SINGLE PASS COLLIDER MEMO CN-355

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TITLE: BETATRON OSCILLATIONS AND ROLLED ACHROMATS

Introduction

Consider a bunch (or just a single electron) that is moving along with a betatron oscillation in one plane - - say x - - in some Achromat. If the next Achromat is rolled with respect to the first one, the bunch will have a different betatron oscillation in the x-coordinate, and, in addition, an oscillation will be introduced in the y-coordinate. This note calculates these effects.

Basic Relations

In each Achromat, I describe the betatron oscillations in a "local" coordinate system - - in which x is in the "horizontal" symmetry plane, and y is at right-angles to it. In the ideal Achromat, the oscillations in the two planes are uncoupled and can be described by

$$x(s) = a_x\sqrt{\beta_x(s)}\ cos\ [\phi_x(s)\ +\ \Phi_x] \tag{1}$$

$$y(s) = a_y\sqrt{\beta_y(s)}\ cos\ [\phi_y(s)\ +\ \Phi_y] \tag{2}$$

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where s is the distance from the start of the Achromat, a_x and a_y are constants in the Achromat, and β_x and β_y are the usual betatron functions. The phases ϕ_x and ϕ_y are the betatron phase advance from the beginning of the Achromat and Φ_x and Φ_y are the initial phases of the bunch oscillations - - at the start of the Achromat. Inasmuch as the ideal Achromat has a phase advance of 6π, Φ_x and Φ_y are also the phases at the exit of the Achromat.

Effect of a Roll

Now suppose that in Achromat-1 there is a betatron oscillation in x only and described by a_{x1} and Φ_{x1} (a_{y1} is zero). And, suppose that Achromat-2 (the next one) is "rolled" by the angle θ with respect to Achromat-1. By that I mean that the x_2-axis at the entrance to Achromat-2 is rotated by the angle θ with respect to the x_1-axis at the exit of Achromat-1 - - with θ taken as positive for a rotation toward y_1. This rotation gives rise to a modified betatron oscillation in Achromat-2.

I will give here just the results for the modified oscillations and leave their calculation to an Appendix. The amplitudes a_{x2} and a_{y2} and the phases Φ_{x2} and Φ_{y2} on entering Achromat-2 are

$$a_{x2} = a_{x1} \cos\theta \tag{3}$$

$$a_{y2} = M a_{x1} |\sin\theta| \tag{4}$$

$$\Phi_{x2} = \Phi_{x1} \tag{5}$$

$$\tan\Phi_{y2} = \tan\Phi_{x1} - \beta'_x \tag{6}$$

in which $\beta'_x = d\beta_x/ds = -d\beta_y/ds$ is the value at the entrance to an Achromat, and the "magnification factor" M is given by

$$M = [1 - 2\beta'_x \sin \Phi_{x1} \cos \Phi_{x1} + \beta'^2_x \cos^2 \Phi_{x1}]^{1/2} \quad (7)$$

I should emphasize that this result applies only to the case that a roll occurs between two regular F- and D- magnets of the Arcs (where $\beta_y = \beta_x$ and $\beta'_y = -\beta'_x$).

Since there is no change in Φ_x across the roll, we no longer need the subscripts 1 and 2 on Φ_x. Note that the evaluation of Φ_{y2} with Eq. (6) leaves an ambiguity of ± 180 deg. This ambiguity is resolved by satisfying Eq. (A5) (in the Appendix) and has been done in the Figures given below.

For the SLC Arcs, $\beta'_x = 5.30$ at the entrance to an F-type Achromat (one that begins with a focussing magnet), and $\beta'_x = -5.30$ at the entrance to a D-type Achromat (one that begins with a defocussing magnet).

I show in the attached Figures M and Φ_{y2} as a function of Φ_x for both an F-Type Achromat and a D-type Achromat. (One gets from one type to the other simply by reversing the scales of both Φ_x and Φ_{y2}.) The curves given apply to a <u>positive</u> roll angle θ. If θ is negative, the oscillation y is reversed; that is, Φ_{y2} changes by $\pm 180^o$. In the North Arc, Achromats 1 through 7 are F-type, and Achromats 9 through 23 are D-type. In the South Arc, the opposite is true (except for Achromat 3 which does not exist). The following comments are in order:

a) The change in the x-oscillation is as one might suspect. It is as if the oscillation were "projected" onto the new x-direction with no change in phase.

