number of inequivalent B-type branes in a Gepner model.

$$
\begin{aligned}
n+r \text{ even}: \quad & 2\times H \times \prod_{k_i \text{ odd}} \Big(\frac{k_i+1}{2}\Big) \times \prod_{k_i \text{ even}} \frac{k_i}{2} \\
& + \sum_{I,\,|I| \text{ odd}} H \times \prod_{k_i \text{ odd}} \Big(\frac{k_i+1}{2}\Big) \times \prod_{k_i \text{ even},\, i\notin I} \frac{k_i}{2} \\
& + \sum_{I,\,|I| \text{ even}} 2\times H \times \prod_{k_i \text{ odd}} \Big(\frac{k_i+1}{2}\Big) \times \prod_{k_i \text{ even},\, i\notin I} \frac{k_i}{2} \\
n+r \text{ odd}: \quad & H \times \prod_{k_i \text{ odd}} \Big(\frac{k_i+1}{2}\Big) \times \prod_{k_i \text{ even}} \frac{k_i}{2} \\
& + \sum_{I,\,|I| \text{ odd}} 2\times H \times \prod_{k_i \text{ odd}} \Big(\frac{k_i+1}{2}\Big) \times \prod_{k_i \text{ even},\, i\notin I} \frac{k_i}{2} \\
& + \sum_{I,\,|I| \text{ even}} H \times \prod_{k_i \text{ odd}} \Big(\frac{k_i+1}{2}\Big) \times \prod_{k_i \text{ even},\, i\notin I} \frac{k_i}{2}
\end{aligned}
\tag{4.103}
$$

It is not difficult to see from the expressions (4.87) and (4.103) that the number of Ishibashi states with s even (*i.e.*, in the NS sector) is precisely equal to the number of branes (*i.e.*, boundary conditions with S even). Both counts can be written in the form

$$
H \times \left[\prod_{k_i \text{ odd}} \Big(\frac{k_i+1}{2}\Big) \times \left[\prod_{k_i \text{ even}} \Big(\frac{k_i}{2}+1\Big) + \sum_{I} \prod_{k_i \text{ even},\, i\notin I} \frac{k_i}{2} \right]\right], \tag{4.104}
$$

where the sum is over $I \subset \{1,\ldots,r\}$ with k_i even for $i \in I$, and with $|I|$ odd for $n+r$ odd, and $|I|$ even for $n+r$ even (now including also $|I|=0$ as even).

That the number of Ishibashis agrees with the number of boundary conditions is a good cross-check on the results. But the counting and labelling is really different! Ishibashi labels are of the form

$$
(\lambda, m, s, \mathrm{f}) \quad \text{with} \quad \mathrm{f} = \prod_{i\in I_{\mathrm{f}}} f_i \in \mathcal{S}^X_{(\lambda,m)}\,, \tag{4.105}
$$

where I_{f} is identified with I in (4.85). Labels for boundary conditions read

$$
(\Lambda, M, S, \Psi) \quad \text{with} \quad \Psi \in \mathcal{U}^*_\Lambda\,, \tag{4.106}
$$

where $\mathcal{U}_\Lambda$ depends on I_Λ as in eqs. (4.101) and (4.102). Although formally this might look similar, the label m is different from the label M. While m describes the position on the u-orbit of an allowed field, M gives the monodromy charge of an entire u-orbit. More drastically, notice that on the side of the Ishibashis, the parities of the l_i's are not independent, while on the side of the boundary states, the L_i's are. Also s is not restricted in the same way as S.

The last piece of information about the boundary conditions that will be displayed here are the reflection coefficients. They are given by the general formula (2.40) in section 2.2. The fixed point matrices are given in (4.35). Putting everything together yields the following reflection coefficients:

$$B_{(\lambda,m,s,\mathrm{f}),(\Lambda,M,S,\Psi)} = \sqrt{\frac{|\mathcal{G}_{\mathrm{mirr}}|}{|\mathcal{S}_\Lambda|\,|\mathcal{U}_\Lambda|}} \prod_{i\notin I_{\mathrm{f}}} 2\sqrt{\frac{2}{h_i}} \sin\pi\frac{(l_i+1)(L_i+1)}{h_i} \times \\ \Big(\prod_{i=1}^{r} \frac{1}{2\sqrt{2h_i}}\Big)\, \mathrm{e}^{-2\pi\mathrm{i}\,(S\,s+(r-1)\,S^2\,s^2)/4}\, \mathrm{e}^{2\pi\mathrm{i}\,Mm/2H} \times \\ \Psi(\mathrm{f}) \prod_{i\in I_{\mathrm{f}}} \mathrm{e}^{-2\pi\mathrm{i}\,3k_i/16}\,. \qquad (4.107)$$

Intersection index

The intersection index of B-type boundary conditions can be computed along the same lines as for A-type, simply by replacing the cyclic group $\mathbb{Z}_H$ with the Greene-Plesser group of phase symmetries, $\mathcal{G}_{\mathrm{mirr}}$. The analog of (4.64) is

$$\mathcal{I}^{\mathrm{B}}_{00} \otimes \mathbf{1}_{\tilde{H}} = \Big[\sum_{\substack{(\epsilon,\alpha_i) \\ \epsilon/2-\sum\alpha_i/h_i=0 \bmod \mathbb{Z}}} (-1)^{\epsilon} \otimes_i \big(g_i\big)^{\alpha_i} \Big] \otimes_{i=1}^{r} (1-g_i)\,, \qquad (4.108)$$

where the size of the reduction matrix is $\tilde{H} = \big(\prod h_i\big)/H$. Thus, $\mathcal{I}^{\mathrm{B}}_{00}$ is an $H\times H$ matrix. It is easy to see that the square bracket effectively sets $g_i = g^{w_i}$, where $g\equiv g_H$ and $w_i = H/h_i$. Straightforwardly, if $n+r$ is even, one obtains,

$$\mathcal{I}^{\mathrm{B}}_{00} = \prod_{i=1}^{r} (1-g^{w_i})\,. \qquad (4.109)$$

If $n+r$ odd, there are phase symmetries of the form $\epsilon=1$, $\alpha_i = h_i/2$, so that

$$\mathcal{I}^{\mathrm{B}}_{00} = (1-g^{H/2}) \prod_{i=1}^{r} (1-g^{w_i})\,. \qquad (4.110)$$

The extra factor $1 - g^{H/2}$ simply is a reflection of the fact that brane and antibrane lie on the same "$\mathbb{Z}_H$-orbit". [13]

As before, the states with $\Lambda = 0$ can be viewed as generators of the charge lattice, and the expansion coefficients (RR charges) of the $\Lambda \neq 0$ states are

$$Q^{(\mathrm{RR})}_{\mathrm{CFT}} = \prod_{i=1}^{r} t_{L_i}\,, \tag{4.112}$$

where t_{L_i} is as in eq. (4.41) with $g_{2h_i} = \big(g_{2H}\big)^{w_i}$.

4.3.4 Connecting CFT with geometry

By now, there exist at least two independent chains of arguments that one might employ to identify the RR charge lattices in BCFT and in the geometric description. The first of those, proposed in [32] makes crucial use of mirror symmetry. Basically, the idea is to compare the intersection form computed in some basis in the BCFT with the geometric intersection form at large volume. In BCFT, any natural basis will reflect the discrete symmetries that the Gepner model has, but which are invisible at large volume, where one uses some convenient basis of $\oplus_i H^{i,i}(X)$. The natural geometric basis with these symmetries is associated with the mirror manifold. Indeed, it is one of the bases of periods that appears in mirror symmetry computations. Thus, mirror symmetry provides the link between the symmetric basis in CFT and the geometric basis at large volume.

In fact, the prescription of [32] for the identification of bases has received further justification by the results of [29]. It is well-known that minimal models, and as a consequence Gepner models, also have a description as conformal fixed points of certain Landau-Ginzburg models. One of the main results of [29] was to identify the Cardy boundary conditions in minimal models with geometric boundary conditions in the corresponding Landau-Ginzburg model. This identification is completely explicit and in principle even goes beyond the topological sector. The happy coincidence is that

[13] On may use this to further reduce the size of the intersection matrix to the form $\prod(1-\gamma^{w_i})$, where γ is the $H/2 \times H/2$-matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.111}$$

the special Lagrangian cycles corresponding to the minimal model boundary states are identical with the cycles used for the computation of the basis of periods of the mirror manifold which is the natural basis at the Gepner point.

An independent approach connecting large and small volume has appeared recently that avoids mirror symmetry completely [118, 121]. Instead of dealing with quantum corrections by considering the mirror, one uses the connection between non-linear σ-models and the gauged linear σ-model of ref. [123]. Such a linear description is achieved by embedding the Calabi-Yau hypersurface (more generally, one can describe toric varieties and also non-Abelian generalizations thereof) as the vacuum manifold in the much larger field space of a certain two-dimensional gauge theory. One of the advantages of the approach is that the description is global and hence allows a simple tracking of the moduli of the theory. It is a natural idea to extend this powerful tool to also study open strings and their boundary conditions. Results in this direction have been obtained in [29, 118, 121, 90], and more recently, in [124, 125, 126]. The strategy of [118, 121] is to propose natural objects in the gauged linear σ-model that are related to boundary states in CFT at small volume and can easily be identified with geometric D-branes at large volume. The computations are dramatically simpler than, and the results reassuringly consistent with, the ones using mirror symmetry.

The linear σ-model approach to D-branes on Calabi-Yaus will not be expanded here, and only the approaches that use mirror symmetry will be described in somewhat more detail now, focusing on a specific example.

An explicit example: The K3-fibrations $\mathbb{P}^4_{1,1,2,2,6}[12]$ and $\mathbb{P}^4_{1,1,2,2,2}[8]$

The Calabi-Yau hypersurfaces in weighted projective space, $Y_1 = \mathbb{P}^4_{1,1,2,2,6}[12]$ and $Y_2 = \mathbb{P}^4_{1,1,2,2,2}[8]$ have the structure of K3-fibrations. At the respective Gepner points in moduli space the defining equations take the form

$$Y_1 = \{[z_1, \ldots, z_5] \in \mathbb{P}^4_{1,1,2,2,6};\, z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 = 0\} \tag{4.113}$$

$$Y_2 = \{[z_1, \ldots, z_5] \in \mathbb{P}^4_{1,1,2,2,2};\, z_1^{8} + z_2^{8} + z_3^4 + z_4^4 + z_5^4 = 0\}\,, \tag{4.114}$$

and the manifolds have a special value of the Kähler class. At these points, the exact solution of the σ-models are given by Gepner models with minimal models at levels $(4, 4, 10, 10)$ and $(2, 2, 2, 6, 6)$, respectively. After blowing up the singularities in the weighted projective space, each of these manifolds has two Kähler parameters, denoted by t_1 and t_2. They belong to a natural basis of Kähler classes, $J_1, J_2 \in H^{1,1}(Y)$ (for $Y = Y_1, Y_2$).

