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Polynomial Associative Algebras of Quantum Superintegrable Systems*

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Abstract—The integrals of motion of classical two-dimensional superintegrable systems, with polynomial integrals of motion, close in a restrained polynomial Poisson algebra; the general form of the quadratic case is investigated. The polynomial Poisson algebra of the classical system is deformed into a quantum associative algebra of the corresponding quantum system, and the finite-dimensional representations of this algebra are calculated by using a deformed parafermion oscillator technique. The finite-dimensional representations of the algebra are determined by the energy eigenvalues of the superintegrable system. The calculation of energy eigenvalues is reduced to the roots of algebraic equations in the quadratic case.

1. INTRODUCTION

In classical mechanics, an integrable system is a system possessing a number of constants of motion equal to the dimensionality of the space. A comprehensive review of two-dimensional integrable classical systems is given by Hietarinta [1], who assumed the space to be a flat real one. The case of a nonflat space is under current investigation [2–7]. An interesting subset of the totality of integrable systems is the set of systems that possess a maximum number of integrals; these systems are referred to as superintegrable ones.

The Hamiltonian of a classical system is a quadratic function of momenta. All "nondegenerate" superintegrable systems with quadratic integrals of motion in a complex flat space were classified by Kalnins, Miller, and Pogosyan [8]. In that paper, the term "nondegenerate" means that the potential depends on four independent parameters. These potentials are simultaneously separable in more than two orthogonal coordinate systems [9]. The notions of the multiseparability and superintegrability do not coincide. The most illustrative example is that of an anisotropic harmonic oscillator with a rational ratio of frequencies. The integrals of motion of a two-dimensional superintegrable system in flat space close in a restrained classical Poisson algebra [4, 8, 10–12]. The general form of the Poisson algebra was studied in [8, 12]. In the case of potentials with two quadratic integrals of motion, the Poisson algebra is a quadratic Poisson algebra. In [8], these quadratic Poisson algebras are listed for all superintegrable systems in a complex flat space. In [7], the quadratic algebras for systems superintegrable on a sphere are given for all classified cases. The general form of this algebra is given in [12]. The deformation of the classical Poisson algebra to a polynomial associative algebra with three generators implies a deformation of the parameters of the quadratic algebra [8, 12]. In [13], a three-generator polynomial algebra can be realized by nonlinear combinations of the generators of the $sl(3,R)$ algebra. In [10–19], it was conjectured that the energy eigenvalues correspond to finite-dimensional representations of latent quadratic algebras. Granovskii *et al.* [14] studied the representations of the quadratic Askey–Wilson algebras $QAW(3)$. Using the ladder representation proposed there, they calculated finite-dimensional representations. This method was applied to several superintegrable systems in [15, 17, 19]. Another method [10–12] for calculating finite-dimensional representations consists in the use of the deformed oscillator algebra and their finite-dimensional version, which are referred to as "generalized deformed parafermionic algebras" [20]. The main task of this paper is to reduce the calculations of eigenvalues to a system of two algebraic equations with two parameters to be determined. These equations are universal equations, which are valid for all superintegrable systems, with quadratic integrals of the motion.

2. QUADRATIC POISSON ALGEBRA

Let us consider a two-dimensional superintegrable system. The general form of the Hamiltonian

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is

$$H = a(q_1, q_2)p_1^2 + 2b(q_1, q_2)p_1p_2 + c(q_1, q_2)p_2^2 + V(q_1, q_2); \quad (1)$$

this Hamiltonian is a quadratic form of momenta. The system is superintegrable; therefore, there are two additional integrals of the motion, A and B. In this section, we assume that these integrals of motion are quadratic functions of momenta; i.e., they are given by

$$A = A(q_1, q_2, p_1, p_2) = c(q_1, q_2)p_1^2 + 2d(q_1, q_2)p_1p_2 + e(q_1, q_2)p_2^2 + Q(q_1, q_2).$$

The integral B of the motion is indeed assumed to be a quadratic form that is analogous to the above one:

$$B = B(q_1, q_2, p_1, p_2) = h(q_1, q_2)p_1^2 + 2k(q_1, q_2)p_1p_2 + l(q_1, q_2)p_2^2 + S(q_1, q_2).$$

By definition, the following relations are satisfied:

$$\{H, A\}_{\mathrm{P}} = \{H, B\}_{\mathrm{P}} = 0, \quad (2)$$

where $\{\ .\ ,\ .\ \}_{\mathrm{P}}$ is the usual Poisson bracket.