b) The new y-oscillation is <u>not</u> just the projection of an x_1-motion onto the y_2-direction as one might have guessed, although such a projection would correspond to the term $a_{x1} \sin \theta$ that <u>does</u> appear in a_{y2}. There is, however, the additional factor M. Also, the new y-phase may be quite different from the original x-phase. You will also note that the value of M depends on the phase with which the oscillation arrives at the roll. (For the ideal Achromat, the phase advance ϕ is $3\times\ 2\pi$, so that Φ is both the starting and ending phase.) The maximum value of M is 5.48, and the minimum is the inverse of that, 0.182. Then, for a roll angle of, say 10 deg., the y-oscillation can be just as large as the x-oscillation! Notice also that the phase Φ_{y2} prefers strongly to be near $+90^o$ or -90^o.

c) There is no "conservation of oscillation energies." That is, $(a_x^2 + a_y^2)$ is not conserved in a roll. We see that "longitudinal energy" can be thrown into "transverse energy" at a roll.

d) The transformation from one Achromat to the next is linear, so if we have oscillations in both x and y in Achromat-1, we can use the results here to find the oscillations produced in Achromat-2 by each component, treated separately, and then add the two contributions. (For an oscillation initially in y, there are relations corresponding to Eqs. (3) to (6), with some adjustments of sign.)

e) It is amusing to note that even the average of M over all phases Φ_1 is not 1, but, rather,

$$\langle M\rangle \ = \ [1 + \frac{\beta\prime_x^2}{2}]^{1/2} \qquad (8)$$

General Case

It is, of course, possible to work out the general result for the rotation of a transport system at an arbitrary location in these terms. It is probably not very interesting, so I give only the result - - which can be obtained by the method use in the Appendix. Equations (3) and (5) do not change, but Eqs. (6) and

(7) become

$$M^2 = \frac{\beta_y}{\beta_x} + [\frac{\beta_x}{\beta_y}(1 + \frac{\beta_y'^2}{4}) - \frac{\beta_y}{\beta_x}(1 - \frac{\beta_x'^2}{4}) - \frac{\beta_x'\beta_y'}{2}] \cos^2 \Phi_{x1}$$

$$+ (\beta_y' - \frac{\beta_y\beta_x'}{\beta_x}) \sin \Phi_{x1} \cos \Phi_{x1} \qquad (9)$$

$$\tan \Phi_{y2} = \frac{\beta_y}{\beta_x}(\tan \Phi_{x1} - \frac{\beta_x'}{2}) + \frac{\beta_y'}{2} \qquad (10)$$

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W. Weng provided useful discussions and information.

APPENDIX

The results given in the body of this Note are derived as follows:

From Eq. (1) it follows that the slope of the trajectory at any s can be written as

$$x' = \frac{a_x}{\sqrt{\beta_x}} \{-\sin(\phi_x + \Phi_x) + \frac{\beta_x'}{2} \cos(\phi_x + \Phi_x)\} \qquad (A1)$$

We can use this equation together with Eq. (1) to relate the amplitude and phase of the oscillation at any s to x and x' there. We get that

$$a_x^2 = \frac{x^2}{\beta_x} + (\sqrt{\beta_x}x' - \frac{\beta_x'}{2\sqrt{\beta_x}}x)^2 \qquad (A2)$$

and

$$\tan(\phi_x + \Phi_x) = -\frac{\beta_x x'}{x} + \frac{\beta_x'}{2} \qquad (A3)$$

The extension to y is evident.

Now, consider that x_1 and x'_1 are given at the exit to Achromat 1. The corresponding a_{x1} and Φ_{x1} are determined by Eqs. (A2) and (A3). The same x and x'_1 also determine x_2 and x'_2 and y_2 and y'_2 the coordinates and slopes measured with respect to Achromat 2. Indeed, we have

$$x_2 = x_1 \cos\theta \qquad x'_2 = x'_1 \cos\theta \tag{A4}$$

$$y_2 = -x_1 \sin\theta \qquad y'_2 = -x'_1 \sin\theta \tag{A5}$$

The values of β_x and β'_x do not change across the boundary between two Achromats. And, if we take (as usual) the boundary between Achromats at the symmetry point between F and D magnets, then it follows that at the boundary

$$\beta_y = \beta_x \quad ; \beta'_y = -\beta'_x \tag{A6}$$

Using these relations Eqs. (3) and (5) are obtained from Eqs. (A2) and (A3) by inspection. Since x and x' are changed by the same factor, a will be changed by that factor also, and Φ_x will not change at all.

