Scalar cosmological perturbations in the Gauss-Bonnet braneworld

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Abstract

We study scalar cosmological perturbations in a braneworld model with a bulk Gauss-Bonnet term. For an anti-de Sitter bulk, the five-dimensional perturbation equations share the same form as in the Randall-Sundrum model, which allows us to obtain metric perturbations in terms of a master variable. We derive the boundary conditions for the master variable from the generalized junction conditions on the brane. We then investigate several limiting cases in which the junction equations are reduced to a feasible level. In the low energy limit, we confirm that the standard result of four-dimensional Einstein gravity is reproduced on large scales, whereas on small scales we find that the perturbation dynamics is described by the four-dimensional Brans-Dicke theory. In the high energy limit, all the non-local contributions drop off from the junction equations, leaving a closed system of equations on the brane. We show that, for inflation models driven by a scalar field on the brane, the Sasaki-Mukhanov equation holds on the high energy brane in its original four-dimensional form. This article is a short summary of [1].

1 Introduction

One of the simplest realizations of braneworld is proposed by Randall and Sundrum (RS) [2], assuming that the bulk involves five-dimensional (5D) Einstein gravity with a negative cosmological constant. The RS model can be naturally extended to include the *Gauss-Bonnet* (GB) term:

$$\mathcal{L}_{\rm GB} := \mathcal{R}^2 - 4\mathcal{R}_{AB}\mathcal{R}^{AB} + \mathcal{R}_{ABCD}\mathcal{R}^{ABCD}, \tag{1}$$

where $\mathcal{R}$, $\mathcal{R}_{AB}$, and $\mathcal{R}_{ABCD}$ denote the Ricci scalar, Ricci tensor, and Riemann tensor in five dimensions, respectively. Cosmology on a GB brane is important because one of the possible ways to test the braneworld idea is studying cosmological perturbations from inflation. In this direction, Minamitsuji and Sasaki [3] have examined linearized effective gravity on a de Sitter brane, and Dufaux *et al.* [4] investigated tensor and scalar perturbations generated from de Sitter inflation in the GB braneworld (The authors of [4] have performed an exact analysis for the tensor perturbations, but they have neglected bulk effects for the scalar perturbations without any justification). In the present article, we study scalar cosmological perturbations on a more general (flat) Friedmann-Robertson-Walker cosmological brane.

2 Gauss-Bonnet braneworld

We start with providing the basic equations that describe the GB braneworld. Our action is

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left[\mathcal{R} - 2\Lambda + \alpha\mathcal{L}_{\rm GB}\right] + \int d^4x\sqrt{-q}\left[2K + \frac{4\alpha}{3}Q + \mathcal{L}_m - \sigma\right], \tag{2}$$

where Λ is the cosmological constant in the bulk, $\mathcal{L}_m$ is the matter Lagrangian on the brane, and σ is the brane tension. The GB Lagrangian $\mathcal{L}_{\rm GB}$ was already defined in Eq. (1) and the coupling constant α has dimension of (length)^2. The surface term is given by $2K + (4\alpha/3)Q$, where K is the trace of the extrinsic curvature $K_\mu^{\ \nu}$ of the brane and $Q := Q_\mu^{\ \mu}$ with $Q_\mu^{\ \nu}$ defined below in Eq. (6).

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The 5D field equations following from the above action are

$$\mathcal{G}_{AB} - \frac{\alpha}{2}\mathcal{H}_{AB} = -\Lambda g_{AB}, \tag{3}$$

where $\mathcal{G}_{AB} := \mathcal{R}_{AB} - \mathcal{R}g_{AB}/2$ is the Einstein tensor and $\mathcal{H}_{AB}$ is the GB tensor defined by

$$\mathcal{H}_{AB} := \mathcal{L}_{\rm GB}g_{AB} - 4\left(\mathcal{R}\mathcal{R}_{AB} - 2\mathcal{R}_{AC}\mathcal{R}^{C}_{\ B} - 2\mathcal{R}_{ACBD}\mathcal{R}^{CD} + \mathcal{R}_{ACDE}\mathcal{R}_{B}^{\ CDE}\right). \tag{4}$$

