

GAP listing of the finite subgroups of $U(3)$ of order smaller than 2000

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We have sorted the `SmallGroups` library of all the finite groups of order smaller than 2000 to identify the groups that possess a faithful 3D irreducible representation (“irrep”) and that cannot be written as the direct product of a smaller group times a cyclic group. Using the computer algebra system GAP, we have scanned all the 3D irreps of each of these groups to identify those that are subgroups of $SU(3)$; we have labeled each of these subgroups of $SU(3)$ by using the extant complete classification of the finite subgroups of $SU(3)$. Turning to the subgroups of $U(3)$ that are not subgroups of $SU(3)$, we have found the generators of all of them and classified most of them in series according to their generators and structure.
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1. Introduction

Many high-energy physicists are thrilled by the prospect that the numerical entries of the leptonic mixing matrix (PMNS (Pontecorvo–Maki–Nakagawa–Sakata) matrix) might be related to some small (or maybe not so small) finite group. Many specific finite groups have been considered, such as, for instance, A_4 [1]–[11], S_4 [12]–[19], S_3 [20]–[25], T_7 [26]–[30], A_5 [31]–[36], $\Delta(27)$ [37]–[42], the group series $\Delta(6n^2)$ [43,44], the groups $\Sigma(n\varphi)$ [45], and so on. Most of the finite groups considered are subgroups of $SU(3)$; these subgroups are especially inviting because a complete classification of them, and their generators, has been known for over a century [46–49]. In contrast, there is no complete classification of the finite subgroups of $U(3)$,¹ though a few series of these subgroups have been derived in Ref. [50]. At least one finite subgroup of $U(3)$ has already been utilized in particle physics [51,52].

Although a full theoretical study of each individual group can always be undertaken, for large groups such a study becomes impractical and it is convenient to have recourse to the computer algebra system GAP, which is tailored to deal with finite groups and can readily furnish the structure, irreducible representations (“irreps”), character table, and so on of each of them. GAP is supplemented by the `SmallGroups` library, which contains, in particular, all the finite groups of order smaller than 2000. In that library each finite group has an identifier $[o, j]$, where $o \geq 1$ is the order, i.e., the number of elements, of the group and $j \geq 1$ is an integer that distinguishes between the nonisomorphic groups

¹In this paper, whenever we use the expression “finite subgroups of $U(3)$ ” we usually mean only the subgroups of $U(3)$ that are not subgroups of $SU(3)$.

of identical order. For instance, the group with `SmallGroups` identifier [4, 1] is the cyclic group² $\mathbb{Z}_4 \cong \{1, i, -1, -i\}$ while the group with `SmallGroups` identifier [4, 2] is the direct product of cyclic groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$; `SmallGroups` informs us that there are, in fact, only these two nonisomorphic groups with four elements. A `SmallGroups` listing of all the finite groups of order up to 100, together with their structure,³ was published in Ref. [54]. A `SmallGroups` listing of the finite groups of order up to 512 that have a faithful 3D irrep and that are not the direct product of a cyclic group and some other group was published in Ref. [50].

However, `SmallGroups` lists groups of the same order in a way that does not allow one to extract much information on them. For instance,

- the group [12, 3] $\cong A_4$ is a subgroup of $SU(3)$ and has a 3D faithful irrep;
- the groups [12, 1] and [12, 4] $\cong D_6$ are subgroups of $SU(3)$ but do not possess 3D irreps;
- the group [12, 2] $\cong \mathbb{Z}_{12}$ is a subgroup of $U(1) \subset U(3)$;
- the group [12, 5] $\cong \mathbb{Z}_6 \times \mathbb{Z}_2$ is a subgroup of $U(1) \times U(1)$ but not of $U(3)$.

The first step in this work was to survey the whole `SmallGroups` list of groups of order smaller than 2000 in order to identify the ones that

- have at least one faithful 3D irreducible representation;
- cannot be written as the direct product of a smaller group and a cyclic group.

The second step in this work was to pick each of the groups above and ask GAP to compute the determinant of each of the matrices in each of its 3D representations. If there is a 3D representation in which all the matrices have unit determinant, then the group is a subgroup of $SU(3)$; otherwise the group is not a subgroup of $SU(3)$ but it is a subgroup of $U(3)$ —because every representation of a finite group is equivalent to a representation through unitary matrices. In this way, we have separated the subgroups of $SU(3)$ from the subgroups of $U(3)$.

A complete classification of all the finite subgroups of $SU(3)$ has long existed [46–49]. There are groups (so-called type A) of diagonal matrices, i.e., Abelian groups; they may be written as direct products of cyclic factors and do not concern us here. Then there are the subgroups of $U(2)$, which are called type B; their 3D representations are (just like the ones of type A subgroups) reducible and therefore they do not concern us either. Of interest to us are the type C and type D groups, which were best characterized in Ref. [55], and also the “exceptional” groups. In this work we give the `SmallGroups` identifiers of all the $SU(3)$ subgroups of types C and D, together with their classification according to Ref. [55], and also the `SmallGroups` identifiers of the exceptional subgroups. This is done in Sect. 3.

There is no theoretical classification of all the finite subgroups of $U(3)$. We feel that having a complete listing of all those subgroups of order less than 2000, together with their generators, may be a useful step towards achieving such a classification; at the very least, it allows one to get a feeling for what it might look like. Therefore, in this work we give the `SmallGroups` identifiers of all the finite $U(3)$ subgroups, together with their generators. We also partially unite those subgroups in

² `SmallGroups` uses C_n to denote the cyclic group of order n , instead of the more usual notation \mathbb{Z}_n . `SmallGroups` uses the notation $E(n)$ for the n th root of unity.

³ `SmallGroups` informs us about the structure of each group. This is given in terms of direct products (denoted “ \times ”), semidirect products (denoted “ \rtimes ”), or group extensions (denoted “.”). A pedagogical explanation of these concepts may be found, for instance, in Ref. [53].

series, namely, in sets of groups that have related generators depending on one, two, or sometimes three integers. This is done in Sect. 4.

We also give, for every finite subgroup of $U(3)$, the dimensions of all its inequivalent irreps, as determined by GAP.

In Sect. 2 we explain our procedure. In the appendix we provide tables of all the finite subgroups of $U(3)$ that have a faithful 3D irrep and are not isomorphic to the direct product of a smaller group and a cyclic group. We give separate tables for the groups that are subgroups of $SU(3)$ and for the groups that are not subgroups of $SU(3)$. In these tables, we order the groups according to their `SmallGroups` classification, namely, in increasing order first of o and then of j in their $[o, j]$ identifiers.

2. GAP procedures

GAP [56] is a computer algebra system that provides a programming language, including many functions that implement algebraic algorithms. It is supplemented by many libraries containing a large amount of data on algebraic objects. Using GAP, it is possible to study groups and their representations, display the character tables, find the subgroups of larger groups, identify groups given through their generating matrices, and so on.

GAP allows access to the `SmallGroups` library through the `SmallGroups` package [57]. This library contains *all* the finite groups of “small” orders,⁴ namely, less than a certain upper bound, and also orders whose prime factorization is small in some sense. The groups are ordered by their orders; for each of the available orders, a complete list of nonisomorphic groups is given. `SmallGroups` contains all the groups of order less than 2000 except order 1024, because there are many thousands of millions of groups of order 1024. `SmallGroups` also contains other groups with some specific orders larger than 2000.

The `SmallGroups` library has an identification function that returns the `SmallGroups` identifier of any given group. For each generic group in the library there are effective recognition algorithms available. To identify encoded and insoluble groups, two approaches are used: one is a general algorithm to solve the isomorphism problem for p -groups,⁵ the second one uses the invariants⁶ of stored groups [58]. Using these methods, it is possible to identify all the groups in the library, except for orders 512, 1536, and some orders above 2000. For the identification of groups we use GAP command

$$\text{IdGroup}(.). \quad (1)$$

In our work, firstly we scanned the `SmallGroups` library and extracted therefrom all the groups with 3D irreps. Using the GAP command

$$G := \text{SmallGroup}([o, j]), \quad (2)$$

⁴ The order of a finite group is the number of its elements.

⁵ A p -group, where p is a prime number, is a group in which each element has a power of p as its order. That is, for each element g of a p -group, there is a nonnegative integer n such that the product of p^n copies of g , and not less, is equal to the identity element e . (However, the integer n is in general different for different elements g of the group.)

⁶ In the `SmallGroups` library there is a list of distinguishing invariants for all encoded groups except those of orders 512 and 1536. This list of invariants is compressed. It provides an efficient approach to identify any encoded group in the library.

one lets G denote the group with identifier $[o, j]$ in the `SmallGroups` library. The command

$$\text{NumberSmallGroups}(o) \quad (3)$$

allows one to find out how many groups there are for a chosen order o and thus automates the scanning of library. For a given group G , GAP offers the possibility to calculate the irreps by using the command

$$\text{repG} := \text{IrreducibleRepresentations}(G). \quad (4)$$

It is possible to display all the irreps by using the GAP command

$$\text{Display}(\text{CharacterTable}(G)) \quad (5)$$

too; however, the labeling of the irreps may differ from the labeling received through the command

$$\text{IrreducibleRepresentations}(G). \quad (6)$$

It is convenient to select all the 3D irreps by using the command

$$\text{repG3} := \text{Filtered}(\text{repG}, x \rightarrow \text{Length}(\text{Identity}(G)^x) = 3). \quad (7)$$

One may select all the elements of a given group G through the command

$$\text{elG} := \text{Elements}(G). \quad (8)$$

Then, the command

$$\text{elGlist} := \text{List}(\text{elG}, x \rightarrow x^{\text{repG3}[i]}), \quad (9)$$

where the integer i parameterizes the loop, allows one to list all the elements of the chosen irrep. We have selected the groups from the `SmallGroups` library that have at least one faithful⁷ 3D irrep. Then, by using the GAP command that gives the structure of a group, namely,

$$\text{StructureDescription}(G), \quad (10)$$

we have discarded the groups that are direct products with a cyclic group.

There are 10 494 213 groups of order 512 and 408 641 062 groups of order 1536. However, the groups of order 512 do not possess 3D irreps because 512 cannot be divided by three; therefore, we did not need to consider them. On the other hand, the number of groups of order 1536 is too large for all of them to be scanned in the way described above. Therefore, we have used the conjecture in Ref. [59] that both nilpotent groups and groups with a normal Sylow 3-subgroup⁸ do not have 3D faithful irreps. Utilizing the command

$$\text{SmallGroupsInformation}(o) \quad (11)$$

⁷ In order to identify the faithful irreps, we have compared all the matrices in each irrep. If different elements of the group are represented by different matrices in the irrep, then the irrep is faithful.

⁸ These two concepts of group theory have been explained in Ref. [60].

one obtains information about the arrangement of the groups of a given order. Using this information, we have determined the scanning range of groups of order 1536. To check whether the group is nilpotent, the command

$$\text{IsNilpotentGroup}(G) \quad (12)$$

may be used, while

$$\text{NilpotencyClassOfGroup}(G) \quad (13)$$

gives the nilpotency class of the group G . The Sylow 3-subgroups of a group G may be found by typing the command

$$\text{SylowSubgroup}(G, 3). \quad (14)$$

We have found that only four groups of order 1536 have faithful 3D irreps and cannot be written as the direct product of a smaller group and a cyclic group.

For groups that have faithful 3D irreps, we asked GAP to compute the determinant of each of the matrices in each of its 3D representations. This was done through the command

$$\text{DeterminantMat}(\text{elGlist}[i]). \quad (15)$$

If there is a 3D representation in which all the matrices have unit determinant, then the group is a subgroup of $SU(3)$; if there is no such representation, then the group is not a subgroup of $SU(3)$, but it is a subgroup of $U(3)$ because it has a 3D representation and because all the representations of finite groups are equivalent to representations through unitary matrices.

We have used different methods in order to classify the groups in the lists of subgroups of $U(3)$ and $SU(3)$. One of the methods is the analysis of the generators of the 3D irreps. The command

$$\text{genG} := \text{GeneratorsOfGroup}(G) \quad (16)$$

returns a list of generators of the group G . The generators of the 3D irreps may be listed through the command

$$\text{List}(\text{genG}, x \rightarrow x^{\wedge} \text{repG3}[i]). \quad (17)$$

By looking at these lists we have tried to find regularities in the generators. Another strategy was looking at the structures of the groups and sorting groups with analogous structures.

When one has some generators, say three matrices $M1$, $M2$, and $M3$, a group G may be generated through the command

$$G := \text{Group}([M1, M2, M3]). \quad (18)$$

Afterwards this group may be identified by finding its order, using the command

$$\text{Order}(G), \quad (19)$$

or by counting the elements of the group through

$$\text{Size}(\text{elG}). \quad (20)$$

Afterwards one may discover the `SmallGroups` identifier of G by using the command

$$\text{IdGroup}(G). \quad (21)$$

The identification of some groups with large order may require a long computational time; therefore, some hints about the group classification may be acquired by analyzing the group structure—using the command (10)—or by comparing the traces of the group matrices, determined through the command

$$\text{List}(\text{elG}, x \rightarrow \text{Trace}(x)). \quad (22)$$

3. Finite subgroups of $SU(3)$

In this section we give the generators and the `SmallGroups` identifiers of all the finite subgroups of $SU(3)$ that

- have a faithful 3D irrep,
- cannot be written as the direct product of a smaller group and a cyclic group,
- have less than 2000 elements.

3.1. Generators

We firstly define a few 3×3 matrices that act as generators of the various $SU(3)$ subgroups. All these matrices have, of course, unit determinant.

The matrices

$$E \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (23a)$$

$$I \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (23b)$$

are especially useful. Let $n \geq 1$ be an integer. Then,

$$L_n \equiv \text{diag}(1, \nu, \nu^{-1}), \quad \text{where } \nu = \exp(2i\pi/n). \quad (24)$$

Let $n \geq 1$ and $k \geq 1$ be integers. We define

$$B_{n,k} \equiv \text{diag}(\nu, \nu^k, \nu^{-1-k}), \quad \text{where } \nu = \exp(2i\pi/n). \quad (25)$$

Let $n \geq 1$ and $r \geq 1$ be integers. We define

$$G_{n,r} \equiv \text{diag}(1, \nu^{-r}, \nu^r), \quad \text{where } \nu = \exp(2i\pi/n), \quad (26)$$

i.e., $G_{n,r} = (L_n)^{-r}$.