The mirror manifolds of Y_1 and Y_2 can be obtained by the ("geometric") Greene-Plesser method, and are denoted by $\tilde{Y}_1$ and $\tilde{Y}_2$, respectively. Mirror symmetry relates the complex structure and Kähler moduli spaces of mirror pairs. For the manifolds at hand, the necessary computations have been performed in [127, 128]. In particular, these results show how the complex structure moduli space of $\tilde{Y}$ is parametrized by t_1 and t_2. But this is not enough for connecting the complete bases of the RR charge lattices (which have rank 6) in CFT and large volume.

At large volume, there are always two natural bases for these lattices. One is a basis of $H^{\mathrm{diag}}(Y)$, naturally generated by the two Kähler classes $J_1, J_2 \in H^{1,1}(Y)$, the other is a symplectic basis, B_{sympl}, of $H_3(\tilde{Y})$. The map between the two bases is obtained by comparing the central charge of a brane as a function on moduli space.

Namely, for a RR charge $Q^{(\mathrm{RR})} \in H^3(\tilde{Y})$, one can express the central charge as

$$Z(Q^{(\mathrm{RR})}) = \sum_i Q_i^{(\mathrm{RR})}\, \Pi^i(t_1, t_2)\,, \tag{4.115}$$

where $Q_i^{(\mathrm{RR})}$ are the coefficients of $Q^{(\mathrm{RR})}$, and $\Pi^i(t_1, t_2)$ the "period vector", with respect to the basis B_{sympl}.

If the same charge $Q^{(\mathrm{RR})}$ is viewed as element of $H^{\mathrm{diag}}(Y)$, one may compute the central charge as

$$Z(Q^{(\mathrm{RR})}) = \int \mathrm{e}^{-K} Q^{(\mathrm{RR})}\,, \tag{4.116}$$

where $K = t_1 J_1 + t_2 J_2$ is the Kähler class of Y, and the relation between characteristic classes and $Q^{(\mathrm{RR})}$ is given by eq. (3.12). Comparison of (4.115) and (4.116) then yields the following expressions for the characteristic classes ch_i of a vector bundle $\mathcal{V}$ in terms of the charge vector $(Q_i^{(\mathrm{RR})}) = (n_6, n_4^{(1)}, n_4^{(2)}, n_0, n_2^{(1)}, n_2^{(2)})$.

$$\begin{aligned}
Y_1: \qquad r &= n_6 \\
\mathrm{ch}_1 &= n_4^{(1)}\, J_1 + n_4^{(2)}\, J_2 \\
\mathrm{ch}_2 &= \left(\tfrac{1}{2} n_2^{(1)} - n_2^{(2)}\right) J_1 J_2 + \tfrac{1}{2} n_2^{(2)}\, (J_1)^2 \\
\mathrm{ch}_3 &= -\tfrac{1}{4}\left(n_0 + \tfrac{13}{3} n_4^{(1)} + 2 n_4^{(2)}\right) (J_1)^3\,,
\end{aligned} \tag{4.117}$$

$$\begin{aligned}
Y_2: \qquad r &= n_6\,, \\
\mathrm{ch}_1 &= n_4^{(1)}\, J_1 + n_4^{(2)}\, J_2 \\
\mathrm{ch}_2 &= \left(\tfrac{1}{4} n_2^{(1)} - \tfrac{1}{2} n_2^{(2)}\right) J_1 J_2 + \tfrac{1}{4} n_2^{(2)}\, (J_1)^2 \\
\mathrm{ch}_3 &= -\tfrac{1}{8}\left(n_0 + \tfrac{14}{3} n_4^{(1)} + 2 n_4^{(2)}\right) J_1^3\,.
\end{aligned} \tag{4.118}$$

Equations (4.117) and (4.118) describe explicitly the isomorphism between $H^{\mathrm{diag}}(Y)$ and $H^3(\tilde{Y})$.

Of particular interest (for example for string dualities) are the branes that wrap only the K3-fibers (called fiber-branes in [93]). In general, to find the characteristic classes of the bundles corresponding to branes wrapped on submanifolds, one has to take into account that the induced charge vector is modified by the $\hat{A}$ genus of the normal bundle of the submanifold inside the Calabi-Yau. For the K3's inside the Calabi-Yau's at hand, one obtains

$$\begin{aligned} Y_1: \quad r &= n_4^{(2)} \\ \mathrm{ch}_1 &= \tfrac{1}{2} n_2^{(1)} \\ \mathrm{ch}_2 &= -n_0 - 2n_4^{(2)}\,, \end{aligned} \tag{4.119}$$

$$\begin{aligned} Y_2: \quad r &= n_4^{(2)} \\ \mathrm{ch}_1 &= \tfrac{1}{4} n_2^{(1)} \\ \mathrm{ch}_2 &= -\tfrac{1}{2} n_0 - n_4^{(2)}\,, \end{aligned} \tag{4.120}$$

where the characteristic classes are expanded in both cases with respect to the bases $(1, J_1, \frac{1}{2}(J_1)^2)$.

As indicated above, there is a third basis for the RR charge lattice, which is induced by the natural basis of periods at the Gepner point. Specifically, the periods of the holomorphic three-form on (the mirror of) a Calabi-Yau are computed by solving a set of linear differential equations (the Picard-Fuchs system) satisfied by the periods.

At the Gepner point in moduli space, the natural solutions of the Picard-Fuchs equations are a set of H functions, $(\varpi_1, \ldots, \varpi_H)$ related to each other by the $\mathbb{Z}_H$ monodromy around the origin (where $H = \mathrm{l.c.m.}(h_i)$). Although naturally symmetric, the set $(\varpi_1, \ldots, \varpi_H)$ is not a basis of solutions of the Picard-Fuchs system. Rather than being independent, these functions satisfy a set of "period relations" that can be derived combinatorially from the weights $h_1, \ldots, h_5$. Related to $(\varpi_1, \ldots, \varpi_H)$ is a set of cycles, $\tilde{B}_\omega$, satisfying the same relations. The main work in mirror symmetry computations following [13] is to connect $\tilde{B}_\omega$ with the symplectic basis B_{sympl} at infinity by analytic continuation of periods. With this in hand, one can derive the intersection matrix of the cycles $\tilde{B}_\omega$.

Exploiting the $\mathbb{Z}_H$ symmetry, the proposal of [32] was to connect the basis of cycles $\tilde{B}_\omega$ used for the computation of the periods with the collection of $\Lambda = 0$ boundary conditions in the Gepner model. These $\Lambda = 0$ states can be thought of as a set of (dependent) generators B_{CFT} of the charge lattice. This proposal is substantiated by the fact that B_{CFT} satisfy the same relations as the $\tilde{B}_\omega$, namely the period relations (this is purely combinatorial data). Moreover, there is a simple relation between the intersection matrices.

Explicitly, for the two K3-fibrations discussed above, one finds the analytic continuation matrices

$$Y_1: \quad m = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{4.121}$$

$$Y_2: \quad m = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{4.122}$$

between the symplectic basis B_{sympl} and the projection of $\tilde{B}_\omega$ to a linearly independent set, which is found by using the "intertwiner" T that implements the period relations[14].

This allows to compute the intersection form on $\tilde{B}_\omega$, $\mathcal{I}^{\tilde{B}_\omega}$, and compare with the CFT intersection matrix $\mathcal{I}^{B_{\mathrm{CFT}}} = \mathcal{I}^{\mathrm{B}}_{00}$ (see eq. (4.109)). In all cases that were investigated by these methods, it was suspiciously found that one has

$$\mathcal{I}^{B_{\mathrm{CFT}}} = (1-g)\,\mathcal{I}^{\tilde{B}_\omega}\,(1-g)^t\,. \tag{4.124}$$

[14]These "intertwiners" are given explicitly by

$$Y_1:\ T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad Y_2:\ T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix}. \tag{4.123}$$

In view of the results [29], this is not surprising any longer. Indeed, it turns out that the cycles $\tilde{B}_\omega$ that are conventionally used for the computations of the overcomplete set of solutions of the Picard-Fuchs system [129] are almost identical to the concatenation of the cycles that appear in the Landau-Ginzburg description of (the mirrors of) the individual minimal models. The only discrepancy between [129] and the prescription derived from [29] is a factor of $(1-g)$, exactly as it appears in (4.124).

All in all, one obtains the big basis transformation between the RR charges in CFT and the charges in the basis B_{sympl} at large volume,

$$Q_\infty^{(\text{RR})} = Q_{\text{CFT}}^{(\text{RR})}(1-g)Tm^{-1}\,, \tag{4.125}$$

where $Q_{\text{CFT}}^{(\text{RR})}$ are the expansion coefficients of a CFT state in terms of the basic $\Lambda = 0$ states, see eq. (4.112). Inserting this into eqs. (4.117) or (4.118) yields the characteristic classes of a vector bundle.

Results for fiber-branes in the two models Y_1 and Y_2, before fixed point resolution, are summarized in table 4.2.

Fixed point resolution and enhanced gauge symmetry

The last question in the comparison between Gepner model boundary states and geometry that will be considered here is the interpretation of the fixed points and their resolution. The proposal is as follows.

Considering the boundary states before fixed point resolution shows that in the open string amplitude, one finds that the number ν of vacua is equal to the order of the stabilizer and in general larger than one. This is of course typical of fixed points, and is actually one of the reasons why they *have* to be resolved into elementary boundary states. Physically, these additional vacua give rise to extra gauge fields on the world-volume. Mathematically, one should think of a degenerate bundle or sheaf.

Given such a degenerate configuration, one can ask how many U(1) factors the gauge group contains. If this number $\tilde{\nu}$ (where $1 \le \tilde{\nu} \le \nu$) is larger than 1, the configuration should be considered as reducible because each U(1) corresponds to an independent center-of-mass degree of freedom of a multi-brane system. The structure of the gauge group can in principle be analyzed by studying correlation functions of the ν gauge bosons. However, this is unnecessary here, given the origin of the fixed points and the combinatorial structure of their resolution.

