Infrared Limit of $\mathcal{N} = 4$ Gluon Amplitudes at Strong Coupling

Evgeny I. Buchbinder

Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5, Canada

Abstract

We discuss our proposal that the infrared structure of $\mathcal{N} = 4$ gluon amplitudes at strong coupling can be fully extracted from a local consideration near cusps. This is consistent with field theory and correctly reproduces the infrared divergences of the four-gluon amplitude at strong coupling calculated recently by Alday and Maldacena.

1 Introduction

In the recent paper [1], Alday and Maldacena made a substantial progress in applying ideas of the AdS/CFT correspondence [2] to study scattering amplitudes of gluons at strong coupling. One of the crucial ingredients of [1] is dimensional regularization on the gravity dual side. The importance of it is motivated by the fact that many field theory results on gluon amplitudes have been obtained in dimensional regularization scheme and it is necessary to use the same regularization if one intends to provide an unambiguous comparison between the gravity and field theory sides. In particular, Alday in Maldacena [1] computed the four-gluon amplitude at strong coupling and found a perfect agreement with the infrared structure in field theory. In addition, they also found an agreement with the conjecture of Bern, Dixon and Smirnov [3] (see also an earlier paper [4]) regarding the all-loop iterative structure of gluon amplitudes. Furthermore, the analysis of [1] made a prediction for the strong coupling behavior of the cusp anomalous dimension. Their result agreed with [5], [6] where the same behavior was established by different methods.

It is well-known that in field theory, gluon amplitudes have to satisfy several consistency conditions such as unitarity cuts, infrared behavior, collinear and soft gluon limits. It is very interesting to understand what they are translated on the AdS side to. The simplest limit that one can consider is the infrared divergences. In field theory they arise from a very special set of Feynman diagrams. This makes them summable to all orders in perturbation theory [3,7–9]. This suggests that on the AdS side the infrared structure also arises from some very special minimal worldsheets whose study does not require the complete answer for the n-gluon amplitude. In [10] it was proposed that the infrared behavior can be completely derived from the worldsheet whose boundary is momenta of two neighboring gluons meeting at a cusp. We show that this is consistent with the infrared structure on the field theory side. When applied to the case of four gluons, this proposal gives the same answer as the infrared divergent contribution to the four-gluon amplitude found in [1].

2 Infrared Behavior in $\mathcal{N} = 4$ Super Yang-Mills Theory

The infrared behavior of the gluon amplitudes in $\mathcal{N} = 4$ Super Yang-Mills Theory is known to all orders in perturbation theory and has a nice exponential structure [3,7–9]. In

our review below, we will follow section IV of [3]. We will assume that we have performed the color decomposition and study the leading-color partial amplitudes. First, let us define

$$M_n^{(L)}(\epsilon) = \frac{\mathcal{A}_n^{(L)}(\epsilon)}{\mathcal{A}_n^{tree}}. \tag{1}$$

In this equation, $\mathcal{A}_n^{tree}$ and $\mathcal{A}_n^{(L)}(\epsilon)$ are the tree-level and L-loop n-gluon amplitudes respectively. The dependence on ϵ indicates that $M_n^{(L)}$ is evaluated in dimensional regularization. Furthermore, let us define

$$M_n(\epsilon) = 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon), \tag{2}$$

where

$$a = \lambda (4\pi e^{-\gamma})^{-\epsilon}, \tag{3}$$

λ is the t'Hooft coupling and γ is the Euler constant. For any neighboring pair of gluons $i, i+1$ with momenta k_i and k_{i+1} we introduce

$$s_{i,i+1} = (k_i + k_{i+1})^2, \quad i = 1, \ldots, n, \quad s_{n,n+1} \equiv s_{n1}. \tag{4}$$

The leading-color all-loop infrared behavior of $M_n(\epsilon)$ can be expressed as follows

$$\ln M_n(\epsilon) \sim \sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon) \hat{I}_n^{(1)}(l\epsilon). \tag{5}$$

Here $\hat{I}_n^{(1)}(\epsilon)$ represents the infrared behavior at one loop and has the following additive structure

$$\hat{I}_n^{(1)}(\epsilon) = \sum_{i=1}^{n} \hat{I}_n^{(1)}(s_{i,i+1}, \epsilon), \tag{6}$$

where

$$\hat{I}_n^{(1)}(s_{i,i+1}, \epsilon) = -\frac{1}{2\epsilon^2} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon}. \tag{7}$$