From the integrals A and B of the motion, we can construct the integral of motion

$$C = \{A, B\}_{\mathrm{P}}\,. \quad (3)$$

The integral C of motion is not a new independent integral of motion that is a cubic function of the momenta. As will be shown later, the integral C is not independent of the integrals H, A, and B. Starting from the integral of motion C, we can construct the (nonindependent) integrals $\{A, C\}_{\mathrm{P}}$ and $\{B, C\}_{\mathrm{P}}$. These integrals are quartic functions of momenta, i.e., functions of fourth order. Therefore, these integrals could be expressed as quadratic combinations of the integrals H, A, and B. After translations and rotations, the integrals A, B, and C satisfy the quadratic Poisson algebra:

$$\{A, B\}_{\mathrm{P}} = C, \quad (4)$$
$$\{A, C\}_{\mathrm{P}} = \alpha A^2 + 2\gamma AB + \delta A + \epsilon B + \zeta,$$
$$\{B, C\}_{\mathrm{P}} = aA^2 - \gamma B^2 - 2\alpha AB + dA - \delta B + z,$$

where α, γ, and a are constants and

$$\delta = \delta(H) = \delta_0 + \delta_1 H,$$
$$\epsilon = \epsilon(H) = \epsilon_0 + \epsilon_1 H,$$
$$\zeta = \zeta(H) = \zeta_0 + \zeta_1 H + \zeta_2 H^2,$$
$$d = d(H) = d_0 + d_1 H,$$
$$z = z(H) = z_0 + z_1 H + z_2 H^2,$$

with δ_i, ϵ_i, ζ_i, d_i, and z_i being constants. The associative algebra whose generators satisfy Eqs. (4) is a general form of the closed Poisson algebra of the integrals of superintegrable systems with integrals quadratic in momenta.

The quadratic Poisson algebra (4) possesses a Casimir operator that is a function of momenta of degree six and which is given by

$$K = C^2 - 2\alpha A^2 B - 2\gamma AB^2 - 2\delta AB - \epsilon B^2 - 2\zeta B + \frac{2}{3}aA^3 + dA^2 + 2zA = k_0 + k_1 H + k_2 H^2 + k_3 H^3. \quad (5)$$

Obviously, we have

$$\{K, A\}_{\mathrm{P}} = \{K, B\}_{\mathrm{P}} = \{K, C\}_{\mathrm{P}} = 0.$$

Therefore, the integrals of motion of a superintegrable two-dimensional system, with quadratic integrals of motion, close a constrained classical quadratic Poisson algebra (4), corresponding to a Casimir operator equal at most to a cubic function of the Hamiltonian in (5).

In the general case of a superintegrable system, the integrals are not necessarily quadratic functions of the momenta, but they are rather polynomial functions of the momenta. The case of systems with a quadratic and cubic integral of motion were studied by Tsiganov [21]. The general form of the Poisson algebra of the generators A, B, and C is characterized by a polynomial function $h(A, B)$:

$$\{A, B\}_{\mathrm{P}} = C, \quad \{A, C\}_{\mathrm{P}} = \partial h/\partial B, \quad (6)$$
$$\{C, B\}_{\mathrm{P}} = \partial h/\partial A.$$

The above general forms of the Poisson algebra were introduced by Kalnins, Miller, and Pogosyan [8]. The Casimir operator of the algebra is given by

$$K = K(H) = C^2 - 2h(A, B), \quad (7)$$
$$\{K, A\}_{\mathrm{P}} = \{K, B\}_{\mathrm{P}} = 0,$$

where $h(A, B)$ is a polynomial function of the integrals A and B of the motion. These relations were also discussed in [8] in a slightly different context.