Table 1. The SmallGroups identifiers of the groups $\Delta(3n^2)$ with order smaller than 2000.

n	2	3	4	5
identifier	[12, 3]	[27, 3]	[48, 3]	[75, 2]
n	6	7	8	9
identifier	[108, 22]	[147, 5]	[192, 3]	[243, 26]
n	10	11	12	13
identifier	[300, 43]	[363, 2]	[432, 103]	[507, 5]
n	14	15	16	17
identifier	[588, 60]	[675, 12]	[768, 1083 477]	[867, 2]
n	18	19	20	21
identifier	[972, 122]	[1083, 5]	[1200, 384]	[1323, 43]
n	22	23	24	25
identifier	[1452, 34]	[1587, 2]	[1728, 1291]	[1875, 16]

Table 2. The SmallGroups identifiers of the groups $\Delta(6n^2)$ with order smaller than 2000.

n	2	3	4	5
identifier	[24, 12]	[54, 8]	[96, 64]	[150, 5]
n	6	7	8	9
identifier	[216, 95]	[294, 7]	[384, 568]	[486, 61]
n	10	11	12	13
identifier	[600, 179]	[726, 5]	[864, 701]	[1014, 7]
n	14	15	16	17
identifier	[1176, 243]	[1350, 46]	[1536, 408 544 632]	[1734, 5]
n	18			
identifier	[1944, 849]			

3.2. The groups $\Delta(3n^2)$ and $\Delta(6n^2)$

For $n \geq 1$, the groups $\Delta(3n^2)$ have structure $(\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$ and order $3n^2$;⁹ the groups $\Delta(6n^2)$ have structure $[(\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_2$ and order $6n^2$. The group $\Delta(3n^2)$ is generated by the matrices E and L_n ; the group $\Delta(6n^2)$ is generated by the matrices E , I , and L_n . The SmallGroups identifiers of the groups $\Delta(3n^2)$ of order smaller than 2000 are given in Table 1;¹⁰ the SmallGroups identifiers of the groups $\Delta(6n^2)$ of order smaller than 2000 are given in Table 2.¹¹

The group $\Delta(3 \times 2^2)$ is isomorphic to A_4 , the group of even permutations of four objects, and also to the symmetry group of the regular tetrahedron. The group $\Delta(6 \times 2^2)$ is isomorphic to S_4 , the group of all the permutations of four objects, and also to the symmetry group of the cube and of the regular octahedron.

⁹ We adopt the convention that \mathbb{Z}_1 is the trivial group, i.e., the group that has only one element, namely, the identity element e .

¹⁰ The group $\Delta(3 \times 1^2) \cong \mathbb{Z}_3 \cong [3, 1]$ is not included in Table 1 because it is a cyclic group.

¹¹ The group $\Delta(6 \times 1^2) \cong S_3 \cong [6, 1]$ is not included in Table 2 because its 3D representations are reducible.

Table 3. The lowest values of r , and the corresponding values of k and l , that produce groups $C_{rl,l}^{(k)}$ with order $3rl^2 < 2000$.

r	3	7	13	19	21	31	37
k	1	2	3	7	4	5	10
l	3, 6, 9, 12	1 to 9	1 to 7	1 to 5	3	1 to 4	1 to 4
r	39	43	49	57	61	67	73
k	16	6	18	7	13	29	8
l	3	1, 2, 3	1, 2, 3	3	1, 2, 3	1, 2, 3	1, 2, 3
r	79	91	97	103	109	127	133
k	23	9, 16	35	46	45	19	11, 30
l	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2
r	139	151	157	163	169	181	
k	42	32	12	58	22	48	
l	1, 2	1, 2	1, 2	1, 2	1	1	

When n cannot be divided by three, the group $\Delta(3n^2)$ has three singlet irreps and $(n^2 - 1)/3$ triplet irreps; when n is a multiple of three, $\Delta(3n^2)$ has nine inequivalent singlet irreps and $n^2/3 - 1$ inequivalent triplet irreps. The group $\Delta(6n^2)$ has [43,44,46–49,61], for any $n \geq 2$, two inequivalent singlet irreps and $2(n - 1)$ inequivalent triplet irreps. When n is not a multiple of three, $\Delta(6n^2)$ has one doublet irrep and $(n - 1)(n - 2)/6$ 6D irreps; when n is a multiple of three, $\Delta(6n^2)$ has four inequivalent doublet irreps and $n(n - 3)/6$ 6D irreps.

3.3. The groups $C_{n,l}^{(k)}$

We use the notation of Ref. [55]. The groups $C_{n,l}^{(k)}$ have structure $(\mathbb{Z}_n \times \mathbb{Z}_l) \rtimes \mathbb{Z}_3$ and order $3nl$. The integer l is positive. The integer n may be written $n = rl$, where r is another positive integer. The integer r may be either

- (1) a product of prime numbers p_1, p_2, \dots that are of the form $p_j = 6i_j + 1$, where the numbers i_j are integers, or
- (2) three times a product of prime numbers as in Eq. (1).

In case (1), l may be any positive integer; in case (2), l must be a multiple of three. The integer k is a function of r defined by $1 + k + k^2 = 0 \pmod{r}$ and $k \leq (r - 1)/2$. For most values of r there is only one possible k , but for some r more than one (usually two) k are possible. The values of r , k , and l that produce groups $C_{n,l}^{(k)}$ with order smaller than 2000 are given in Tables 3 and 4. There is a very large number of groups $C_{n,l}^{(k)}$ of order smaller than 2000; therefore, we opt for giving their SmallGroups identifiers only in the appendix.

The generators of $C_{n,l}^{(k)}$ are the matrices E in Eq. (23a), $B_{n,k}$ in Eq. (25), and $G_{n,r}$ in Eq. (26), where $r = n/l$.¹²

The groups $C_{n,l}^{(k)}$ only have singlet and triplet irreps. The number of inequivalent singlet irreps is three when l cannot be divided by three and nine when l is a multiple of three.

¹² For almost all the groups $C_{n,l}^{(k)}$ of order smaller than 2000, the third generator $G_{n,r}$ is not really needed, i.e., one may generate the group by using solely E and $B_{n,k}$.

Table 4. Continuation of Table 3: other values of r and k that produce groups $C_{r,l}^{(k)}$ with order $3r < 2000$. (For the values of r in this table, only $l = 1$ produces group orders smaller than 2000.)

r	193	199	211	217	223	229
k	84	92	14	25, 67	39	94
r	241	247	259	271	277	283
k	15	68, 87	100, 121	28	116	44
r	301	307	313	331	337	343
k	79, 135	17	98	31	128	18
r	349	361	367	373	379	397
k	122	68	83	88	51	34
r	403	409	421	427	433	439
k	87, 191	53	20	74, 135	198	171
r	457	463	469	481	487	499
k	133	21	37, 163	100, 211	232	139
r	511	523	541	547	553	559
k	81, 137	60	129	40	23, 102	165, 178
r	571	577	589	601	607	613
k	109	213	87, 273	24	210	65
r	619	631	637	643	661	
k	252	43	165, 263	177	296	

3.4. The groups $D_{3l,l}^{(1)}$

We continue to use the notation of Ref. [55]. For an integer l that is a multiple of three, the groups $D_{3l,l}^{(1)}$ have structure $[(\mathbb{Z}_{3l} \times \mathbb{Z}_l) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_2$ and order $18l^2$. They are generated by the matrices E , I , and $B_{3l,1} = \text{diag}(\nu, \nu, \nu^{-2})$ for $\nu = \exp[2i\pi/(3l)]$. There are only three groups $D_{3l,l}^{(1)}$ of order smaller than 2000:

$$D_{9,3}^{(1)} \cong [162, 14], \quad (27a)$$

$$D_{18,6}^{(1)} \cong [648, 259], \quad (27b)$$

$$D_{27,9}^{(1)} \cong [1458, 659]. \quad (27c)$$

The groups $D_{3l,l}^{(1)}$ have six inequivalent singlets and three inequivalent doublets for any value of l . In addition, they have $6(l-1)$ inequivalent triplet irreps and $l(l-3)/2 + 1$ inequivalent six-plets.

3.5. The exceptional subgroups of $SU(3)$

The groups $\Delta(3n^2)$ and $C_{n,l}^{(k)}$ form the class C of finite subgroups of $SU(3)$. The groups $\Delta(6n^2)$ and $D_{3l,l}^{(1)}$ form the class D of finite subgroups of $SU(3)$. Both classes C and D contain infinite numbers of subgroups. Besides these infinite classes of subgroups, $SU(3)$ has six “exceptional” subgroups,^{13,14} their SmallGroups identifiers are given in Table 5. The generators of the exceptional subgroups

¹³ The groups $\Sigma(36 \times 1)$, $\Sigma(72 \times 1)$, $\Sigma(216 \times 1)$, and $\Sigma(360 \times 1)$ are subgroups of $PSU(3)$, i.e., of $SU(3)$ divided by its \mathbb{Z}_3 center. They are *not* subgroups of $SU(3)$.

¹⁴ The group $\Sigma(60)$ is in fact a subgroup of $SO(3)$, i.e., it may be represented through real 3×3 matrices.

Table 5. The `SmallGroups` identifiers of the exceptional subgroups of $SU(3)$.

$\Sigma(60)$	$\Sigma(36 \times 3)$	$\Sigma(168)$	$\Sigma(72 \times 3)$	$\Sigma(216 \times 3)$	$\Sigma(360 \times 3)$
[60, 5]	[108, 15]	[168, 42]	[216, 88]	[648, 532]	[1080, 260]

Table 6. The number of inequivalent p -dimensional irreducible representations of the exceptional subgroups of $SU(3)$.

group	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$\Sigma(60)$	1	0	2	1	1	0
$\Sigma(36 \times 3)$	4	0	8	2	0	0
$\Sigma(168)$	1	0	2	0	0	1
$\Sigma(72 \times 3)$	4	1	8	0	0	2
$\Sigma(216 \times 3)$	3	3	7	0	0	6
$\Sigma(360 \times 3)$	1	0	4	0	2	2
group	$p = 7$	$p = 8$	$p = 9$	$p = 10$	$p = 15$	
$\Sigma(60)$	0	0	0	0	0	
$\Sigma(36 \times 3)$	0	0	0	0	0	
$\Sigma(168)$	1	1	0	0	0	
$\Sigma(72 \times 3)$	0	1	0	0	0	
$\Sigma(216 \times 3)$	0	3	2	0	0	
$\Sigma(360 \times 3)$	0	2	3	1	2	

are given, for instance, in Ref. [50] and references therein.

The group $\Sigma(60)$ is isomorphic to A_5 , the group of even permutations of five objects, and to the symmetry group of the regular icosahedron and regular dodecahedron. The group $\Sigma(168)$ is isomorphic to the projective special linear group $PSL(2, 7)$ and also to the general linear group $GL(3, 2)$.

The number of inequivalent p -dimensional irreps of the exceptional finite subgroups of $SU(3)$ is given in Table 6 [62].

4. Finite subgroups of $U(3)$

In this section we give the generators and the `SmallGroups` identifiers of all the finite subgroups of $U(3)$ that

- are not subgroups of $SU(3)$,
- have a faithful 3D irrep,
- cannot be written as the direct product of a smaller group and a cyclic group,
- have less than 2000 elements.

For most groups, we also give the numbers of inequivalent irreps of each dimension.

There is at present no mathematical classification of the finite subgroups of $U(3)$. Therefore, we will just classify the various subgroups that we have found using the `SmallGroups` library and GAP, by constructing “series” of subgroups that have generators, structures, and numbers of irreps related among themselves. Unfortunately, there is some degree of ambiguity in this task, since any group may always be generated by different sets of generators. It is moreover often found that groups

Table 7. The lowest possible values of r and the corresponding values of k .

r	7	13	19	31	37	43	49	61	67
k	2	3	7	5	10	6	18	13	29
r	73	79	91	97	103	109	127	133	139
k	8	23	9, 16	35	46	45	19	11, 30	42
r	151	157	163	169	181	193	199	211	217
k	32	12	58	22	48	84	92	14	25, 67

with related generators end up having quite different structures. Still, we hope to be able to shed some light on the possible types of subgroups of $U(3)$.

4.1. The generators

We firstly define some 3×3 matrices that often appear as generators of the $U(3)$ subgroups.

Let

- r be a product of prime numbers p_1, p_2, \dots that are of the form $p_j = 6i_j + 1$, where the numbers i_j are integers;
- k be an integer that is a function of r defined by $1 + k + k^2 = 0 \pmod{r}$ and $k \leq (r-1)/2$. For most values of r there is only one possible k , but for some r more than one k is possible.

The lowest r and the corresponding k are given in Table 7. In this section, whenever we let r and k denote a pair of integers, we will be referring to one of the pairs in Table 7. The matrix

$$B_{r,k} = \text{diag}(\rho, \rho^k, \rho^{-1-k}), \quad \text{where } \rho = \exp(2i\pi/r), \quad (28)$$

appears as generator of many $U(3)$ subgroups. Notice that $B_{r,k} \in SU(3)$.

We use the definition of L_n in Eq. (24). Notice that $L_n \in SU(3)$. The matrix

$$L_2 = \text{diag}(1, -1, -1) \quad (29)$$

is especially useful. We will also encounter

$$L_3 = \text{diag}(1, \omega, \omega^2), \quad \text{where } \omega = \exp(2i\pi/3). \quad (30)$$

Let m be an integer. We define

$$E_m \equiv \begin{pmatrix} 0 & \mu & 0 \\ 0 & 0 & \mu \\ \mu & 0 & 0 \end{pmatrix}, \quad (31a)$$

$$Z_m \equiv \begin{pmatrix} 0 & 0 & \mu \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (31b)$$

$$T_1(m) \equiv \text{diag}(1, \mu, \mu^2), \quad (31c)$$

$$T_2(m) \equiv \text{diag}(1, \mu^2, \mu), \quad \text{where } \mu = \exp[2i\pi / (3^m)]. \quad (31d)$$

The matrix $E \equiv E_0$ in Eq. (23a) is especially useful. Both E_0 and E_1 have unit determinant, but $E_m \notin SU(3)$ for $m > 1$.