The function $f^{(l)}(\epsilon)$ has a perturbative expansion in ϵ

$$f^{(l)}(\epsilon) = f_0^{(l)} + f_1^{(l)}\epsilon + f_2^{(l)}\epsilon^2 + \ldots. \tag{8}$$

The leading term $f_0^{(l)}$ is known to coincide, up to a constant, with the cusp anomalous dimension $\gamma^{(l)}$,

$$f_0^{(l)} = \frac{1}{4}\gamma^{(l)}. \tag{9}$$

Substituting $\hat{I}_n^{(1)}(\epsilon)$ into eq. (5) we obtain that the right hand side has the following functional dependence

$$\ln M_n(\epsilon) \sim \frac{1}{\epsilon^2} \sum_{i=1}^{n} F\left(\lambda \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon}, \epsilon \right) \tag{10}$$

for some function F which goes to zero for finite negative ϵ as $s_{i,i+1}$ goes to zero. To find another implication of the general infrared behavior (5) let us consider the terms of order

ϵ^0. It follows from eqs. (6), (7) and (8) that up to an additive constant $\ln M_n(\epsilon)$ contains the the terms of the form

$$\ln M_n(\epsilon)|_{\epsilon^0} \sim -\frac{1}{16} f(\lambda) \sum_{i=1}^{n} \ln^2 \left(\frac{\mu^2}{-s_{i,i+1}} \right) - \frac{1}{4} g(\lambda) \sum_{i=1}^{n} \ln \left(\frac{\mu^2}{-s_{i,i+1}} \right), \tag{11}$$

where

$$f(\lambda) = 4 \sum_{l=1}^{\infty} \lambda^l f_0^{(l)} = \sum_{l=1}^{\infty} \lambda^l \gamma^{(l)} \tag{12}$$

is the all-loop cusp anomalous dimension and

$$g(\lambda) = 2 \sum_{l=1}^{\infty} \frac{\lambda^l f_1^{(l)}}{l} - 8\pi e^{-\gamma} \sum_{l=1}^{\infty} f_0^{(l)}. \tag{13}$$

Note that the definition of $g(\lambda)$ depends on the infrared scale μ. As we change μ, $\mu \to \mu\kappa$, we have

$$g(\lambda) \to g(\lambda) + 2 f(\lambda) \ln \kappa. \tag{14}$$

On the other hand, the coefficient in front of the $\ln^2 \left(\frac{\mu^2}{-s_{i,i+1}} \right)$ term is always the cusp anomalous dimension.

3 Infrared Behavior and Cusps in AdS

In [1], Alday and Maldacena developed an approach to study the strong coupling limit of $\mathcal{N} = 4$ gluon amplitudes using string theory in AdS space. As explained in [1], scattering of open string states happens at large proper AdS momenta. Therefore, similarly to the flat space case [11], the scattering amplitude to the leading order is determined by the appropriate classical solution

$$\mathcal{A} \sim e^{iS}, \tag{15}$$

where S is the action evaluated on the classical solution, which is just the area of the worldsheet. The prefactor was determined in [12]. Furthermore, similarly to the flat space case [11], $\mathcal{A}$ depends only on the momenta of the scattered particles and not on any other additional data like the helicity structure. All information about the momenta is encoded in the boundary conditions. To describe it, it is convenient to use the T-dual coordinates [13] (see [1] for details). In this coordinates, the metric is also AdS

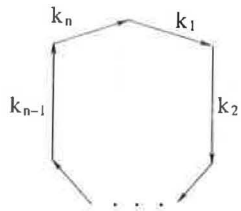

Figure 1: The boundary of the worldsheet of the n-gluon amplitude. For simplicity, we removed the factors of 2π multiplying each k_i.

$$ds^2 = R^2 \frac{dy_\mu dy^\mu + dr^2}{r^2}, \quad \mu = 0, \ldots, 3, \tag{16}$$

where R is the radius of AdS space. The boundary conditions are imposed at $r = 0$. In the T-dual coordinates the fact that a state has momentum k^μ translates into the statement that it has a winding

$$\Delta y^\mu = 2\pi k^\mu. \tag{17}$$

Then, the boundary conditions are such that as $r \to 0$ the worldsheet describing the n-particle amplitude ends on the vectors $2\pi k_1^\mu, \ldots, 2\pi k_n^\mu$. The ordering of the vectors corresponds to the particular color ordered amplitude. From momentum conservation it follows that the above vectors form a closed loop.

