TUNNELING OF CHARGED AND MAGNETIZED FERMIONS FROM A ROTATING DYONIC TAUB-NUT BLACK HOLE

KAUSARI SULTANA

Department of Mathematics,
Shahjalal University of Science and Technology, Sylhet-3100, Bangladesh
Email: kausarisu@gmail.com

Abstract. We investigate tunneling of charged and magnetized Dirac particles from a rotating dyonic Taub-NUT (TN) black hole (BH) called the Kerr-Newman-Kasuya-Tub-NUT (KNKTN) BH endowed with electric as well as magnetic charges. We derive the tunneling probability of outgoing charged particles by using the semiclassical WKB approximation to the covariant Dirac equation and obtain the corresponding Hawking temperature. The emission spectrum deviates from the purely thermal spectrum with the leading term exactly the Boltzman factor, if energy conservation and the backreaction of particles to the spacetime are considered. The results provides a quantum-corrected radiation temperature depending on the BH background and the radiation particles energy, angular momentum, and charges. The results are consistent with those already available in literature.

Key words: Dyonic black hole – Hawking radiation – Quantum Tunneling – Backreaction – NUT parameter.

1. INTRODUCTION

Hawking's discovery that a BH can radiate thermally (Hawking, 1974, 1975) has attracted lots of physicists' attention and many papers have appeared to deeply discuss the quantum radiation of BHs via different methods (Damour and Ruffini, 1976; Hawking and Gibbons, 1977; Zhu *et al.*, 1995; Jing, 2001; Wu and Cai, 2000). Much interests have grown up in exploring quantum phenomenon of Hawking radiation from BHs as a tunneling technique of emitting quantum particles. The tunneling rate with the WKB approximation takes the form: $\Gamma \propto \exp[-2ImI]$ with I the classical action of the trajectory. Therefore, it is important for this tunneling method to calculate the imaginary part of the action. There are two universal methods to compute particle's action. The first one is the null geodesic method developed by Parikh and Wilczek (PW) (Parikh and Wilczek, 2000; Parikh, 2004), following the work of Kraus and Wilczek (Kraus and Wilczek, 1995), in which the imaginary part of the action is regarded as the only contribution of the momentum p_r of the emitted null s-wave. The barrier is created by the outgoing particles themselves and a corrected spectrum can be derived when self-gravitation of particles is taken into account. The second tunneling method, called Hamilton-Jacobi (HJ) ansatz, is proposed by Srinivasan and Padmanabhan (Srinivasan and Padmanabhan, 1999; Shankar*et al.*, 2001) and has been developed by Angheben et al. (Angheben *et al.*, 2005), which successfully derives the imaginary part of the action by solving the HJ equation. In this method, the same conclusion as the first method can be drawn.

Kerner and Mann (Kerner and Mann, 2006) investigated quantum tunneling methods for calculating temperatures of KN and TN BHs, using both the null-geodesic and HJ methods. Subsequently, using the WKB approximation to the covariant Dirac equation, Kerner and Mann (KM) developed the calculations of the tunneling process for the spin-1/2 particle's emission from nonrotating BHs (Kerner and Mann, 2008a) and investigated the Hawking temperature for the KN BH (Kerner and Mann, 2008b). This model has been extended to study the tunneling of charged and magnetized fermions from the RN (Li and Han, 2008) and KN-AdS (Zeng and Li, 2009a) BHs with magnetic charges. HJ method and KM tunneling approach have also been exploited in investigating tunneling of scalar and Dirac particles from the TN-AdS BH (Zeng and Li, 2009b). There are some works of investigating the tunneling phenomenon for different BHs (Sharif and Javed, 2010, 2012a,b,c, 2013a) (including accelerating as well as rotating (Gillani *et al.*, 2011; Gillani and Saifullah, 2011; Rehman and Saifullah, 2011; Sharif and Javed, 2013b)) by employing the aforementioned methods.

My previous work, with my honorable supervisor during Ph.D.degree (Ali and Sultana, 2013a), is a study of charged particles Hawking radiation via tunneling of both horizons from the Reissner-Nordstrm-Taub-NUT (RNTN) black holes. All these works are in agreement with Parikhs work and show no loss of information. Also in (Ali and Sultana, 2013b) we calculate, following Zhou and Liu (Zhou and Liu, 2008), the temperature of the inner horizon of the dyonic KNKTN-AdS black hole, which is a rotating RNTN black hole in AdS space and prove the existence of thermal characters of the inner horizon. Like as in the RNTN black hole case (Ali and Sultana, 2013a), the inner horizon of the KNKTN-AdS black hole emits positive energy particles inside the inner horizon (towards the singularity) with a positive temperature. In order to maintain a local energy balance, antiparticles with negative energy are emitted away from the singularity through the inner horizon. This is a process analogous to that takes place at the outer horizon according to the Hawking effect at the outer horizon antiparticles go in and particles come out. The real particle remains inside the inner horizon and finally meets with the singularity. But the antiparticle enters the intermediate region between the horizons. Traveling across the intermediate region this antiparticle finally comes out from the white hole horizon, if the backscattering effects are neglected.