Let m and j be integers. We define

$$F_{m,j} \equiv \begin{pmatrix} 0 & 0 & -\xi \\ 0 & -\xi & 0 \\ -\xi & 0 & 0 \end{pmatrix}, \quad \text{where } \xi = \exp[2i\pi / (3^m 2^j)]. \quad (32)$$

Notice that $F_{m,j} \notin SU(3)$ for $m \geq 2$ or $j \geq 1$. The matrix $I \equiv F_{0,0}$ in Eq. (23b) has already been useful; also useful is

$$I' \equiv F_{0,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -I. \quad (33)$$

Let $\omega = \exp(2i\pi/3)$ and $\mu = \exp[2i\pi / (3^m)]$. We define

$$X_1(m) \equiv \text{diag}(\mu\omega, \mu\omega, \mu\omega^2), \quad (34a)$$

$$X_2(m) \equiv \text{diag}(\mu\omega^2, \mu\omega^2, \mu\omega), \quad (34b)$$

$$Y_1(m) \equiv \text{diag}(\mu, \mu\omega, \mu\omega^2), \quad (34c)$$

$$Y_2(m) \equiv \text{diag}(\mu, \mu\omega^2, \mu\omega), \quad (34d)$$

$$X_3(m) = Y_3(m) \equiv \text{diag}(\mu, \mu, \mu). \quad (34e)$$

Let $\omega = \exp(2i\pi/3)$. We define

$$K \equiv \frac{-i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \quad (35)$$

Notice that $K \in SU(3)$. Furthermore, let $\xi = \exp[2i\pi / (3^m 2^j)]$. We define

$$Q_{m,j} \equiv \frac{-i\xi}{\sqrt{3}} \begin{pmatrix} 1 & \omega^2 & \omega^2 \\ \omega^2 & \omega^2 & 1 \\ 1 & \omega & 1 \end{pmatrix}. \quad (36)$$

Notice that $\det Q_{m,j} = \xi^3 \neq 1$ in general.

4.2. The series of groups discovered by Ludl

Ludl [50] has proved the existence of the following series of finite subgroups of $U(3)$.

Groups $T_r^{(k)}(m)$ The group $T_r^{(k)}(m)$, where m is an integer larger than 1,¹⁵ has structure $\mathbb{Z}_r \rtimes \mathbb{Z}_{3^m}$ and order $3^m r$. The groups $T_r^{(k)}(m)$ of order smaller than 2000 are given in Table 8. Each of these groups has two generators, which may be chosen to be $B_{r,k}$ in Eq. (28) and E_m in Eq. (31a).

The groups $T_r^{(k)}(m)$ have 3^m inequivalent singlet irreps; all the remaining irreps of these groups are triplets.

¹⁵ If $m = 1$, then $T_r^{(k)}(1) \cong C_{r,1}^{(k)}$ is a subgroup of $SU(3)$.

Table 8. The SmallGroups identifiers of the groups $T_r^{(k)}(m)$ with order smaller than 2000.

$T_r^{(k)}(m)$	$T_7^{(2)}(2)$	$T_7^{(2)}(3)$	$T_7^{(2)}(4)$	$T_7^{(2)}(5)$	$T_{13}^{(3)}(2)$
identifier	[63, 1]	[189, 1]	[567, 1]	[1701, 68]	[117, 1]
$T_r^{(k)}(m)$	$T_{13}^{(3)}(3)$	$T_{13}^{(3)}(4)$	$T_{19}^{(7)}(2)$	$T_{19}^{(7)}(3)$	$T_{19}^{(7)}(4)$
identifier	[351, 1]	[1053, 16]	[171, 1]	[513, 1]	[1539, 16]
$T_r^{(k)}(m)$	$T_{31}^{(5)}(2)$	$T_{31}^{(5)}(3)$	$T_{37}^{(10)}(2)$	$T_{37}^{(10)}(3)$	$T_{43}^{(6)}(2)$
identifier	[279, 1]	[837, 1]	[333, 1]	[999, 1]	[387, 1]
$T_r^{(k)}(m)$	$T_{43}^{(6)}(3)$	$T_{49}^{(18)}(2)$	$T_{49}^{(18)}(3)$	$T_{61}^{(13)}(2)$	$T_{61}^{(13)}(3)$
identifier	[1161, 6]	[441, 1]	[1323, 1]	[549, 1]	[1647, 6]
$T_r^{(k)}(m)$	$T_{67}^{(29)}(2)$	$T_{67}^{(29)}(3)$	$T_{73}^{(8)}(2)$	$T_{73}^{(8)}(3)$	$T_{79}^{(23)}(2)$
identifier	[603, 1]	[1809, 6]	[657, 1]	[1971, 6]	[711, 1]
$T_r^{(k)}(m)$	$T_{91}^{(9)}(2)$	$T_{91}^{(16)}(2)$	$T_{97}^{(35)}(2)$	$T_{103}^{(46)}(2)$	$T_{109}^{(45)}(2)$
identifier	[819, 4]	[819, 3]	[873, 1]	[927, 1]	[981, 1]
$T_r^{(k)}(m)$	$T_{127}^{(19)}(2)$	$T_{133}^{(11)}(2)$	$T_{133}^{(30)}(2)$	$T_{139}^{(42)}(2)$	$T_{151}^{(32)}(2)$
identifier	[1143, 1]	[1197, 3]	[1197, 4]	[1251, 1]	[1359, 1]
$T_r^{(k)}(m)$	$T_{157}^{(12)}(2)$	$T_{163}^{(58)}(2)$	$T_{169}^{(22)}(2)$	$T_{181}^{(48)}(2)$	$T_{193}^{(84)}(2)$
identifier	[1413, 1]	[1467, 1]	[1521, 1]	[1629, 1]	[1737, 1]
$T_r^{(k)}(m)$	$T_{199}^{(92)}(2)$	$T_{211}^{(14)}(2)$	$T_{217}^{(25)}(2)$	$T_{217}^{(67)}(2)$	
identifier	[1791, 1]	[1899, 1]	[1953, 3]	[1953, 4]	

Table 9. The SmallGroups identifiers of the groups $\Delta(3n^2, m)$ with order smaller than 2000.

n, m	2, 2	2, 3	2, 4	2, 5	
identifier	[36, 3]	[108, 3]	[324, 3]	[972, 3]	
n, m	4, 2	4, 3	4, 4	5, 2	5, 3
identifier	[144, 3]	[432, 3]	[1296, 3]	[225, 3]	[675, 5]
n, m	7, 2	7, 3	8, 2	8, 3	10, 2
identifier	[441, 7]	[1323, 14]	[576, 3]	[1728, 3]	[900, 66]
n, m	11, 2	13, 2	14, 2		
identifier	[1089, 3]	[1521, 7]	[1764, 91]		

Groups $\Delta(3n^2, m)$ The group $\Delta(3n^2, m)$, where the integer n cannot be divided by 3 and $m > 1$,¹⁶ has structure $(\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_{3m}$ and order $3^m n^2$. The groups $\Delta(3n^2, m)$ of order less than 2000 are listed in Table 9. The group $\Delta(3n^2, m)$ is generated by the matrices L_n in Eq. (24) and E_m in Eq. (31a).

The groups $\Delta(3n^2, m)$ have 3^m inequivalent singlet irreps; all the remaining irreps of these groups are triplets.

Groups $S_4(j)$ The group $S_4(j)$, where $j > 1$,¹⁷ has structure $A_4 \rtimes \mathbb{Z}_{2j}$ and order $3 \times 2^{j+2}$.¹⁸ There are six groups $S_4(j)$ of order smaller than 2000; they are given in Table 10. The group $S_4(j)$ is

¹⁶ If $m = 1$, then $\Delta(3n^2, 1) \cong \Delta(3n^2)$ is a subgroup of $SU(3)$.

¹⁷ The group $S_4(1) \cong \Delta(6 \times 2^2)$ is a subgroup of $SU(3)$.

¹⁸ The group A_4 has structure $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$.

Table 10. The SmallGroups identifiers of the groups $S_4(j)$ with order smaller than 2000.

j	2	3	4
identifier	[48, 30]	[96, 65]	[192, 186]
j	5	6	7
identifier	[384, 581]	[768, 1085 351]	[1536, 408 544 687]

Table 11. The SmallGroups identifiers of the groups $\Delta(6n^2, j)$ with order smaller than 2000.

n, j	3, 2	3, 3	3, 4
identifier	[108, 11]	[216, 17]	[432, 33]
n, j	3, 5	3, 6	4, 2
identifier	[864, 69]	[1728, 185]	[192, 182]
n, j	4, 3	4, 4	4, 5
identifier	[384, 571]	[768, 1085 333]	[1536, 408 544 678]
n, j	5, 2	5, 3	5, 4
identifier	[300, 13]	[600, 45]	[1200, 183]
n, j	6, 2	6, 3	6, 4
identifier	[432, 260]	[864, 703]	[1728, 2855]
n, j	7, 2	7, 3	8, 2
identifier	[588, 16]	[1176, 57]	[768, 1085 335]
n, j	8, 3	9, 2	9, 3
identifier	[1536, 408 544 641]	[972, 64]	[1944, 70]
n, j	10, 2	11, 2	12, 2
identifier	[1200, 682]	[1452, 11]	[1728, 2847]

generated by the matrices E in Eq. (23a), L_2 in Eq. (29), and $-F_{0,j}$, where $F_{m,j}$ is given in Eq. (32).

The group $S_4(j)$ has 2^j inequivalent singlet irreps, 2^{j-1} inequivalent doublet irreps, 2^j inequivalent triplet irreps, and no other irreps.

Groups $\Delta(6n^2, j)$ The group $\Delta(6n^2, j)$, where¹⁹ $n > 1$ and $j > 1$,²⁰ has structure $[(\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_{2j}$ and order $3 \times 2^j n^2$. But, for $n = 2$, $\Delta(6n^2, j) = S_4(j)$; therefore, we only need to take into account $n \geq 3$; there are then the 24 groups $\Delta(6n^2, j)$ with order less than 2000 given in Table 11. The generators of $\Delta(6n^2, j)$ are the matrices E in Eq. (23a), L_n in Eq. (24), and $-F_{0,j}$. It is clear that $\Delta(6n^2, j)$ is just a generalization of $S_4(j)$ for $n > 2$.

The groups $\Delta(6n^2, j)$ have 2^j inequivalent singlet irreps and $2^j(n-1)$ inequivalent triplet irreps for any value of n . When n cannot be divided by three, these groups have, in addition, 2^{j-1} doublet irreps; when n is a multiple of three, the number of inequivalent doublet irreps is 2^{j+1} . All the remaining irreps of these groups are six-plets.

¹⁹ In Ludl's paper the existence of $\Delta(6n^2, j)$ has been proved for integers n not divisible by 3. We have verified, though, that $\Delta(6n^2, j)$ exists for every $n > 1$, at least when $3 \times 2^j n^2 < 2000$.

²⁰ The groups $\Delta(6n^2, 1) \cong \Delta(6n^2)$ are subgroups of $SU(3)$.

Table 12. SmallGroups identifiers of the groups $\Delta' (6n^2, m, j)$ with order smaller than 2000.

n, m, j	3, 2, 1	3, 2, 2	3, 2, 3	3, 2, 4
identifier	[162, 44]	[324, 102]	[648, 244]	[1296, 647]
n, m, j	3, 3, 1	3, 3, 2	3, 3, 3	3, 4, 1
identifier	[486, 164]	[972, 348]	[1944, 746]	[1458, 1354]
n, m, j	6, 2, 1	6, 2, 2	6, 3, 1	9, 2, 1
identifier	[648, 563]	[1296, 2113]	[1944, 2415]	[1458, 1371]

Table 13. SmallGroups identifiers of the groups $L_r^{(k)}(n, m)$ with order smaller than 2000.

$L_r^{(k)}(n, m)$	$L_7^{(2)}(2, 2)$	$L_7^{(2)}(2, 3)$	$L_7^{(2)}(4, 2)$	$L_7^{(2)}(5, 2)$
identifier	[252, 11]	[756, 11]	[1008, 57]	[1575, 7]
$L_r^{(k)}(n, m)$	$L_{13}^{(3)}(2, 2)$	$L_{13}^{(3)}(2, 3)$	$L_{13}^{(3)}(4, 2)$	$L_{19}^{(7)}(2, 2)$
identifier	[468, 14]	[1404, 14]	[1872, 60]	[684, 11]
$L_r^{(k)}(n, m)$	$L_{31}^{(5)}(2, 2)$	$L_{37}^{(10)}(2, 2)$	$L_{43}^{(6)}(2, 2)$	$L_{49}^{(18)}(2, 2)$
identifier	[1116, 11]	[1332, 14]	[1548, 11]	[1764, 11]

Groups $\Delta' (6n^2, m, j)$ These groups, where n can be divided by 3, $m \geq 2$,²¹ and $j \geq 1$, have structure²² $[(\mathbb{Z}_{3^{m-1}n} \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_{2j}$ and order $3^m 2^j n^2$. There are 12 groups with order less than 2000 in Table 12. The generators of $\Delta' (6n^2, m, j)$ are the matrices E in Eq. (23a), L_n in Eq. (24), and $-F_{m,j}$ in Eq. (32).

The groups $\Delta' (6n^2, m, j)$ have $3^{m-1} 2^j$ inequivalent singlet irreps and $3^{m-1} 2^{j+1}$ inequivalent doublet irreps. There are also $(n-1) 3^{m-1} 2^j$ inequivalent triplets; the remaining irreps are six-plets.