Indeed, it suffices to realize that the algebra of open string states of the unresolved fixed point is given by the twisted group algebra of the stabilizer, $\mathbb{C}_{F_\Lambda^X}(\mathcal{S}_\Lambda)$, where F_Λ^X is the simple-current twist (4.99). This group algebra

L_i	$Q^{(\mathrm{RR})}(\mathcal{V}) = \big(r, \mathrm{ch}_1(\mathcal{V}), \mathrm{ch}_2(\mathcal{V})\big)$			ν	$\tilde{\nu}$	G
[1,0,0,0]	(1,0,0)	(1,-1,1)	(2,-1,-1)	1	1	U(1)
[3,0,0,0]	(1,-1,-1)	(1,0,-2)	(0,1,-1)	1	1	U(1)
[3,0,1,0]	(1,1,-3)	(1,-2,0)	(2,-1,-3)	1	1	U(1)
[3,0,1,1]	(3,-3,-3)	(3,0,-6)	(0,3,-3)	1	1	U(1)
[5,0,0,0]	(2,0,-2)	(2,-2,0)	(0,0,-2)	2	2	U(1) × U(1)
[5,0,1,0]	(2,-2,-2)	(2,0,-4)	(0,2,-2)	2	2	U(1) × U(1)
[5,0,1,1]	(2,2,-6)	(2,-4,0)	(4,-2,-6)	2	2	U(1) × U(1)
[5,0,2,0]	(4,-2,-4)	(0,2,0)	(0,2,-4)	4	1	U(2)
[5,0,2,1]	(4,-4,-4)	(4,0,-8)	(0,4,-4)	4	1	U(2)
[5,0,2,2]	(4,-2,-8)	(4,2,-8)	(4,-6,0)	8	2	U(2) × U(2)

L_i	$Q^{(\mathrm{RR})}(\mathcal{V}) = \big(r, \mathrm{ch}_1(\mathcal{V}), \mathrm{ch}_2(\mathcal{V})\big)$				ν	$\tilde{\nu}$	G
[1,0,0,0,0]	(1,0,0)	(3,-2,0)	(1,-1,1)	(3,-1,-1)	1	1	U(1)
[3,0,0,0,0]	(2,-1,-1)	(0,1,-1)			1	1	U(1)
[3,0,1,0,0]	(2,-2,0)	(2,0,-2)			2	2	U(1) × U(1)
[3,0,1,1,0]	(4,-2,-2)	(0,2,-2)			4	1	U(2)
[3,0,1,1,1]	(4,-4,0)	(4,0,-4)			8	2	U(2) × U(2)

Table 4.2: These tables show the characteristic classes of coherent sheaves corresponding to the RR charges of B-type boundary states in Gepner models (before fixed point resolution). The levels in the Gepner model are $(10, 10, 4, 4)$ for the top and $(6, 6, 2, 2, 2)$ for the bottom table, respectively. Boundary states are labelled by collections of L_i's and a $\mathbb{Z}_H$ label M, which however is not shown explicitly. The models correspond geometrically to the K3-fibrations $\mathbb{P}^4_{1,1,2,2,6}[12]$ and $\mathbb{P}^4_{1,1,2,2,2}[8]$, respectively, and only states that have non-trivial charges only on the K3 are displayed. The rightmost columns of the tables refer to fixed point data, as described in the text.

splits into the direct sum of full matrix algebras

$$\mathbb{C}_{F^X_\Lambda}(\mathcal{S}_\Lambda) = \oplus_{i=1}^{|\mathcal{U}_\Lambda|} \mathrm{Mat}_N(\mathbb{C}) \,, \tag{4.126}$$

where $|\mathcal{S}_\Lambda| = N^2|\mathcal{U}_\Lambda|$. This split is the origin of the fixed point resolution.

Given all this, it is natural to identify the order of the untwisted stabilizer with the number of U(1) gauge bosons, $\tilde{\nu} = |\mathcal{U}_\Lambda|$, and N with the order of the unbroken gauge group. It should be stressed that the presence of the simple-current twist makes it impossible to split the boundary states further, *i.e.*, it is not possible to "pull apart" building blocks of the brane, in spite of the fact that its world-volume does have an enlarged gauge group. The physical picture underlying the equation

$$\nu = N^2\tilde{\nu} \tag{4.127}$$

thus is that the collection of ν gauge fields splits into $\tilde{\nu}$ families, each containing N^2 gauge fields carrying the adjoint representation of $\mathrm{U}(N)$.

According to this interpretation, and in view of the formula for the reflection coefficients (4.107), it is easy to see that the RR charges of the resolved states is simply $1/\tilde{\nu}$ times the charges of the unresolved states, which are the charges appearing in table 4.2.

The enhancement of gauge symmetry discussed above is reminiscent of orbifolds with discrete torsion [106, 130]. Specifically, it was found in [106, 130] that discrete torsion in a closed string orbifold should be accompanied, in the open string sector, by a projective representation on the Chan-Paton labels. Thus, the regular representation (*i.e.*, the twisted group algebra) splits according to a rule analogous to (4.127) $|\Gamma| = \sum_{i=1}^{N_R}(d_{R_i})^2$, where Γ is the orbifold group and d_{R_i} the dimensions of the irreducible projective representations of Γ. Consistency conditions such as charge quantization then seem to require that the minimal D-brane charge is larger than expected by a factor of d_{R_i}, leading to an enhanced gauge symmetry. The author of ref. [130] further argues that discrete torsion might be attributed to a flat but non-trivial B-field on a torsion 2-cycle, and that consistency requires a minimal wrapping number larger than one.

4.4 From $\mathcal{N} = 2$ coset models to Grassmannians

4.4.1 Introduction

As was shown in previous sections, the theory of boundary conditions in $\mathcal{N} = 2$ superconformal field theories is a powerful tool to explore the quantum geometry of D-branes. But still, the general picture of D-branes in regimes of large curvature is quite incomplete. In the recent literature, the analysis of exactly solvable CFT models such as orbifolds [131, 132], and Gepner models, has given hints as to what algebraic structures one should try to use for such a general description of D-branes in the small volume regimes.

Thus, while at large volume, D-branes correspond to geometric objects—submanifolds supporting vector bundles, or, more generally, coherent sheaves, and therefore have a microscopic description as geometric boundary conditions in a σ-model, the most appropriate description at small volume seems to be in terms of "quiver theory" [131, 132]. This point of view has recently received further support from the work in [116], where it was shown that compact D-branes on the non-compact Calabi-Yau manifold $\mathcal{O}_{\mathbb{P}^2}(-3)$ can be constructed following Beilinson [133], and are hence classified as the representations of a quiver. Most recently it was shown in [118, 119, 120, 121] how the descriptions at large and small volume are related in general by a form of McKay correspondence [134, 135], which gives a precise map between the large radius bundle data and the quiver group theory data at small radius.

In order to test some of these recent ideas about D-branes in small volume regimes, the purpose of the present section is to extend the analysis of exactly solvable $\mathcal{N}=2$ CFTs to the class of the Kazama-Suzuki models. In particular, the objects of study will be $\mathcal{N}=2$ superconformal field theories based on cosets $\mathrm{SU}(n+1)_k/\mathrm{U}(n)$ [136], which generalize the minimal models (for which $n=1$). From the CFT point of view, these models are on a similar footing as the minimal models, so that it is a natural question to ask about the properties of the boundary states of these models. On the other hand, from a geometrical point of view the models correspond to isolated singularities that are not necessarily of orbifold type, so one may expect to find novel features with regard to generalizations of the McKay correspondence. Indeed these models have an abundantly rich mathematical structure (related to Grassmannians $\mathrm{Gr}(n, n+k) \cong \mathrm{U}(n+k)/\mathrm{U}(n)\times\mathrm{U}(k)$) that has been analyzed in great detail in the past, see refs. [10, 137, 138, 139, 140, 141, 142], as far as the bulk physics is concerned. One of the questions one might ask is whether such connections persist after inclusion of boundary sectors. Here, the primary focus will be on the intrinsic, algebraic aspects of coset CFT with boundaries[15]. A few brief comments about the relation to sheaves and helices on Grassmannians will be made at the end.

Some of the general ideas relating coset CFTs to geometric information about singularities were explained in section 4.2 for the minimal models. For Kazama-Suzuki models, the strategy will be very similar. The main ingredient is the intersection index of boundary states, $\mathcal{I}_{ab} = \mathrm{tr}_{\mathcal{H}_{ab}}(-1)^F$, see section 3.2.3. In the present case, the intersection structure—encoding information

[15]See [143] for some other aspects of D-branes in Kazama-Suzuki models.

about the "quiver algebra" or "boundary ring"—turns out to be given by the fusion ring of $\mathrm{U}(n)$. Thus, the generalized McKay correspondence apparently does not involve discrete groups [144, 145]. But still a close link seems to emerge between the intersection homology of the resolution of the isolated singularity corresponding to the coset model [146, 147], and the boundary ring.

Another aspect of the analysis is that the class of "Cardy" boundary states covers only a very small subset of all possible quiver representations. This could be expected from the fact that a general Kazama-Suzuki model is irrational over the $\mathcal{N} = 2$ algebra, while Cardy's construction always preserves a rational chiral symmetry algebra. These results can hence be viewed as a consistency check of the presently available, and limited, ideas and methods in conformal field theory with boundaries.

4.4.2 $\mathcal{N} = 2$ coset models

The starting point for Kazama-Suzuki (KS) models [136] are rational $\mathcal{N} = 2$ superconformal field theories defined by the coset construction, of the form,

$$\left(\frac{\mathfrak{g} \times \mathfrak{so}(2d)}{\mathfrak{h} \times \mathfrak{u}(1)}\right)_k . \tag{4.128}$$

Here, k is the level for the untwisted affine Lie algebra with horizontal subalgebra the simple Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. The $\mathfrak{so}(2d)$ factor arises from bosonization of the fermions and is at level 1. Furthermore, $2d = \dim \mathfrak{g} - \dim \mathfrak{h}$. One often finds the notations $\mathfrak{g}_k/\mathfrak{h}$ or $\mathfrak{g}_k/\mathfrak{h} \times \mathfrak{u}(1)$ as a shorthand for (4.128). It turns out [136] that such a coset model will have its supersymmetry extended to $\mathcal{N} = 2$—and hence be a good starting point for superstring models—precisely if the corresponding coset space of Lie groups, G/H, is Kählerian. The main interest here is in the simplest class of models, namely where $\mathfrak{g}$ is simply laced, at level one, and the underlying coset space is a hermitian symmetric space (the SLOHSS models). Other models can be treated by similar methods, but require more computational power and also somewhat more care due to field identification fixed points [10, 148].