In this paper, I apply the KM tunneling approach to calculate emission rate of charged and magnetized fermions from a rotating dyonic TN BH namely the KNK BH generalized NUT or magnetic monopole parameter. This BH is a more general background to be investigated. Conservation of energy and the particles' backreaction lead to the same terminations as the previous works. The result can be treated as tunneling radiation at a quantum corrected temperature, which is dependent on not only the BH background, but also the tunneling particle's energy, angular momentum, and charges. The interest in the possibility of dyonic BHs has grown since magnetic monopoles have been predicted in various extensions of the standard model of particle physics. The magnetic monopole hypothesis in general relativity was put forward by Dirac (Dirac, 1948) relatively long ago. His ingenious suggestion of existing magnetic monopole in nature was neglected due to the failure to detect such objects. In recent years, however, the development of gauge theories has shed new light on it. The string theory (Mignemi, 1995) admits the existence of such objects. By exhausting the energies related to rotation and charges, the KNKTN BH may reduce to the interesting TN BH, which plays an important role in the conceptional development of general relativity and in the construction of brane solutions in string theory and M-theory (Cherkis and Hashimoto, 2002; Clarkson *et al.*, 2004a,b). According to Misner the TN spacetime has the interpretation of being a counter example to almost anything (Misner, 1967). It has drawn a particular interest in recent years, because it plays the role in furthering our understanding of the AdS/CFT correspondence (Kerner and Mann, 2006; Hawking *et al.*, 1999; Chamblin *et al.*, 1999; Mann, 1999) and in this regard, the thermodynamics of various TN solutions has become a subject of intense study. The Dirac field in TN background has been analyzed (Comtet and Horvathy, 1995; Cotuaescu and Visinescu, 2001a,b). Of course, the existence of the closed time-like geodesics in the TN spacetime violates the causality condition. Nevertheless, one can explore the half-closed time-like geodesics of Taub area in the NUT area, so the naked singularity does exist. In the meantime, it admits no angular velocity and no superradiation occurs at the event horizon. Hawking radiation from the TN BH was found to agree with Parikh and Wilczek's result (Chen and Zu, 2008).

The paper is organized as follows. In Sec 2, we describe the background geometry of the KNKTN BH with magnetic charges and perform the dragging coordinate transformation. In Sec 3, we provide Dirac equation of charged and magnetized particles and compute the tunneling probability as well as the corresponding temperature across the event horizon. A precise construction of the particles action has also been done. In Sec 4, we calculate the emitting rate by considering backreaction of the radiation. Finally, we give our concluding remarks in Sec 5.

2. DYONIC KNKTN BLACK HOLE

The metric of the dyonic KNKTN BH in Boyer-Lindquist coordinates can be written as:

$$ds^2 = -\frac{\Delta_r}{\Sigma}(dt - \eta d\varphi)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}(adt - \rho^2 d\varphi)^2, \tag{1}$$

where $\Sigma = r^2 + (n + a\cos\theta)^2$, $\eta = a\sin^2\theta - 2n\cos\theta$, $\Delta_r = \rho^2 - 2(Mr + n^2) + Q^2 + P^2$, $\rho^2 = r^2 + a^2 + n^2$ and M, $a(= J/M)$, Q, P, n are respectively the mass, angular momentum per unit mass parameter, electric charge, magnetic charge and NUT (magnetic mass) parameters of the BH. The presence of the NUT parameter makes the spacetime asymptotically nonflat. In a recent work, Aliev (Aliev, 2008) interpreted the NUT parameter as generating a "rotational effect". The associated "specific angular momentum" is $J_M = nM$, which we have incorporated in the following analysis. The gauge potential associated with the metric eq(1) is (Cazares, 2013)

$$\mathcal{A} = A_\mu + iB_\mu = \left[-\frac{Qr}{\Sigma}(dt - \eta d\varphi) - \frac{P(n + a\cos\theta)}{a\Sigma}(adt - \rho^2 d\varphi)\right] + i\left[-\frac{Pr}{\Sigma}(dt - \eta d\varphi) + \frac{Q(n + a\cos\theta)}{a\Sigma}(adt - \rho^2 d\varphi)\right].$$