4.3. New series of groups that we have discovered

Ludl [50] derived the existence of the series of groups in the previous subsection by applying mathematical theorems that he demonstrated. We have discovered some further series of groups through a careful inspection of the list of all the finite subgroups of $U(3)$ of order smaller than 2000 that we have produced, together with some guesswork. Clearly, since there are no theorems supporting our method, we cannot be sure that our series of groups extend to groups of order larger than 2000. Still, the series of groups in this subsection seem to us to be on firm standing, since they are quite large and display no exceptions up to group order 2000.

Groups $L_r^{(k)}(n, m)$ For an integer n that cannot be divided by 3 and for $m > 1$, these are groups with structure $(\mathbb{Z}_{rn} \times \mathbb{Z}_n) \rtimes \mathbb{Z}_{3^m}$ and order $3^m r n^2$. While the groups $T_r^{(k)}(m)$ are generated by the matrices $B_{r,k}$ and E_m , and the groups $\Delta(3n^2, m)$ are generated by the matrices L_n and E_m , the groups $L_r^{(k)}(n, m)$ are generated by all three matrices $B_{r,k}$, L_n , and E_m . Thus, the groups $L_r^{(k)}(n, m)$ simultaneously generalize $T_r^{(k)}(m) = L_r^{(k)}(1, m)$ and $\Delta(3n^2, m) = L_1^{(0)}(n, m)$. The groups $L_r^{(k)}(n, m)$ of order smaller than 2000 are listed in Table 13.

²¹ The groups $\Delta' (6n^2, 1, j)$ are the same as the groups $\Delta(6n^2, j)$.

²² The exception is $\Delta' (6 \times 9^2, 2, 1)$, which has structure $[(\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_2$ instead of $[(\mathbb{Z}_{27} \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_2$.

Table 14. The SmallGroups identifiers of the groups $P_r^{(k)}(m)$, $Q_r^{(k)}(m)$, and $Q_r^{(k)'}(m)$ with order smaller than 2000.

$P_r^{(k)}(m)$	$\binom{(2)}{7}(2)$	$\binom{(3)}{13}(2)$	$\binom{(7)}{19}(2)$	$\binom{(5)}{31}(2)$	$\binom{(10)}{37}(2)$
$Q_r^{(k)}(m)$	[189, 7]	[351, 7]	[513, 8]	[837, 7]	[999, 8]
$Q_r^{(k)'}(m)$	[189, 4]	[351, 4]	[513, 5]	[837, 5]	[999, 6]
$Q_r^{(k)'}(m)$	[189, 5]	[351, 5]	[513, 6]	[837, 4]	[999, 5]
$P_r^{(k)}(m)$	$\binom{(6)}{43}(2)$	$\binom{(18)}{49}(2)$	$\binom{(13)}{61}(2)$	$\binom{(29)}{67}(2)$	$\binom{(8)}{73}(2)$
$Q_r^{(k)}(m)$	[1161, 12]	[1323, 7]	[1647, 17]	[1809, 17]	[1971, 17]
$Q_r^{(k)'}(m)$	[1161, 10]	[1323, 4]	[1647, 10]	[1809, 10]	[1971, 11]
$Q_r^{(k)'}(m)$	[1161, 11]	[1323, 5]	[1647, 11]	[1809, 11]	[1971, 10]
$P_r^{(k)}(m)$	$\binom{(2)}{7}(3)$	$\binom{(3)}{13}(3)$	$\binom{(7)}{19}(3)$	$\binom{(2)}{7}(4)$	
$Q_r^{(k)}(m)$	[567, 7]	[1053, 27]	[1539, 27]	[1701, 128]	
$Q_r^{(k)'}(m)$	[567, 4]	[1053, 26]	[1539, 26]	[1701, 127]	
$Q_r^{(k)'}(m)$	[567, 5]	[1053, 25]	[1539, 25]	[1701, 126]	

The groups $L_r^{(k)}(n, m)$ have 3^m inequivalent singlets; the remaining irreps are triplets.

Groups $P_r^{(k)}(m)$, $Q_r^{(k)}(m)$, and $Q_r^{(k)'}(m)$ These groups exist for integer $m > 1$ and have order $3^{m+1}r$. The groups $P_r^{(k)}(m)$ have structure $(\mathbb{Z}_r \times \mathbb{Z}_{3^m}) \rtimes \mathbb{Z}_3$; the groups $Q_r^{(k)}(m)$ and $Q_r^{(k)'}(m)$ have structure $\mathbb{Z}_{3^m r} \rtimes \mathbb{Z}_3$. The groups of order smaller than 2000 are listed in Table 14. The group $P_r^{(k)}(m)$ is generated by $B_{r,k}$ together with L_3 and Z_{m-1} . The groups $Q_r^{(k)}(m)$ and $Q_r^{(k)'}(m)$ are generated by the matrices $B_{r,k}$ and E together with $Y_1(m)$ for $Q_r^{(k)}(m)$ or $Y_2(m)$ for $Q_r^{(k)'}(m)$.

The groups $P_r^{(k)}(m)$, $Q_r^{(k)}(m)$, and $Q_r^{(k)'}(m)$ have 3^m inequivalent singlets; all their remaining irreps are triplets.

Groups $X(n)$ There are several groups that have a 3D irrep where all the matrices are of one of the following types [50]:

$$R(n, a, b, c) \equiv \begin{pmatrix} 0 & 0 & v^a \\ v^b & 0 & 0 \\ 0 & v^c & 0 \end{pmatrix}, \quad (37a)$$

$$V(n, a, b, c) \equiv \begin{pmatrix} 0 & v^a & 0 \\ 0 & 0 & v^b \\ v^c & 0 & 0 \end{pmatrix}, \quad (37b)$$

$$W(n, a, b, c) \equiv \begin{pmatrix} v^a & 0 & 0 \\ 0 & v^b & 0 \\ 0 & 0 & v^c \end{pmatrix}, \quad (37c)$$

where $v = \exp(2i\pi/n)$. We call them “groups RVW”. The groups $X(n)$ are groups RVW where

- n is a multiple of 3,
- the matrices $R(n, a, b, c)$ have $a + b + c = (n/3) \bmod n$,
- the matrices $V(n, a, b, c)$ have $a + b + c = (2n/3) \bmod n$,
- the matrices $W(n, a, b, c)$ have $a + b + c = 0 \bmod n$.

Table 15. The SmallGroups identifiers of the groups $X(n)$ with order smaller than 2000.

n	3	6	9	12
identifier	[27, 4]	[108, 21]	[243, 27]	[432, 102]
n	15	18	21	24
identifier	[675, 11]	[972, 123]	[1323, 42]	[1728, 1290]

Table 16. The SmallGroups identifiers of the groups $S_r^{(k)}(m)$, $S_r^{(k)'}(m)$, $Y_r^{(k)}(m)$, and $V_r^{(k)}(m)$ with order smaller than 2000.

$S_7^{(2)}(2)$	$S_{13}^{(3)}(2)$	$S_{19}^{(7)}(2)$	$S_7^{(2)}(3)$
[567, 36]	[1053, 47]	[1539, 47]	[1701, 240]
$S_7^{(2)'}(2)$	$S_{13}^{(3)'}(2)$	$S_{19}^{(7)'}(2)$	$S_7^{(2)'}(3)$
[567, 12]	[1053, 32]	[1539, 32]	[1701, 115]
$Y_7^{(2)}(2)$	$Y_{13}^{(3)}(2)$	$Y_{19}^{(7)}(2)$	$Y_7^{(2)}(3)$
[567, 23]	[1053, 29]	[1539, 29]	[1701, 261]
$V_7^{(2)}(2)$	$V_{13}^{(3)}(2)$	$V_{19}^{(7)}(2)$	$V_7^{(2)}(3)$
[567, 14]	[1053, 37]	[1539, 37]	[1701, 138]

The groups $X(n)$ have order $3n^2$; the identifiers of the groups of order less than 2000 are shown in Table 15. The structure of $X(n)$ is $[(\mathbb{Z}_{n/3} \times \mathbb{Z}_{n/3}) \rtimes \mathbb{Z}_9] \rtimes \mathbb{Z}_3$ provided n is not a multiple of 9; otherwise it is more complicated. The groups $X(n)$ are generated by the matrices L_n in Eq. (24) and Z_1 in Eq. (31b).

The groups $X(n)$ have nine inequivalent singlets; their remaining irreps are all triplets.

4.4. Tentative series of groups

We have found a few more series of groups through inspection of the list of the finite subgroups of $U(3)$ of order less than 2000. However, these series have few groups each and we can hardly ascertain whether and how they extend to groups of order larger than 2000.

Groups $S_r^{(k)}(m)$, $S_r^{(k)'}(m)$, $Y_r^{(k)}(m)$, and $V_r^{(k)}(m)$ These groups exist for $m \geq 2$. The groups $S_r^{(k)}(m)$ and $S_r^{(k)'}(m)$ have structure $(\mathbb{Z}_{3m} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$; the groups $Y_r^{(k)}(m)$ have structure $(\mathbb{Z}_{3m-1} \times \mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$; the groups $V_r^{(k)}(m)$ have structure $\mathbb{Z}_r \rtimes [(\mathbb{Z}_{3m-1} \times \mathbb{Z}_3) \cdot (\mathbb{Z}_3 \times \mathbb{Z}_3)]$; they all have order $3^{m+2}r$. The generators are the matrices $B_{r,k}$ in Eq. (28), together with

- E in Eq. (23a), L_3 in Eq. (30), and $X_3(m)$ in Eq. (34e) for $S_r^{(k)}(m)$;
- E , L_3 , and $X_1(m)$ in Eq. (34a) for $S_r^{(k)'}(m)$;
- E , $X_1(m-1)$, and $X_3(m-1)$ for $Y_r^{(k)}(m)$;
- $X_2(2)$ in Eq. (34b), Z_1 in Eq. (31b), and L_{3m-1} for $V_r^{(k)}(m)$.

The groups $S_r^{(k)}(m)$, $S_r^{(k)'}(m)$, $Y_r^{(k)}(m)$, and $V_r^{(k)}(m)$ of order less than 2000 are shown in Table 16.

The groups $S_r^{(k)}(m)$ have 3^{m+1} inequivalent singlets. The groups $S_r^{(k)'}(m)$ and $Y_r^{(k)}(m)$ have 3^m inequivalent singlets. The groups $V_r^{(k)}(m)$ have nine inequivalent singlets. All the remaining irreps of all these groups are triplets.

Table 17. The SmallGroups identifiers of the groups $M_r^{(k)}$, $M_r^{(k)'}$, and $J_r^{(k)}$ with order smaller than 2000.

$M_7^{(2)}$	$M_7^{(2)'}$	$J_7^{(2)}$	$M_{13}^{(3)}$	$M_{13}^{(3)'}$	$J_{13}^{(3)}$
[756, 113]	[756, 114]	[756, 116]	[1404, 137]	[1404, 138]	[1404, 140]

Table 18. SmallGroups identifiers of the groups $W(n, m)$ with order smaller than 2000.

n, m	1, 2	1, 3	1, 4	1, 5
identifier	[27, 4]	[81, 6]	[243, 24]	[729, 94]
n, m	2, 2	2, 3	2, 4	4, 2
identifier	[108, 19]	[324, 43]	[972, 117]	[432, 100]
n, m	4, 3	5, 2	7, 2	8, 2
identifier	[1296, 220]	[675, 9]	[1323, 40]	[1728, 1286]

Groups $M_r^{(k)}$, $M_r^{(k)'}$, and $J_r^{(k)}$ These groups have order $108r$. The groups $M_r^{(k)}$ and $M_r^{(k)'}$ have structure $(\mathbb{Z}_{18r} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$; the groups $J_r^{(k)}$ have structure $(\mathbb{Z}_{2r} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9 \rtimes \mathbb{Z}_3$. The generators are the matrices $B_{r,k}$ in Eq. (28) and L_2 in Eq. (29) together with

- E in Eq. (23a) and $Y_1(2)$ in Eq. (34c) for $M_r^{(k)}$,
- E and $Y_2(2)$ in Eq. (34d) for $M_r^{(k)'}$,
- L_3 in Eq. (30) and Z_1 in Eq. (31b) for $J_r^{(k)}$.

The groups $M_r^{(k)}$, $M_r^{(k)'}$, and $J_r^{(k)}$ of order less than 2000 are shown in Table 17.

Each of the groups $M_r^{(k)}$, $M_r^{(k)'}$, and $J_r^{(k)}$ has nine inequivalent singlets. All the remaining irreps of these groups are triplets.

Groups $W(n, m)$ The groups $W(n, m)$, where n cannot be divided by 3 and $m > 1$, are generated by the matrices E in Eq. (23a), L_n in Eq. (24), and $Y_1(m)$ in Eq. (34c). They have structure $(\mathbb{Z}_{3^m n} \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$ and order $3^{m+1}n^2$. The groups $W(n, m)$ with order smaller than 2000 are listed in Table 18.

Each of the groups $W(n, m)$ has 3^m inequivalent singlets; the remaining irreps of these groups are triplets.

Groups $Z(n, m)$, $Z'(n, m)$, and $Z''(n, m)$ These groups, where n is a multiple of 3 and $m > 1$, have structure²³ $(\mathbb{Z}_{3^{m-1}n} \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$ and order $3^m n^2$. The groups with order smaller than 2000 are listed in Table 19. The generators of $Z(n, m)$ are just the same as those of $W(n, m)$, namely, E , L_n , and $Y_1(m)$, the only difference being that for $Z(n, m)$ the integer n is a multiple of 3 while for $W(n, m)$ the integer n cannot be divided by 3. The groups $Z'(n, m)$ are generated by the matrices E , L_n , and $X_1(m)$. The groups $Z''(n, m)$ are generated by the matrices E , L_n , and $X_2(m)$. Notice that, for $m = 2$, $Z''(n, m)$ is generated by matrices with unit determinant and therefore it is a subgroup of $SU(3)$.

