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LATTICE QCD CALCULATION OF THE VECTOR FORM FACTOR FOR $K_{\ell 3}$ SEMILEPTONIC DECAYS

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Abstract

We present a quenched lattice calculation of the vector form factor at zero-momentum transfer, $f_+(0)$, relevant for the determination of $|V_{us}|$ from semileptonic $K \to \pi\ell\nu$ decays. Our final result is $f_+^{K^0\pi^-}(0) = 0.960 \pm 0.005_{\rm stat} \pm 0.007_{\rm syst}$, in good agreement with the old quark model estimate made by Leutwyler and Roos. The impact of our result on the extraction of $|V_{us}|$ is discussed by taking into account the new experimental determinations.

1 Introduction

The most precise determination of the CKM matrix element $|V_{us}|$ is presently obtained from the semileptonic weak decays of kaons. The analysis of the experimental data on $K \to \pi\ell\nu$ ($K_{\ell 3}$) decays gives access to the quantity $|V_{us}| \cdot f_+(0)$, where $f_+(0)$ is the vector form factor at zero-momentum transfer.

Vector current conservation guarantees that, in the SU(3)-symmetric limit, $f_+(0) = 1$. A good theoretical control on these transitions is obtained via the Ademollo-Gatto (AG) theorem, which states that $f_+(0)$ is renormalized only by terms of at least second order in the breaking of the SU(3)-flavor symmetry. The estimate of the difference of $f_+(0)$ from its SU(3)-symmetric value represents the main source of theoretical uncertainty and it presently dominates the error in the determination of $|V_{us}|$.

The amount of SU(3) breaking due to light quark masses can be investigated within Chiral Perturbation Theory (CHPT) by performing a systematic expansion of the type $f_+(0) = 1 + f_2 + f_4 + \ldots$, where $f_n = \mathcal{O}[M^n_{K,\pi}/(4\pi f_\pi)^n]$. Thanks to the AG theorem, the first non-trivial term in the chiral expansion, f_2, does not receive contributions of local operators appearing in the effective theory and can be computed unambiguously in terms of M_K, M_π and f_π ($f_2 = -0.023$ in the $K^0 \to \pi^-$ case [1]). The higher-order terms of the chiral expansion, instead, involve the coefficients of local chiral operators, that are difficult to estimate. The next-to-leading correction, f_4, has been evaluated many years ago by Leutwyler and Roos (LR) in the quark model framework [1], by using a general parameterization of the SU(3) breaking structure of the pseudoscalar meson wave functions. Their result is $f_4 = -(0.016 \pm 0.008)$ and this value still represents the estimate of reference [2].

The two-loop CHPT calculation of f_4 has been recently completed [3, 4]. The whole result is the sum of a loop contribution, expressed in terms of chiral logs and the $\mathcal{O}(p^4)$ low-energy constants, plus an analytic term that involves a single combination of the (unknown) $\mathcal{O}(p^6)$ chiral coefficients. Furthermore, the separation between non-local and local contribution quantitatively depends on the choice of the renormalization scale, only the whole result for f_4 being scale independent. An important observation by Bijnens and Talavera [3] is that, in principle, the combination of low-energy constants entering in f_4 could be constrained by experimental data on the slope and curvature of the scalar form factor; the required level of experimental precision, however, is far from the presently available one. Thus, one is left with the LR result, and the large scale dependence of the $\mathcal{O}(p^6)$ loop calculations seems to indicate that its 0.008 error might well be underestimated [5].

Very recently [6] the SU(3)-breaking effects on $f_+(0)$ have been computed with lattice QCD simulations. Within this non-perturbative approach, which is only based on the fundamental theory, a new strategy has been proposed and successfully applied, in the quenched approximation, in order to reach the challenging goal of a $\approx 1\%$ error on $f_+(0)$. In this paper we present this result, we discuss its impact on the determination of $|V_{us}|$ and briefly explain the strategy of the lattice calculation.

2 Lattice result and phenomenological implications

The procedure developed in Ref. [6] to compute $f_+(0)$ on the lattice is described in the next section. Here we anticipate our final result for the form factor at zero momentum transfer and briefly discuss its phenomenological implications in the light, in particular, of the new experimental results on $K_{\ell 3}$ decays.

