DISCUSSION

THIRRING: I think always in these quantized space time theories certain conceptual difficulties arise, namely the coordinates x are no longer commuting quantities; they are operators. On what Hilbert space do they act, and in particular what is the physical significance of the eigenstates of the operators x? Do they simply mean that we measured that there was a point x?

TODOROV: Until now in Kadyshevsky's work the x space is almost completely excluded from consideration. The author tries to work only in momentum space. It is wellknown that in present day quantum field theory the co-ordinates x and the momenta p play quite different roles, although in quantum mechanics we say that x- and p-spaces are equivalent. We are usually working in p-space, and for example, the Lorentz invariance in p-space is much better experimentally checked than Lorentz invariance in x-space. We do not even know what is the physical meaning of a point in x-space. We know for example that if we try to localize a particle at a point, we have to give it a big momentum and we will find, instead of one particle, a multiplicity of pairs.

GENERALISED COMMUTATION RELATIONS AND STATISTICS

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Before going into my talk, I must mention that exactly the same conclusion has been reached before independently, in a way somewhat different from ours, by V. Glaser and M. Fierz. This report is, therefore, to be taken as a contribution from these authors and ourselves.

We would like to present here a general method of field quantisation and discuss its application to elementary particle physics. The question we ask ourselves is the following: when we require a field operator $\psi(x)$ to satisfy the relation

$$\frac{\partial\psi(x)}{\partial x_\mu} = i[\psi(x), P_\mu]\,, \tag{1}$$

(where P_μ is the energy-momentum four vector) what are the most general commutation relations for the operator ψ?

It is convenient to discuss the problem in momentum space by introducing Fourier coefficients of ψ's a_k and a_k^+. To determine the commutation relations between the a_k and a_k^+ in a self-consistent fashion we consider the following infinitesimal transformation

$$\begin{aligned} a_k \to a_k' &= a_k - i\sum_m a_m\xi_{km} - i\sum_m a_m^+\eta_{km}\,, \\ a_k^+ \to a_k'^+ &= a_k^+ + i\sum_m a_m^+\xi_{km}^* + i\sum_m a_m\zeta_{mk}\;(\eta_{km}^* = \zeta_{mk})\,, \end{aligned} \tag{2}$$

and distinguish the following two cases (R) and (S). Case (R): The transformations (2) belong to an infinite-dimensional *rotation* group, provided

$$\begin{aligned} \xi_{km}^* &= \xi_{mk}\,, \\ \eta_{km}+\eta_{mk} &= 0\,. \end{aligned} \tag{3}$$

Case (S): The transformations (2) belong to an infinite-dimensional *symplectic* group, provided

$$\begin{aligned} \xi^*_{km} &= \xi_{mk} , \\ \eta_{km} - \eta_{mk} &= 0 . \end{aligned} \tag{3'}$$

The transformations (2) can be generated by means of the infinitesimal unitary transformation

$$G \equiv 1 - i \sum_{l,\, m} N_{lm}\xi_{lm} - \frac{i}{2} \sum_{l,\, m} L_{lm}\eta_{lm} - \frac{i}{2} \sum_{l,\, m} M_{lm}\zeta_{lm} . \tag{4}$$

Thus, from $a'_k = G^{-1} a_k G$, etc., one obtains

$$\begin{aligned} [a_k , N_{lm}] &= \delta_{kl} a_m , \\ [a_k , L_{lm}] &= \delta_{kl} a_m^+ \mp \delta_{km} a_l^+ , \\ [a_k , M_{lm}] &= 0 . \end{aligned} \tag{5}$$

Here and also in what follows, the upper and lower signs refer to the cases (R) and (S), respectively. One can derive, by standard methods, the commutation relations between the generators N_{lm} , L_{lm} and M_{lm} , which thus determine the structure constants of our Lie groups:

$$\begin{aligned} [N_{kl} , N_{mn}] &= \delta_{lm} N_{kn} - \delta_{kn} N_{ml} , \\ [L_{kl} , L_{mn}] &= [M_{kl} , M_{mn}] = 0 , \\ [L_{kl} , N_{mn}] &= -\delta_{kn} L_{ml} \pm \delta_{ln} L_{mk} , \\ [M_{kl} , N_{mn}] &= \delta_{km} M_{nl} \mp \delta_{lm} M_{nk} , \\ [L_{kl} , M_{mn}] &= -\delta_{km} N_{ln} \pm \delta_{kn} N_{lm} - \delta_{ln} N_{km} \pm \delta_{lm} N_{kn} . \end{aligned} \tag{6}$$

Our problem is now reduced to investigating some representations of the groups which are suitable for our present purpose. These can be obtained by taking

$$\begin{aligned} N_{kl} &= \tfrac{1}{2}(a_k^+ a_l \mp a_l a_k^+) , \\ L_{kl} &= \tfrac{1}{2}(a_k^+ a_l^+ \mp a_l^+ a_k^+) , \\ M_{kl} &= \tfrac{1}{2}(a_k a_l \mp a_l a_k) . \end{aligned} \tag{7}$$

It should be noticed here that the relations (5), together with the explicit forms (7) for N_{kl} , etc., actually lead to our fundamental relation (1).