Each of the groups $Z(n, m)$ and $Z''(n, m)$ has 3^{m+1} inequivalent singlets. The groups $Z'(n, m)$ have 3^m inequivalent singlets. All the remaining irreps of all these groups are triplets.

²³ An exception are the groups $Z(9, 2)$ and $Z'(9, 2)$, which happen to be isomorphic and are of the form $(\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ instead of $(\mathbb{Z}_{27} \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3$.

Table 19. SmallGroups identifiers of the groups $Z(n, m)$, $Z'(n, m)$, and $Z''(n, m)$ with order smaller than 2000.

n, m	3, 2	6, 2	9, 2	12, 2
$Z(n, m)$	[81, 14]	[324, 128]	[729, 397]	[1296, 1499]
$Z'(n, m)$	[81, 8]	[324, 49]	[729, 397]	[1296, 227]
$Z''(n, m)$	$C_{9,3}^{(1)}$	$C_{18,6}^{(1)}$	$\Delta(3 \times 9^2)$	$C_{36,12}^{(1)}$
n, m	3, 3	6, 3	3, 4	
$Z(n, m)$	[243, 50]	[972, 520]	[729, 393]	
$Z'(n, m)$	[243, 20]	[972, 152]	[729, 64]	
$Z''(n, m)$	[243, 19]	[972, 153]	[729, 63]	

Table 20. The SmallGroups identifiers of the groups $Z(n, m, j)$ and $Z'(n, m, j)$ with order smaller than 2000.

n, m, j	3, 2, 1	3, 2, 2	3, 2, 3	3, 2, 4
$Z(n, m, j)$	[162, 12]	[324, 15]	[648, 21]	[1296, 37]
$Z'(n, m, j)$	$D_{9,3}^{(1)}$	[324, 17]	[648, 23]	[1296, 39]
n, m, j	3, 3, 1	3, 3, 2	3, 3, 3	3, 4, 1
$Z(n, m, j)$	[486, 28]	[972, 31]	[1944, 37]	[1458, 618]
$Z'(n, m, j)$	[486, 26]	[972, 29]	[1944, 35]	[1458, 615]
n, m, j	6, 2, 1	6, 2, 2	6, 3, 1	
$Z(n, m, j)$	[648, 260]	[1296, 689]	[1944, 833]	
$Z'(n, m, j)$	$D_{18,6}^{(1)}$	[1296, 688]	[1944, 832]	

Groups $Z(n, m, j)$ and $Z'(n, m, j)$ These groups, where

- n is a multiple of 3,
- $m > 1$,
- j is an integer,

have order $3^m 2^j n^2$. The groups $Z(n, m, j)$ and $Z'(n, m, j)$ with order smaller than 2000 are shown in Table 20.²⁴ The groups $Z(n, m, j)$ and $Z'(n, m, j)$ are generated by the same matrices as the groups $Z'(n, m)$ and $Z''(n, m)$, respectively, with the addition of the further generator $-F_{1,j}$, where $F_{m,j}$ is given in Eq. (32). Notice that there are no groups $Z'(n, 2, 1)$ in Table 20, because all the matrices generating $Z'(n, 2, 1)$, namely, E , L_n , $X_2(2)$, and $-F_{1,1}$, have unit determinant and therefore $Z'(n, 2, 1)$ is a subgroup of $SU(3)$.

The groups $Z(n, m, j)$ and $Z'(n, m, j)$ have the same numbers of irreps of each dimension: $3^{m-1} 2^j$ inequivalent singlet irreps, $3^{m-1} 2^{j-1}$ inequivalent doublet irreps, $(n-1) 3^{m-1} 2^j$ triplet irreps, and $(n-1)(n-2) 3^{m-2} 2^{j-2}$ six-plet irreps.

Groups $H(n, m, j)$ When we use generators E , L_n , $X_1(2)$, and $-F_{m,j}$ with $m > 1$, we obtain groups that we call $H(n, m, j)$, listed in Table 21. The groups $H(n, m, 1)$ have structure

²⁴ The group $Z(9, 2, 1)$ is isomorphic to the group $\Delta'(6 \times 9^2, 2, 1)$ and we omit it from Table 20, since it has numbers of irreps quite inconsistent with those of the other groups $Z(n, m, j)$.

Table 21. The SmallGroups identifiers of the groups $H(n, m, j)$ of order smaller than 2000.

n, m, j	3, 2, 1	3, 2, 2	3, 2, 3	3, 3, 1	6, 2, 1
identifier	[486, 125]	[972, 309]	[1944, 707]	[1458, 1095]	[1944, 2363]

Table 22. The SmallGroups identifiers of the groups $G(m, j)$ with order smaller than 2000.

m, j	1, 2	2, 2	1, 3	2, 3	1, 4
identifier	[324, 13]	[972, 309]	[648, 19]	[1944, 707]	[1296, 35]

$[(\mathbb{Z}_{3^{m-1}n} \times \mathbb{Z}_n \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_2$ and order $3^{m+1} \times 2n^2$. The groups $H(n, m, j)$ with $j > 1$ are described in the paragraph on groups $G(m, j)$ below.

The groups $H(n, m, j)$ have exactly the same number of inequivalent irreps of each dimension as the groups $Z(n, m+1, j)$ and $Z'(n, m+1, j)$.

Groups $Y(m, j)$ The groups $Y(m, j)$, where $m \geq 2$ and $j \geq 1$, have structure $[(\mathbb{Z}_{2^j} \times \mathbb{Z}_{2^j}) \rtimes \mathbb{Z}_{3^{m+1}}] \rtimes \mathbb{Z}_3$ and order $3^{m+2} 4^j$. There are only three groups $Y(m, j)$ of order smaller than 2000:

$$Y(2, 1) \cong [324, 45], \quad (38a)$$

$$Y(3, 1) \cong [972, 147], \quad (38b)$$

$$Y(2, 2) \cong [1296, 222]. \quad (38c)$$

The groups $Y(m, j)$ are generated by L_3 in Eq. (30), L_{2^j} in Eq. (24), and Z_m in Eq. (31b).

The groups $Y(m, j)$ only have singlet and triplet irreps: 3^{m+1} inequivalent singlets and $3^m 4^j - 3^{m-1}$ inequivalent triplets.

Groups $G(m, j)$ and [1296, 699] The groups $G(m, j)$, where $m \geq 1$ and $j \geq 2$, have structure $\{[(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_{2^j}\} \rtimes \mathbb{Z}_3$ and order $3^{m+3} 2^j$. The groups $G(m, j)$ of order smaller than 2000 are shown in Table 22. (Notice that the groups [972, 309] and [1944, 707] appear in Table 21 too.) The groups $G(m, j)$ are generated by the matrices $E, -F_{m,j}$, where $F_{m,j}$ is given in Eq. (32), and $\text{diag}(1, 1, \omega)$. For $m = 1$ and $j = 2$ one may add a fourth generator L_2 , given in Eq. (29), to obtain the group [1296, 699], which has structure $\{[(\mathbb{Z}_6 \times \mathbb{Z}_6) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_4\} \rtimes \mathbb{Z}_3$.

The groups $G(m, j)$ have exactly the same number of inequivalent irreps of each dimension as the groups $Z(3, m+1, j)$ and $Z'(3, m+1, j)$.

Groups $Y(j)$ and $\tilde{Y}(j)$ The groups $Y(j)$ have order 81×4^j and structure $(\mathbb{Z}_{3 \times 2^j} \times \mathbb{Z}_{3 \times 2^j} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. There are three groups $Y(j)$ with order smaller than 2000:

$$[81, 7] \text{ with } j = 0, \quad (39a)$$

$$[324, 60] \text{ with } j = 1, \quad (39b)$$

$$[1296, 237] \text{ with } j = 2. \quad (39c)$$

The group $Y(0)$ coincides with the group $\Sigma(81)$ or $\Sigma(3 \times 3^3)$ of Refs. [51,52]. The generators of $Y(j)$ are the matrix E of Eq. (23a) together with the matrix

$$\text{diag}(\xi, \xi, \xi^2), \quad \text{where } \xi = \exp[2i\pi / (3 \times 2^j)]. \quad (40)$$

The groups $\tilde{Y}(j)$ have structure $[(\mathbb{Z}_{3 \times 2^j} \times \mathbb{Z}_{3 \times 2^j} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_2$ and order 162×4^j . There are two groups $\tilde{Y}(j)$ with order smaller than 2000:

$$[162, 10] \text{ with } j = 0, \quad (41a)$$

$$[648, 266] \text{ with } j = 1. \quad (41b)$$

The generators of $\tilde{Y}(j)$ are those of $Y(j)$ together with the additional matrix I' in Eq. (33).

The groups $Y(j)$ have nine inequivalent singlet irreps; all their remaining irreps are triplets. The groups $\tilde{Y}(j)$ have six singlet and three doublet irreps; their remaining irreps are 12 triplets and one six-plet for $\tilde{Y}(0)$, 30 triplets and ten six-plets for $\tilde{Y}(1)$.

Groups $U(n, m, j)$ The groups $U(n, m, j)$, where n is a multiple of 3, $m > 1$, and $1 < j \leq m$, have structure $(\mathbb{Z}_{3^{m-1}n} \times \mathbb{Z}_n \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ and order $3^{m+1}n^2$. We have found the following groups $U(n, m, j)$ with order smaller than 2000:

$$[243, 55] \text{ with } n = 3, m = 2, j = 2, \quad (42a)$$

$$[729, 86] \text{ with } n = 3, m = 3, j = 2, \quad (42b)$$

$$[729, 284] \text{ with } n = 3, m = 3, j = 3, \quad (42c)$$

$$[972, 550] \text{ with } n = 6, m = 2, j = 2. \quad (42d)$$

The generators of $U(n, m, j)$ are the matrix E together with

$$\text{diag}(\nu, \nu, \nu^2), \quad \text{where } \nu = \exp(2i\pi / n), \quad (43)$$

and

$$\mu T_1(m - j + 1), \quad \text{where } \mu = \exp[2i\pi / (3^m)]. \quad (44)$$

Notice that, when $j = m$, which happens in three out of the four groups $U(n, m, j)$ in Eqs. (42), the matrix (44) reduces to the matrix $Y_1(m)$ in Eq. (34c).

The groups $U(n, m, j)$ possess 3^{j+1} inequivalent singlet irreps; all their remaining irreps are triplets.

Groups $L(m)$ and [1701, 102] The groups $L(m)$ have order 3^{m+3} and structure $(\mathbb{Z}_{3^{m+1}} \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. They are generated by the matrices $X_1(2)$, Z_m , and L_3 . There are the following groups $L(m)$ of order smaller than 2000:

$$L(2) \cong [243, 16], \quad (45a)$$

$$L(3) \cong [729, 62]. \quad (45b)$$

Adding the matrix $B_{7,2}$ to the matrices Z_2 , $X_1(2)$, and L_3 generates a group with structure $(\mathbb{Z}_7 \rtimes \mathbb{Z}_{27}) \rtimes \mathbb{Z}_3 \rtimes \mathbb{Z}_3$ and SmallGroups identifier [1701, 102].

The groups $L(m)$ have 3^{m+1} singlet irreps; their other irreps are triplets.

Groups $V(j)$ The groups $V(j)$ have order 81×4^j and structure

$$(\mathbb{Z}_{2^j} \times \mathbb{Z}_{2^j}) \rtimes \{\mathbb{Z}_3 \cdot [(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] = (\mathbb{Z}_3 \times \mathbb{Z}_3) \cdot (\mathbb{Z}_3 \times \mathbb{Z}_3)\}. \quad (46)$$

There are three groups $V(j)$ with order smaller than 2000:

$$V(0) \cong [81, 10], \quad (47a)$$

$$V(1) \cong [324, 51], \quad (47b)$$

$$V(2) \cong [1296, 226]. \quad (47c)$$

The generators of $V(j)$ are the matrices Z_1 , $X_2(2)$, and L_{2^j} .

The groups $V(j)$ have nine singlet irreps. All their other irreps are triplets.

Groups $D(j)$ The groups $D(j)$ have structure $(\mathbb{Z}_{9 \times 2^j} \times \mathbb{Z}_{9 \times 2^j}) \rtimes \mathbb{Z}_3$ and order 243×4^j . They are generated by the matrices E_2 , L_{2^j} , and $T_1(2)$. There are two groups of order smaller than 2000:

$$D(0) \cong [243, 25], \quad (48a)$$

$$D(1) \cong [972, 121]. \quad (48b)$$

Both these groups have nine inequivalent singlets; their other irreps are triplets.

Groups $J(m)$ The groups $J(m)$ have structure $\mathbb{Z}_{3^m} \cdot [(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3]$ and order 81×3^m . They are generated by the matrices Z_m and L_9 . There are two groups of order smaller than 2000:

$$J(1) \cong [243, 27], \quad (49a)$$

$$J(2) \cong [729, 80]. \quad (49b)$$

Notice that $J(1)$ coincides with $X(9)$ in Table 15. The groups $J(m)$ have 3^{m+1} singlets; their other irreps are triplets.

4.5. The generators of a few more groups

In this subsection we collect a few more groups together with their generators.

Three groups of order 729 Both groups [729, 97] and [729, 98] have structure $(\mathbb{Z}_{27} \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3$. Group [729, 96] has the more complicated structure $\mathbb{Z}_3 \cdot [(\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3]$. They are generated by the matrix Z_1 together with

$$\hat{\mu} \text{ diag } (\tilde{\mu}^2, \omega, \omega) \text{ for } [729, 96], \quad (50a)$$

$$\hat{\mu} \text{ diag } (\omega, \omega\tilde{\mu}, \omega\tilde{\mu}) \text{ for } [729, 97], \quad (50b)$$

$$\hat{\mu} \text{ diag } (\omega^2, \omega\tilde{\mu}, \omega\tilde{\mu}) \text{ for } [729, 98], \quad (50c)$$

where $\omega = \exp(2i\pi/3)$, $\tilde{\mu} = \exp(2i\pi/9)$, and $\hat{\mu} = \exp(2i\pi/27)$.

Each of these three groups of order 729 possesses nine singlet and 80 triplet irreps.