In our analysis we write the electric potential A_μ and the magnetic-like potential B_μ as

$$A_\mu = -\frac{Qr}{\Sigma}(dt - \eta d\varphi), \quad B_\mu = -\frac{Pr}{\Sigma}(dt - \eta d\varphi), \tag{2}$$

considering the appropriate limiting case. The event (outer) horizon r_+ and Cauchy (inner) horizon r_- of the BH are given by $r_\pm = M \pm \sqrt{M^2 - Q^2 - P^2 - a^2 + n^2}$. Since the outer infinite red-shift surface obtained from $g^{\mu\nu}\partial_\mu\partial_\nu f = 0$ of the metric eq(1) doesn't coincide with the event horizon, the geometrical optics limit cannot be used at the horizon and the semi-classical WKB approximate is invalid there. In order to view the Hawking radiation of fermions, we need to choose one coordinate system in which they will be coincident. We can perform this by either selecting the dragging coordinate system (t, r, θ) with $d\varphi = -\frac{g_{03}}{g_{33}}dt$, or introducing a new coordinate system (t, r, θ, χ). We adopt the latter one and set $\chi = \varphi - \Omega t$, where $\Omega = -\frac{g_{03}}{g_{33}} = \frac{\rho^2 a\sin^2\theta - \Delta_r\eta}{\rho^4\sin^2\theta - \Delta_r\eta^2}$ is the dragged angular velocity of the BH. The metric eq(1) then takes the form

$$ds^2 = -Fdt^2 + G^{-1}dr^2 + Hd\theta^2 + Kd\chi^2, \tag{3}$$

where

$$F(r,\theta) = \frac{\Delta_r\sin^2\theta(\rho^2 - a\eta)^2}{\Sigma(\rho^4\sin^2\theta - \Delta_r\eta^2)}, \quad G(r,\theta) = \frac{\Delta_r}{\Sigma},$$

$$K(r,\theta) = \frac{1}{\Sigma}(\rho^4 \sin^2\theta - \Delta_r \eta^2), \quad H(r,\theta) = \Sigma, \tag{4}$$

with the corresponding gauge potential of electric and magnetic fields

$$A_\mu = -\frac{Qr}{\Sigma} \cdot \frac{\rho^2 \sin^2\theta(\rho^2 - a\eta)}{\rho^4 \sin^2\theta - \Delta_r \eta^2} dt + \frac{Qr\eta}{\Sigma} d\chi,$$

$$B_\mu = -\frac{Pr}{\Sigma} \cdot \frac{\rho^2 \sin^2\theta(\rho^2 - a\eta)}{\rho^4 \sin^2\theta - \Delta_r \eta^2} dt + \frac{Pr\eta}{\Sigma} d\chi. \tag{5}$$

In metric eq(3), the outer (inner) horizons coincide with the outer (inner) infinite red-shift surfaces. The Landau's condition of the coordinate clock synchronization is also satisfied. So, we can investigate the radiation of fermions for this BH.

3. CHARGED AND MAGNETIZED PARTICLES TUNNELING

In order to study the tunneling of charged and magnetized fermion of mass μ_o from a KNKTN BH, we consider the Dirac equation in covariant form as (Sharif and Javed, 2013b)

$$\gamma^\mu \left[i\hbar \left(\partial_\mu + \frac{i}{2} \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta} \right) + q_{\mathrm{e}} A_\mu + q_{\mathrm{m}} B_\mu \right] \Psi + \mu_o \Psi = 0, \tag{6}$$

where $\Gamma_\mu^{\alpha\beta} = \mathrm{g}^{\beta\gamma}\Gamma^\alpha_{\mu\gamma}$, $\Sigma_{\alpha\beta} = \frac{1}{4} i [\gamma^\alpha, \gamma^\beta]$ and γ^μ marices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\mathrm{g}^{\mu\nu} I$. We choose the γ^μ matrices as

$$\gamma^t = \frac{1}{\sqrt{F(r,\theta)}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \gamma^r = \sqrt{G(r,\theta)} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix},$$

$$\gamma^\theta = \frac{1}{\sqrt{H(r,\theta)}} \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \gamma^\chi = \frac{1}{\sqrt{K(r,\theta)}} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \tag{7}$$

where σ^i $(i = 1,2,3)$ are Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

Dirac matrices imply that $[\gamma^\alpha, \gamma^\beta] = 0$ or $-[\gamma^\beta, \gamma^\alpha]$ according as $\alpha = \beta$ or $\alpha \neq \beta$. Consequently, contribution of the term containing $\Sigma_{\alpha\beta}$ in eq(6) is zero. The eigenvectors of σ^3 are denoted by $\xi_{\uparrow/\downarrow}$. Then measuring spin in the r-direction we have two spin states for the spinor wave function Ψ (related to the particle's action): *spin-up* in $+$ve r-direction and *spin-down* in $-$ve r-direction. Accordingly, we assume the two following ansatz for the spin-1/2 Dirac field (Kerner and Mann, 2008a):