The results for a_{y2} and Φ_{y2} are different, because β'_y is not equal to β'_x. Substituting (A5) into (A2) and factoring out $\sin^2\theta$, we get

$$a_{y2}^2 = \sin^2\theta \, [\frac{x_1^2}{\sqrt{\beta_y}} + (\sqrt{\beta_y} x'_1 - \frac{\beta'_y}{2\sqrt{\beta_y}} x_1)^2] \tag{A7}$$

Now we use (A6) to get

$$a_{y2}^2 = \sin^2\theta [\frac{x_1^2}{\sqrt{\beta_x}} + (\sqrt{\beta_x}\, x'_1 + \frac{\beta'_x}{2\sqrt{\beta_y}} x_1)^2] \tag{A8}$$

The quantity in the square brackets differs from a_x^2 only by the sign of β_x'. Expanding the squared term and comparing a_{y2}^2 to a_{x1}^2 (A2), we see that

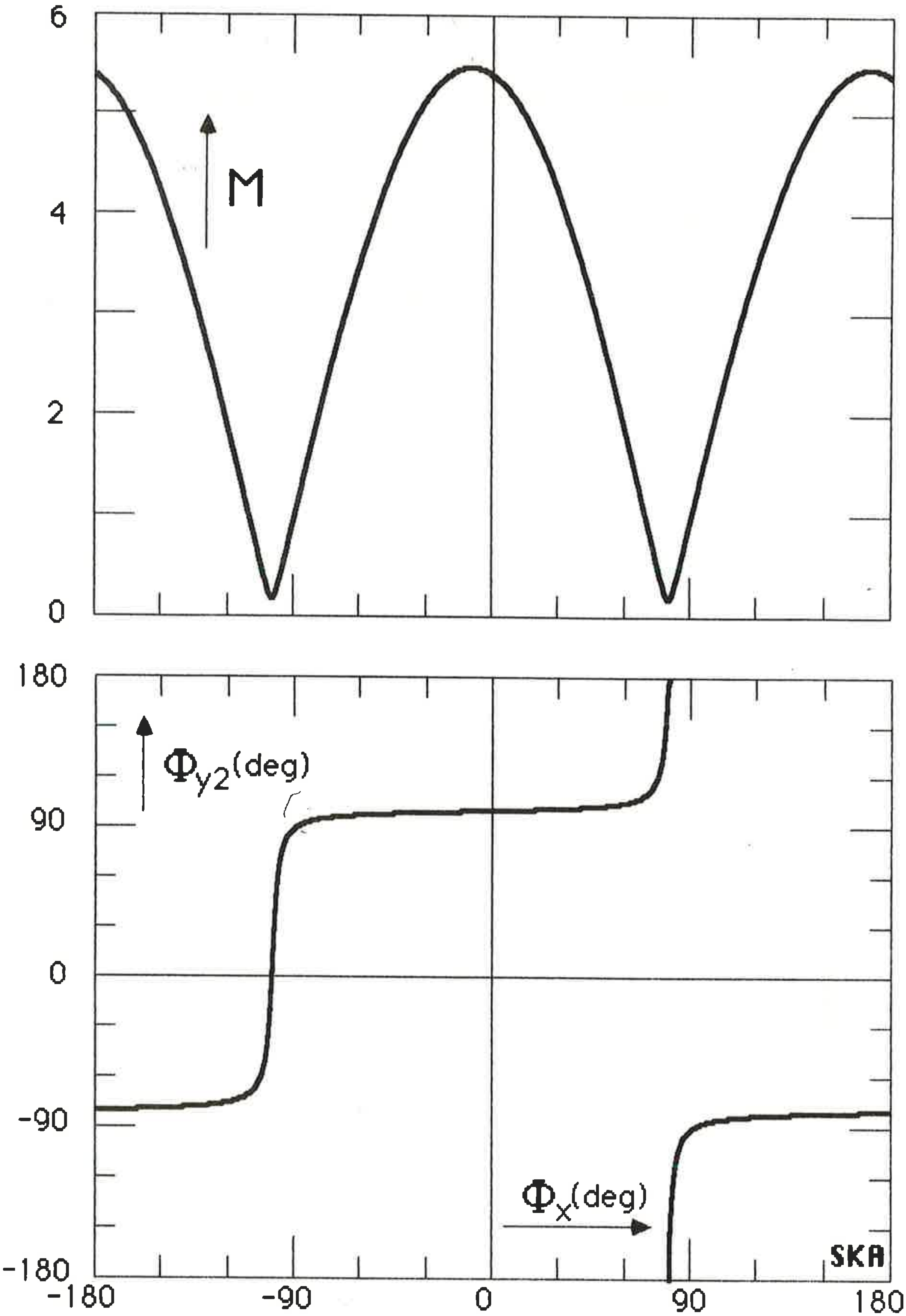
$$a_{y2}^2 = sin^2\,\theta\,[a_{x1}^2 + 2\beta_x'\,x\,x'] \qquad (A9)$$

Or, putting x and x' in terms of a_{x1} and Φ_{x1} we find Eqs. (4) and (7).