A group of order 972 The group [972, 170] is generated by the matrices L_2 , Z_2 , and $\text{diag}(1, 1, \omega)$. It has structure $\{[(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_{27}] \rtimes \mathbb{Z}_3\} \rtimes \mathbb{Z}_3$.

Two groups of order 1458 The groups [1458, 663] and [1458, 666] have structure $[(\mathbb{Z}_{27} \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_2$. They are generated by the matrices E, L_3, I , and (50b) for group [1458, 663], (50c) for group [1458, 666].²⁵ Each of these groups has six singlets, three doublets, 48 triplets, and 28 six-plets.

Three groups of order 1701 The groups with SmallGroups identifiers [1701, 112], [1701, 130], and [1701, 131] have structure $(\mathbb{Z}_{7 \times 3^m} \times \mathbb{Z}_{3^n}) \rtimes \mathbb{Z}_3$, where $m = 3$ and $n = 1$ for [1701, 112] and $m = n = 2$ for [1701, 130] and [1701, 131]. They are generated by the matrices $B_{7,2}, E, X_2(m)$, and

$$T_1(n) \text{ for [1701, 112] and [1701, 130],} \quad (51a)$$

$$T_2(n) \text{ for [1701, 131].} \quad (51b)$$

4.6. Other finite subgroups of $U(3)$

It is clear from the forms of the generators of the groups of matrices in Sects. 4.2–4.5 that those groups are formed by matrices that are all of one of the forms $R(n, a, b, c)$, $V(n, a, b, c)$, $W(n, a, b, c)$ in Eqs. (37), or, possibly, also of the forms [50]

$$S(n, a, b, c) \equiv \begin{pmatrix} \nu^a & 0 & 0 \\ 0 & 0 & \nu^b \\ 0 & \nu^c & 0 \end{pmatrix}, \quad (52a)$$

$$T(n, a, b, c) \equiv \begin{pmatrix} 0 & 0 & \nu^a \\ 0 & \nu^b & 0 \\ \nu^c & 0 & 0 \end{pmatrix}, \quad (52b)$$

$$U(n, a, b, c) \equiv \begin{pmatrix} 0 & \nu^a & 0 \\ \nu^b & 0 & 0 \\ 0 & 0 & \nu^c \end{pmatrix}, \quad (52c)$$

where $\nu = \exp(2i\pi/n)$, for some value of n . It can also be seen that groups RVW only have singlet and triplet irreps, while groups that include matrices of types (52) also have doublet and six-plet irreps; no group in Sects. 4.2–4.5 has irreps of any other dimension. There are, however, finite subgroups of $U(3)$ that possess no 3D faithful irrep consisting solely of matrices of the forms (37) and (52). Those groups have irreps of dimensions other than just one, two, three, and six; they are analogous to the exceptional subgroups of $SU(3)$. We present in this section the $U(3)$ subgroups of that type that have order smaller than 2000.

Groups $\Xi(m, j)$ and $\hat{\Xi}(m, j)$ The groups $\Xi(m, j)$, where $m \geq 1$ and $j \geq 2$, have structure $[(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_{2^j}$ and order $3^{m+2} 2^j$. They are generated by the matrices E in Eq. (23a) and $iQ_{m,j}$, where $Q_{m,j}$ is the matrix in Eq. (36). The groups $\Xi(m, j)$ of order smaller than 2000 are shown in Table 23. Notice that, since $\det(iQ_{m,j}) = -i \exp[2i\pi / (3^{m-1} 2^j)]$, the group $\Xi(1, 2) \cong [108, 15] \cong \Sigma(36 \times 3)$ is a subgroup of $SU(3)$.

The groups $\hat{\Xi}(m, j)$ have structure $\{[(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_{2^j}\} \rtimes \mathbb{Z}_2$ and order $3^{m+2} 2^{j+1}$. The generators of $\hat{\Xi}(m, j)$ are the same as the generators of $\Xi(m, j)$ together with the additional matrix

²⁵ Using Eq. (50a) leads to the group [1458, 659] $\cong D_{27,9}^{(1)}$, which is a subgroup of $SU(3)$.

Table 23. The SmallGroups identifiers of the groups $\Xi(m, j)$ with order smaller than 2000.

m, j	1, 2	1, 3	1, 4	1, 5	1, 6
identifier	$\Sigma(36 \times 3)$	[216, 25]	[432, 57]	[864, 194]	[1728, 953]
m, j	2, 2	2, 3	2, 4	3, 2	3, 3
identifier	[324, 111]	[648, 352]	[1296, 1239]	[972, 411]	[1944, 1123]

Table 24. The SmallGroups identifiers of the groups $\hat{\Xi}(m, j)$ with order smaller than 2000.

m, j	1, 3	1, 4	1, 5	2, 3
identifier	[432, 273]	[864, 737]	[1728, 2929]	[1296, 2203]

Table 25. The SmallGroups identifiers of the groups $\Pi(m, j)$ with order smaller than 2000.

m, j	1, 2	1, 3	1, 4	2, 2
identifier	[432, 239]	[864, 675]	[1728, 2785]	[1296, 1995]

I' in Eq. (33). The groups $\hat{\Xi}(m, j)$ of order smaller than 2000 are shown in Table 24. Notice that the groups $\hat{\Xi}(m, 2)$ have structure $\{[(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_4\} \times \mathbb{Z}_2$; these groups may be written as *direct* products of \mathbb{Z}_2 and smaller groups, hence they are not included in Table 24.

The groups $\Xi(m, j)$ possess $3^{m-1}2^j$ inequivalent singlet irreps, $3^{m-1}2^{j+1}$ inequivalent triplet irreps, and $3^{m-1}2^{j-1}$ inequivalent quadruplet irreps. The groups $\hat{\Xi}(m, j)$ have twice as many irreps of each dimension as the groups $\Xi(m, j)$.

Groups $\Pi(m, j)$ The groups $\Pi(m, j)$, where $m \geq 1$ and $j \geq 2$, have structure $\{[(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_4\} \rtimes \mathbb{Z}_{2^j}$ and order $3^{m+2}2^{j+2}$. They are generated by the matrices E , K , and $Q_{m,j}$. The groups $\Pi(m, j)$ of order smaller than 2000 are shown in Table 25.

The groups $\Pi(m, j)$ have $3^{m-1}2^{j+1}$ singlet irreps, $3^{m-1}2^{j-1}$ doublet irreps, $3^{m-1}2^{j+2}$ triplet irreps, $3^{m-1}2^j$ six-plet irreps, and $3^{m-1}2^{j-1}$ eight-plet irreps.

Groups $\Theta(m)$ These groups have structure $[(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes Q_8$, where Q_8 is the quaternion group. Since Q_8 has eight elements, the order of $\Theta(m)$ is 72×3^m . The generators of $\Theta(m)$ are E , K , and either $Q_{m,1}$ or $Q_{m,0}$. Since $\det Q_{m,0} = \exp[2i\pi / (3^{m-1})]$, the group $\Theta(1)$ is a subgroup of $SU(3)$. Notice that the groups $\Theta(m)$ have the same generators as hypothetical groups $\Pi(m, 1)$ would have had; but they have a slightly different structure. There are three groups $\Theta(m)$ of order smaller than 2000:

$$\Theta(1) \cong \Sigma(72 \times 3), \quad (53a)$$

$$\Theta(2) \cong [648, 551], \quad (53b)$$

$$\Theta(3) \cong [1944, 2333]. \quad (53c)$$

The groups $\Theta(m)$ have as many inequivalent irreps of each dimension as groups $\Pi(m, 1)$.

Groups $\Upsilon(m)$ and $\Upsilon'(m)$ These groups have structure

$$\{[(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes Q_8\} \cdot \mathbb{Z}_{3^{m-1}} = \mathbb{Z}_{3^{m-1}} \cdot \{[(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8] \rtimes \mathbb{Z}_3\} \quad (54)$$

and order 72×3^m . The generators are E , $Q_{0,0}$, and $X_1(m)$ for $\Upsilon(m)$ or $X_2(m)$ for $\Upsilon'(m)$. There are the following groups of order smaller than 2000:

$$\Upsilon(2) \cong [648, 531], \quad (55a)$$

$$\Upsilon(3) \cong [1944, 2293], \quad (55b)$$

$$\Upsilon'(2) \cong \Sigma(216 \times 3), \quad (55c)$$

$$\Upsilon'(3) \cong [1944, 2294]. \quad (55d)$$

Notice that all three generators of $\Upsilon'(2)$ have unit determinant and therefore $\Upsilon'(2)$ is a subgroup of $SU(3)$.

The groups $\Upsilon(m)$ and $\Upsilon'(m)$ have 3^{m-1} singlets, 3^{m-1} doublets, $7 \times 3^{m-2}$ triplets, $2 \times 3^{m-1}$ six-plets, 3^{m-1} eight-plets, and $2 \times 3^{m-2}$ nine-plets.

Groups $\Omega(m)$ These groups have structure $\{[(\mathbb{Z}_{3^m} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3] \rtimes Q_8\} \rtimes \mathbb{Z}_3$ and order $72 \times 3^{m+1}$. They are generated by the matrices $Q_{m,0}$ and Z_1 . There are the following groups of order smaller than 2000:

$$\Omega(1) \cong [648, 533], \quad (56a)$$

$$\Omega(2) \cong [1944, 3448]. \quad (56b)$$

The groups $\Omega(m)$ have exactly as many inequivalent irreps of each dimension as the groups $\Upsilon(m+1)$ and $\Upsilon'(m+1)$.

5. Conclusion

In this paper we have used the `SmallGroups` library to search for all the finite subgroups of $U(3)$ of order less than 2000 that have a faithful 3D irreducible representation and that cannot be written as the direct product of some smaller group and a cyclic group. We have found that there are three types of finite subgroups of $U(3)$:

- Groups that have a 3D representation consisting solely of matrices of the forms (37) for some value of n . Those groups only have singlet and triplet irreducible representations.
- Groups that have a 3D representation consisting solely of matrices of the forms (37) and (52) for some value of n . Those groups only have singlet, doublet, triplet, and six-plet irreducible representations.
- Groups that do *not* have a 3D representation consisting solely of matrices of the forms (37) and (52). Those groups have irreducible representations of other dimensions, such as, for instance, four-plets, eight-plets, or nine-plets. Their generators include matrices $Q_{m,j}$ and possibly K in Eqs. (35), (36). These groups include as special cases the exceptional $SU(3)$ subgroups $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, and $\Sigma(216 \times 3)$.²⁶

We were able to group most finite subgroups of $U(3)$ in many series depending on one, two, or sometimes three integers; the groups in each series have related generators and related numbers of irreps of each dimension. Unfortunately, many of these series have very few groups and we do not

²⁶ It seems likely to us that the $SU(3)$ subgroup $\Sigma(360 \times 3)$ is also a special case of a series of $U(3)$ subgroups; the other groups of that series, though, surely have order larger than 2000.

know whether and how they extend to groups of order higher than 2000. It is possible (and it would be desirable) that some of these series may be further unified among themselves.

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Appendix. Full tables

In this appendix we present tables of all the groups of order smaller than 2000 that have a faithful 3D irrep and that cannot be written as the direct product of some smaller group and a cyclic group. The groups are ordered according to increasing values of firstly o and then j in their `SmallGroups` identifier $[o, j]$. Tables A1–A7 show the groups that are subgroups of $SU(3)$; Tables A8–A18 show the groups that are not subgroups of $SU(3)$.

Table A1. The finite subgroups of $SU(3)$. Part 1: groups with order through 201.

Identifier	Classification
[12, 3]	$\Delta(3 \times 2^2)$
[21, 1]	$C_{7,1}^{(2)}$
[24, 12]	$\Delta(6 \times 2^2)$
[27, 3]	$\Delta(3 \times 3^2)$
[39, 1]	$C_{13,1}^{(3)}$
[48, 3]	$\Delta(3 \times 4^2)$
[54, 8]	$\Delta(6 \times 3^2)$
[57, 1]	$C_{19,1}^{(7)}$
[60, 5]	$\Sigma(60)$
[75, 2]	$\Delta(3 \times 5^2)$
[81, 9]	$C_{9,3}^{(1)}$
[84, 11]	$C_{14,2}^{(2)}$
[93, 1]	$C_{31,1}^{(5)}$
[96, 64]	$\Delta(6 \times 4^2)$
[108, 15]	$\Sigma(36 \times 3)$
[108, 22]	$\Delta(3 \times 6^2)$
[111, 1]	$C_{37,1}^{(10)}$
[129, 1]	$C_{43,1}^{(6)}$
[147, 1]	$C_{49,1}^{(18)}$
[147, 5]	$\Delta(3 \times 7^2)$
[150, 5]	$\Delta(6 \times 5^2)$
[156, 14]	$C_{26,2}^{(3)}$
[162, 14]	$D_{9,3}^{(1)}$
[168, 42]	$\Sigma(168)$
[183, 1]	$C_{61,1}^{(13)}$
[189, 8]	$C_{21,3}^{(2)}$
[192, 3]	$\Delta(3 \times 8^2)$
[201, 1]	$C_{67,1}^{(29)}$

Table A2. The finite subgroups of $SU(3)$. Part 2: groups with $216 \leq \text{order} \leq 486$.