Since $\mathcal{N} = 4$ gluon amplitudes are infrared divergent they have to be regulated. In order to be able to compare the string and field theory results, one has to use the same regularization scheme. The most convenient is to use dimensional regularization. The regulated AdS metric looks as follows [1]

$$ds^2 = \sqrt{c_D \lambda_D} \left(\frac{dy_D^2 + dr^2}{r^{2+\epsilon}} \right), \tag{18}$$

where

$$\lambda_D = \frac{\lambda \mu^{2\epsilon}}{(4\pi e^{-\gamma})^\epsilon}, \quad c_D = 2^{4\epsilon} \pi^{3\epsilon} \Gamma(2+\epsilon),$$
$$D = 4 - 2\epsilon. \tag{19}$$

The parametrization of λ_D in terms of the IR scale μ is chosen to match the field theory side. In notation (18), the worldsheet action becomes

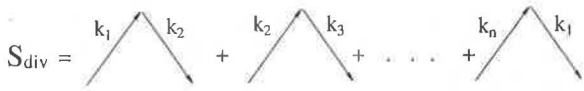

Figure 2: The graphical representation of the infrared divergences. Each cusp represents the boundary of the minimal worldsheet.

$$S = \frac{\sqrt{\lambda_D c_D}}{2\pi} \int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}, \tag{20}$$

where $\mathcal{L}_{\epsilon=0}$ is the Lagrangian evaluated at $\epsilon = 0$, that is using the metric (16) where the AdS radius is set to unity.

Let us now consider the boundary of the worldsheet corresponding to the n-gluon amplitude as in Figure 1. The boundary consists of n vectors $2\pi k_1^\mu, \ldots, 2\pi k_n^\mu$. We will denote by $\mathcal{C}_i$ the cusp where the vectors $2\pi k_i^\mu$ and $2\pi k_{i+1}^\mu$ meet. In this note, we propose that the infrared behavior of the n-gluon amplitude at strong coupling is fully captured by the local behavior of the worldsheet near the n cusps. To be more precise, we propose that the infrared divergences have the structure as in Figure 2. That is, [1]

$$\ln M_n \sim i S_{div}(\epsilon) = \sum_{i=1}^{n} i S_{i,i+1}(\epsilon). \tag{21}$$

[1] In eq. (21), it does not matter whether we write $\ln \mathcal{A}_n$ or $\ln M_n$ since the difference between them, $\ln \mathcal{A}_n^{tree}$ is independent of λ and, thus, is subleading at strong coupling.

The summation is over all the pairs of neighboring gluons or, equivalently, over all cusps. The i-th term in the right hand sides in eq. (21) and in Figure 2 represents the area $S_{i,i+1}(\epsilon)$ of the minimal worldsheet whose boundary is just the two vectors $2\pi k_i^\mu$ and $2\pi k_{i+1}^\mu$. As a trivial consistency check, we note that eq. (21) has the same additive structure as eqs. (5), (6) on the field theory side.

Since eq. (21) and Figure 2 have an additive structure, it is sufficient to single out only one cusp $\mathcal{C}_i$ and consider the problem of finding a minimal worldsheet which ends on the vectors $2\pi k_i^\mu$ and $2\pi k_{i+1}^\mu$. We want to point out that our calculation is universal and does not depend on global structure of the amplitude. Without loss of generality, we can assume that the worldsheet is located in the subspace parametrized by (y_0, y_1, y_2, r) and set $y_3 = 0$. [2] It is convenient to introduce the light-cone coordinates in the (y_0, y_1) plane

$$y_- = y_0 - y_1, \quad y_+ = y_0 + y_1. \tag{22}$$

We need to find a solution that turns into two intersecting lines whose directions are specified by k_i and k_{i+1} as $r \to 0$. We can choose the coordinate system in such a way that one of the vectors, say k_i, is located in the (y_-, y_+) plane. Moreover, we can chose k_i to lie along the y_+ direction. We parametrize it as