$$\Psi_\uparrow = \begin{bmatrix} A\xi_\uparrow \\ B\xi_\uparrow \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right] = [A, 0, B, 0]' \exp\left[\frac{i}{\hbar} I_\uparrow\right], \tag{9}$$

$$\Psi_{\downarrow} = \begin{bmatrix} C\xi_{\downarrow} \\ D\xi_{\downarrow} \end{bmatrix} \exp\left[\frac{i}{\hbar} I_{\downarrow}\right] = [0, C, 0, D]' \exp\left[\frac{i}{\hbar} I_{\downarrow}\right], \tag{10}$$

where the action of the radiant spin particles $I_{\uparrow/\downarrow}$ and the wave modes represented by A, B, C, D are all functions of (t, r, θ, χ). In this paper, we only analyze the spin-up case because the final result is the same as the spin-down case, as can be presented by using the methods described below. Inserting eq(9) into the Dirac Equation eq(6) and applying WKB approximation, we find

$$-\left(\frac{iA}{\sqrt{F(r,\theta)}}(\partial_t I_{\uparrow} - q_{\mathrm{e}} A_t - q_{\mathrm{m}} B_t) + B\sqrt{G(r,\theta)}\partial_r I_{\uparrow}\right) + \mu_o A = 0, \tag{11}$$

$$-B\left(\frac{1}{\sqrt{H(r,\theta)}}\partial_\theta I_{\uparrow} + \frac{i}{\sqrt{K(r,\theta)}}(\partial_\chi I_{\uparrow} + q_{\mathrm{e}} A_\chi + q_{\mathrm{m}} B_\chi)\right) = 0, \tag{12}$$

$$\left(\frac{iB}{\sqrt{F(r,\theta)}}(\partial_t I_{\uparrow} - q_{\mathrm{e}} A_t - q_{\mathrm{m}} B_t) - A\sqrt{G(r,\theta)}\partial_r I_{\uparrow}\right) + \mu_o B = 0, \tag{13}$$

$$-A\left(\frac{1}{\sqrt{H(r,\theta)}}\partial_\theta I_{\uparrow} + \frac{i}{\sqrt{K(r,\theta)}}(\partial_\chi I_{\uparrow} + q_{\mathrm{e}} A_\chi + q_{\mathrm{m}} B_\chi)\right) = 0. \tag{14}$$

It is quite difficult to immediately determine the action. Nevertheless, taking into account the existence of time-like killing vector $(\frac{\partial}{\partial t})^a$ and space-like killing vector $(\frac{\partial}{\partial \varphi})^a$ in the stationary space time, we carry out the separation variable as

$$I_{\uparrow} = -\omega t + W(r) + \tilde{j}\varphi + \Theta(\theta), \tag{15}$$

where ω is the energy and $\tilde{j} = \tilde{j}(j, j_M)$ is the magnetic quantum number of the particle. Using eq.(15) in eq.(11)–eq.(14) and Taylor's expansion of $F(r,\theta)$ near the horizon r_+, we obtain with $iA = B$, $iB = A$,

$$-B\left(\frac{-\omega + \omega_o}{\sqrt{(r - r_+)\partial_r F(r_+,\theta)}} + \sqrt{(r - r_+)\partial_r G(r_+,\theta)}(\partial_r W)\right) + \mu_o A = 0, \tag{16}$$

$$-B\left(\frac{1}{\sqrt{H(r_+,\theta)}}\partial_\theta \Theta + \frac{i}{\sqrt{K(r_+,\theta)}}\tilde{j} + q_{\mathrm{e}} A_\chi(r_+) + q_{\mathrm{m}} B_\chi(r_+))\right) = 0, \tag{17}$$

$$A\left(\frac{-\omega + \omega_o}{\sqrt{(r - r_+)\partial_r F(r_+,\theta)}} - \sqrt{(r - r_+)\partial_r G(r_+,\theta)}(\partial_r W)\right) + \mu_o B = 0, \tag{18}$$

$$-A\left(\frac{1}{\sqrt{H(r_+,\theta)}}\partial_\theta \Theta + \frac{i}{\sqrt{K(r_+,\theta)}}\tilde{j} + q_{\mathrm{e}} A_\chi(r_+) + q_{\mathrm{m}} B_\chi(r_+))\right) = 0, \tag{19}$$

where $\omega_o = \frac{\tilde{j}a}{r_+^2 + a^2 + n^2} + \frac{q_{\mathrm{e}} Q r_+}{r_+^2 + a^2 + n^2} + \frac{q_{\mathrm{m}} P r_+}{r_+^2 + a^2 + n^2}$.