More specifically, the models of interest are based on Grassmannians $\mathrm{Gr}(n, n+k)$, for which the following equivalences hold:

$$\frac{\mathfrak{su}(n+k)_1}{\mathfrak{su}(n) \times \mathfrak{su}(k) \times \mathfrak{u}(1)} \cong \frac{\mathfrak{su}(n+1)_k}{\mathfrak{su}(n) \times \mathfrak{u}(1)} \cong \frac{\mathfrak{su}(k+1)_n}{\mathfrak{su}(k) \times \mathfrak{u}(1)} . \tag{4.129}$$

Quantities pertaining to these models will be labelled by a superscript $[n, k]$, and, as a convention, $n \leq k$ is assumed.

The definition of the coset (4.128) includes the specification of the embedding of $\mathfrak{h}$ into $\mathfrak{g}$, and is accompanied by specific selection rules and field identifications. Field identification fixed points do not occur for the models under consideration, so this complication can be neglected. It should be pointed out, however, that since fixed point resolution affects the modular data and fusion rules in a non-trivial way, it will have interesting consequences for the intersection index of boundary states in theories with fixed points.

Primary (with respect to the bosonic algebra) fields in the coset CFT are labelled by quadruples $(\Lambda, \lambda, m, \sigma)$, where Λ stands for an integrable highest weight of $\mathfrak{g}_k$, λ for a weight of $\mathfrak{h}$ and m for the $\mathfrak{u}(1)$ charge. Furthermore, σ is a weight of the $\mathfrak{so}(2d)$ factor, which is the scalar, 0, or the vector, v, representation in the NS sector, and the spinor, s, or the conjugate spinor, c, representation in the R sector. The restrictions and identifications on the labels depend on the particular coset one is considering. For present purposes, they can be formally implemented by considering a simple-current extension [148] of the tensor product,

$$[\mathfrak{g} \times \mathfrak{h}^* \times \mathfrak{u}(1)^* \times \mathfrak{so}(2d)]_{\text{extended}} \,. \tag{4.130}$$

At least for the modular properties of the model, this extended tensor product is equivalent to the coset model. Since only modular data and with it the fusion rules enters the construction of Cardy boundary states [47], this will be sufficient.

As a concrete example, consider the cosets $\mathfrak{su}(n+1)_k/\mathfrak{su}(n)$. The extension is by the simple current

$$\mathcal{J} = (J^{(n+1)}, J^{(n)}, h, v) \tag{4.131}$$

in the tensor product (4.130). Here, $J^{(n+1)}$ (respectively $J^{(n)}$) denotes the generator of the simple-current group of $\mathfrak{su}(n+1)_k$ (respectively $\mathfrak{su}(n)_{k+1}$). Its monodromy charge, $Q_{J^{(n+1)}}(\Lambda) = \tau_{n+1}(\Lambda)/(n+1)$ measures the $(n+1)$-ality of the representation Λ (analogously, $\tau_n(\lambda)$ stands for the n-ality of the representation λ). Moreover $J^{(n+1)}$ acts on Λ, to yield $J^{(n+1)}\Lambda$, by rotating clockwise the Dynkin labels of the corresponding highest weight of the affine Lie algebra $\mathfrak{su}(n+1)_k$ (and similarly for $\mathfrak{su}(n)$). And $h := k+n+1$. Extension by the simple current $\mathcal{J}$ is equivalent to the selection rule

$$Q_{(J^{(n+1)},J^{(n)},h,v)}(\Lambda, \lambda, m, \sigma) = \frac{\tau_{n+1}(\Lambda)}{n+1} + \frac{\tau_n(\lambda)}{n} + \frac{m}{n(n+1)} + Q_v(\sigma) \, = 0 \,, \tag{4.132}$$

where $Q_v(\sigma)$ is 0 in the NS sector and 1/2 in the R sector, and to the order $n(n+1)$ identification $(\Lambda, \lambda, m, \sigma) \equiv \mathcal{J}(\Lambda, \lambda, m, \sigma) \equiv (J^{(n+1)}\Lambda, J^{(n)}\lambda, m + h, v\sigma)$. Further details can be found in the cited literature.

It is known [10] that the ring of chiral primary fields of any one of these models (4.129) is isomorphic to the cohomology ring of the underlying Grassmannian,

$$\mathcal{R}^{[n,k]} \cong H^*_{\bar\partial}\big(\tfrac{\mathrm{SU}(n+k)}{\mathrm{SU}(n)\times\mathrm{SU}(k)\times\mathrm{U}(1)}\, , \mathbb{R} \big)\, , \tag{4.133}$$

with dimension

$$\dim(\mathcal{R}^{[n,k]}) = \binom{n+k}{n}\, . \tag{4.134}$$

The relations in this ring can be integrated to a potential $W^{[n,k]}(x_i)$, which can be interpreted as superpotential of a Landau-Ginzburg model with fields x_i, $i = 1, \ldots, n$ (with U(1) charges $q(i) = i/(n+k+1)$). The superpotentials were explicitly given in [10, 137], and can be compactly characterized by the following generating function:

$$-\log\Big[\sum_{i=1}^{n-1}(-t)^i x_i\Big] \;=\; \sum_{k=-n+1}^{\infty} t^{n+k}\, W^{[n,k]}(x_i)\, . \tag{4.135}$$

The quasi-homogeneous superpotentials $W^{[n,k]}(x_i)$ represent isolated singularities that can be viewed as generalizations of the A_{k+1} simple singularities. Those were mentioned in section 4.2 and correspond to $W^{[1,k]}(x_i)$. In analogy to the minimal models and their relationship to ALE spaces, one expects that the CFT of the coset models should be compared with the D-brane geometry of the resolved singularities, described by the superpotential

$$W^{[n,k]}(x_i, \mu) \;=\; W^{[n,k]}(x_i) + \mu\, . \tag{4.136}$$

This particular resolution is distinguished in that it preserves the discrete $\mathbb{Z}_h$ $(h \equiv n+k+1)$ "Coxeter" symmetry that is intrinsic to the coset models.

The resolved potential (4.136) can be viewed as the inhomogeneous form of a Landau-Ginzburg potential for a non-compact Calabi-Yau space. The most natural way to form such a space is to tensor the coset model with a matching, generalized Liouville theory with n fields z_i (with charges $q(i) = -i/h$). The combined system has central charge

$$\hat c(n, h) + \hat c(n, -h) = 2n\, , \tag{4.137}$$

(where $\hat c(n, h) \equiv (h-n-1)n/h$), and corresponds to a non-compact $2n$-fold.

The intersection indices $\mathcal{I}_{a,b} \equiv \mathrm{tr}_{\mathcal{H}_{a,b}}(-1)^F$ between boundary states a, b, computed in the next subsection, will then gain a concrete geometrical meaning after taking the non-compact piece into account. This produces symmetric generalized Cartan matrices,

$$\hat{C}^{[n,k]} = \hat{\mathcal{I}}^{[n,k]} + \left(\hat{\mathcal{I}}^{[n,k]}\right)^t, \tag{4.138}$$

and makes contact with the proposals of [146], proven in [147], that the fusion coefficients of $\mathfrak{su}(n+1)_k$ are naturally related to the intersection form on the homology of the resolved singularity. Nevertheless, the main concern here will be the intrinsic properties of the boundary states of the $\mathcal{N}=2$ coset models. (The hats in (4.138) indicate that these are "extended generalized Cartan matrices" associated with over-complete, $\mathbb{Z}_h$ symmetric homology bases.)

4.4.3 Boundary conditions and intersection index

The class of boundary conditions under consideration here are the Cardy states. Thus these states will preserve the complete chiral algebra (without twist) of the $\mathcal{N}=2$ coset models (these chiral algebras are known to be $\mathcal{N}=2$ $\mathcal{W}$-algebras), and do not exhaust all possible $\mathcal{N}=2$ superconformal boundary conditions. To be precise, the Cardy construction yields A-type (with respect to the $\mathcal{N}=2$ algebra) boundary conditions, using the charge conjugation modular invariant in the closed string sector (see table 3.1). In the coset models, the Cardy boundary states are labelled in the same way as the primary fields are, namely by (orbits of) $(\Lambda, \lambda, m, \sigma)$ with the same selection and identification rules.

Recall from section 3.2.3 that the intersection index can be written in terms of the annulus coefficients, A_{ab}^m, as follows:

$$\mathcal{I}_{ab} = \sum_{m \text{ Rgs}} A_{ab}^{s^{-1}m} - A_{ab}^{vs^{-1}m}, \tag{4.139}$$

where v denotes the simple current corresponding to the worldsheet supercurrent and s the simple current corresponding to spectral flow by half a unit. The sum in (4.139) is over all Ramond ground states m. Thus, the $s^{-1}m$ are chiral primary fields. In the cases of present interest, the expression (4.139) simplifies further since the annulus coefficients are identical to the fusion coefficients, *i.e.*, to the structure constants of the Verlinde algebra of the coset model. Modulo field identification fixed points, those are given by the products of fusion coefficients of the factors in (4.130), restricted to allowed fields, and summed over field identification orbits. The fusion coefficients of $\mathfrak{g}$ and $\mathfrak{h}$ will be denoted by ${}^G\!N$ and ${}^H\!N$, respectively. The fusion coefficients of the $\mathfrak{u}(1)$ factor are conveniently encoded in a shift matrix g, of size h or $n(n+1)h$, depending on the context (see the comments around (4.37) in section 4.2. The fusion coefficients of the $\mathfrak{so}(2d)$ factor are given by $vs = c$, $v^2 = 0$, $s^2 = v^d$.