Identifier	Classification
[216, 88]	$\Sigma (72 \times 3)$
[216, 95]	$\Delta (6 \times 6^2)$
[219, 1]	$C_{73,1}^{(8)}$
[228, 11]	$C_{38,2}^{(7)}$
[237, 1]	$C_{79,1}^{(23)}$
[243, 26]	$\Delta (3 \times 9^2)$
[273, 3]	$C_{91,1}^{(16)}$
[273, 4]	$C_{91,1}^{(9)}$
[291, 1]	$C_{97,1}^{(35)}$
[294, 7]	$\Delta (6 \times 7^2)$
[300, 43]	$\Delta (3 \times 10^2)$
[309, 1]	$C_{103,1}^{(46)}$
[324, 50]	$C_{18,6}^{(1)}$
[327, 1]	$C_{109,1}^{(45)}$
[336, 57]	$C_{28,4}^{(2)}$
[351, 8]	$C_{39,3}^{(3)}$
[363, 2]	$\Delta (3 \times 11^2)$
[372, 11]	$C_{62,2}^{(5)}$
[381, 1]	$C_{127,1}^{(19)}$
[384, 568]	$\Delta (6 \times 8^2)$
[399, 3]	$C_{133,1}^{(11)}$
[399, 4]	$C_{133,1}^{(30)}$
[417, 1]	$C_{139,1}^{(42)}$
[432, 103]	$\Delta (3 \times 12^2)$
[444, 14]	$C_{74,2}^{(10)}$
[453, 1]	$C_{151,1}^{(32)}$
[471, 1]	$C_{157,1}^{(12)}$
[486, 61]	$\Delta (6 \times 9^2)$

Table A3. The finite subgroups of $SU(3)$. Part 3: groups with $489 \leq \text{order} \leq 756$.

Identifier	Classification
[489, 1]	$C_{163,1}^{(58)}$
[507, 1]	$C_{169,1}^{(22)}$
[507, 5]	$\Delta(3 \times 13^2)$
[513, 9]	$C_{57,3}^{(7)}$
[516, 11]	$C_{86,2}^{(6)}$
[525, 5]	$C_{35,5}^{(2)}$
[543, 1]	$C_{181,1}^{(48)}$
[567, 13]	$C_{63,3}^{(4)}$
[579, 1]	$C_{193,1}^{(84)}$
[588, 11]	$C_{98,2}^{(18)}$
[588, 60]	$\Delta(3 \times 14^2)$
[597, 1]	$C_{199,1}^{(92)}$
[600, 179]	$\Delta(6 \times 10^2)$
[624, 60]	$C_{52,4}^{(3)}$
[633, 1]	$C_{211,1}^{(14)}$
[648, 259]	$D_{18,6}^{(1)}$
[648, 532]	$\Sigma(216 \times 3)$
[651, 3]	$C_{217,1}^{(25)}$
[651, 4]	$C_{217,1}^{(67)}$
[669, 1]	$C_{223,1}^{(39)}$
[675, 12]	$\Delta(3 \times 15^2)$
[687, 1]	$C_{229,1}^{(94)}$
[723, 1]	$C_{241,1}^{(15)}$
[726, 5]	$\Delta(6 \times 11^2)$
[729, 95]	$C_{27,9}^{(1)}$
[732, 14]	$C_{122,2}^{(13)}$
[741, 3]	$C_{247,1}^{(87)}$
[741, 4]	$C_{247,1}^{(68)}$
[756, 117]	$C_{42,6}^{(2)}$

Table A4. The finite subgroups of $SU(3)$. Part 4: groups with $768 \leq \text{order} \leq 1080$.

Identifier	Classification
[768, 1083 477]	$\Delta (3 \times 16^2)$
[777, 3]	$C_{259,1}^{(121)}$
[777, 4]	$C_{259,1}^{(100)}$
[804, 11]	$C_{134,2}^{(29)}$
[813, 1]	$C_{271,1}^{(28)}$
[831, 1]	$C_{277,1}^{(116)}$
[837, 8]	$C_{93,3}^{(5)}$
[849, 1]	$C_{283,1}^{(44)}$
[864, 701]	$\Delta (6 \times 12^2)$
[867, 2]	$\Delta (3 \times 17^2)$
[876, 14]	$C_{146,2}^{(8)}$
[903, 5]	$C_{301,1}^{(135)}$
[903, 6]	$C_{301,1}^{(79)}$
[912, 57]	$C_{76,4}^{(7)}$
[921, 1]	$C_{307,1}^{(17)}$
[939, 1]	$C_{313,1}^{(98)}$
[948, 11]	$C_{158,2}^{(23)}$
[972, 122]	$\Delta (3 \times 18^2)$
[975, 5]	$C_{65,5}^{(3)}$
[993, 1]	$C_{331,1}^{(31)}$
[999, 9]	$C_{111,3}^{(10)}$
[1011, 1]	$C_{337,1}^{(128)}$
[1014, 7]	$\Delta (6 \times 13^2)$
[1029, 6]	$C_{343,1}^{(18)}$
[1029, 9]	$C_{49,7}^{(2)}$
[1047, 1]	$C_{349,1}^{(122)}$
[1053, 35]	$C_{117,3}^{(16)}$
[1080, 260]	$\Sigma (360 \times 3)$

Table A5. The finite subgroups of $SU(3)$. Part 5: groups with $1083 \leq \text{order} \leq 1389$.

Identifier	Classification
[1083, 1]	$C_{361,1}^{(68)}$
[1083, 5]	$\Delta(3 \times 19^2)$
[1092, 68]	$C_{182,2}^{(9)}$
[1092, 69]	$C_{182,2}^{(16)}$
[1101, 1]	$C_{367,1}^{(83)}$
[1119, 1]	$C_{373,1}^{(88)}$
[1137, 1]	$C_{379,1}^{(51)}$
[1161, 9]	$C_{129,3}^{(6)}$
[1164, 14]	$C_{194,2}^{(35)}$
[1176, 243]	$\Delta(6 \times 14^2)$
[1191, 1]	$C_{397,1}^{(34)}$
[1200, 384]	$\Delta(3 \times 20^2)$
[1209, 3]	$C_{403,1}^{(87)}$
[1209, 4]	$C_{403,1}^{(191)}$
[1227, 1]	$C_{409,1}^{(53)}$
[1236, 11]	$C_{206,2}^{(46)}$
[1263, 1]	$C_{421,1}^{(20)}$
[1281, 3]	$C_{427,1}^{(135)}$
[1281, 4]	$C_{427,1}^{(74)}$
[1296, 228]	$C_{36,12}^{(1)}$
[1299, 1]	$C_{433,1}^{(198)}$
[1308, 14]	$C_{218,2}^{(45)}$
[1317, 1]	$C_{439,1}^{(171)}$
[1323, 8]	$C_{147,3}^{(18)}$
[1323, 43]	$\Delta(3 \times 21^2)$
[1344, 393]	$C_{56,8}^{(2)}$
[1350, 46]	$\Delta(6 \times 15^2)$
[1371, 1]	$C_{457,1}^{(133)}$
[1389, 1]	$C_{463,1}^{(21)}$

Table A6. The finite subgroups of $SU(3)$. Part 6: groups with $1404 \leq \text{order} \leq 1701$.

Identifier	Classification
[1404, 141]	$C_{78,6}^{(3)}$
[1407, 3]	$C_{469,1}^{(163)}$
[1407, 4]	$C_{469,1}^{(37)}$
[1425, 5]	$C_{95,5}^{(7)}$
[1443, 3]	$C_{481,1}^{(100)}$
[1443, 4]	$C_{481,1}^{(211)}$
[1452, 34]	$\Delta(3 \times 22^2)$
[1458, 659]	$D_{27,9}^{(1)}$
[1461, 1]	$C_{487,1}^{(232)}$
[1488, 57]	$C_{124,4}^{(5)}$
[1497, 1]	$C_{499,1}^{(139)}$
[1524, 11]	$C_{254,2}^{(19)}$
[1533, 3]	$C_{511,1}^{(137)}$
[1533, 4]	$C_{511,1}^{(81)}$
[1536, 408 544 632]	$\Delta(6 \times 16^2)$
[1539, 35]	$C_{171,3}^{(7)}$
[1569, 1]	$C_{523,1}^{(60)}$
[1587, 2]	$\Delta(3 \times 23^2)$
[1596, 55]	$C_{266,2}^{(11)}$
[1596, 56]	$C_{266,2}^{(30)}$
[1623, 1]	$C_{541,1}^{(129)}$
[1641, 1]	$C_{547,1}^{(40)}$
[1647, 9]	$C_{183,3}^{(13)}$
[1659, 3]	$C_{553,1}^{(23)}$
[1659, 4]	$C_{553,1}^{(102)}$
[1668, 11]	$C_{278,2}^{(42)}$
[1677, 3]	$C_{559,1}^{(165)}$
[1677, 4]	$C_{559,1}^{(178)}$
[1701, 135]	$C_{63,9}^{(2)}$

Table A7. The finite subgroups of $SU(3)$. Part 7: groups with $1713 \leq \text{order} < 2000$.

Identifier	Classification
[1713, 1]	$C_{571,1}^{(109)}$
[1728, 1291]	$\Delta(3 \times 24^2)$
[1731, 1]	$C_{577,1}^{(213)}$
[1734, 5]	$\Delta(6 \times 17^2)$
[1767, 3]	$C_{589,1}^{(87)}$
[1767, 4]	$C_{589,1}^{(273)}$
[1776, 60]	$C_{148,4}^{(10)}$
[1803, 1]	$C_{601,1}^{(24)}$
[1809, 9]	$C_{201,3}^{(29)}$
[1812, 11]	$C_{302,2}^{(32)}$
[1821, 1]	$C_{607,1}^{(210)}$
[1839, 1]	$C_{613,1}^{(65)}$
[1857, 1]	$C_{619,1}^{(252)}$
[1875, 16]	$\Delta(3 \times 25^2)$
[1884, 14]	$C_{314,2}^{(12)}$
[1893, 1]	$C_{631,1}^{(43)}$
[1911, 3]	$C_{637,1}^{(165)}$
[1911, 4]	$C_{637,1}^{(263)}$
[1911, 14]	$C_{91,7}^{(3)}$
[1929, 1]	$C_{643,1}^{(177)}$
[1944, 849]	$\Delta(6 \times 18^2)$
[1956, 11]	$C_{326,2}^{(58)}$
[1971, 9]	$C_{219,3}^{(8)}$
[1983, 1]	$C_{661,1}^{(296)}$

Table A8. The finite subgroups of $U(3)$. Part 1: groups with order ≤ 225 .

Identifier	Classification
[27, 4]	$X(3), W(1, 2)$
[36, 3]	$\Delta(3 \times 2^2, 2)$
[48, 30]	$S_4(2)$
[63, 1]	$T_7^{(2)}(2)$
[81, 6]	$W(1, 3)$
[81, 7]	$Y(0), \Sigma(3 \times 3^3)$
[81, 8]	$Z'(3, 2)$
[81, 10]	$V(0)$
[81, 14]	$Z(3, 2)$
[96, 65]	$S_4(3)$
[108, 3]	$\Delta(3 \times 2^2, 3)$
[108, 11]	$\Delta(6 \times 3^2, 2)$
[108, 19]	$W(2, 2)$
[108, 21]	$X(6)$
[117, 1]	$T_{13}^{(3)}(2)$
[144, 3]	$\Delta(3 \times 4^2, 2)$
[162, 10]	$\tilde{Y}(0)$
[162, 12]	$Z(3, 2, 1)$
[162, 44]	$\Delta'(6 \times 3^2, 2, 1)$
[171, 1]	$T_{19}^{(7)}(2)$
[189, 1]	$T_7^{(2)}(3)$
[189, 4]	$Q_7^{(2)}(2)$
[189, 5]	$Q_7^{(2)'}(2)$
[189, 7]	$P_7^{(2)}(2)$
[192, 182]	$\Delta(6 \times 4^2, 2)$
[192, 186]	$S_4(4)$
[216, 17]	$\Delta(6 \times 3^2, 3)$
[216, 25]	$\Xi(1, 3)$
[225, 3]	$\Delta(3 \times 5^2, 2)$

Table A9. The finite subgroups of $U(3)$. Part 2: groups with $243 \leq \text{order} \leq 384$.

Identifier	Classification
[243, 16]	$L(2)$
[243, 19]	$Z''(3, 3)$
[243, 20]	$Z'(3, 3)$
[243, 24]	$W(1, 4)$
[243, 25]	$D(0)$
[243, 27]	$X(9), J(1)$
[243, 50]	$Z(3, 3)$
[243, 55]	$U(3, 2, 2)$
[252, 11]	$L_7^{(2)}(2, 2)$
[279, 1]	$T_{31}^{(5)}(2)$
[300, 13]	$\Delta(6 \times 5^2, 2)$
[324, 3]	$\Delta(3 \times 2^2, 4)$
[324, 13]	$G(1, 2)$
[324, 15]	$Z(3, 2, 2)$
[324, 17]	$Z'(3, 2, 2)$
[324, 43]	$W(2, 3)$
[324, 45]	$Y(2, 1)$
[324, 49]	$Z'(6, 2)$
[324, 51]	$V(1)$
[324, 60]	$Y(1)$
[324, 102]	$\Delta'(6 \times 3^2, 2, 2)$
[324, 111]	$\Xi(2, 2)$
[324, 128]	$Z(6, 2)$
[333, 1]	$T_{37}^{(10)}(2)$
[351, 1]	$T_{13}^{(3)}(3)$
[351, 4]	$Q_{13}^{(3)}(2)$
[351, 5]	$Q_{13}^{(3)'}(2)$
[351, 7]	$P_{13}^{(3)}(2)$
[384, 571]	$\Delta(6 \times 4^2, 3)$
[384, 581]	$S_4(5)$

Table A10. The finite subgroups of $U(3)$. Part 3: groups with $387 \leq \text{order} \leq 576$.