$$2\pi k_i = z_1(0, 1, 0) \tag{23}$$

in the (y_-, y_+, y_2) coordinates. The parameter z_1 is arbitrary as long as it is non-zero and finite. Similarly, we parametrize the vector k_{i+1} as

$$2\pi k_{i+1} = z_2(\alpha, 1, \sqrt{\alpha}), \tag{24}$$

where α is the tangent of the angle between the lines when they are projected to the (y_-, y_+) plane and z_2, like z_1, is an arbitrary non-zero finite parameter.From eqs. (23) and (24) it follows that

$$(2\pi)^2 s_{i,i+1} = -\alpha z_1 z_2. \tag{25}$$

The solution to all orders in ϵ with the appropriate boundary conditions was found in [10] to be

$$r(y_-, y_+) = \sqrt{1+\epsilon/2}\sqrt{2}\sqrt{y_-(y_+ - \frac{1}{\alpha}y_-)}, \quad y_2(y_-, y_+) = \frac{1}{\sqrt{\alpha}}y_-. \tag{26}$$

To continue, it is convenient to change the variables from (y_-, y_+) to (Y_-, Y_+) so that the (Y_-, Y_+) components of $2\pi k_i$ and $2\pi k_{i+1}$ become $(0, 1)$ and $(1, 0)$. The transformation is the following

$$y_- = \alpha z_2 Y_-, \quad y_+ = z_1 Y_+ + z_2 Y_-. \tag{27}$$

In these new variables we get

$$r(Y_-, Y_+) = \sqrt{1+\epsilon/2}\sqrt{2}\sqrt{Y_- Y_+}\sqrt{-(2\pi)^2 s_{i,i+1}}, \quad y_2(Y_-, Y_+) = \sqrt{\alpha} z_2 Y_-. \tag{28}$$

Substituting eqs. (27), (28) into the action (20), we obtain

$$iS_{i,i+1}(\epsilon) = -\frac{\sqrt{\lambda_D c_D}}{4\pi}\frac{\sqrt{1+\epsilon}}{(1+\epsilon/2)^{1+\epsilon/2}}(-(2\pi)^2 s_{i,i+1})^{-\epsilon/2}\int_0^1 \frac{dY_- dY_+}{(2Y_- Y_+)^{1+\epsilon/2}}. \tag{29}$$

[2] In principle, one can choose a coordinate system in which two light-like vectors lie in a two-plane. The choice of the coordinate system is a matter of convenience. The physical conclusions studied below are, of course, independent of this choice.

Assuming that $\epsilon < 0$ and performing the integral, we find that

$$iS_{i,i+1}(\epsilon) = -\frac{1}{\epsilon^2}\frac{\sqrt{\lambda}}{2\pi}\sqrt{\frac{\mu^{2\epsilon}}{(-s_{i,i+1})^{\epsilon}}}C(\epsilon), \tag{30}$$

where $C(\epsilon)$ is given by

$$C(\epsilon) = \frac{\sqrt{c_D}}{2^{\epsilon/2}}\frac{(2\pi)^{-\epsilon}}{(4\pi e^{-\gamma})^{\epsilon/2}}\frac{\sqrt{1+\epsilon}}{(1+\epsilon/2)^{1+\epsilon/2}}. \tag{31}$$

Eq. (30) represents our final answer for $S_{i,i+1}(\epsilon)$ to all orders in ϵ. Note that it depends only on the kinematic invariant $s_{i,i+1}$ and not separately on α, z_1 and z_2. Furthermore, note that eq. (30) is consistent with the general properties of the infrared behavior of $\mathcal{N} = 4$ gluon amplitudes reviewed in the previous section. It is of the form (10), where the function $F\left(\lambda\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon}, \epsilon\right)$ is given by

$$F\left(\lambda\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon}, \epsilon\right) = -\frac{1}{2\pi}\sqrt{\lambda\left(\frac{\mu^2}{(-s_{i,i+1})}\right)^{\epsilon}}C(\epsilon). \tag{32}$$

To compare eq. (30) with eq. (11) on the field theory side we have to expand $C(\epsilon)$ to the linear order in ϵ. Up to a constant, this yields the following ϵ-independent term in $iS_{i,i+1}(\epsilon)$

$$-\frac{1}{16}f(\lambda)\ln^2\left(\frac{\mu^2}{-s_{i,i+1}}\right) - \frac{1}{4}g(\lambda)\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right), \tag{33}$$

where

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} \tag{34}$$

and

$$g(\lambda) = \frac{\sqrt{\lambda}}{2\pi}(1 - \ln 2). \tag{35}$$

Since the general structure of the infrared divergences implies that the coefficient at $\ln^2\left(\frac{\mu^2}{-s_{i,i+1}}\right)$ is the cusp anomalous dimension, we find that it behaves as $\sqrt{\lambda}$ at strong coupling which is in agreement with [1, 5, 6].

