Optical analogues of spherically symmetric black hole spacetimes

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Abstract. We have given an analytical formalism for developing optical analogues of spherically symmetric black hole spacetimes, and demonstrated the exact similarity between the electromagnetic wave equations in an inhomogeneous medium in flat spacetime and in a general relativistic curved spacetime. The permittivity and permeability of the inhomogeneous optical medium act as the metric components of an effective optical spacetime. Manifest properties of black holes, like curved trajectories of light rays and quasi-normal modes follow directly from our formalism. It is also applied to the specific case of Schwarzschild spacetime. The formalism would facilitate table-top experiments to investigate black hole phenomena.

1. Introduction

Advances in Transformation Optics and Analogue Gravity have led to systems that closely resemble those engendered by the general theory of relativity such as the black holes. Models that have come under the category of Analogue Gravity [1] make use of sound waves in flowing medium [2] like super fluids in BEC [3, 4], to mimic certain aspects of a black hole. Such systems developed until now are not exact analogues of any real spacetimes with corresponding metric tensors. On the other hand, Transformation Optics has utilized the formalism of general relativity [5] to build gradient index static media in order to gain excellent control over the light paths in those media [6]. Although, much of the efforts in Transformation Optics have been directed towards developing technological applications, the results can be used effectively to investigate and construct models of optical analogues of gravity. In this context, we have given a general analytical formalism for developing optical black hole analogues using static optical media. Our purpose is to simulate, in as much detail as possible, a gravitational black hole and its manifestly characteristic phenomena, such as the existence of a 'no-return' surface, namely, the event horizon, the bending of light rays, and the quasi-normal modes. To achieve this, we must have: (i) A region outside the horizon that replicates the black hole spacetime, and (ii) A perfect light absorber, which can act as the event horizon.

In the following sections, we have described how a particular profile of permittivity and/or permeability of the optical analogue can lead to an 'optical spacetime' that has the black hole metric, and therefore, reproduces the gravitational phenomena outside the event horizon. Within the framework of our formalism, we further have discussed the broadband omni-directional light absorber developed by Narimanov and Khildishev [7] that serves exactly as a perfect absorber, or a black hole.

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2. Electromagnetic wave equation in an inhomogeneous medium

The macroscopic electromagnetic field strength tensor in a continuous medium is defined as [2]:

$$I^{\mu\nu} = Z^{\mu\nu\sigma\tau} F_{\sigma\tau}, \tag{1}$$

where $F_{0\mu} = -F_{\mu 0} = -E_\mu$, $F_{ij} = \epsilon_{ijk} B^k$, $I^{0\nu} = I^{\nu 0} = D^\nu$, and $I^{ij} = \epsilon^{ijk} H_k$. Here, $F_{\sigma\tau}$ is the field strength tensor in free space and $Z^{\mu\nu\sigma\tau}$ is a fourth rank tensor whose components are the permittivity and permeability tensors. It is antisymmetric in the first two (μ, ν), and the last two (σ, τ) indices. ϵ^{ijk} is the Levi-Civita symbol. The Roman indices span 1, 2, 3, while the Greek indices range over 0, 1, 2, 3. Eq. (1) is the succinct form of the two known equations: $D = \varepsilon E$ and $H = \mu^{-1} B$. For an optical medium in flat spacetime, the Maxwell's equations involving sources are given by:

$$(Z^{\mu\nu\sigma\tau} F_{\sigma\tau})_{,\nu} = 0. \tag{2}$$

The tensor Z is related to the permittivity and permeability tensors in the following way:

$$Z^{0i0j} = -Z^{0ij0} = Z^{i0j0} = -Z^{i00j} = -\frac{1}{2}\varepsilon^{ij} \ , \quad and \tag{3}$$

$$Z^{ijkl} = \frac{1}{2}\epsilon^{ijm}\epsilon^{kln}{\mu_{mn}}^{-1}. \tag{4}$$

For a curved spacetime with metric $g^{\mu\nu}$, Maxwell's equations are given by:

$$\frac{\partial}{\partial x^\nu}\left[\sqrt{-g}(g^{\mu\sigma}g^{\nu\tau} - g^{\mu\tau}g^{\nu\sigma})F_{\sigma\tau}\right] = 0. \tag{5}$$