Identifier	Classification
[387, 1]	$T_{43}^{(6)}(2)$
[432, 3]	$\Delta(3 \times 4^2, 3)$
[432, 33]	$\Delta(6 \times 3^2, 4)$
[432, 57]	$\Xi(1, 4)$
[432, 100]	$W(4, 2)$
[432, 102]	$X(12)$
[432, 239]	$\Pi(1, 2)$
[432, 260]	$\Delta(6 \times 6^2, 2)$
[432, 273]	$\hat{\Xi}(1, 3)$
[441, 1]	$T_{49}^{(18)}(2)$
[441, 7]	$\Delta(3 \times 7^2, 2)$
[468, 14]	$L_{13}^{(3)}(2, 2)$
[486, 26]	$Z'(3, 3, 1)$
[486, 28]	$Z(3, 3, 1)$
[486, 125]	$H(3, 2, 1)$
[486, 164]	$\Delta'(6 \times 3^2, 3, 1)$
[513, 1]	$T_{19}^{(7)}(3)$
[513, 5]	$Q_{19}^{(7)}(2)$
[513, 6]	$Q_{19}^{(7)'}(2)$
[513, 8]	$P_{19}^{(7)}(2)$
[549, 1]	$T_{61}^{(13)}(2)$
[567, 1]	$T_7^{(2)}(4)$
[567, 4]	$Q_7^{(2)}(3)$
[567, 5]	$Q_7^{(2)'}(3)$
[567, 7]	$P_7^{(2)}(3)$
[567, 12]	$S_7^{(2)'}(2)$
[567, 14]	$V_7^{(2)}(2)$
[567, 23]	$Y_7^{(2)}(2)$
[567, 36]	$S_7^{(2)}(2)$
[576, 3]	$\Delta(3 \times 8^2, 2)$

Table A11. The finite subgroups of $U(3)$. Part 4: groups with $588 \leq \text{order} \leq 729$.

Identifier	Classification
[588, 16]	$\Delta(6 \times 7^2, 2)$
[600, 45]	$\Delta(6 \times 5^2, 3)$
[603, 1]	$T_{67}^{(29)}(2)$
[648, 19]	$G(1, 3)$
[648, 21]	$Z(3, 2, 3)$
[648, 23]	$Z'(3, 2, 3)$
[648, 244]	$\Delta'(6 \times 3^2, 2, 3)$
[648, 260]	$Z(6, 2, 1)$
[648, 266]	$\tilde{Y}(1)$
[648, 352]	$\Xi(2, 3)$
[648, 531]	$\Upsilon(2)$
[648, 533]	$\Omega(1)$
[648, 551]	$\Theta(2)$
[648, 563]	$\Delta'(6 \times 6^2, 2, 1)$
[657, 1]	$T_{73}^{(8)}(2)$
[675, 5]	$\Delta(3 \times 5^2, 3)$
[675, 9]	$W(5, 2)$
[675, 11]	$X(15)$
[684, 11]	$L_{19}^{(7)}(2, 2)$
[711, 1]	$T_{79}^{(23)}(2)$
[729, 62]	$L(3)$
[729, 63]	$Z''(3, 4)$
[729, 64]	$Z'(3, 4)$
[729, 80]	$J(2)$
[729, 86]	$U(3, 3, 2)$
[729, 94]	$W(1, 5)$
[729, 96]	see Sect. 4.5
[729, 97]	see Sect. 4.5
[729, 98]	see Sect. 4.5
[729, 284]	$U(3, 3, 3)$

Table A12. The finite subgroups of $U(3)$. Part 5: groups with $729 \leq \text{order} \leq 972$.

Identifier	Classification
[729, 393]	$Z(3, 4)$
[729, 397]	$Z(9, 2)$
[756, 11]	$L_7^{(2)}(2, 3)$
[756, 113]	$M_7^{(2)}$
[756, 114]	$M_7^{(2)'}$
[756, 116]	$J_7^{(2)}$
[768, 1085 333]	$\Delta(6 \times 4^2, 4)$
[768, 1085 335]	$\Delta(6 \times 8^2, 2)$
[768, 1085 351]	$S_4(6)$
[819, 3]	$T_{91}^{(16)}(2)$
[819, 4]	$T_{91}^{(9)}(2)$
[837, 1]	$T_{31}^{(5)}(3)$
[837, 4]	$Q_{31}^{(5)'}(2)$
[837, 5]	$Q_{31}^{(5)}(2)$
[837, 7]	$P_{31}^{(5)}(2)$
[864, 69]	$\Delta(6 \times 3^2, 5)$
[864, 194]	$\Xi(1, 5)$
[864, 675]	$\Pi(1, 3)$
[864, 703]	$\Delta(6 \times 6^2, 3)$
[864, 737]	$\hat{\Xi}(1, 4)$
[873, 1]	$T_{97}^{(35)}(2)$
[900, 66]	$\Delta(3 \times 10^2, 2)$
[927, 1]	$T_{103}^{(46)}(2)$
[972, 3]	$\Delta(3 \times 2^2, 5)$
[972, 29]	$Z'(3, 3, 2)$
[972, 31]	$Z(3, 3, 2)$
[972, 64]	$\Delta(6 \times 9^2, 2)$
[972, 117]	$W(2, 4)$
[972, 121]	$D(1)$

Table A13. The finite subgroups of $U(3)$. Part 6: groups with $972 \leq \text{order} \leq 1143$.

Identifier	Classification
[972, 123]	$X(18)$
[972, 147]	$Y(3, 1)$
[972, 152]	$Z'(6, 3)$
[972, 153]	$Z''(6, 3)$
[972, 170]	see Sect. 4.5
[972, 309]	$H(3, 2, 2), G(2, 2)$
[972, 348]	$\Delta'(6 \times 3^2, 3, 2)$
[972, 411]	$\Xi(3, 2)$
[972, 520]	$Z(6, 3)$
[972, 550]	$U(6, 2, 2)$
[981, 1]	$T_{109}^{(45)}(2)$
[999, 1]	$T_{37}^{(10)}(3)$
[999, 5]	$Q_{37}^{(10)'}(2)$
[999, 6]	$Q_{37}^{(10)}(2)$
[999, 8]	$P_{37}^{(10)}(2)$
[1008, 57]	$L_7^{(2)}(4, 2)$
[1053, 16]	$T_{13}^{(3)}(4)$
[1053, 25]	$Q_{13}^{(3)'}(3)$
[1053, 26]	$Q_{13}^{(3)}(3)$
[1053, 27]	$P_{13}^{(3)}(3)$
[1053, 29]	$Y_{13}^{(3)}(2)$
[1053, 32]	$S_{13}^{(3)'}(2)$
[1053, 37]	$V_{13}^{(3)}(2)$
[1053, 47]	$S_{13}^{(3)}(2)$
[1089, 3]	$\Delta(3 \times 11^2, 2)$
[1116, 11]	$L_{31}^{(5)}(2, 2)$
[1143, 1]	$T_{127}^{(19)}(2)$

Table A14. The finite subgroups of $U(3)$. Part 7: groups with $1161 \leq \text{order} \leq 1296$.

Identifier	Classification
[1161, 6]	$T_{43}^{(6)}(3)$
[1161, 10]	$Q_{43}^{(6)}(2)$
[1161, 11]	$Q_{43}^{(6)'}(2)$
[1161, 12]	$P_{43}^{(6)}(2)$
[1176, 57]	$\Delta(6 \times 7^2, 3)$
[1197, 3]	$T_{133}^{(11)}(2)$
[1197, 4]	$T_{133}^{(30)}(2)$
[1200, 183]	$\Delta(6 \times 5^2, 4)$
[1200, 682]	$\Delta(6 \times 10^2, 2)$
[1251, 1]	$T_{139}^{(42)}(2)$
[1296, 3]	$\Delta(3 \times 4^2, 4)$
[1296, 35]	$G(1, 4)$
[1296, 37]	$Z(3, 2, 4)$
[1296, 39]	$Z'(3, 2, 4)$
[1296, 220]	$W(4, 3)$
[1296, 222]	$Y(2, 2)$
[1296, 226]	$V(2)$
[1296, 227]	$Z'(12, 2)$
[1296, 237]	$Y(2)$
[1296, 647]	$\Delta'(6 \times 3^2, 2, 4)$
[1296, 688]	$Z'(6, 2, 2)$
[1296, 689]	$Z(6, 2, 2)$
[1296, 699]	see Sect. 4.4
[1296, 1239]	$\Xi(2, 4)$
[1296, 1499]	$Z(12, 2)$
[1296, 1995]	$\Pi(2, 2)$
[1296, 2113]	$\Delta'(6 \times 6^2, 2, 2)$
[1296, 2203]	$\hat{\Xi}(2, 3)$

Table A15. The finite subgroups of $U(3)$. Part 8: groups with $1323 \leq \text{order} \leq 1536$.

Identifier	Classification
[1323, 1]	$T_{49}^{(18)}(3)$
[1323, 4]	$Q_{49}^{(18)}(2)$
[1323, 5]	$Q_{49}^{(18)'}(2)$
[1323, 7]	$P_{49}^{(18)}(2)$
[1323, 14]	$\Delta(3 \times 7^2, 3)$
[1323, 40]	$W(7, 2)$
[1323, 42]	$X(21)$
[1332, 14]	$L_{37}^{(10)}(2, 2)$
[1359, 1]	$T_{151}^{(32)}(2)$
[1404, 14]	$L_{13}^{(3)}(2, 3)$
[1404, 137]	$M_{13}^{(3)}$
[1404, 138]	$M_{13}^{(3)'}$
[1404, 140]	$J_{13}^{(3)}$
[1413, 1]	$T_{157}^{(12)}(2)$
[1452, 11]	$\Delta(6 \times 11^2, 2)$
[1458, 615]	$Z'(3, 4, 1)$
[1458, 618]	$Z(3, 4, 1)$
[1458, 663]	see Sect. 4.5
[1458, 666]	see Sect. 4.5
[1458, 1095]	$H(3, 3, 1)$
[1458, 1354]	$\Delta'(6 \times 3^2, 4, 1)$
[1458, 1371]	$\Delta'(6 \times 9^2, 2, 1)$
[1467, 1]	$T_{163}^{(58)}(2)$
[1521, 1]	$T_{169}^{(22)}(2)$
[1521, 7]	$\Delta(3 \times 13^2, 2)$
[1536, 408 544 641]	$\Delta(6 \times 8^2, 3)$
[1536, 408 544 678]	$\Delta(6 \times 4^2, 5)$
[1536, 408 544 687]	$S_4(7)$

Table A16. The finite subgroups of $U(3)$. Part 9: groups with $1539 \leq \text{order} \leq 1701$.

Identifier	Classification
[1539, 16]	$T_{19}^{(7)}(4)$
[1539, 25]	$Q_{19}^{(7)'}(3)$
[1539, 26]	$Q_{19}^{(7)}(3)$
[1539, 27]	$P_{19}^{(7)}(3)$
[1539, 29]	$Y_{19}^{(7)}(2)$
[1539, 32]	$S_{19}^{(7)'}(2)$
[1539, 37]	$V_{19}^{(7)}(2)$
[1539, 47]	$S_{19}^{(7)}(2)$
[1548, 11]	$L_{43}^{(6)}(2, 2)$
[1575, 7]	$L_7^{(2)}(5, 2)$
[1629, 1]	$T_{181}^{(48)}(2)$
[1647, 6]	$T_{61}^{(13)}(3)$
[1647, 10]	$Q_{61}^{(13)}(2)$
[1647, 11]	$Q_{61}^{(13)'}(2)$
[1647, 12]	$P_{61}^{(13)}(2)$
[1701, 68]	$T_7^{(2)}(5)$
[1701, 102]	see Sect. 4.4
[1701, 112]	see Sect. 4.5
[1701, 115]	$S_7^{(2)'}(3)$
[1701, 126]	$Q_7^{(2)'}(4)$
[1701, 127]	$Q_7^{(2)}(4)$
[1701, 128]	$P_7^{(2)}(4)$
[1701, 130]	see Sect. 4.5
[1701, 131]	see Sect. 4.5
[1701, 138]	$V_7^{(2)}(3)$
[1701, 240]	$S_7^{(2)}(3)$
[1701, 261]	$Y_7^{(2)}(3)$

Table A17. The finite subgroups of $U(3)$. Part 10: groups with $1728 \leq \text{order} \leq 1944$.

Identifier	Classification
[1728, 3]	$\Delta(3 \times 8^2, 3)$
[1728, 185]	$\Delta(6 \times 3^2, 6)$
[1728, 953]	$\Xi(1, 6)$
[1728, 1286]	$W(8, 2)$
[1728, 1290]	$X(24)$
[1728, 2785]	$\Pi(1, 4)$
[1728, 2847]	$\Delta(6 \times 12^2, 2)$
[1728, 2855]	$\Delta(6 \times 6^2, 4)$
[1728, 2929]	$\hat{\Xi}(1, 5)$
[1737, 1]	$T_{193}^{(84)}(2)$
[1764, 11]	$L_{49}^{(18)}(2, 2)$
[1764, 91]	$\Delta(3 \times 14^2, 2)$
[1791, 1]	$T_{199}^{(92)}(2)$
[1809, 6]	$T_{67}^{(29)}(3)$
[1809, 10]	$Q_{67}^{(29)}(2)$
[1809, 11]	$Q_{67}^{(29)'}(2)$
[1809, 12]	$P_{67}^{(29)}(2)$
[1872, 60]	$L_{13}^{(3)}(4, 2)$
[1899, 1]	$T_{211}^{(14)}(2)$
[1944, 35]	$Z'(3, 3, 3)$
[1944, 37]	$Z(3, 3, 3)$
[1944, 70]	$\Delta(6 \times 9^2, 3)$
[1944, 707]	$H(3, 2, 3), G(2, 3)$
[1944, 746]	$\Delta'(6 \times 3^2, 3, 3)$
[1944, 832]	$Z'(6, 3, 1)$
[1944, 833]	$Z(6, 3, 1)$
[1944, 1123]	$\Xi(3, 3)$

Table A18. The finite subgroups of $U(3)$. Part 11: groups with $1944 \leq \text{order} < 2000$.

Identifier	Classification
[1944, 2293]	$\Upsilon(3)$
[1944, 2294]	$\Upsilon'(3)$
[1944, 2333]	$\Theta(3)$
[1944, 2363]	$H(6, 2, 1)$
[1944, 2415]	$\Delta'(6 \times 6^2, 3, 1)$
[1944, 3448]	$\Omega(2)$
[1953, 3]	$T_{217}^{(25)}(2)$
[1953, 4]	$T_{217}^{(67)}(2)$
[1971, 6]	$T_{73}^{(8)}(3)$
[1971, 10]	$\mathcal{Q}_{73}^{(8)'}(2)$
[1971, 11]	$\mathcal{Q}_{73}^{(8)}(2)$
[1971, 12]	$P_{73}^{(8)}(2)$

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