In the general case of a two-dimensional superintegrable system with a quadratic Hamiltonian, one integral A of order m in momenta, and one integral B of order n $(n \geq m)$, the function $h(A, B)$, in most cases, can be represented as

$$h(A, B) = h_0(A) + h_1(A)B + h_2(A)B^2,$$

where $h_i(A)$ are polynomials of the integrals A and H.

3. QUADRATIC ASSOCIATIVE ALGEBRA

The quantum counterparts of classical systems that have been studied in Section 2 are quantum superintegrable systems. The quadratic classical Poisson algebra (4) possesses a quantum counterpart that is a quadratic associative algebra of operators. The form of the quadratic algebra is similar to that of the classical Poisson algebra, the constants involved are generally functions of $\hbar$, and they should coincide with the classical constants in the case of $\hbar \to 0$:

$$[A, B] = C, \tag{8}$$

$$[A, C] = \alpha A^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta, \tag{9}$$

$$[B, C] = aA^2 - \gamma B^2 - \alpha \{A, B\} + dA - \delta B + z. \tag{10}$$

The Casimir operator of this algebra is given by

$$K = C^2 - \alpha \{A^2, B\} - \gamma \{A, B^2\} + (\alpha\gamma - \delta) \{A, B\} + (\gamma^2 - \epsilon)B^2 + (\gamma\delta - 2\zeta)B + \frac{2a}{3}A^3 + \left(d + \frac{a\gamma}{3} + \alpha^2\right) A^2 + \left(\frac{a\epsilon}{3} + \alpha\delta + 2z\right) A. \tag{11}$$

This quadratic algebra has many similarities to the Racah algebra $QR(3)$, which is a special case of the Askey–Wilson algebra $QAW(3)$. The algebra specified by Eqs. (8)–(10) does not coincide with the Racah algebra $QR(3)$ if $a \neq 0$ in relation (10). A representation theory can be constructed by following the same procedures as those described by Granovskii, Lutzenko, and Zhedanov in [14, 15]. In this paper, we shall give another realization of this algebra using the deformed-oscillator techniques [22]. The finite-dimensional representations of the algebra given by (8)–(10) will be constructed by constructing a realization of the algebra with the generalized parafermionic algebra introduced by Quesne [20].

Let us now consider a realization of the algebra given by (8)–(10) by using the deformed-oscillator technique, i.e., by using a deformed-oscillator algebra [22] $\{b^\dagger, b, \mathcal{N}\}$, which satisfies

$$\left[\mathcal{N}, b^\dagger\right] = b^\dagger, \quad [\mathcal{N}, b] = -b, \quad b^\dagger b = \Phi(\mathcal{N}), \quad bb^\dagger = \Phi(\mathcal{N} + 1), \tag{12}$$

where the function $\Phi(x)$ is a "well-behaved" real function that satisfies the boundary condition

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(x) > 0 \quad \text{for} \quad x > 0. \tag{13}$$

As is well known [22], this constraint entails the existence of a Fock-type representation of the deformed-oscillator algebra; i.e., there is a Fock basis $|n\rangle$, $n = 0, 1, \ldots$, such that

$$\mathcal{N}|n\rangle = n|n\rangle, \tag{14}$$

$$b^\dagger|n\rangle = \sqrt{\Phi(n+1)}|n+1\rangle, \quad n = 0, 1, \ldots,$$
$$b|0\rangle = 0,$$
$$b|n\rangle = \sqrt{\Phi(n)}|n-1\rangle, \quad n = 1, 2, \ldots.$$

In the case of nilpotent deformed-oscillator algebras, there is a positive integer p such that

$$b^{p+1} = 0, \quad (b^\dagger)^{p+1} = 0.$$

The above equations imply that

$$\Phi(p+1) = 0. \tag{15}$$

In that case, the deformed oscillator (12) has a finite-dimensional representation of dimension equal to $p+1$. This kind of oscillator is called a deformed parafermion oscillator of order p. The structure function $\Phi(\mathcal{N})$ has the general form [20]

$$\Phi(\mathcal{N}) = \mathcal{N}(p + 1 - \mathcal{N})(a_0 + a_1\mathcal{N} + a_2\mathcal{N}^2 + \cdots + a_{p-1}\mathcal{N}^{p-1}).$$

A systematic study and applications of the parafermionic oscillator are given in [20, 23–25].