The similarity of Eqs. (2) and (5), suggest an analogy between an inhomogeneous medium and a curved spacetime, and hence, we can write:

$$Z^{\mu\nu\sigma\tau} = \frac{\sqrt{-\bar{g}}}{\sqrt{\gamma}}\Big(\bar{g}^{\mu\sigma}\bar{g}^{\nu\tau} - \bar{g}^{\mu\tau}\bar{g}^{\nu\sigma}\Big). \tag{6}$$

$\bar{g}^{\mu\sigma}$ is a function of the permittivity and permeability, and is used with a bar on the top to distinguish it from the actual metric. Here, γ is the determinant of the spatial metric in the ambient curvilinear coordinate system (r, θ, ϕ), defined by $\gamma_{\alpha\beta} = -g_{\alpha\beta} + (g_{0\alpha}g_{0\beta})/g_{00}$, where g is the metric of the ambient flat spacetime. In spherical polar coordinate system on a flat spacetime, $\sqrt{\gamma} = r^2\ sin\theta$.

The field strength $F_{\sigma\tau}$ is written in terms of the four-potentials A_μ, which are expanded in the vector spherical harmonics as follows [9]:

$$\begin{aligned} F_{\sigma\tau} &= A_{\tau,\sigma} - A_{\sigma,\tau}, \\ A_\mu &= \sum_{lm}\left[\left(0, 0, \frac{a^{lm}}{\sin\theta}\frac{\partial Y^{lm}}{\partial\phi}, -a^{lm}\sin\theta\frac{\partial Y^{lm}}{\partial\theta}\right)\right. \\ &+ \left.\left(f^{lm}Y^{lm}, h^{lm}Y^{lm}, k^{lm}\frac{\partial Y^{lm}}{\partial\theta}, k^{lm}\frac{\partial Y^{lm}}{\partial\phi}\right)\right], \end{aligned} \tag{7}$$

where a^{lm} are odd parity functions, and f^{lm}, h^{lm}, k^{lm} are even parity functions of r and t. $Y^{lm}(\theta, \phi)$ are scalar spherical harmonics.

Using the above in Eq. (5), we get the following equation for odd parity functions:

$$\left(\bar{\mu}^{-1}_{\theta\theta}a^{lm}_{,r}\right)_{,r} - \bar{\varepsilon}^{\theta\theta}\frac{\partial^2 a^{lm}}{\partial t^2} - \frac{l(l+1)}{r^2}a^{lm} = 0, \tag{8}$$

and for even parity functions, we get:

$$\left(\bar{\mu}_{\theta\theta}^{-1} b_{,r}^{lm}\right)_{,r} - \bar{\varepsilon}^{\theta\theta} \frac{\partial^2 b^{lm}}{\partial t^2} - \frac{l(l+1)}{r^2} b^{lm} = 0. \tag{9}$$

These equations are similar to those developed for the gravitational case, by J A Wheeler and D Brill [8], and also by R Ruffini, J Tiomno and C V Vishveshwara [9] (who included the source term), which are shown below:

$$\left(g^{rr} a_{,r}^{lm}\right)_{,r} - g^{00} \frac{\partial^2 a^{lm}}{\partial t^2} - \frac{l(l+1)}{r^2} a^{lm} = 0, \tag{10}$$

and for even parity functions, it is:

$$\left(g^{rr} b_{,r}^{lm}\right)_{,r} - g^{00} \frac{\partial^2 b^{lm}}{\partial t^2} - \frac{l(l+1)}{r^2} b^{lm} = 0. \tag{11}$$

The similarity in Eqs. (9), (10) and (11) gives rise to the similarity in a number of phenomena, which depend on the form of the wave equation. One can formulate an optical spacetime with an effective optical metric, whose components are related to the permittivity and the permeability tensors as defined previously, and given as: $\bar{g}^{rr} = \bar{\mu}_{\theta\theta}^{-1} = \bar{\mu}_{\phi\phi}^{-1}$, $\bar{g}^{00} = -\bar{\varepsilon}^{\theta\theta} = -\bar{\varepsilon}^{\phi\phi}$, and $\bar{\varepsilon}^{\theta\theta}\bar{\mu}_{\theta\theta}^{-1} = 1$. Metamaterials can be used to realize such an optical spacetime. Such a system has been studied, particularly, for the Schwarzschild spacetime in [10].