As in previous sections, it is natural to view the intersection numbers of boundary states with representatives $(\Lambda_1, \lambda_1, m_1, \sigma_1)$ and $(\Lambda_2, \lambda_2, m_2, \sigma_2)$, for fixed Λ_1 and Λ_2, as a matrix in λ_1, m_1 and λ_2, m_2. Consider also fixed $\sigma_1 = \sigma_2 = 0$. From (4.139) one obtains,

$$\left(\hat{\mathcal{I}}_{\Lambda_1,\Lambda_2}\right)^{\lambda_1,m_1}_{\lambda_2,m_2} = \sum_{(\Lambda,\lambda,m,\sigma)\text{ ch. prim.}} {}^{\text{coset}}N^{(\Lambda_1,\lambda_1,m_1,0)}_{(\Lambda,\lambda,m,\sigma)(\Lambda_2,\lambda_2,m_2,0)} - {}^{\text{coset}}N^{(\Lambda_1,\lambda_1,m_1,0)}_{v(\Lambda,\lambda,m,\sigma)(\Lambda_2,\lambda_2,m_2,0)}\,, \tag{4.140}$$

where the sum is over all chiral primary field representatives. Inserting the fusion coefficients of $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{u}(1)$, and $\mathfrak{so}(2d)$ then gives

$$\left(\hat{\mathcal{I}}_{\Lambda_1,\Lambda_2}\right)^{\lambda_1,m_1}_{\lambda_2,m_2} = \sum_{\Lambda} {}^{G}N^{\Lambda_1}_{\Lambda\Lambda_2} \times \left[\sum_{\substack{\lambda,m \\ (\Lambda,\lambda,m,0)\text{ ch. prim.}}} {}^{H}N^{\lambda_1}_{\lambda\lambda_2}(g^{-m})^{m_1}_{m_2} - \sum_{\substack{\lambda,m \\ (\Lambda,\lambda,m,v)\text{ ch. prim.}}} {}^{H}N^{\lambda_1}_{\lambda\lambda_2}(g^{-m})^{m_1}_{m_2}\right], \tag{4.141}$$

Thus, it is necessary to know which λ, m labels yield, for fixed Λ, a representative of a chiral primary field. To this end, use the fact [138] that any Ramond ground state has a representative $(\Lambda, \lambda, m, \sigma)$ with

$$(\lambda, m) + (\rho_{\mathfrak{h}}, 0) = w(\Lambda + \rho_{\mathfrak{g}})\,, \tag{4.142}$$

where $\rho_{\mathfrak{h}}$ and $\rho_{\mathfrak{g}}$ are the Weyl vectors, and where w runs over the minimal length representatives, $\mathrm{W}(\mathfrak{g}/\mathfrak{h})$, of the Weyl group coset $\mathrm{W}(G)/\mathrm{W}(H)$. The $w \in \mathrm{W}(\mathfrak{g}/\mathfrak{h})$ can also be uniquely characterized by the fact that λ in eq. (4.142) is an integrable highest weight of $\mathfrak{h}$ at the level of interest. In (4.142), m is determined by the embedding of the $\mathfrak{u}(1)$ factor in $\mathfrak{g}$, and the $\mathfrak{so}(2d)$ representation σ is the spinor, s, or conjugate spinor, c, if the sign of w is $+1$ or -1, respectively. Using spectral flow to the NS sector, given by $(0, 0, m_0, s)$, for a particular m_0, a solution of (4.142) is seen to contributes in (4.141) with a sign equal to $\mathrm{sign}(w)$.

However, not all Ramond ground states representatives are of the form (4.142). One also has to implement the identification rules that do not change a given Λ. These identifications introduce an additional sign if they act non-trivially on the $\mathfrak{so}(2d)$ label. Summing up, one can write (4.141) in the compact form

$$\left(\hat{\mathcal{I}}_{\Lambda_1,\Lambda_2}\right)^{\lambda_1,m_1}_{\lambda_2,m_2} = \sum_{\Lambda} {}^{G}\!N^{\Lambda_1}_{\Lambda\Lambda_2} \sum_{w\in \mathrm{W}(\mathfrak{g}/\mathfrak{h})} \sum_{(\lambda,m)}{}' \epsilon \,\mathrm{sign}(w)\; {}^{H}\!N^{\lambda_1}_{\lambda\lambda_2}\, (g^{-m+m_0})^{m_1}_{m_2}\,. \tag{4.143}$$

where $\sum'$ is over all those (λ, m) that are related to (4.142) by a field identification in the denominator and in the $\mathfrak{so}(2d)$ factor (which determines the additional sign $\epsilon = \pm 1$).

Examples

As a first example, reconsider the intersection of the $\Lambda \equiv L = 0$ states of the $\mathcal{N} = 2$ minimal models, $\mathfrak{su}(2)_k \times \mathfrak{so}(2)_1/\mathfrak{u}(1)_{2h}$. Here, $\mathrm{W}(G/H) = \mathrm{W}(\mathrm{SU}(2))$ consists just of two elements, namely of the identity $w_0(l) = l$ and of $w_1(l) = -l$. Furthermore, $m_0 = 1$, and $w_0(0 + \rho_{\mathfrak{su}(2)}) - m_0 = 0$, $w_1(0 + \rho_{\mathfrak{su}(2)}) - m_0 = -2$, so that there are two terms in the intersection matrix,

$$\hat{\mathcal{I}}^{[1,k]}_{0,0} = 1 - g^2\,. \tag{4.144}$$

This reproduces the result (4.39) (modulo reducing the size of the matrix $g = g_{2h} \equiv g_{2(k+2)}$ in order to avoid redundancy).

The second example are the models $\mathfrak{su}(3)_k/\mathfrak{u}(2)$. The full coset reads

$$\frac{\mathfrak{su}(3)_k \times \mathfrak{so}(4)_1}{\mathfrak{su}(2)_{k+1} \times \mathfrak{u}(1)_{6h}}\,, \tag{4.145}$$

where $h = k + 3$. Primary fields in the coset are labelled by allowed field identification orbits of

$$((l_1, l_2), \lambda, m, \sigma)\ , \tag{4.146}$$

where $l_1, l_2, \lambda \geq 0$, $l_1 + l_2 \leq k$, $\lambda \leq k + 1$, m is defined modulo $6h$ and σ is scalar (0) or vector (v) in the NS sector and spinor (s) or conjugate spinor (c) in the R sector.

Fix (l_1', l_2') and (l_1'', l_2''), and consider boundary states with varying λ and m, $\sigma = 0$. Then the intersection matrix of those states is

$$\hat{\mathcal{I}}_{(l_1',l_2'),(l_1'',l_2'')} = \sum_{(l_1,l_2)} \mathcal{N}_{(l_1,l_2)\,(l_1'',l_2'')}^{\quad (l_1',l_2')}\ \tilde{\mathcal{I}}_{(l_1,l_2)}\ , \tag{4.147}$$

where the $\mathcal{N}$'s are the $\mathfrak{su}(3)_k$ fusion coefficients, and $\tilde{\mathcal{I}}_{(l_1,l_2)}$ is the contribution of all ground states in the open string R sector that can occur for fixed (l_1, l_2), modulo field identification. This reads explicitly,

$$\begin{aligned}\tilde{\mathcal{I}}_{(l_1,l_2)} &= N_{l_1} g^{-l_1-2l_2} - N_{l_1+l_2+1} g^{-l_1+l_2+3} \\ &\quad + N_{l_2} g^{2l_1+l_2+6} - N_{k+1-l_1} g^{-l_1-2l_2+3h} \\ &\qquad + N_{k-l_1-l_2} g^{-l_1+l_2+3+3h} - N_{k+1-l_2} g^{2l_1+l_2+6+3h} \\ &= \big(N_{l_1} g^{-l_1-2l_2} - N_{l_1+l_2+1} g^{-l_1+l_2+3} + N_{l_2} g^{2l_1+l_2+6}\big)(1 - N_{k+1} g^{3h})\,.\end{aligned} \tag{4.148}$$

Here and from now on, the N's will be reserved to denote the $\mathfrak{su}(2)$ fusion matrices. The matrix g is $6h \times 6h$ dimensional. The terms on the RHS of (4.148) correspond, respectively, to the occurrence of the fields

$$\begin{aligned}&\big((l_1, l_2), l_1, l_1 + 2l_2, 0\big) \\ &\big((l_1, l_2), l_1 + l_2 + 1, l_1 - l_2 - 3, v\big) \\ &\qquad\qquad \equiv \big((k - l_1 - l_2, l_1), k - l_1 - l_2, k + l_1 - l_2, 0\big) \\ &\big((l_1, l_2), l_2, -2l_1 - l_2 - 6, 0\big) \\ &\qquad\qquad \equiv \big((l_2, k - l_1 - l_2), l_2, 2k - 2l_1 - l_2, 0\big) \\ &\big((l_1, l_2), k + 1 - l_1, l_1 + 2l_2 + 3h, v\big) \\ &\qquad\qquad \equiv \big((l_1, l_2), l_1, l_1 + 2l_2, 0\big) \\ &\big((l_1, l_2), k - l_1 - l_2, l_1 - l_2 - 3 + 3h, 0\big) \\ &\qquad\qquad \equiv \big((k - l_1 - l_2, l_1), k - l_1 - l_2, k + l_1 - l_2, 0\big) \\ &\big((l_1, l_2), k + 1 - l_2, -2l_1 - l_2 - 6 + 3h, v\big) \\ &\qquad\qquad \equiv \big((l_2, k - l_1 - l_2), l_2, 2k - 2l_1 - l_2, 0\big)\end{aligned} \tag{4.149}$$

in the open string sector. According to (3.51), the fields with $\sigma = 0$ contribute with a plus sign and the fields with $\sigma = v$ with a minus sign; this explains the signs in (4.148). The structure of (4.148) is as expected from (4.143). The first bracket is the sum over the relative Weyl group, while the second implements the identification which are trivial in the numerator of the coset.

4.4.4 Some properties of the intersection index

The intersection index of boundary states in Kazama-Suzuki models has some rather interesting properties, which nicely illustrate the general structure of $\mathcal{N} = 2$ BCFT. Full details will now be worked out for the $\mathfrak{su}(3)_k/\mathfrak{u}(2)$ models. The generalization to other models should be straightforward.