We deal first with eq.(16)–eq.(19) for massless ($\mu_o = 0$) case. Accordingly, eq.(16) and eq.(18) imply that

$$W'(r) = W'_+(r) = -W'_-(r) = \left[\frac{(r_+^2 + a^2 + n^2)(\omega - \omega_o)}{(r - r_+)(r_+ - r_-)}\right], \tag{20}$$

where W_+ corresponds to outward solutions and W_- corresponds to the incoming solutions. Thus the particle's tunneling probability from inside to outside the horizon is

$$\Gamma = \frac{\text{Prob[out]}}{\text{Prob[in]}} = \frac{\exp[-2(ImW_+ + Im\Theta)]}{\exp[-2(ImW_- + Im\Theta)]} = \exp[-4ImW_+], \tag{21}$$

where W_+ is the integral of eq.(20). There is a pole at the horizon, $r = r_+$, in eq.(20). After integrating around the pole, we find

$$W_+ = \left[\frac{\pi i(r_+^2 + a^2 + n^2)(\omega - \omega_o)}{r_+ - r_-}\right], \tag{22}$$

and consequently, the resultant tunneling probability to leading order in $\hbar$ is given by

$$\Gamma = \exp\left[\frac{-4\pi(r_+^2 + a^2 + n^2)(\omega - \omega_o)}{r_+ - r_-}\right] = \exp[-2\pi(\omega - \omega_o)/\kappa]. \tag{23}$$

Hence, the Hawking temperature of the KNKTN BH is recovered as

$$T = \frac{1}{\beta} = \frac{\kappa}{2\pi}, \quad \kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2 + n^2)}. \tag{24}$$

This is fully in consistence with that obtained by other method. For $n = 0$, it reduces to the temperature of the KN BH (Kerner and Mann, 2008a), which further reduces to the temperature of the RN BH for $a = 0$ (Li and Han, 2008). In absence of charge ($Q = 0 = P$), it exactly becomes the Hawking temperature of the Schwarzschild BH (Gillani *et al.*, 2011). In the massive particles ($\mu_o \neq 0$) case, adopting the same process, we can find the same temperature. Thus, when a BH radiates massless particles and massive particles, the tunneling probability and Hawking temperature remain the same and are not related to the kind of particles. In the case of the spin-down case, adopting the corresponding (spin-down) wave function and analyzing it again as the analogous process, we can find the same result.

The action $I_\uparrow$ in the spin-up case is obtained by solving eq.(16)–eq.(19). For outgoing particles, eq.(16) gives on integration

$$W(r) = W_+(r)$$

$$= \int \frac{\mu_o A}{B\sqrt{(r - r_+)\partial_r G(r_+,\theta)}} dr - \frac{-\omega + \omega_o}{\sqrt{\partial_r F(r_+,\theta)\partial_r G(r_+,\theta)}} \ln(r - r_+), \tag{25}$$

while for the incoming particles, eq.(18) yields

$$W(r) = W_-(r)$$
$$= \int \frac{\mu_o B}{A\sqrt{(r-r_+)\partial_r G(r_+,\theta)}} dr + \frac{-\omega+\omega_o}{\sqrt{\partial_r F(r_+,\theta)\partial_r G(r_+,\theta)}} \ln(r-r_+). \tag{26}$$

Equations eq.(17) and eq.(19) imply that

$$\Theta = -i\left[\tilde{j}\ln\tan\frac{\theta}{2} + \frac{a-(q_{\mathrm{e}}Q+q_{\mathrm{m}}P)r_+}{r_+^2+a^2+n^2}(a\cos\theta + 2n\ln\sin\theta)\right]. \tag{27}$$

Equations eq.(25) and eq.(27) with eq.(15) determine the outgoing massive particles' action. This expression reduces to the massless particles' action for $\mu_o = 0$. Likewise, one can find the action for the ingoing particle either massive or massless.