We shall show that there is a realization of the quadratic algebra such that

$$A = A(\mathcal{N}), \tag{16}$$

$$B = b(\mathcal{N}) + b^\dagger \rho(\mathcal{N}) + \rho(\mathcal{N})\, b, \tag{17}$$

where $A(x)$, $b(x)$, and $\rho(x)$ are functions that will be determined. In this case, (8) implies that

$$C = [A, B] \;\Rightarrow\; C = b^\dagger \Delta A(\mathcal{N})\, \rho(\mathcal{N}) - \rho(\mathcal{N})\, \Delta A(\mathcal{N})\, b, \tag{18}$$

where

$$\Delta A(\mathcal{N}) = A(\mathcal{N} + 1) - A(\mathcal{N}).$$

Using Eqs. (16), (17), and (9) we find

$$(\Delta A(\mathcal{N}))^2 = \gamma (A(\mathcal{N} + 1) + A(\mathcal{N})) + \epsilon, \tag{19}$$

$$\alpha A(\mathcal{N})^2 + 2\gamma A(\mathcal{N})\, b(\mathcal{N}) + \delta A(\mathcal{N}) + \epsilon b(\mathcal{N}) + \zeta = 0, \tag{20}$$

while the function $\rho(\mathcal{N})$ can be arbitrarily determined. In fact, this function can be fixed in order to have a polynomial structure function $\Phi(x)$ for the deformed-oscillator algebra (12). Solutions to Eqs. (19) depend on the value of the parameter γ, while the function $b(\mathcal{N})$ is uniquely determined by Eq. (20) (provided that at most one of the parameters γ or ϵ is not zero). At this stage, the cases of $\gamma \neq 0$ or $\gamma = 0$ should be treated separately.

Case 1: $\gamma \neq 0$. In this case, solutions to Eqs. (19) and (20) are given by

$$A(\mathcal{N}) = \frac{\gamma}{2}\left((\mathcal{N} + u)^2 - 1/4 - \frac{\epsilon}{\gamma^2}\right), \tag{21}$$

$$b(\mathcal{N}) = -\frac{\alpha\left((\mathcal{N}+u)^2-1/4\right)}{4} + \frac{\alpha\,\epsilon-\delta\,\gamma}{2\,\gamma^2} - \frac{\alpha\,\epsilon^2-2\,\delta\,\epsilon\,\gamma+4\,\gamma^2\,\zeta}{4\,\gamma^4}\frac{1}{((\mathcal{N}+u)^2-1/4)}. \qquad (22)$$

Case 2: $\gamma = 0,\ \epsilon \neq 0$. Solutions to Eqs. (19) and (20) are given by

$$A(\mathcal{N}) = \sqrt{\epsilon}\,(\mathcal{N}+u)\,, \qquad (23)$$

$$b(\mathcal{N}) = -\alpha\,(\mathcal{N}+u)^2 - \frac{\delta}{\sqrt{\epsilon}}\,(\mathcal{N}+u) - \frac{\zeta}{\epsilon}. \qquad (24)$$

The constant u will be determined later.