Our result is [6]

$$f_+^{K^0\pi^-}(0) = 0.960 \pm 0.005_{\rm stat} \pm 0.007_{\rm syst} \qquad (1)$$

where the systematic error does not include an estimate of quenched effects beyond $\mathcal{O}(p^4)$. The value (1) compares well with $f_+^{K^0\pi^-}(0) = 0.961 \pm 0.008$, quoted by the PDG [2] and based on the LR estimate of f_4 [1].

By averaging the old experimental results for $K_{\ell 3}$ decays with the recent measurement of the E865 experiment [7], and by using the LR determination of the vector form factor, the PDG quotes $|V_{us}| = 0.2200 \pm 0.0026$ [2]. This estimate, once combined with the accurate determination of $|V_{ud}|$ from nuclear $0^+ \to 0^+$ and nucleon beta decays, $|V_{ud}| = 0.9740 \pm 0.0005$ [8], implies a $\sim 2\,\sigma$ deviation from the CKM unitarity condition, $|V_{us}|^{\rm Unit.} \simeq \sqrt{1-|V_{ud}|^2} = 0.2265 \pm 0.0022$. In this respect, our lattice determination of the vector form factor, being in agreement with the LR estimate, does not modify the picture.

A significant novelty, however, is introduced by the new experimental results for both charged and neutral $K_{\ell 3}$ decays recently obtained by the E865 [7], KTeV [9], NA48 [10] and KLOE [11] collaborations. The corresponding determinations of $|V_{us}|\cdot f_+(0)$ are shown in Fig.1 [12], together with the averages of the old results quoted by the PDG.

Remarkably, the average of the new results, represented in the plot by the gray band, turns out to be in good agreement with the unitarity prediction, once either the LR or the lattice determination of the vector form factor is taken into account. The unitarity prediction is illustrated by the yellow band in Fig.1.

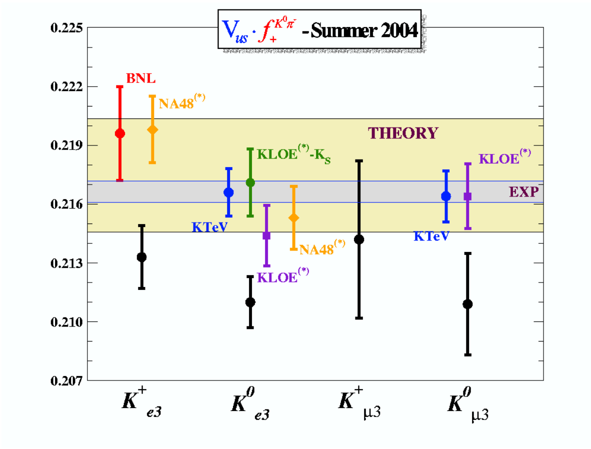


Figure 1: *Experimental results for* $|V_{us}| \cdot f_+(0)$*. The gray band indicates the average of the new experimental results, whereas the yellow band represents the unitarity prediction combined with our determination of the vector form factor.*

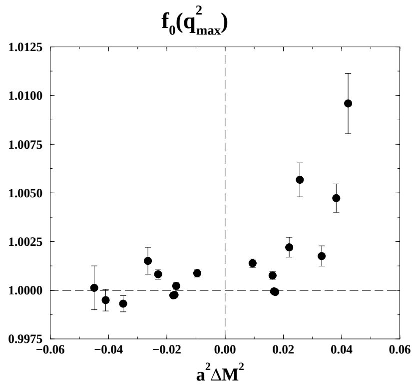
In terms of $|V_{us}|$, our determination of the vector form factor combined with the new experimental results implies $|V_{us}| = 0.2256 \pm 0.0022$.

We also note that the recent theoretical estimate [3] $f_+(0) = 0.976 \pm 0.010$, based on two loops CHPT and the LR quark model calculation, implies $|V_{us}| = 0.2219 \pm 0.0022$, which represent a $\sim 1.5\sigma$ deviation from the unitarity prediction.

3 Strategy of the lattice calculation

In this section we briefly illustrate the strategy to compute $f_+(0)$ with $\approx 1\%$ of accuracy, by referring to Ref. [6] for all details. This strategy is based on three main steps.