4. BACKREACTION IN THE TUNNELING PROCESS

In this section, we consider the emitting particles' backreaction on the space-time. When a particle with energy ω_i, charge q_{ie}, magnet q_{im}, and angular momentum $\tilde{j}_i$ tunnels out of the BH, the parameters M, Q, P, a, n should be substituted by $(M-\omega_i)$, $(Q-q_{ie})$, $(P-q_{im})$, $a_i = \frac{Ma-\tilde{j}_i}{M-\omega_i}$ and $n_i = \frac{Mn-\tilde{j}_i}{M-\omega_i}$, respectively. Then, the emission rate is

$$\Gamma_i = \exp[-2\pi(\omega_i - \omega_{i0})/\kappa_i], \tag{28}$$

where

$$\omega_{i0} = \tilde{j}_i\Omega_{i+} + q_{ie}A_{i+} + q_{im}B_{i+}$$
$$= \frac{\tilde{j}_i a_i}{r_{i+}^2+a_i^2+n_i^2} + \frac{q_{ie}(Q-q_{ie})r_{i+}}{r_{i+}^2+a_i^2+n_i^2} + \frac{q_{im}(P-q_{im})r_{i+}}{r_{i+}^2+a_i^2+n_i^2},$$
$$r_{i\pm} = (M-\omega_i) \pm \sqrt{(M-\omega_i)^2 - a_i^2 - z_i^2 + n_i^2},$$
$$z_i^2 = (Q-q_{ie})^2 + (P-q_{im})^2, \quad \kappa_i = \frac{(r_{i+}-r_{i-})}{2(r_{i+}^2+a_i^2+n_i^2)}. \tag{29}$$

For emission of many particles and thinking that they radiate one by one, we get

$$\Gamma = \prod_i \Gamma_i = \exp\left[\sum_i (-2\pi(\omega_i - \omega_{i0}))/\kappa_i\right]. \tag{30}$$

If the emission is regarded as a continuous procession, the sum in eq.(30) could be replaced by integration

$$\Gamma = \exp[-2\pi\int(d\omega' - \Omega'_+ d\tilde{j}' - A'_+ dq'_{\mathrm{e}} - B'_+ dq'_{\mathrm{m}})/\kappa']. \tag{31}$$

We consider the entropy $S = A/4 = \pi(r_+^2 + a^2 + n^2)$ and obtain the difference between the entropies of the horizon before and after the emission, $\Delta S = S_f - S_i$:

$$\begin{aligned}\Delta S = \pi\Big[&2(M-\omega)^2 - (Q-q_\mathrm{e})^2 - (P-q_\mathrm{m})^2 + 2n'^2\\ &+2(M-\omega)\sqrt{(M-\omega)^2-(Q-q_\mathrm{e})^2-(P-q_\mathrm{m})^2-a'^2+n'^2}\\ &-2M^2+Q^2+P^2-2n^2-2M\sqrt{M^2-Q^2-P^2-a^2+n^2}\Big].\end{aligned} \tag{32}$$

We find

$$\begin{aligned}\frac{1}{2\pi}\frac{\partial(\Delta S)}{\partial\omega'} &= -\frac{((M-\omega')+\sqrt{(M-\omega')^2-z'^2-a'^2+n'^2})^2}{\sqrt{(M-\omega')^2-z'^2-a'^2+n'^2}}\\ &\quad -\frac{(M-\omega')a'^2+n'^2[(M-\omega')-2r'_+]}{(M-\omega')\sqrt{(M-\omega')^2-z'^2-a'^2+n'^2}}\\ &= -2\frac{r'^2_+ + a'^2+n'^2}{r'_+-r'_-} - \frac{4n'^2r'_+}{(M-\omega')(r'_+-r'_-)} \simeq -\frac{1}{\kappa'},\\ \frac{1}{2\pi}\frac{\partial(\Delta S)}{\partial q'_\mathrm{e}} &= \frac{(Q-q'_\mathrm{e})((M-\omega')+\sqrt{(M-\omega')^2-z'^2-a'^2+n'^2})}{\sqrt{(M-\omega')^2-z'^2-a'^2+n'^2}} = \frac{A'_+}{\kappa'},\\ \frac{1}{2\pi}\frac{\partial(\Delta S)}{\partial q'_\mathrm{m}} &= \frac{(Q-q'_\mathrm{m})((M-\omega')+\sqrt{(M-\omega')^2-z'^2-a'^2+n'^2})}{\sqrt{(M-\omega')^2-z'^2-a'^2+n'^2}} = \frac{B'_+}{\kappa'},\\ \frac{1}{2\pi}\frac{\partial(\Delta S)}{\partial \tilde{j}'} &= \frac{a'}{\sqrt{(M-\omega')^2-z'^2-a'^2+n'^2}} = \frac{\Omega'_+}{\kappa'},\end{aligned} \tag{33}$$

where $a' = \left(\frac{Ma-\tilde{j}'}{M-\omega'}\right)$, $n' = \left(\frac{Mn-\tilde{j}'}{M-\omega'}\right)$ and $z'^2 = (Q-q'_\mathrm{e})^2 + (P-q'_\mathrm{m})^2$. From eq.(31) and eq.(33), we deduce that the emission rate is connected with the change in Bekenstein-Hawking entropy:

$$\Gamma = e^{\int d(\Delta S)} = e^{\Delta S}. \tag{34}$$

Expanding the emission rate Γ in ω, q_e, q_m and $\tilde{j}$, one finds

$$\Gamma = \exp[-\beta(\omega-\omega_0) + \mathcal{O}(\omega, q_\mathrm{e}, q_\mathrm{m}, \tilde{j})^2]. \tag{35}$$