Using the above definitions of $A(\mathcal{N})$ and $b(\mathcal{N})$, we find that the left-hand side and the right-hand side of Eq. (10) give the equation

$$2\,\Phi(\mathcal{N}+1)\left(\Delta A\,(\mathcal{N}) + \frac{\gamma}{2}\right)\rho(\mathcal{N}) - 2\,\Phi(\mathcal{N})\left(\Delta A\,(\mathcal{N}-1) - \frac{\gamma}{2}\right)\rho(\mathcal{N}-1) = aA^2\,(\mathcal{N}) - \gamma b^2(\mathcal{N}) - 2\alpha A\,(\mathcal{N})\,b(\mathcal{N}) + dA\,(\mathcal{N}) - \delta b(\mathcal{N}) + z. \qquad (25)$$

Equation (11) gives the relation

$$K = \Phi(\mathcal{N}+1)\big(\gamma^2-\epsilon-2\gamma A\,(\mathcal{N}) - \Delta A^2\,(\mathcal{N})\big)\rho(\mathcal{N}) + \Phi(\mathcal{N})\big(\gamma^2-\epsilon - 2\gamma A\,(\mathcal{N}) - \Delta A^2\,(\mathcal{N}-1)\big)\rho(\mathcal{N}-1) - 2\alpha A^2\,(\mathcal{N})\,b(\mathcal{N}) + \big(\gamma^2-\epsilon-2\gamma A\,(\mathcal{N})\big)\,b^2(\mathcal{N}) + 2\,(\alpha\gamma-\delta)\,A\,(\mathcal{N})\,b(\mathcal{N}) + (\gamma\delta-2\zeta)\,b(\mathcal{N}) + \frac{2}{3}aA^3\,(\mathcal{N}) + \left(d+\frac{1}{3}a\gamma+\alpha^2\right)A^2\,(\mathcal{N}) + \left(\frac{1}{3}a\epsilon+\alpha\delta+2z\right)A\,(\mathcal{N})\,. \qquad (26)$$

Equations (25) and (26) are linear functions of the expressions $\Phi(\mathcal{N})$ and $\Phi(\mathcal{N}+1)$. Then, the function $\Phi\,(\mathcal{N})$ can be determined if the function $\rho(\mathcal{N})$ is given. A solution of this system, i.e., the function $\Phi\,(\mathcal{N})$, depends on two parameters, u and K, and is given by the following formulas:

Case 1: $\gamma \neq 0$.

$$\rho(\mathcal{N}) = \frac{1}{3\cdot 2^{12}\cdot\gamma^8(\mathcal{N}+u)(1+\mathcal{N}+u)(1+2(\mathcal{N}+u))^2}$$

and

$$\Phi(\mathcal{N}) = -3072\gamma^6K(-1+2(\mathcal{N}+u))^2 - 48\gamma^6(\alpha^2\epsilon-\alpha\delta\gamma+a\epsilon\gamma-d\gamma^2) \times(-3+2(\mathcal{N}+u))(-1+2(\mathcal{N}+u))^4 \times(1+2(\mathcal{N}+u)) + \gamma^8(3\alpha^2+4a\gamma)(-3+2(\mathcal{N}+u))^2(-1+2(\mathcal{N}+u))^4 \times(1+2(\mathcal{N}+u))^2 + 768(\alpha\epsilon^2-2\delta\epsilon\gamma+4\gamma^2\zeta)^2 + 32\gamma^4(-1+2(\mathcal{N}+u))^2(-1-12(\mathcal{N}+u) + 12(\mathcal{N}+u)^2)(3\alpha^2\epsilon^2-6\alpha\delta\epsilon\gamma+2a\epsilon^2\gamma+2\delta^2\gamma^2 - 4d\epsilon\gamma^2+8\gamma^3z+4\alpha\gamma^2\zeta) - 256\gamma^2(-1+2(\mathcal{N}+u))^2 \times(3\alpha^2\epsilon^3-9\alpha\delta\epsilon^2\gamma+a\epsilon^3\gamma+6\delta^2\epsilon\gamma^2-3d\epsilon^2\gamma^2+2\delta^2\gamma^4 + 2d\epsilon\gamma^4+12\epsilon\gamma^3z-4\gamma^5z + 12\alpha\epsilon\gamma^2\zeta-12\delta\gamma^3\zeta+4\alpha\gamma^4\zeta). \qquad (27)$$

Case 2: $\gamma = 0,\ \epsilon \neq 0$.