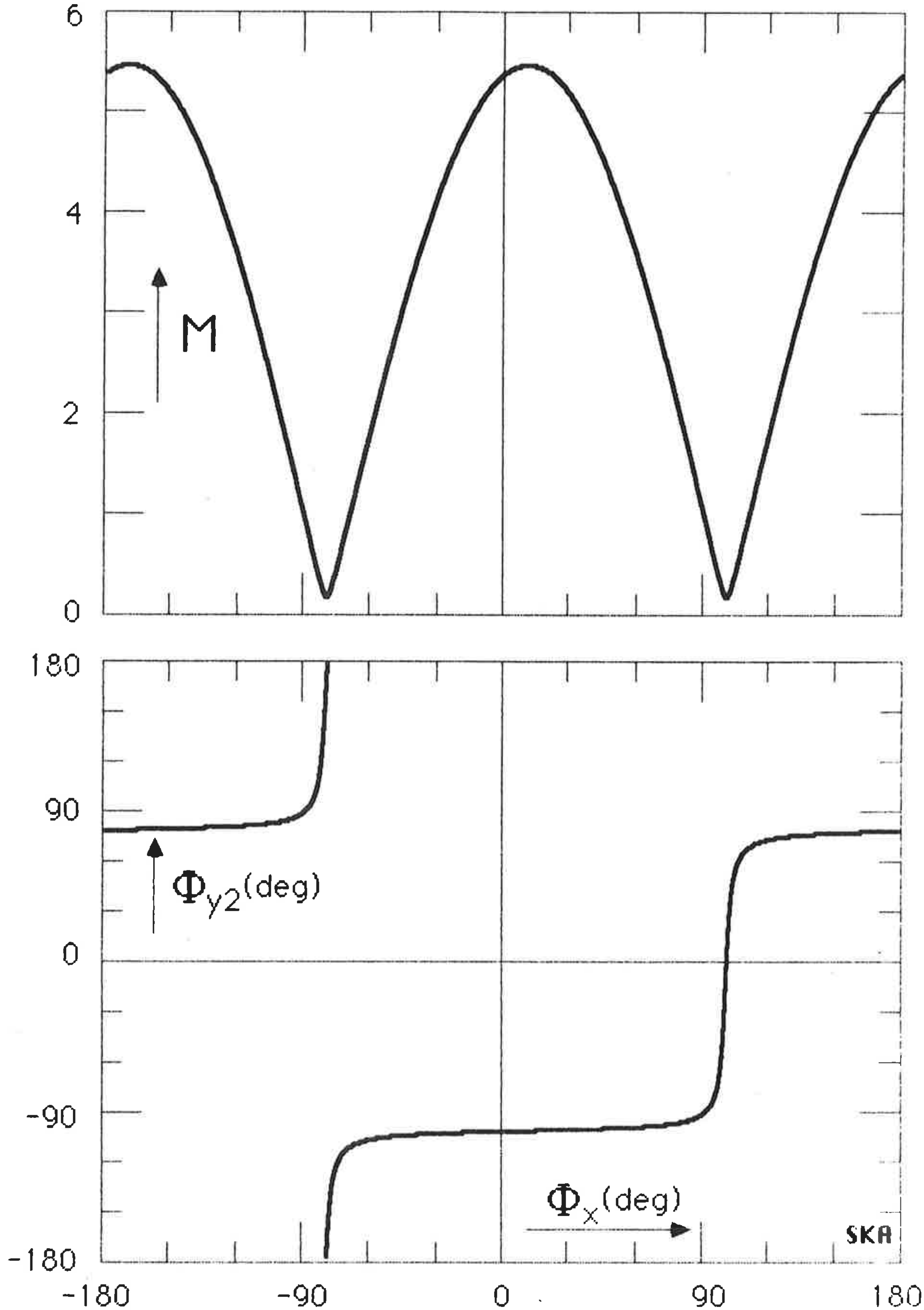
Translating Eq. (A3) to y (with $\Phi_y = 0$ at the beginning of an Achromat) and then using (A5) and (A6), we get that

$$tan\;\Phi_y = -\frac{\beta_y y'}{y} + \frac{\beta_y'}{2} = -\frac{\beta_x x'}{x} - \frac{\beta_x'}{2} \qquad (A10)$$

Comparing this result with (A4) for $\Phi_x = 0$, we obtain the result of Eq. (5).



F-Type Achromat



D-Type Achromat