The Cardy construction yields a list of boundary states labelled by the primary fields of the coset, and the intersection index $\hat{\mathcal{I}}$ between any pair of them, as computed above. The intersection index gives the set of boundary conditions the structure of an integral lattice. In string theory, this lattice is naturally interpreted as the lattice of RR charges, of rank equal to the dimension of the relevant chiral ring, eq. (4.134). From unitarity of the matrix of Cardy coefficients, it follows that the set of Cardy states span this lattice, but *a priori*, it is not clear that they contain an integer basis. It turns out, however, that such an integral basis is provided by the states with $\Lambda = 0$. Indeed, as far as RR charges are concerned, all other states can be considered as integral linear combinations of (a subset of) the $\Lambda = 0$ states. These are thus the analogs of the basic $L = 0$ states of the minimal models, and in fact they can be viewed as the D-brane states with lowest mass if one resolves the singularity by switching on μ in (4.136).

In order to simplify notation, notice that from the formulae above, it is obvious that a state with (representative) label $(\Lambda, \lambda, m, 0)$ intersects all other states with a minus sign relative to the state (Λ, λ, m, v) (brane and anti-brane). Thus, one can immediately restrict attention to, say, $\sigma = 0$ states. Furthermore, in many instances there are identification rules that are trivial in the numerator of the coset, and this leads to a further reduction of the labels among $\Lambda = 0$ representatives.

Consider the favorite example, $\mathfrak{su}(3)_k/\mathfrak{u}(2)$. From (4.147)and (4.148), one deduces the basic intersection matrix of the states with $\Lambda = 0$ representatives,

$$\begin{aligned}\hat{\mathcal{I}}^{[2,k]} \equiv \hat{\mathcal{I}}^{[2,k]}_{(0,0)(0,0)} &= 1 - N_1 g^3 + g^6 - N_{k+1} g^{3h} + N_k g^{3h+3} - N_{k+1} g^{3h+6} \\ &= (1 - N_{k+1} g^{3h})(1 - N_1 g^3 + g^6)\,. \end{aligned} \tag{4.150}$$

Suppressing the $\Lambda = (0,0)$ label, the remaining labels are $(\lambda, m, 0)$. Note that for the $\Lambda = 0$ states, m is always a multiple of three, and one may therefore reduce the size of the g-matrix accordingly: $g = g_{6(k+3)} \to g_{2(k+3)}$. The coset rules require $\lambda/2 + m/6$ to be integer, and moreover identify $(\lambda, m, 0)$ with $(k+1-\lambda, m+3h, v)$. Therefore, the following "standard" range is natural,

$$\lambda = 0, \ldots, k+1,\; m = 3m' \text{ with } m' = \lambda, \lambda+2, \ldots, 2k+2-\lambda. \tag{4.151}$$

Now let

$$\begin{aligned} l'_1 &= \lambda \\ l'_2 &= \frac{m' - \lambda}{2}\,. \end{aligned} \tag{4.152}$$

The standard range can then be more concisely expressed as,

$$l'_1, l'_2 \geq 0\,, \qquad l'_1 + l'_2 \leq k+1\,. \tag{4.153}$$

This looks like the labels of the integrable representations of $\mathfrak{su}(3)_{k+1}$ (where the level is by one higher than what appears in the coset [137]). This point of view is sometimes convenient, but as will be clear later, the labelling in terms of λ and m reflects more naturally the underlying algebraic structure, which is related to the $\mathfrak{u}(2)$ fusion ring.

It is easy to see that restricting the labels to $l_1'+l_2' \leq k$, which corresponds to the integrable representations of $\mathfrak{su}(3)_k$, and ordering the states according to increasing l_2' and l_1', the reduced intersection form, denoted by $\mathcal{I}^{[2,k]} \equiv \mathcal{I}^{[2,k]}_{(0,0)(0,0)}$, is upper triangular with 1 on the diagonal. Its rank is $(k+1)(k+2)/2$, which is equal to the dimension of the chiral ring of the coset model.

The $\Lambda = 0$ boundary states with $l_1' + l_2' \leq k$ thus yield a complete basis of the charge lattice, and what remains to be shown is that all other boundary states can be obtained from them via integral linear combinations. As far as the rest of the $\Lambda = 0$ states is concerned, namely the ones with $l_1'+l_2' = k+1$, this can be seen in the following way. Simply observe that the formal sums of states

$$(0, l_2') + (1, l_2') + ... + (k+1, l_2') \tag{4.154}$$

(assuming they are mapped back to the standard range with an appropriate minus sign) do not intersect with any other state, and so correspond to null eigenvectors of $\hat{\mathcal{I}}$. This shows in a direct way that (the charges of) the states with $l_1' + l_2' = k+1$ can be written as integral linear combinations of the states with $l_1' + l_2' \leq k$. To show the analogous statement for the states with $\Lambda > 0$, it is convenient to use again matrix notation for the charges. Thus, one seeks matrices of charge vectors, ${Q_{(l_1,l_2)}}^{\lambda',m'}_{\lambda,m}$ with (l_1, l_2) fixed, satisfying

$$\hat{\mathcal{I}}_{(l_1',l_2')(l_1'',l_2'')} = Q^T_{(l_1',l_2')}\ \hat{\mathcal{I}}_{(0,0)(0,0)}\ Q_{(l_1'',l_2'')}\,. \tag{4.155}$$

These charge vectors can be obtained as follows. First define

$$\begin{aligned}\tilde{Q}_{(l_1,l_2)} = N_{l_1} g^{-l_1-2l_2} + N_{l_1+1} g^{-l_1-2l_2+3} + \dots \\ \dots + N_{l_1+l_2} g^{-l_1+l_2} + N_{l_1+l_2-1} g^{-l_1+l_2+3} + \dots \\ \dots + N_{l_2} g^{l_2+2l_1}\,.\end{aligned} \tag{4.156}$$

Then, if $l_1 \geq l_2$

$$Q_{(l_1,l_2)} = \tilde{Q}_{(l_1,l_2)} + \tilde{Q}_{(l_1-1,l_2-1)} + \dots + \tilde{Q}_{(l_1-l_2,0)}\,, \tag{4.157}$$

and the analogous expression if $l_2 \geq l_1$. Indeed, a simple computation shows

$$\begin{aligned}\hat{\mathcal{I}}_{(0,0)(0,0)}\ \tilde{Q}_{(l_1,l_2)} &= \left(1 - N_{k+1}g^{3h}\right)\Big[N_{l_1}g^{-l_1-2l_2} - N_{l_1+l_2+1}g^{-l_1+l_2+3} \\ &\qquad + N_{l_2}g^{l_2+2l_1+6} - N_{l_1-1}g^{-l_1-2l_2+3} \\ &\qquad + N_{l_1+l_2-1}g^{-l_1+l_2+3} - N_{l_2-1}g^{l_2+2l_1+3}\Big] \\ &= \hat{\mathcal{I}}_{(0,0)(l_1,l_2)} - \hat{\mathcal{I}}_{(0,0)(l_1-1,l_2-1)}\,,\end{aligned} \tag{4.158}$$

where the second term is absent if $l_1 = 0$ or $l_2 = 0$. Therefore, summing up $\tilde{Q}$ as in (4.157), one obtains,

$$\hat{\mathcal{I}}_{(0,0)(0,0)}\ Q_{(l_1,l_2)} = \hat{\mathcal{I}}_{(0,0)(l_1,l_2)}\,. \tag{4.159}$$

With some more effort, one can check that indeed the Q's satisfy (4.155).

The above considerations can be made more transparent by associating a graph with the basic intersection index (4.150), whose nodes correspond to boundary states and oriented signed links between them encode their intersection. Such a graph (omitting the arrows) is shown for $k = 2$ in fig. 4.2. In this picture, the fat lines denote the sub-graph $\mathcal{I}^{[2,2]}$ of the integral homology basis, which corresponds to the fusion graph of $\mathfrak{su}(3)_2$ (by change of basis it can be put into the form of the D_6 Dynkin diagram, which reflects the equivalence of the KS model $\mathfrak{su}(3)_2/\mathfrak{u}(2)$ with the minimal model of type D_6). Note that the extended graph looks similar to the fusion graph of the integrable representations of $\mathfrak{su}(3)_3$, but in fact, the dashed links really make it into a fusion graph of $\mathfrak{u}(2)$. It is also quite instructive to represent the charge vectors (4.157) of the $\Lambda > 0$ states graphically. Fig. 4.3 shows those $\Lambda = 0$ states whose charges add up to the charge $\tilde{Q}_{(l_1,l_2),\lambda,m}$.

The generalization of (4.150) to all KS models of the form $\mathfrak{su}(n+1)_k/\mathfrak{u}(n)$ is straightforward. The $\Lambda = 0$, $\sigma = 0$ states intersect as

$$\begin{aligned}\hat{\mathcal{I}}^{[n,k]} \equiv \hat{\mathcal{I}}^{[n,k]}_{0,0} =&\ 1 - N_{[1]}g^{n+1} + N_{[2]}g^{2(n+1)} + ... + (-1)^n g^{n(n+1)} \\ &+ (-1)^{n+1}N_J g^{-(n+1)h} + (-1)^{n+2}N_{J[1]}g^{-(n+1)h+(n+1)} + \dots \\ &\qquad + (-1)^{2n+1}N_J g^{-(n+1)h+n(n+1)} \\ &\vdots \\ &+ (-1)^{(n+1)(n-1)}N_{J^{n-1}}g^{-(n+1)(n-1)h} + \dots \\ &\qquad + (-1)^{(n+1)(n-1)+n}N_{J^{n-1}}g^{-(n+1)(n-1)h+n(n+1)} \\ =&\ \left(1 - N_{[1]}g^{n+1} + N_{[2]}g^{2(n+1)} + ... + (-1)^n g^{n(n+1)}\right) \\ &\times \left(1 + (-1)^{n+1}N_J g^{-(n+1)h} + \dots + (-1)^{(n+1)}N_{J^{n-1}}g^{-(n+1)(n-1)h}\right)\end{aligned} \tag{4.160}$$

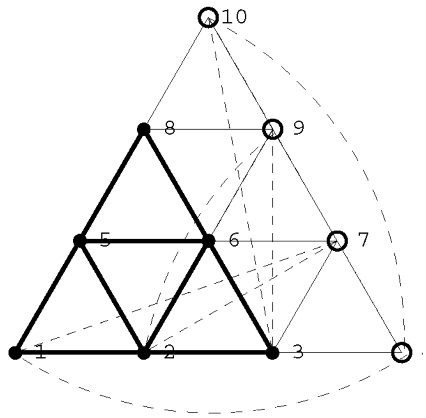

Figure 4.2: The intersection graph $\hat{\mathcal{I}}^{[2,2]}$ of $\Lambda = 0$ boundary states of the $\mathfrak{su}(3)_2/\mathfrak{u}(2)$ KS model. The fat lines represent the sub-graph $\mathcal{I}^{[2,2]}$ of the integral homology basis, which coincides with the fusion graph of $\mathfrak{su}(3)_2$. The open dots denote extending nodes, which give the fusion graph of $\mathfrak{su}(3)_3$; the dashed links extend this further to the fusion graph of $\mathfrak{u}(2)$.