Assuming a Z_2 symmetry across the brane, the junction conditions at the brane are given by [5]

$$K_{\mu}^{\ \nu} - K\delta_{\mu}^{\ \nu} = -\frac{\kappa^2}{2}\left(T_{\mu}^{\ \nu} - \sigma\delta_{\mu}^{\ \nu}\right) - 2\alpha\left(Q_{\mu}^{\ \nu} - \frac{1}{3}Q\delta_{\mu}^{\ \nu}\right), \tag{5}$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor and

$$\begin{aligned} Q_{\mu}^{\ \nu} \ &:= \ 2KK_{\mu}^{\ \alpha}K_{\alpha}^{\ \nu} - 2K_{\mu}^{\ \alpha}K_{\alpha}^{\ \beta}K_{\beta}^{\ \nu} + \left(K_{\alpha}^{\ \beta}K_{\beta}^{\ \alpha} - K^2\right)K_{\mu}^{\ \nu} \\ &\quad +2KR_{\mu}^{\ \nu} + RK_{\mu}^{\ \nu} - 2K_{\alpha}^{\ \beta}R_{\mu\beta}^{\ \ \ \nu\alpha} - 2R_{\mu}^{\ \alpha}K_{\alpha}^{\ \nu} - 2R_{\alpha}^{\ \nu}K_{\mu}^{\ \alpha}, \end{aligned} \tag{6}$$

with $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$ and R being the Riemann tensor, Ricci tensor and Ricci scalar with respect to the 4D induced metric. The main difference in the junction conditions from those in Einstein gravity is that they include intrinsic curvature terms as well as external ones. Using the Codacci equation, we can show that the conservation law holds on the brane.

The field equations (3) admit an anti-de Sitter (AdS) bulk with the curvature radius ℓ $(=: \mu^{-1})$. The 5D cosmological constant and μ are related by $\Lambda = -6\mu^2\left(1 - 2\alpha\mu^2\right)$. It is useful to define a dimensionless parameter $\beta := 4\alpha\mu^2$. In this paper, we assume the parameter range $0 \le \beta < 1$. To present a cosmological background solution which has a flat 3D geometry [6], we write the AdS metric in the Gaussian normal coordinates as $g^{(0)}_{AB}dx^A dx^B = -n^2(t,y)dt^2 + a^2(t,y)\delta_{ij}dx^i dx^j + dy^2$. We may set $n(t,0) = 1$, so that t is the proper time on the brane at $y = y_b = 0$ and $a_b(t) := a(t,0)$ is the scale factor. Then the 5D field equations are solved to give

$$n(t,y) \ = \ \dot{a}(t,y)/\dot{a}_b(t), \tag{7}$$

$$a(t,y) \ = \ a_b(t)\left[\cosh(\mu y) - \sqrt{1 + \frac{H^2}{\mu^2}}\sinh(\mu y)\right]. \tag{8}$$

Although the 5D field equations include the GB term, the metric functions $n(t,y)$ and $a(t,y)$ have the same form as in the cosmological solution in the RS braneworld based on the Einstein-Hilbert action. What is manifestly different is the Friedmann equation that relates the Hubble expansion rate H and the energy-momentum components on the brane. The Friedmann equation derived from the generalized junction conditions at the brane is [6]

$$2\sqrt{H^2+\mu^2}\left(3 - \beta + 2\beta\frac{H^2}{\mu^2}\right) = \kappa^2(\rho + \sigma). \tag{9}$$

The critical brane tension, which allows for a Minkowski brane, is obtained by setting $H \to 0$ as $\rho \to 0$: $\kappa^2\sigma = 2\mu(3 - \beta)$. There are three regimes for the dynamical history of the GB brane universe, two of which are basically the same as those found in the context of the RS braneworld. When $H^2 \ll \mu^2/\beta [= (4\alpha)^{-1}]$, we recover the RS-type Friedmann equation, $H^2 \simeq 8\pi G/3\left(\rho + \rho^2/2\sigma\right)$, where we defined the 4D gravitational constant as $8\pi G := \kappa^2\mu/(1+\beta)$. At very high energies, $H^2 \gg \mu^2/\beta$, the effect of the GB term becomes prominent. In this regime, we find $H^2 \simeq \left(\kappa^2\mu^2\rho/4\beta\right)^{2/3}$.