Since Hermitian operators $N_k \equiv N_{kk}$ commute among themselves, one can use these operators to specify representations. We shall discuss the two cases separately.

Case (R): Finite dimensional representations are possible in this case. If we denote by s the dimension of the representation, the N_k has eigenvalues

$$-\frac{(s-1)}{2} , -\frac{(s-1)}{2}+1 , \dots , \frac{(s-1)}{2}-1 , \frac{(s-1)}{2} .$$

Thus, a new operator defined by $n_k \equiv N_k + \frac{(s-1)}{2}$ has the spectrum 0, 1, 2, ..., (s−1), and it can, therefore, be identified with the number operator. In this case we get a generalized Fermi-Dirac statistics, in which $n_{max}\,(=(s-1))$ particles can occupy one and the same state.

Case (S): In this case there does not exist any finite-dimensional representation. N_k has therefore the eigen-values $n_0 = \frac{(s-1)}{2}$, n_0+1, n_0+2, ..., ∞. An operator defined by $n_k \equiv N_k - \frac{(s-1)}{2}$ has the spectrum of the number operator 0, 1, 2, ..., ∞. This case, therefore, leads to generalized Bose-Einstein statistics.

Explicit forms of the commutation relations for a_k and a_k^+ with s = 1, 2, 3, and 4 are given as follows:

$$\text{(i) } s = 1: \qquad a_k = a_k^+ = 0 . \tag{8}$$

This is a trivial case

$$\text{(ii) } s = 2: \qquad \begin{aligned} [a_k , a_l^+]_\pm &= \delta_{kl} , \\ [a_k , a_l]_\pm &= 0 . \end{aligned} \tag{9}$$

This is the case of the Fermi-Dirac (Bose-Einstein) statistics.

(iii) s = 3:

$$\begin{aligned} [a_k , a_l^+ , a_m]_\pm &= 2\delta_{kl} a_m \pm 2\delta_{lm} a_k , \\ [a_k , a_l , a_m^+]_\pm &= 2\delta_{lm} a_k , \\ [a_k , a_l , a_m]_\pm &= 0 . \end{aligned} \tag{10}$$

This is the case, previously discussed by Green and Volkov [1].

(iv) $s=4$:

$$a_k[a_l\,,\ a_m\,,\ a_n]_\pm+[a_m\,,\ a_l\,,\ a_n]_\pm a_k=0\,,$$

$$[a_m\,,[a_l,a_k^+,a_n]_\pm]_\pm=\delta_{km}[a_n,a_l]_\pm+3\delta_{kl}[a_m,a_n]_\pm+3\delta_{kn}[a_l,a_m]_\pm\,,$$

$$a_m[a_k^+,a_l,a_n]_\pm+[a_l,a_k^+,a_n]_\pm a_m=\delta_{km}[a_l,a_n]_\pm+3\delta_{kl}[a_n,a_m]_\pm\pm3\delta_{kn}a_la_m+\delta_{kn}a_ma_l\,,$$

$$a_k^+[a_l,a_m,a_n]_\pm+[a_m,a_l,a_n]_\pm a_n^+=\delta_{kl}[a_n,a_m]_\pm+\delta_{km}[a_l,a_n]_\pm+\delta_{kn}[a_m,a_l]_\pm\,,\tag{11}$$

$$[a_k^+,[a_l,a_n^+,a_m]_\pm]_\pm+a_m[a_n^+,a_l,a_k^+]_\pm+[a_n^+,a_k^+,a_l]_\pm a_m=$$
$$=4\delta_{nm}[a_l,a_k^+]_\pm+4\delta_{kl}[a_m,a_n^+]_\pm+2\delta_{km}[a_n^+,a_l]_\pm\pm6\delta_{nl}a_ma_k^+ +4\delta_{nl}a_k^+a_m-6(\pm\delta_{km}\delta_{ln}+\delta_{kl}\delta_{mn})\,,$$

$$a_k^+[a_l,a_m,a_n^+]_\pm+[a_l,a_n^+,a_m]_\pm a_k^+ +a_m[a_k^+,a_l,a_n^+]_\pm+[a_l,a_k^+,a_n^+]_\pm a_m=$$
$$=4\delta_{nm}[a_k^+,a_l]_\pm+4\delta_{kl}[a_n^+,a_m]_\pm+2\delta_{mk}[a_l,a_n^+]_\pm+6\delta_{nl}a_ma_k^+\pm2\delta_{nl}a_k^+a_m-6(\delta_{km}\delta_{ln}\pm\delta_{kl}\delta_{mn})\,.$$

Here, we have used the notations $[A, B]_\pm=AB\pm BA$, $[A, B, C]_\pm=ABC\pm CBA$.

One of the specific features of field theories quantized by the above general method is that for many particle states there arises a degeneracy. Thus, the sub-space of n-particle state vectors forms a basis of certain representations of the symmetric group of degree n, corresponding to Young diagrams with more than one row and column.