$$\rho(\mathcal{N}) = 1,$$

$$\Phi(\mathcal{N}) = \frac{1}{4}\left(-\frac{K}{\epsilon} - \frac{z}{\sqrt{\epsilon}} - \frac{\delta}{\sqrt{\epsilon}}\frac{\zeta}{\epsilon} + \frac{\zeta^2}{\epsilon^2}\right) - \frac{1}{12}\left(3d - a\sqrt{\epsilon} - 3\alpha\frac{\delta}{\sqrt{\epsilon}} + 3\left(\frac{\delta}{\sqrt{\epsilon}}\right)^2 - 6\frac{z}{\sqrt{\epsilon}} + 6\alpha\frac{\zeta}{\epsilon} - 6\frac{\delta}{\sqrt{\epsilon}}\frac{\zeta}{\epsilon}\right)(\mathcal{N}+u) + \frac{1}{4}\left(\alpha^2+d-a\sqrt{\epsilon}-3\alpha\frac{\delta}{\sqrt{\epsilon}} + \left(\frac{\delta}{\sqrt{\epsilon}}\right)^2 + 2\alpha\frac{\zeta}{\epsilon}\right) \times(\mathcal{N}+u)^2 - \frac{1}{6}\left(3\alpha^2 - a\sqrt{\epsilon} - 3\alpha\frac{\delta}{\sqrt{\epsilon}}\right) \times(\mathcal{N}+u)^3 + \frac{1}{4}\alpha^2(\mathcal{N}+u)^4. \qquad (28)$$

The above formula is valid for $\epsilon > 0$.

Let us consider a representation of the quadratic algebra that is diagonal in the generator A and the Casimir operator K. Using the parafermionic realization defined by Eqs. (16) and (17), we see that this is a representation diagonal in the parafermionic number operator $\mathcal{N}$ and the Casimir operator K. The basis of this representation corresponds to the Fock basis of the parafermionic oscillator; i.e., the vectors $|k,\,n\rangle,\ n = 0, 1, \ldots$, of the carrier Fock space satisfy the equations

$$\mathcal{N}|k,\,n\rangle = n|k,\,n\rangle, \quad K|k,\,n\rangle = k|k,\,n\rangle.$$

The structure function (27) [or, respectively, (28)] depends on the eigenvalues of the parafermionic number operator $\mathcal{N}$ and the Casimir operator K. If the deformed oscillator corresponds to a deformed parafermionic oscillator of order p, then the two parameters of the calculation, k and u, should satisfy the constraints (13) and (15) of the system:

$$\Phi(0,u,k) = 0 \quad\text{and}\quad \Phi(p+1,u,k) = 0. \qquad (29)$$

Then, the parameter $u = u(k,p)$ is a solution to the set of Eqs. (29). Generally, there are many solutions to the above set of equations, but a unitary representation of the deformed parafermionic oscillator entails the additional restriction

$$\Phi(x) > 0 \quad \text{for} \quad x = 1, 2, \ldots, p.$$

We must indicate that the set of Eqs. (29) corresponds to a representation of dimension equal to $p+1$. The proposed method for calculating the representation of the quadratic algebra is an alternative to the method given by Granovskii *et al.* [14, 15] and reduces the search for the representations to solving the set of polynomial Eqs. (29). Also, it is applied to an algebra not included in the cases of the algebras that are treated in the above references.

4. QUADRATIC ALGEBRAS FOR QUANTUM SUPERINTEGRABLE SYSTEMS

In this section, we shall give two examples of the calculation of eigenvalues for a superintegrable two-dimensional system using the methods of the preceding section. The calculation by an empirical method was performed in [11], and solving the same problem by a separation of variables was studied in [4]. In order to show the effects of the quantization procedure, we do not use here units in which $\hbar = 1$.