It depicts that the emission rate in the tunneling approach, up to first order in ω, retrieves the Boltzmann factor of the form $\exp[-\beta\omega]$ with β the inverse Hawking temperature. The higher order terms of ω, q_e, q_m, $\tilde{j}$ describe self-interaction effects resulting from the energy conservation. They are a deviation from a purely thermal spectrum. So, some information can be brought out of the BH with the corrected spectrum. This can give an explanation to the information loss paradox. Equation eq.(35) can be put in the form

$$\Gamma = \exp[-\beta'(\omega-\omega_0)], \quad \beta' = \frac{1}{T'} = \beta\left[1 - \frac{\mathcal{O}(\omega, q_\mathrm{e}, q_\mathrm{m}, \tilde{j})^2}{\beta(\omega-\omega_0)}\right], \tag{36}$$

where β' can be treated as an inverse quantum-corrected temperature. This result goes over to the KN BH case (Zhou and Liu, 2008) in the limit $n = 0 = P$.

In quantum mechanics the emitting rate is given by $\Gamma(i \to f) = |A_{fi}|^2 \delta_p$ where A_{fi} is the amplitude for the tunneling action. The *phase space factor* δ_p is the average of the number of microstates of the initial state and the number of microstates of the final state. Since the number of microstates of the initial and final states are the exponent of the initial and final entropies, $\Gamma = e^{S_f}/e^{S_i} = e^{\Delta S}$. Manifestly, this is consistent with our result. Thus, our result satisfies the underlying unitary theory in quantum mechanics and thereby provides a might explanation to the BH information puzzle.

5. CONCLUDING REMARKS

Our concern in this study is to apply the KM fermion tunneling method (Kerner and Mann, 2008a,b) to charged and magnetized fermion cases for a rotating dyonic TN BH described by the KNKTN metric eq.(1). In this semiclassical method the horizon plays a role of two way energy barrier for a pair of positive and negative energy particles. Although classically a particle can only fall inside the event horizon, there exists a crossing of a particle's energy level near the event horizon in the semiclassical approach. The energy of a negative energy particle can be larger than the energy of the nearby positive energy level. As a result, it can travel across the forbidden region because of the quantum tunneling effect and becomes a positive energy particle that can move out. In our investigation we have therefore computed tunneling probabilities for both incoming as well as outgoing particles. We find the tunneling probability of emission for the spin-up particle case at the event horizon and recover the corresponding Hawking temperature. The result shows that the tunneling probability depends upon fermion's charges but not upon its mass. The corresponding Hawking temperature depends upon mass, rotation and NUT parameters as well as electric and magnetic charges of the BH. One can find the result for the spin-down particle case with the equations of the spin-up case by only excluding a negative sign. The Hawking temperature implies that the tunneling rates of the spin-up and spin-down particles are the same and it remains invariant as well for both massive and massless cases (Kerner and Mann, 2008a).

We also have calculated corrections to the fermion emission temperature by computing corrections to the tunneling probability with fully taking into account conservation of energy and considering self-gravitational interaction and backreaction of radiant particles. The result shows that the tunneling probability of fermion is related to the change of the Bekenstein-Hawking entropy. We find that the quantum-corrected radiation temperature is dependent on the BH background and the radiation particle's energy, angular momentum and charges. The result implies that the emission spectrum is not purely thermal anymore and the leading term is exactly the Boltzman factor. This is consistent with the previous works done by using PW or HJ method. Our analysis gives as well results in agreement to that obtained by Damour-Ruffini's method in the case of charged Dirac particles' Hawking radiation from a KN BH (Zhou and Liu, 2008). The corrected spectrum of emission can bring some information out of the BH. The underlying unitary theory may be satisfied as well. In the limit $n = 0 = P$ our results reduce to the results of the KN BH (Kerner and Mann, 2008b) and $a = 0 = n$ yields the results of the RN BH (Li and Han, 2008).