3. Discussions

Choosing $\bar{\varepsilon}^{\theta\theta} = \left(1 - \frac{L}{r}\right)^{-1}$ and $\bar{\mu}_{\theta\theta}^{-1} = \left(1 - \frac{L}{r}\right)$, one gets an exact analogue of Schwarzschild black hole spacetime as:

$$\left[\left(1 - \frac{L}{r}\right) a_{,r}^{lm}\right]_{,r} - \left(1 - \frac{L}{r}\right)^{-1} \frac{\partial^2 a^{lm}}{\partial t^2} - \frac{l(l+1)}{r^2} a^{lm} = 0. \tag{12}$$

Here, $r = L$ can be considered to be the location of the event horizon. Therefore, we have the simplified equation as:

$$\frac{\partial^2 a^{lm}}{\partial r_*^2} - \frac{\partial^2 a^{lm}}{\partial t^2} - \left(1 - \frac{L}{r}\right) \frac{l(l+1)}{a^{lm}} a^{lm} = 0. \tag{13}$$

This is exactly similar to the equation obtained in the gravitational case. Here, we have used the Regge-Wheeler tortoise coordinate $r_* = r + L ln(\frac{r}{L} - 1)$. Following are the two metric components of the effective optical spacetime: $g^{00} = -(1 - \frac{L}{r})^{-1}$, and $g^{rr} = (1 - \frac{L}{r})$. The necessary conditions for a spacetime around a black hole are $g_{00} = 0$, $g^{rr} = 0$ at the event horizon. We have clearly seen that the effective metric satisfy these. This indicates that the medium can be used to simulate a Schwarzschild black hole spacetime. The analysis of the gravitational bending of light and quasi-normal modes will be exactly like that in the gravitational case.

3.1. The optical black hole of Narimanov and Khildishev

In order to simulate black hole phenomena, we must have: (i) a medium, which simulates black hole spacetime, and (ii) a perfect absorber, which acts like a black hole, and absorbs the entire incident light. We have already developed the formalism for designing the equivalent optical spacetime around a black hole. The omni-directional, broadband optical absorber developed in [7] can be used to mimic the black hole. The model developed by Narimanov and Khildishev consists of: (i) A central core (C) of complex permittivity, which absorbs the entire incident light, and (ii) An outer shell (S), with some permittivity profile, which bends the incident light, guiding it to the core, and thus, ensuring that entire incident light converges to the core, without any loss. The model with the core and the shell is placed in an external medium. Our formalism can be used to arrive at the Hamiltonian used by Narimanov and Khildishev. Starting from our wave equation and with the Eikonal approximation, we get their Hamiltonian as:

$$\frac{(\bar{\mu}_{\theta\theta})^{-1}}{\bar{\varepsilon}^{\theta\theta}}(\frac{\partial S}{\partial r})^2 + \frac{m^2}{r^2\bar{\varepsilon}^{\theta\theta}} = H. \tag{14}$$

This gives a trajectory of the light ray as:

$$\phi = \phi_0 + \int_{\frac{m}{r_1}}^{\frac{m}{r}} \frac{d\xi}{\sqrt{C_0\bar{\varepsilon}^{\theta\theta}(\frac{m}{\xi}) - \xi^2}}, \qquad \xi = \frac{m}{r}. \tag{15}$$

There is no need to introduce an optical spacetime here, as we have been considering bending of light into the absorbing core, inside the event horizon. Here, the focus is on absorbing the entire incident light rather than mimicking a spacetime. Their permittivity choice $\bar{\varepsilon}^{\theta\theta} = \bar{\varepsilon_0}^{\theta\theta}[1 + (\frac{R}{r})^n]$ serves the purpose well.

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