3 Summary of cosmological perturbation theory in the Gauss-Bonnet braneworld

Now let us consider linear perturbations about the cosmological brane background discussed in the previous section. If the background geometry is given by AdS, the perturbed GB tensor has a following nice property:

$$\delta\mathcal{H}_{A}^{\ B} = 8\mu^2\delta\mathcal{G}_{A}^{\ B} \quad \Rightarrow \quad (1-\beta)\delta\mathcal{G}_{A}^{\ B} = 0, \tag{10}$$

which, aside from the factor $(1-\beta)$, give the same perturbation equations as in Einstein gravity. This allows us to make full use of the previously known results on cosmological perturbations in the RS model.

We write the perturbed metric in an arbitrary gauge as

$$\begin{aligned}\left(g^{(0)}_{AB}+\delta g_{AB}\right)dx^A dx^B &= -n^2(1+2A)dt^2+2a^2B_{,i}dtdx^i+a^2\left[(1-2\psi)\delta_{ij}+2E_{,ij}\right]dx^idx^j \\ &\quad +2nA_ydtdy+2a^2B_{y,i}dx^idy+(1+2A_{yy})dy^2. \qquad (11)\end{aligned}$$

The 5D perturbation equations will be solved most easily in the so-called *5D longitudinal gauge* [7], which is defined by $\tilde{\sigma}=-\tilde{B}+\dot{\tilde{E}}=0$, and $\tilde{\sigma}_y=-\tilde{B}_y+\tilde{E}'=0$. We use a master variable, Ω, which was originally introduced by Mukohyama [8] in the Einstein gravity case. The perturbed 5D field equations are solved if the metric perturbations are written in terms of this master variable:

$$\tilde{A}=-\frac{1}{6a}\left[2\Omega''-\frac{n'}{n}\Omega'-\mu^2\Omega+\frac{1}{n^2}\left(\ddot{\Omega}-\frac{\dot{n}}{n}\dot{\Omega}\right)\right],\quad \tilde{A}_y=\cdots,\quad \tilde{A}_{yy}=\cdots,\quad \tilde{\psi}=\cdots, \qquad (12)$$

where Ω is a solution of the master equation

$$\Omega''+\left(\frac{n'}{n}-3\frac{a'}{a}\right)\Omega'-\frac{1}{n^2}\left[\ddot{\Omega}-\left(\frac{\dot{n}}{n}+3\frac{\dot{a}}{a}\right)\dot{\Omega}\right]+\left(\mu^2+\frac{1}{a^2}\Delta\right)\Omega=0, \qquad (13)$$

with $\Delta:=\delta^{ij}\partial_i\partial_j$.

From the junction conditions (5) we obtain

$$\kappa^2\delta\rho = -6(1-\beta)\delta K_T+\frac{2\beta}{\mu^2}\frac{a_b'}{a_b}\delta G_0{}^0, \qquad (14)$$

$$\kappa^2\delta q_{,i} = -2(1-\beta)\delta K_i{}^0-\frac{2\beta}{\mu^2}\frac{a_b'}{a_b}\delta G_i{}^0, \qquad (15)$$

$$\kappa^2\delta p = 2(1-\beta)\left(\delta K_0{}^0+2\delta K_T\right)+\frac{2\beta}{\mu^2}\left[\frac{1}{3}\left(\frac{a_b'}{a_b}-\frac{n_b'}{n_b}\right)\delta G_0{}^0-\frac{a_b'}{a_b}\delta G_T\right], \qquad (16)$$

$$\kappa^2\delta\pi = -2(1-\beta)\delta K_{TL}-\frac{2\beta}{\mu^2}\frac{1}{a_b^2}\left[\frac{n_b'}{n_b}\Psi-\frac{a_b'}{a_b}\Phi\right]. \qquad (17)$$

The perturbations of the extrinsic curvature and the Einstein tensor can be written in terms of the master variable Ω and the *brane bending* (the perturbed location of the brane) ξ. In Eq. (17) the metric potentials are defined by $\Phi:=\tilde{A}+(n'/n)_b\xi$ and $\Psi:=\tilde{\psi}-(a'/a)_b\xi$. This equation relates Ω with ξ. Thus the above equations (14)–(17) give the boundary conditions for Ω at the brane.