4.1. Potential (i)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{k}{r} + \frac{1}{r}\left(\frac{\mu_1}{r+x} + \frac{\mu_2}{r-x}\right)\right).$$

In [4], the parabolic coordinates were used:

$$x = \frac{1}{2}\left(\xi^2 - \eta^2\right), \quad y = \xi\eta,$$
$$[\xi, p_\xi] = i\hbar, \quad [\eta, p_\eta] = i\hbar,$$

$$H = \frac{1}{\xi^2+\eta^2}\left(\frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) + k + \frac{\mu_1}{\xi^2} + \frac{\mu_2}{\eta^2}\right).$$

This potential has the following independent integrals of motion:

$$A = \frac{1}{2}\left(\frac{1}{2}\left(\eta p_\xi - \xi p_\eta\right)^2 + \left(\xi^2+\eta^2\right)\left(\frac{\mu_1}{\xi^2} + \frac{\mu_2}{\eta^2}\right)\right),$$

$$B = \frac{1}{\xi^2+\eta^2}\left(\frac{1}{2}\left(\xi^2 p_\eta^2 - \eta^2 p_\xi^2\right) + \mu_2\frac{\xi^2}{\eta^2} - \mu_1\frac{\eta^2}{\xi^2} + \frac{k}{2}\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right).$$

The constants of the corresponding quadratic algebra (8)–(10) are given by

$$\alpha = 0, \quad \gamma = 2\hbar^2, \quad \delta = 0, \quad \epsilon = -\hbar^4,$$
$$\zeta = -\hbar^2 k(\mu_1 - \mu_2), \quad a = 0, \quad d = 8\hbar^2 H,$$
$$z = -\hbar^2\left(4(\mu_1+\mu_2)H - k^2/2\right) + \hbar^4 H.$$

The Casimir operator (11) has the form

$$K = -\hbar^2\left(2(\mu_1-\mu_2)^2 H - k^2(\mu_1+\mu_2)\right) - 2\hbar^4\left((\mu_1+\mu_2)H - \frac{k^2}{4}\right) + \hbar^6 H.$$

For the sake of simplicity, we introduce the positive parameters k_1 and k_1:

$$\mu_1 = \frac{\hbar^2}{2}\left(k_1^2 - \frac{1}{4}\right), \quad \mu_2 = \frac{\hbar^2}{2}\left(k_2^2 - \frac{1}{4}\right).$$

The structure function (27) of the deformed parafermionic algebra can be given by the simple form

$$\Phi(x) = 3\cdot 2^{14}\hbar^{16}\left(2x - 1 + k_1 + k_2\right) \times \left(2x - 1 + k_1 - k_2\right)\left(2x - 1 - k_1 + k_2\right) \times \left(2x - 1 - k_1 - k_2\right)\left(8\hbar^2 H x^2 - 8\hbar^2 H x + 2\hbar^2 H + k^2\right).$$

where E is the eigenvalue of the energy. The values of the parameters u and E are determined by the restrictions in (29). There are four acceptable solutions, which correspond to the following values of the parameters u and E:

$$u = \frac{1}{2}\left(2 + \epsilon_1 k_1 + \epsilon_2 k_2\right),$$

$$E = -\frac{k^2}{2\hbar^2\left(2(p+1) + \epsilon_1 k_1 + \epsilon_2 k_2\right)^2},$$

where $\epsilon_i = \pm 1$. The positive sign of the structure function for $x = 1, 2, \ldots, p$ is obtained when

$$\epsilon_1 k_1 > -1, \quad \epsilon_2 k_2 > -1, \quad \text{and} \quad \epsilon_1 k_1 + \epsilon_2 k_2 > -1.$$

4.2. Potential (ii)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{k}{r} + \mu_1\frac{\sqrt{r+x}}{r} + \mu_2\frac{\sqrt{r-x}}{r}\right) = \frac{1}{\xi^2+\eta^2}\left(\frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) + k + \mu_1\xi + \mu_2\eta\right).$$