Here, $N_{[i]}$ is the fusion matrix of the i-th fundamental representation of $\mathrm{SU}(n)$ at level $k = h - n$, and $(0, J^{(n)}, (n+1)h, v^{n+1}) = \mathcal{J}^{n+1}$ is the simple current implementing the coset rules that act only in the denominator, with N_J the fusion matrix of $J \equiv J^{(n)}$. Due to redundancy, the $\mathfrak{u}(1)$ fusion matrix $g \equiv g_{n(n+1)h}$ can be reduced in size by a factor of $n+1$.

Similarly to the $\mathfrak{su}(3)$ example discussed above, the coset identification rules allow the reduction of the $\Lambda = 0$ states to a set of labels in one-to-one correspondence with the integrable representations of $\mathfrak{su}(n+1)_{k+1}$, which is at one level higher than the CFT suggests. The intersection matrix $\hat{\mathcal{I}}^{[n,k]}$ does not have full rank and thus should be viewed as an intersection form of an over-complete basis. Restricting to boundary states corresponding to level k, the resulting reduced intersection matrix $\mathcal{I}^{[n,k]}$ becomes upper triangular and has full rank (given by (4.134)). The vanishing relations are analogous to the $\mathfrak{su}(3)$ case, and the generalizations of the charge vectors (4.157) are rather obvious, in particular in view of the graphical presentation in fig. 4.3. One thus obtains a basis for the charge lattice also in the general case. A more formal understanding of these relations should be rather interesting to develop.

Note that the graph of the symmetrized reduced matrix $\mathcal{I}^{[n,k]}$,

$$C^{[n,k]} = \mathcal{I}^{[n,k]} + \left(\mathcal{I}^{[n,k]}\right)^t, \qquad (4.161)$$

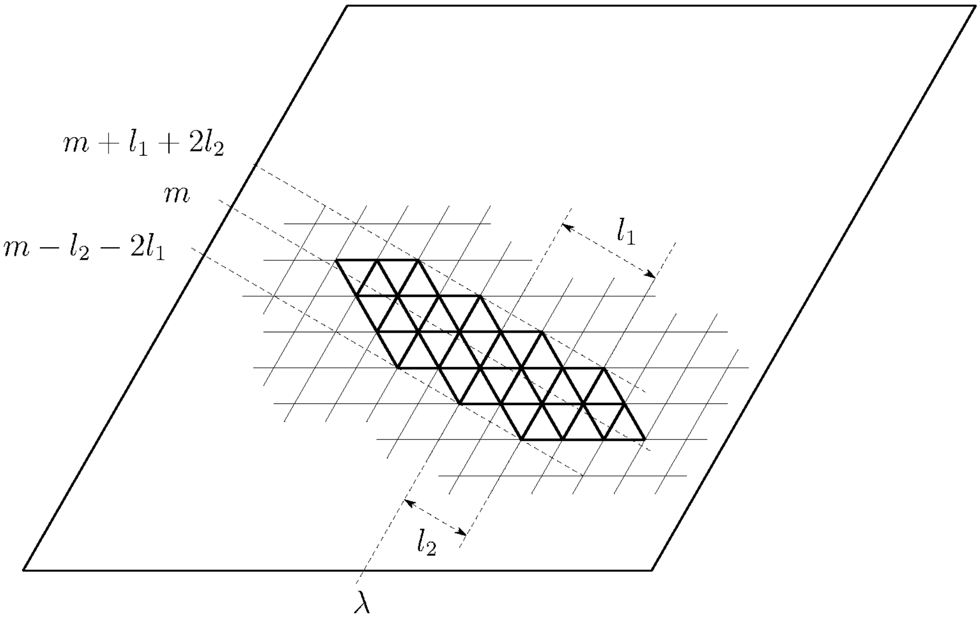

Figure 4.3: Charges of $\Lambda > 0$ states in a Kazama-Suzuki model of type $\mathfrak{su}(3)_k/\mathfrak{u}(2)$. The parallelogram is twice the standard range for $\Lambda = 0$ states and the fat part of the grid inside is a summation region. For a given $((l_1, l_2), \lambda, m)$ state (here $l_1 = 3$, $l_2 = 2$), it shows the expansion of the vector $\tilde{Q}_{(l_1,l_2),\lambda,m}$, eq. (4.156). To obtain the charge $Q_{(l_1,l_2),\lambda,m}$, one has to sum over all regions with the same shape and center, and smaller size as the one shown here.

which represents the intersection index for a complete homology basis, coincides with the fusion graph of $\mathfrak{su}(n+1)_k$; this generalizes the coincidence of the A_{n+1} Dynkin diagram with the $\mathfrak{su}(2)_k$ fusion diagram discussed in section 4.2. It also reproduces and clarifies, from a BCFT point of view, the connection between the resolution of the singularities (4.136) and the Verlinde fusion algebra for $\mathfrak{su}(n+1)_k$. Such a relation had been conjectured by Zuber [146] and others and was proven in [147].

4.4.5 Quiver representations

A quiver (or quiver diagram) is a graph consisting of a set of points and a set of labeled directed arrows between them. The graphs associated to the intersection forms (4.150) or (4.160) are examples of quivers. To any quiver, there is an associated path algebra, and one may study the representation theory of this algebra. The idea underlying the recent proposal of Douglas and coworkers (see [94] and references therein; see also [149, 88] for short introductions and further references to quivers) is that, quite generally, the collection of all D-branes in a string compactification can be built up from the representation theory of an underlying quiver.

Consider as an example again the $\mathfrak{su}(3)_5/\mathfrak{u}(2)$ Kazama-Suzuki model. Fig. 4.4 shows the corresponding quiver in the most natural, manifestly $\mathbb{Z}_5$ symmetric form.

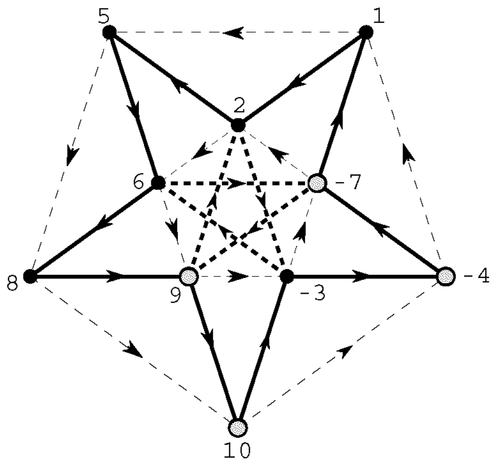

Figure 4.4: The intersection graph of fig. 4.2 in a manifestly $\mathbb{Z}_5$ symmetric form.

Representations of a quiver are partially characterized by a "charge vector", which to each node gives the dimensionality of an associated vector space. The total representation space is the sum of all these vector spaces. In this language, the elementary $\Lambda = 0$ states correspond to the simplest representations of the quiver, in which only a single node has non-zero charge. The states with $\Lambda \neq 0$ then correspond to higher dimensional representations, and their charge vectors are precisely given by eq. (4.157).

It turns out that the quiver in fig. 4.4 has a rather simple representation theory, due to the fact that its generalized Cartan matrix C, eq. (4.161), is positive definite. There are only finitely many indecomposable Schur roots, each yielding a representation without moduli space. This is related to the fact that C can be transformed by change of basis to the Cartan matrix of D_6, and the quiver representations correspond precisely to the roots of the Lie algebra D_6. In CFT language, this simplification is due to the fact that the $\mathfrak{su}(3)_2/\mathfrak{u}(2)$ model is in fact isomorphic to the $\mathfrak{su}(2)_{10}/\mathfrak{u}(1)$ minimal model with $D6$-type modular invariant; in other words, the $\mathcal{N} = 2$ $\mathcal{W}$-algebra is a simple-current extension of the $\mathcal{N} = 2$ Virasoro algebra. The interesting point is that the list of Cardy (and hence symmetry preserving) boundary states in the KS model contains only 40 out of the 60 representations of the quiver. The missing states are symmetry breaking and can be constructed using methods of [49, 50].[16]

The situtation becomes much more involved for models at higher level, which are irrational over the $\mathcal{N} = 2$ algebra. The states obtained from Cardy's construction cover only a very small subset of all possible $\mathcal{N} = 2$ supersymmetric ones. The parallel statement about the quiver is that the unextended Cartan matrix of the quiver becomes indefinite, and hence there are infinitely many irreducible representations.

As a second example, consider the model $\mathfrak{su}(3)_3/\mathfrak{u}(2)$. The ten dimensional unextended Cartan matrix of the corresponding quiver has two zero eigenvalues, and the charge lattice is of type $E_8 \times U \times U$, where U corresponds to a null direction. This is exactly as expected from the geometry of the triangle singularity [150] of type $T_{2,3,6}$, described by the Landau-Ginzburg potential for this Kazama-Suzuki model. The states obtained from CFT turn out to correspond to the roots of E_8 plus a few imaginary roots, compared to the infinite number of positive roots of the hyperbolic algebra associated with the Cartan matrix.

These results are certainly consistent with the idea that the representation theory of quivers organizes boudary conditions in $\mathcal{N} = 2$ superconformal field theories, but more work is clearly needed. A rather important problem in this context would be the reconstruction of boundary states from given quiver representations, which are—at least in some cases—easier to obtain.

4.4.6 Relation to Grassmannians

It was mentionned above that Kazama-Suzuki models also have a well-known relationship to Grassmannians. In the bulk, this is the coincidence [10] of the chiral ring of the KS model $\mathfrak{su}(n+1)_k/\mathfrak{u}(n)$ with the classical cohomology ring of the Grassmannian space $\mathrm{Gr}(n, n+k) \cong \mathrm{U}(n+k)/\mathrm{U}(n) \times \mathrm{U}(k)$. Note that a Grassmanninan has positive Chern class and that therefore the associated σ-model is not conformal. However, a topological A-model and its ring of observables can still be defined, since Kählerity is sufficient for this.