REFERENCES

Ali, M.H., Sultana, K.: 2013, *Int. J. Theor. Phys.* **52**, 2802.
Ali, M.H., Sultana, K.: 2013, *Int. J. Theor. Phys.* **52**, 4537.
Angheben, M., Nadalini, M., Vanzo, L. and Zerbini, S.: 2005, *JHEP* **0505**, 014.
Aliev, A. N.: 2008, *Phys. Rev. D* **77**, 044038.
Cherkis, S., Hashimoto, A.: 2002, *J. High Energy Phys.***11**, 036.
Clarkson, R., Ghezelbash, A. M., Mann, R. B.: 2004, *J. High Energy Phys.***04**, 063.
Clarkson, R., Ghezelbash, A. M., Mann, R. B.: 2004, *J. High Energy Phys.***08**, 025.
Chamblin, A., Emparan, R., Johnson, C. V. and Myers, R. C.: 1999, *Phys. Rev. D* **59**, 064010.
Comtet, A., Horvathy, P. A.: 1995, *Phys. Lett. B* **349**, 49.
Cotuaescu, I. I., Visinescu, M.: 2001, *Int. J. Mod. Phys. A* **16**, 1743.
Cotuaescu, I. I., Visinescu, M.: 2001, *Phys. Lett. B* **502**, 229.
Chen, D., Zu, X.: 2008, *Acta Phys. Pol. B* **39**, 1329.
Cazares, J. A.: 2013, *arXiv:1211.6685v2 [hep-th]*, 25.
Damour, T.,Ruffini, R. : 1976, *Phys.Rev. D* **14**, 332.
Dirac, P. A. M.: 1948, *Phys. Rev.* **74**, 817.
Gillani, U. A., Rehman, M., Saifullah, K.: 2011, *JCAP* **06**, 016.
Gillani, U. A., Saifullah, K.: 2011, *Phys. Lett. B* **699**, 15.
Hawking, S.W.: 1974, *Nature* **248**, 30.
Hawking, S. W.: 1975, *Commun. Math. Phys.* **43**, 199.
Hawking, S. W., Gibbons, G. W.: 1977, *Phys. Rev. D* **15**, 2752.
Hawking, S. W., Hunter, C. J., Page, D. N.: 1999, *Phys. Rev. D***59**, 044033.
Jing, J. L.: 2001, *Chin. Phys.*, **10**, 234.
Kraus, P., Wilczek, F.: 1995, *Nucl. Phys. B* **433**, 403.
Kerner, R. and Mann, R. B.: 2006, *Phys. Rev. D* **73**, 104010.
Kerner, R. and Mann, R. B.: 2008, *Class. Quant. Grav.* **25**, 095014.
Kerner, R. and Mann, R. B.: 2008, *Phys. Lett. B* **665**, 277.
Li, Q., Han, Y. -W.: 2008, *Int. J. Theor. Phys.* **47**, 3248.
Mignemi, S.: 1995, *Phys. Rev. D***51**, 934.
Misner, C. W.: 1967, *Taub-NUT space as a counter example to almost anything, In: Ehlers, J. (ed.) Relativity Theory and Astrophysics 1. Lectures in Applied Mathematics, Am. Math. Soc., Providence* **08**, 160.
Mann, R. B.: 1999, *Phys. Rev. D* **60**, 104047.
Parikh, M.K., Wilczek, F.: 2000, *Phys. Rev. Lett.* **85**, 5042.
Parikh, M.K.: 2004, *Gen. Relativ. Gravit.* **36**, 2419.
Rehman, M., Saifullah, K.: 2011, *JCAP* **03**, 001.

Srinivasan, K., Padmanabhan, T.: 1999, *Phys. Rev. D*, **60**, 024007.
Shankaranarayanan, S., Srinivasan, K. and Padmanabhan, T.: 2001, *Mod. Phys. Lett. A* **16**, 571.
Sharif, M., Javed, W.: 2010, *J. Korean Phys. Soc.* **57**, 217.
Sharif, M., Javed, W.: 2012, *Astrophys. Space Sci.* **337**, 335.
Sharif, M., Javed, W.: 2012, *J. Exp. Theor. Phys.* **114**, 933.
Sharif, M., Javed, W.: 2012, *J. Exp. Theor. Phys.* **115**, 782.
Sharif, M., Javed, W.: 2013, *Can. J. Phys.* **91**, 43.
Sharif, M., Javed, W.: 2013, *Gen. Relativ. Gravit.* **45**, 1051.
Wu, S. Q., Cai, X.: 2000, *Nuovo Cimento B* **115**, 143.
Zhu, J. Y., Bao, A. D. Zhao, Z.: 1995, *Int. J. Theor. Phys.* **34**, 2049.
Zeng, X. -X., Li, Q.: 2009, *Int. J. Theor. Phys.* **48**, 1090.
Zeng, X. -X., Li, Q.: 2009, *Chin. Phys. B* **18**, 4716.
Zhou, S., Liu, W.: 2008, *Phys. Rev. D* **77**, 104021.

Received on 22 September 2017