Let us now apply our method to relativistic field theory. The Fierz-Pauli theorem on the connection between spin and statistics is generalized as follows: the method (R) can be applied only to fields with half-integer spin, whereas the method (S) can be applied only to fields with integer spin. The CTP-theorem still holds. However, since we no longer have strong locality, causality is expressed in a form weaker than the usual one:

$$[[\psi^+(x),\mathrm{O}\psi(x)]_\mp\,,\ \psi(y)]_-=0\ \text{for space-like}\ x-y\,,\tag{12}$$

where O is some matrix. The theorem, due to Nishijima and others [2], about commutativity between different kinds of fields also needs a generalization. Such a generalized theorem imposes certain restrictions on the possible forms of interactions, and one gets the following super-selection rule: except for the ordinary bosons, the numbers of particles which obey the same statistics are conserved (with mod. 2 for tensor particles).

By using this super-selection rule one can determine the statistics of all but two elementary particles. To do this we first accept that e, p and n are Fermi-Dirac particles (F.D.) and γ is a Bose-Einstein particle (B.E.). From $n\to p+e+\bar{\nu}_e$, one concludes that ν_e is F.D. From $\pi^0\to2\gamma$, $\pi^\pm\to e^\pm+\bar{\nu}_e$, all π's are B.E. Similarly one finds that all hyperons are F.D. and K is B.E. The only particles one cannot argue about in this way are μ and ν_μ (we assume here $\nu_\mu\neq\nu_e$: if $\nu_\mu=\nu_e$, μ is F.D.). Thus to determine the statistics of these particles one must compare some quantitative result with experiments.

One can easily see that as far as the one muon problem (such as g-factor or Michel parameter ρ) is concerned, our generalized theory does not differ appreciably from the ordinary one. So, one must look at a two-body problem. The simplest process of this kind is the μ-pair production by γ and the total cross-section is, in the lowest order perturbation theory,

$$\gamma_s = (s-1)\sigma_{F.D.} \tag{13}$$

However, we heard during Session E that the recent experiment shows $(s-1)\sim 1$. This implies that μ is F.D.[(*)] and so is ν_μ.

Our conclusion is therefore that all the known particles are just the ordinary F.D. or B.E. particles.

LIST OF REFERENCES

1*a)* H. S. Green. Phys. Rev. *90*, 270 (1953).
b) D. V. Volkov. Soviet Physics - JETP, *9*, 1107 (1959).
2*a)* K. Nishijima. Prog. Theor. Phys. *5*, 187 (1950).
b) H. Araki. Journal Math. Phys. *2*, 267 (1961) and papers quoted there.

DISCUSSION

SUDARSHAN: I would like to remark that, provided we accept the demonstration regarding the superselection rule, the statistics *of all* " elementary " particles are determined to be ordinary; this may well be the most important contribution that we have heard. The question then arises as to under what condition the demonstration holds: is not the commutativity requirements between different fields simply an additional postulate?

KAMEFUCHI: The answer is yes. When we assume that field operators ψ^a and ψ^b which obey different statistics are variationally independent, then, by applying Schwinger's variational principle [(**)] one can conclude that these field operators either commute or anticommute with each other. Now, in order that one can derive the Euler-Lagrange equation by means of equation (1), the interaction Hamiltonian must have the form

$$H = g[\psi^{-a}, 0_1\psi^b]_\pm[\psi^{-c}, 0_2\psi^d]_\pm \cdots$$

where $S^a = S^b$, $S^c = S^d$. From this we get the superselection rule.

SUDARSHAN: Essentially the same comment has been made in a preprint by Arnowitt and Deser. But are not the commutation properties of the field variations only a sufficient and not a necessary condition for the equivalence of variational and Heisenberg equations?

KAMEFUCHI: It is certainly a sufficient condition, but our assumption seems to me to be quite general. We have also tried to generalize the infinitesimal transformation (2) to cover all the field operators ψ^a, ψ^b,... But, the commutation relations obtained in this way lead to contradiction provided H consists of pairs with $S^a \neq S^b$. So, in both cases we get the same restriction on the form of the interaction.

THIRRING: To me the important contribution appears to be the argument about the photoproduction. This is independent of the question about commutativity or anticommutativity of the variations. This shows that experimentally the intermediate statistics are not realized.

GLASER: Somebody might be interested in proving this rigorously, not starting from a Lagrangian but using the Wightman axioms in the same way as it was done for the alternative Bose or Fermi statistics.

SUDARSHAN: The proof of the spin parastatistics theorem of Greenberg and myself is the by-product of a Wightman formulation of fields obeying parastatistics. Most of the standard Wightman development goes through and the most interesting difference is the appearance of several distinct analytic functions of the same order in place of the " master analytic function ".

(*) This observation has also been made by N. Kroll (private communication from G. Feinberg).

(**) As for a generalization of Schwinger's variational principle, see T. W. B. Kibble and J. C. Polkinghorne, Proc. Roy. Soc. A *242* 252 (1957).