3.1 Low energy limit

3.1.1 Perturbations larger than the bulk curvature radius

We can show that the standard 4D result is reproduced if $H^2\ll\mu^2$ and $\mu^2\Omega\gg\Delta\Omega/a_b^2$.

3.1.2 Small scale perturbations

We now consider scales much smaller than the typical GB scale, $\left|\Delta\Omega/a_b^2\right|\gg\left|\mu^2\Omega/\beta\right|$, by requiring that $\xi\sim(\beta/\mu)\Delta\Omega/a_b^3$. In this regime, the junction equations are equivalent to

$$\delta G_\mu{}^\nu=\frac{1}{2\varphi_0}\delta T_\mu{}^\nu+\frac{1}{\varphi_0}\left(\nabla_\mu\nabla^\nu-\nabla_\lambda\nabla^\lambda\delta_\mu{}^\nu\right)\delta\varphi,\quad \nabla_\lambda\nabla^\lambda\delta\varphi=\frac{1}{6+4\omega}\delta T, \qquad (18)$$

with the identifications

$$\frac{1}{\varphi_0}\to\frac{\kappa^2\mu}{\beta},\quad \frac{\delta\varphi}{\varphi_0}\to-\mu\left(\frac{1-\beta}{\beta}\right)\xi,\quad \omega\to\frac{3\beta}{1-\beta}. \qquad (19)$$

This is nothing but the linearized Brans-Dicke theory with terms of $\mathcal{O}(H^2\delta\varphi)$ neglected.

3.2 High energy limit

Let us take the high energy limit, $H^2 \gg \mu^2/\beta$. In this regime, the right hand side of the junction equations (14)–(16) is dominated by the perturbed 4D Einstein tensor and so we have

$$\delta G_0{}^0 = -\frac{\kappa^2\mu^2}{2\beta H}\delta\rho, \quad \delta G_i{}^0 = \frac{\kappa^2\mu^2}{2\beta H}\delta q_{,i}, \quad \delta G_T = \frac{\kappa^2\mu^2}{2\beta H}\left(\delta p - \frac{\epsilon_H}{3}\delta\rho\right), \tag{20}$$

where we defined $\epsilon_H := -\dot{H}/H^2$. Similarly, it is easy to show that the right hand side of Eq. (17) is dominated by the metric potentials. Thus we obtain

$$(1-\epsilon_H)\Psi - \Phi = \frac{\kappa^2\mu^2}{2\beta H}a_b^2\delta\pi. \tag{21}$$

The set of equations (20) and (21) governs the perturbation dynamics at very high energies.

Consider braneworld inflation driven by a single scalar field ϕ which is confined on the brane. For this background we have $\rho = \dot{\phi}^2/2 + V(\phi)$ and $p = \dot{\phi}^2/2 - V(\phi)$, where $V(\phi)$ is the potential of the inflaton. For perturbations generated by fluctuations of the scalar field, it is quite easy to describe the evolution of perturbations in the high energy limit by introducing the Sasaki-Mukhanov variable and invoking the energy conservation equation. The perturbations of the energy-momentum components are given by $\delta\rho = \dot{\phi}\left(\dot{\delta\phi} - \dot{\phi}\bar{A}_b\right) + (dV/d\phi)\,\delta\phi$, $\delta q = -\dot{\phi}\delta\dot{\phi}$, and $\delta p = \dot{\phi}\left(\dot{\delta\phi} - \dot{\phi}\bar{A}_b\right) - (dV/d\phi)\,\delta\phi$. The equation of motion for the scalar field perturbation $\delta\phi$ follows from the energy conservation equation, $\delta\left(\nabla_\nu T^{\mu\nu}\right) = 0$. Introducing a scalar field perturbation in the spatially flat gauge, $\delta\phi_\psi := \delta\phi + \dot{\phi}/H\bar{\psi}_b$, and defining new variables $v := a_b\delta\phi_\psi, z := a_b\dot{\phi}/H$, the Klein-Gordon equation for the scalar field perturbation can be rewritten in a familiar form

$$v'' + \left(k^2 - \frac{z''}{z}\right)v = 0, \tag{22}$$

where a prime denotes a derivative with respect to conformal time. This exactly coincides with the Sasaki-Mukhanov equation derived in the standard 4D context.

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