There are two crucial differences between the KS model and the Grassmannian. Firstly, the correct structure to consider on the Grassmannian is the quantum cohomology ring, which is a deformation of the classical cohomology ring, and the former reduces to the latter only in the large volume

[16]It is shown in [109] how all of these states can be constructed using methods of [51].

limit. Secondly, even in the large volume limit, the U(1) charges of the ring on the Grassmannian are integer, while they are certain fractional numbers in the KS model. The isomorphism between the two rings apparently has its origin in the group theory that determines them, rather than in an identity of the field theories.

In the classification program for $\mathcal{N}=2$ topological field theories [140], there is besides the chiral ring a second type of invariants, monodromy invariants, that play an important role. For Landau-Ginzburg models, these invariants can be defined as the number of solitons between the vacua of the theory. In geometry, soliton numbers are then also computable from the intersection of vanishing cycles. In the context of σ-models on Kähler manifolds, which are mirror to the Landau-Ginzburg theories [117], the soliton numbers become intersection numbers of certain exceptional collections of bundles (helices) over the Kähler manifold [29]. Quivers are also natural in this context [149]. (See also [124] for investigations of the Grassmannian σ-model and a connection with a generalized McKay correspondence).

The relation between Grassmannians and Kazama-Suzuki models now reappears in the open string sector in the following disguise. If χ_{Grass} is the upper triangular intersection form of the helix on the Grassmannian, and $\chi_{\text{KS}} = \mathcal{I}^{[n,k]}$ the corresponding object in the Kazama-Suzuki model, then the relation

$$\chi_{\text{Grass}} = \big(\chi_{\text{KS}}\big)^n \tag{4.162}$$

turns out to hold. The interesting point is that while a Grassmannian σ-model is certainly not in the same class of $\mathcal{N}=2$ field theories as a Kazama-Suzuki model, the Grassmannian can be viewed as a sort of elder cousin of the Kazama-Suzuki in the sense that the intersection indices are related as in eq. (4.162).

Appendix 4.A Phase symmetries and the labelling of boundary conditions in Gepner models

It is clear from section 4.3.2 and 4.3.3 that the A- and B-type boundary conditions in Gepner models are organized by the group of phase symmetries of the Gepner model, as follows. Denote by

$$\mathcal{G}_{\text{ph}}^{\mathcal{C}^{\text{wsusy}}} = \bigtimes_{i=1}^{r} \mathbb{Z}_{h_i} \tag{4.163}$$

the group of phase symmetries of the minimal models with levels k_i, $i=1,\ldots,r$, after fermion alignment. For simplicity, $n+r$ will be assumed to be even.[17]

Assume that fermions are aligned and Λ and $S=0$ are fixed. Then, in $\mathcal{C}^{\mathrm{wsusy}}$, the range of allowed (M_i) labels for boundary conditions, denoted by

$$\mathcal{M}^{\mathcal{C}^{\mathrm{wsusy}}} = \{(M_i), L_i + M_i = \text{even}\}\,, \tag{4.164}$$

is one-to-one to $\mathcal{G}_{\mathrm{ph}}^{\mathcal{C}^{\mathrm{wsusy}}}$. But $\mathcal{M}^{\mathcal{C}^{\mathrm{wsusy}}}$ is not a group!

The group $\mathcal{G}_{\mathrm{ph}}^{\mathcal{C}^{\mathrm{wsusy}}}$ has a natural pairing X given by monodromy, and $\mathcal{M}^{\mathcal{C}^{\mathrm{wsusy}}}$ has a natural $\mathcal{G}_{\mathrm{ph}}^{\mathcal{C}^{\mathrm{wsusy}}}$ action. "Dividing out" a subgroup $\mathcal{G} \subset \mathcal{G}_{\mathrm{ph}}^{\mathcal{C}^{\mathrm{wsusy}}}$ yields a new theory $\mathcal{C}_{\mathcal{G}}$ with phase symmetries

$$\mathcal{G}_{\mathrm{ph}}^{\mathcal{C}_{\mathcal{G}}} = \{\mathrm{J} \in \mathcal{G}_{\mathrm{ph}}^{\mathcal{C}^{\mathrm{wsusy}}}, X(\mathrm{J},\mathrm{K}) = 0 \text{ for all } \mathrm{K} \in \mathcal{G}_{\mathrm{ph}}^{\mathcal{C}^{\mathrm{wsusy}}}\}\,. \tag{4.165}$$

One the other hand, A-type boundary conditions in $\mathcal{C}_{\mathcal{G}}$ are labelled by orbits,

$$\mathcal{M}^{\mathcal{C}_{\mathcal{G}}} = \{[(M_1, \ldots, M_r)]_{\mathcal{G}}\}\,. \tag{4.166}$$

It is easy to see that the two sets $\mathcal{G}_{\mathrm{ph}}^{\mathcal{C}_{\mathcal{G}}}$ and $\mathcal{M}^{\mathcal{C}_{\mathcal{G}}}$ are still in one-to-one correspondence. In particular, dividing out the maximal phase symmetry group $\mathcal{G}_{\mathrm{mirr}}$ yields the mirror model, and A-type boundary conditions in this model (equivalently, B-type boundary conditions in the original model, obtained by only the U(1) projection) are in one-to-one correspondence to the surviving group $\mathcal{G}_{\mathrm{ph}}^{\mathcal{C}_{\mathcal{G}_{\mathrm{mirr}}}} = \mathbb{Z}_H$.

This generalizes easily to the statement:

> *In a given theory $\mathcal{C}_{\mathcal{G}}$ obtained from $\mathcal{C}^{\mathrm{wsusy}}$ by "dividing out" $\mathcal{G}$, A-type boundary conditions with fixed Λ and $S=0$ are in one-to-one correspondence with the group of surviving phase symmetries, while B-type boundary conditions are in one-to-one correspondence with the complementary group of phase symmetries.*

▪ As always, it is important to notice that this one-to-one correspondence is not canonical.

▪ Also, it should be stressed that this rule does not take care of fixed points. So far it is not known whether there is a universal statement for the appearance of fixed points. However, the procedure in a given case is quite clear, as demonstrated above. It suffices to determine which combinations of f_i's occur in $\mathcal{G}$. Then all stabilizers have the form of a product of $\mathbb{Z}_2$'s, the simple-current twist on the stabilizers is maximal, and the untwisted stabilizer is either trivial or $\mathbb{Z}_2$.

[17] If $n+r$ is odd, one can, for example, append a trivial factor with $k_0 = 0$.

Chapter 5

Conclusions

Since their discovery, D-branes have been playing an ubiquitous role in string theory. It is very likely that this will remain so in the near future, and it is generally expected that D-branes will also enter at some level in a future, more fundamental or "axiomatic", formulation of the theory. Along the way towards this goal, it is therefore an important task to gather as much information about the properties of D-branes as possible.

Adopting the worldsheet perspective, the work presented in this thesis has traced out the way between superconformal boundary conditions in (rational) CFT (chapter 2) and D-branes in string theory. It has thereby led to a good understanding of the various consistency conditions imposed on D-branes from conformal- and super-symmetry, their intuitive geometric interpretation, and their implementation in an abstract algebraic approach, (chapter 3). These results liberate the mathematical power of conformal field theory for many further investigations of D-branes. The general theory has been illustrated in examples in chapter 4. In these examples, it was also shown in several places how the algebraic methods can be linked back to geometry.

One of the outcomes of these investigations is the confirmation that D-branes in the stringy regime can certainly not be described by classical geometry alone. Conformal field theory sheds light on some of the limitations. One example is the truncation of the spectrum of symmetry preserving D-branes to a finite number in theories which are rational over some chiral algebra. In the context of $\mathcal{N} = 2$ minimal models, this reproduces the finiteness of the spectrum of BPS states in SYM theories at strong coupling, see section 4.2. Such a truncation would not be expected purely from classical geometry or classical field theory.

Another interesting and new example stems from the properties of fixed point resolved B-type D-branes in Gepner models in section 4.3. It was argued there that there do exist D-branes configurations which intrinsically carry an enlarged unbroken gauge group but are nevertheless elementary. The main point in the argument is the existence of a "simple-current twist" on the stabilizer of B-type boundary conditions of fixed point type, which forces the minimal wrapping number to be larger than one. From a combinatorial point of view, one might then suspect a relation to torsion in homology or K-theory. But the match does not seem perfect. Also note that the construction involves two different alternating bihomomorphisms on two different groups, the antisymmetric part of X on $\mathcal{G}_{\mathrm{mirr}}$ and the modified simple-current twist F_Λ^X on $\mathcal{S}_\Lambda$. Usually, $X - X^t$ is identified with discrete torsion in orbifolds, which in turn can be related to the existence of a non-trivial B-field background. If one accepts that the combinatorial role of discrete torsion is here played by F_Λ^X rather than $X - X^t$, one is led to suspect an interpretation in terms of twisted K-theory groups, whose relation to D-brane charges in B-field backgrounds have attracted some attention lately (see [96, 151] and references therein).

Of course, it can not be excluded at the present stage that the fixed points and their resolution can be understood using more geometric methods, such as the Beilinson inspired quiver proposal of Douglas et al.[1]. However, a pragmatic attitude is maybe more adequate. Indeed, it must not be expected that all kinematical, let alone the dynamical, properties of D-branes have an interpretation in classical geometry.

But the result about fixed points and their resolution does have a physical relevance, since it provides a new mechanism for obtaining non-Abelian gauge symmetries in type II string compactifications, purely within the conformal field theory of (tensor products of) $\mathcal{N} = 2$ minimal models.

On the other hand, and this has also become clear in the course of this work, conformal field theory does not open every door. In particular, the fact that many constructions so far rely on rationality over some chiral algebra is a rather severe limitation, at least at the practical level. The most pressing problem is that the rational methods always lead to a finite number of boundary conditions, in situations where on general grounds one expects, or for specific reasons knows, that there is an infinite number of branes.

Thus, algebraic and geometric methods give complementary and independent information about D-branes, and allow mutual testing. It is therefore gratifying that certain links between conformal field theory and geometry remain and can be given explicitly. The final version of string theory, of

[1] In fact, this is a very non-trivial test of the proposal.

course, should contain a unified framework also for D-branes. Depending on taste, one expects “quantum geometry”, “quantum algebraic geometry”, “non-commutative geometry”, or simply “M-theory”.

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Research

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