As the last consistency check, let us compare (30) with the infrared divergent contribution to the strong coupling limit of the four-gluon amplitude which was obtain by Alday and Maldacena in [1]. In the case of the four-gluon amplitude we have only two independent kinematic invariants which are usually denoted by s and t,

$$s = s_{12} = s_{34}, \quad t = s_{23} = s_{41}. \tag{36}$$

As the result, eq. (21) becomes

$$S_{div}(\epsilon) = 2S_s(\epsilon) + 2S_t(\epsilon), \tag{37}$$

where $S_s(\epsilon)$ is given by

$$iS_s(\epsilon) = -\frac{1}{\epsilon^2}\frac{\sqrt{\lambda}}{2\pi}\sqrt{\frac{\mu^{2\epsilon}}{(-s)^{\epsilon}}} - \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}(1 - \ln 2)\sqrt{\frac{\mu^{2\epsilon}}{(-s)^{\epsilon}}} + \mathcal{O}(\epsilon^0) \tag{38}$$

and $S_t(\epsilon)$ is given by the similar expression with s replaced with t. Quite remarkably, eqs. (37) and (38) exactly coincide with the infrared behavior of the four-gluon amplitude computed in [1] which represents a non-trivial check of our proposal.

Acknowledgements

The author is very grateful to Freddy Cachazo, Juan Maldacena and Arkady Tseytlin for helpful discussions. The work at the Perimeter Institute is supported by in part by the Government of Canada through NSERC and by the Province of Ontario through MRI.

References

[1] L. F. Alday and J. Maldacena, "Gluon scattering amplitudes at strong coupling," arXiv:0705.0303.

[2] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," Adv.Theor.Math.Phys. 2 (1998) 231-252; Int.J.Theor.Phys. 38 (1999) 1113-1133 [arXiv:hep-th/9711200].

[3] Z. Bern, L. J. Dixon and V. A. Smirnov, "Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond," Phys.Rev. D72 (2005) 085001 [arXiv:hep-th/0505205].

[4] C. Anastasiou, Z. Bern, L. Dixon and D. A. Kosower, "Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory," Phys.Rev.Lett. 91 (2003) 251602 [hep-th/0309040].

[5] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "A semi-classical limit of the gauge/string correspondence," Nucl.Phys. B636 (2002) 99-114 [arXiv:hep-th/0204051].

[6] M. Kruczenski, "A note on twist two operators in N=4 SYM and Wilson loops in Minkowski signature," JHEP 0212 (2002) 024 [arXiv:hep-th/0210115].

[7] L. Magnea and G. Sterman, "Analytic continuation of the Sudakov form-factor in QCD," Phys.Rev.D42:4222-4227,1990.

[8] S. Catani, "The Singular Behaviour of QCD Amplitudes at Two-loop Order," Phys.Lett. B427 (1998) 161-171 [arXiv:hep-ph/9802439].

[9] G. Sterman and M. E. Tejeda-Yeomans, "Multi-loop Amplitudes and Resummation," Phys.Lett. B552 (2003) 48-56 [arXiv:hep-ph/0210130].

[10] E. I. Buchbinder, "Infrared Limit of Gluon Amplitudes at Strong Coupling," [arXiv:0706.2015].

[11] D. J. Gross and P. F. Mende, " The High-Energy Behavior of String Scattering Amplitudes," Phys.Lett.B197 (1987) 129.

[12] S. Abel, S. Forste and V. V. Khoze, "Scattering amplitudes in strongly coupled N=4 SYM from semiclassical strings in AdS," [arXiv:0705.2113].

[13] R. Kallosh and A. A. Tseytlin, "Simplifying superstring action on $AdS_5 \times S^5$," JHEP 9810 (1998) 016 [arXiv:hep-th/9808088].