1) Precise evaluation of the scalar form factor $f_0(q^2)$ at $q^2 = q^2_{max}$.
By following a procedure originally proposed in Ref. [13] to study heavy-light form factors, the scalar form factor $f_0(q^2)$ can be calculated very efficiently at



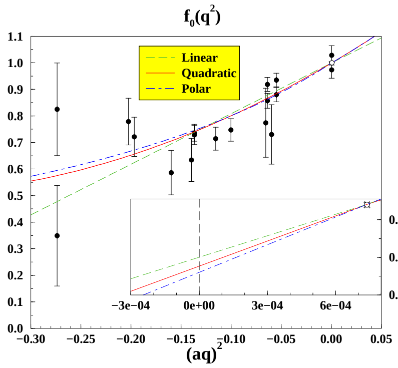


Figure 2: *Left: values of* $f_0(q^2_{\rm max})$ *versus the SU(3)-breaking parameter* $a^2\Delta M^2 \equiv a^2(M_K^2 - M_\pi^2)$. *Right: the form factor* $f_0(q^2)$ *as a function of* q^2 *for one of the quark mass combinations. The dot-dashed, dashed and solid lines correspond to the polar, linear and quadratic fits given in Eq. (3). The inset is an enlargement of the region around* $q^2 = 0$.

$q^2 = q^2_{\rm max} = (M_K - M_\pi)^2$ from the following double ratio of matrix elements:

$$\frac{\langle\pi|\bar{s}\gamma_0 u|K\rangle\,\langle K|\bar{u}\gamma_0 s|\pi\rangle}{\langle K|\bar{s}\gamma_0 s|K\rangle\,\langle\pi|\bar{u}\gamma_0 u|\pi\rangle} = [f_0(q^2_{\rm max})]^2\,\frac{(M_K+M_\pi)^2}{4M_K M_\pi}\,, \tag{2}$$

where all the external particles are taken at rest. There are several crucial advantages in the use of the double ratio (2) which are described in Ref. [6]. From this ratio the values of $f_0(q^2_{\rm max})$ can be determined on the lattice with an uncertainty smaller than 0.1%, as it is illustrated in Fig.2-left.

2) Extrapolation of $\mathbf{f}_0(q^2_{\rm max})$ to $\mathbf{f}_0(0) = f_+(0)$.

For each set of quark masses, hadronic matrix elements can be calculated on the lattice for external mesons with various momenta, in order to extract the q^2 dependence of both $f_0(q^2)$ and $f_+(q^2)$. New suitable double ratios are introduced also in this step, which allows to improve the statistical uncertainties on $f_0(q^2)$. The quality of the results is shown in Fig.2-right for one of the combinations of quark masses used in Ref. [6].

In order to extrapolate the scalar form factor to $q^2 = 0$ three different functional forms have been considered, namely a polar, a linear and a quadratic fit:

$$f_0(q^2) = f_0^{(pol.)}(0)/(1 - \lambda_0^{(pol.)}\, q^2) \quad , \quad f_0(q^2) = f_0^{(lin.)}(0) \cdot (1 + \lambda_0^{(lin.)}\, q^2)\, ,$$
$$f_0(q^2) = f_0^{(quad.)}(0) \cdot (1 + \lambda_0^{(quad.)}\, q^2 + c_0\, q^4)\, . \tag{3}$$

These fits are shown in Fig.2-right and provide values of both $f_0(0)$ and the slope λ_0, which are consistent with each other within the statistical uncertainties. The differences of the results obtained from the various fit are taken into account in the estimate of the systematic error. Our results for the slope λ_0, given in units of $M_{\pi^+}^2$, are: $\lambda_0^{(pol.)} = 0.0122(22)$, $\lambda_0^{(lin.)} = 0.0089(11)$ and $\lambda_0^{(quad.)} = 0.0115(26)$. The "polar" value is consistent with the recent accurate determination from KTeV $\lambda_0^{(pol.)} = 0.01414 \pm 0.00095$ [14] and represents a true theoretical prediction, having been obtained before the KTeV result were published. We also mention that the result for the polar slope of the vector form factor, $\lambda_+ = 0.026 \pm 0.002$ in units of $M_{\pi^+}^2$, is in good agreement with the recent accurate measurement from KTeV, $\lambda_+ = 0.02502 \pm 0.00037$ [14], obtained using a pole parameterization.