This potential has the following independent integrals of motion:

$$A = \frac{1}{2\left(\xi^2+\eta^2\right)}\left(\eta^2 p_\xi^2 - \xi^2 p_\eta^2 + k\left(\eta^2 - \xi^2\right) + 2\xi\eta\left(\mu_1\eta - \mu_2\xi\right)\right),$$

$$B = -\frac{1}{2\left(\xi^2+\eta^2\right)}\left(\xi\eta\left(p_\xi^2 + p_\eta^2\right) - \left(\xi^2+\eta^2\right)p_\xi p_\eta + 2k\xi\eta + \left(\mu_2\xi - \mu_1\eta\right)\left(\eta^2 - \xi^2\right)\right).$$

The constants of the corresponding quadratic algebra (8)–(10) are given by

$$\alpha = 0, \quad \gamma = 0, \quad \delta = 0, \quad \epsilon = -2\hbar^2 H,$$
$$\zeta = \hbar^2 \mu_1 \mu_2 / 2,$$
$$a = 0, \quad d = 2\hbar^2 H,$$
$$z = -\hbar^2(\mu_1^2 - \mu_2^2)/4.$$

The Casimir operator (11) has the form

$$K = \hbar^2 k^2 H/2 + \hbar^2 k(\mu_1^2 + \mu_2^2)/4 + \hbar^4 H^2.$$

For the sake of simplicity, we introduce the parameters

$$\varepsilon = \sqrt{-2E}/\hbar, \quad \lambda = k/\hbar^2,$$
$$\nu_1 = \mu_1/\hbar^2, \quad \nu_2 = \mu_2/\hbar^2, \quad \nu^2 = \nu_1^2 + \nu_2^2.$$

The structure function (28) of the deformed parafermionic algebra can be given by the form

$$\Phi(x) = \frac{\hbar^4}{16\varepsilon^4}\left(\nu_1^2 - \lambda\varepsilon^2 + 2\left(x + u - \frac{1}{2}\right)\varepsilon^3\right)$$
$$\times \left(\nu_2^2 - \lambda\varepsilon^2 - 2\left(x + u - \frac{1}{2}\right)\varepsilon^3\right),$$

where the parameter ε is related to the eigenvalue E of the energy. The values of the parameters u and ε are determined by the restrictions in (29), which become

$$\Phi(0) = 0, \quad \Phi(p+1) = 0.$$

The first condition can be used to determine the acceptable values of the parameter u. Two possible solutions are found to be

$$u = u_1 = \frac{\nu_2^2 - \lambda\varepsilon^2 + \varepsilon^3}{2\varepsilon^3}, \tag{30}$$

$$u = u_2 = -\frac{\nu_1^2 - \lambda\varepsilon^2 - \varepsilon^3}{2\varepsilon^3}. \tag{31}$$

Using these solutions and the condition $\Phi(p+1) = 0$, we find that ε must satisfy two possible cubic equations:

$$u_1 \longrightarrow 2(p+1)\varepsilon^3 - 2\lambda\varepsilon^2 + \nu^2 = 0, \tag{32}$$

$$u_2 \longrightarrow 2(p+1)\varepsilon^3 + 2\lambda\varepsilon^2 - \nu^2 = 0. \tag{33}$$

If ε is a solution to Eq. (32), then $-\varepsilon$ is a solution to Eq. (33); therefore, there is at least one positive solution. This solution leads to the structure function

$$\Phi(x) = \frac{\varepsilon^2}{4} x\,(p + 1 - x)$$

which is positive for $x = 1, 2, \ldots, p$.

5. DISCUSSION

The energy eigenvalues corroborate the results of [4, 13]. The calculation of the energy eigenvalues in [4] was performed by solving the corresponding Schrödinger differential equations, while, in this paper and in [13], the energy eigenvalues are obtained by algebraic methods. The advantage of the proposed method is that the energy eigenvalues are reduced to simple algebraic calculations of the roots of polynomial equations whose form is universally determined by the structure functions (27) and the set of Eqs. (29). These equations are valid for any two-dimensional superintegrable system with integrals of motion that are quadratic functions of momenta. The same equations should be valid in the case of two-dimensional superintegrable systems in a curved space [26]. Superintegrable systems bring up the open problem of the quantization of a Poisson algebra in a well-determined context, because these systems and their quantum counterparts are explicitly known.

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