3) Extrapolation of $\mathbf{f_+(0)}$ to the physical meson masses.

The physical value of $f_+(0)$ is finally determined by extrapolating the lattice results to the physical kaon and pion masses. The problem of the chiral extrapolation is substantially simplified if the AG theorem (holding also in the quenched approximation) is taken into account and if the leading (quenched) chiral logs are subtracted. Thus in [6] the following quantity is introduced

$$R(M_K, M_\pi) = \frac{1 + f_2^q(M_K, M_\pi) - f_+(0; M_K, M_\pi)}{(M_K^2 - M_\pi^2)^2} \tag{4}$$

where f_2^q represents the leading chiral contribution calculated in quenched CHPT [6] and the quadratic dependence on $(M_K^2 - M_\pi^2)$, driven by the AG theorem, is factorized out. After the subtraction of f_2^q we expect that $R(M_K, M_\pi)$ is well suited for a smooth polynomial extrapolation in the meson masses. Indeed, we find that $R(M_K, M_\pi)$ is well described by a simple linear fit:

$$R^{(lin.)}(M_K, M_\pi) = c_{11} + c_{12}[(aM_K)^2 + (aM_\pi)^2]\ , \tag{5}$$

whereas the dependence on $(M_K^2 - M_\pi^2)$ is found to be negligible. In order to check the stability of the results, quadratic and logarithmic fits have been also

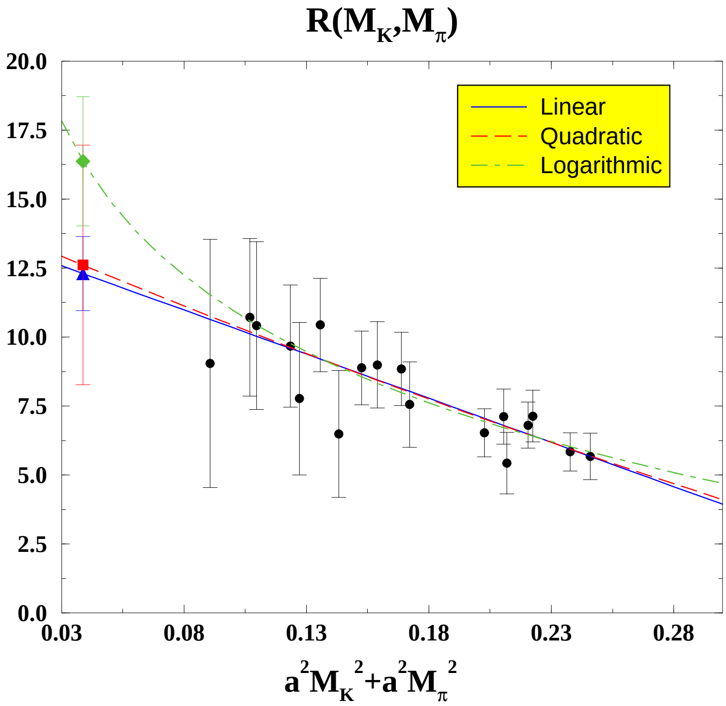


Figure 3: *Comparison among linear, quadratic and logarithmic fits of the ratio* $R(M_K, M_\pi)$ *as a function of* $[a^2 M_K^2 + a^2 M_\pi^2]$.

considered. In Fig.3 it is shown that all these functional forms provide equally good fits to the lattice data with consistent results also at the physical point.

Combining our estimate of $R(M_K, M_\pi)$ at the physical meson masses with the unquenched value of $f_2 = -0.023$ [1], we finally obtain the result quoted in Eq. (1). Note that the systematic error does not include an estimate of quenched effects beyond $\mathcal{O}(p^4)$.

4 Conclusions

We have presented a quenched lattice calculation of the $K_{\ell 3}$ vector form factor at zero-momentum transfer, $f_+(0)$. Our calculation is the first one obtained using a non-perturbative method based only on QCD, except for the quenched approximation. The impact of our result on the determination of $|V_{us}|$ has been also addressed. We find that, once combined with the new experimental determinations, a very good agreement with CKM unitarity is obtained.

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