GAUGE INVARIANCE OF FRACTIONALLY CHARGED QUASIPARTICLES AND HIDDEN TOPOLOGICAL Z_n SYMMETRY

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ABSTRACT

Gauge invariance for fractionally charged anyonic quasiparticles in a 2-dimensional multiply-connected system requires multi-component wave functions, and leads to the emergence of a hidden topological Z_n symmetry and the associated quantum number (the n-ality) for many-body eigenstates. In certain situations, it relates the fractional charge to anyon statistics. The implications for the fractional quantum Hall effect (FQHE) are also discussed.

Gauge invariance plays a fundamental role not only in particle physics, but also in macroscopic quantum phenomena, such as flux quantization[1] in a superconducting ring and the integral quantum Hall effect[2]. Recently, direct evidence[3] has been experimentally available for a quasiparticle of charge $e^* = e/3$ in the FQHE with the filling factor $\nu = 1/3$. This urges the necessity of a theoretical understanding of gauge invariance for fractionally charged quasiparticles.

Consider a cylindrical (or toroidal) system with a magnetic flux Φ through the hole. Gauge invariance[1] implies that all physical properties of the system are periodic functions of Φ with the period (flux quantum) $\Phi_o = hc/e$, where e is the constituent (electron) charge. The problem is under what conditions the system of quasiparticle excitations with fractional charge e^* can have, as required by gauge invariance, a (q times, if $e^* = e/q$) smaller period Φ_o than the naturally expected $\Phi_o^* = hc/e^*$.

To study this problem, we use the braid group formalism[4] on a cylinder[5] (or on a torus[6]), appropriate for anyonic quasiparticles in FQHE. We will show that if the anyon system is described by a one-component wave function, there is no period smaller than Φ_o^*. In fact, on a cylinder, the braid group generators consist of not only usual σ_i (i = 1, ..., N-1) which interchanges the i-th and (i+1)-th particles, but also of additional ρ_j (j = 1, ..., N) representing moving a particle along a simple loop around the hole once with j-1 particles to its left.

1-d unitary BGR's are characterized by two parameters θ and Φ:

$$\sigma_j = \exp(i\theta),$$

$$\rho_j = \exp\{i[2\theta(j-1) + 2\pi e^*\Phi/hc]\} \quad (1)$$

The minimal change in Φ that leads to the same BGR is Φ_o^*. This is compatible with gauge invariance (the existence of another period Φ_o), if and only if e^* is an integral multiple of constituent charge e.

Conversely, when the anyon quasiparticles carry fractional charge e^*, for the system to possess a period smaller than Φ_o^*, the wave function must have more than one component. So we are led to consider an M-dimensional representation for σ_i and ρ_j. Assume the anyons obey scalar statistics: $\sigma_i = e^{i\theta} I_M$ with I_M being the M × M unit matrix. ρ_j are represented by

$$\rho_j(\Phi) = \exp\{i2\pi e^* \Phi/hc\} T_j, \qquad (2)$$

where T_j are Φ-independent M × M matrices satisfying $T_{j+1} = T_j e^{2i\theta}$. If $e^*/e = m/n$ with m, n mutually prime, a smaller period Φ_o implies the minimal period Φ_o^*/n and the unitary equivalence of the two BGR's

$$\rho_j\ (\Phi + \Phi_o^*/n) \simeq \rho_j(\Phi)$$

$$(j = 1, \ldots, N). \qquad (3)$$

Multiplying the eigenvalues of T_1 by $\exp\{i\, 2\pi/n\}$ should just shuffle them, so M must be divisible by n. If M = n, the situation is irreducible under the large gauge transformation which shifts Φ by Φ_o^*/n. The form of $\rho_j(\Phi)$ is determined as

$$\rho_j(\Phi) = \exp\{i\lambda_o + i2\theta(j-1)$$

$$+ i\ 2\pi\Phi e^*/hc\} W \qquad (4)$$

with W is a diagonal n × n matrix given by

$$W = \mathrm{diag}[1, \exp(i\ 2\pi/n), \ldots,$$

$$\exp(i\ 2\pi(n-1)/n)], \qquad (5)$$

or in appropriate basis,

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (6)$$

In such basis, each base state changes into another if one moves one anyon around the hole once, and returns to itself after n rounds (up to a possible phase).

Similarly, we can discuss the torus case, in which anyons require a multi-component wave function[6,7]. If the latter forms an irreducible BGR, gauge invariance relates the anyon charge to its (scalar) statistics θ. In fact, for anyons on a torus with $\sigma_i = e^{i\theta} I_M$, $(\theta/\pi)\ \Phi_o^*$ is always a period for Φ. To see this, we note that on a torus, besides σi there are generators τ_i, ρ_i (i = 1, ..., N) corresponding to moving the i-th particle along one of the fundamental non-contractible loops. They satisfy, among others,

$$\tau_{i+1} = \tau_i e^{-2i\theta} \quad \rho_{i+1} = \rho_i e^{2i\theta} \qquad (7)$$

and $\tau_i(\rho_i)$ have a factorized $\Phi_x(\Phi_y)$-dependence as in (1). Changing Φ_x by $(\theta/\pi)\Phi_o^*$ just gives rise to a phase factor $\exp(2i\theta)$, which shifts τ_i to τ_{i-1}. This only turns the BGR into an equivalent one and therefore does not alter any physical properties. For an irreducible BGR, M = q for $\theta = \pi(p/q)$ with p and q mutually prime, the minimal period is Φ_o^*/q. So the period Φ_o required by gauge invariance must be an integral multiple of the latter, and therefore $e^*/e = m/q$ with integer m.

The above conditions have very profound physical implications. First let us consider the cylinder case. The wave function for fractionally charged quasiparticles has to have n-components with n > 1. To specify their many-body states, besides the positions one needs an extra index, the index of components (or "sheets") s(=1, ..., n). We emphasize that this index generally is not associated to individual quasiparticles. Eq. (4) shows that the operation of moving one anyon around the hole is given by, up to some phase, the winding operator W which acts on the sheet indices. Note that $W^n = 1$. Normally, the Hamiltonian H, no matter how complicated it may be, with various interactions, impurities or defects or external field all included, always commutes with W. So the eigenvalues of W, $\exp(i2\pi k/n)$ or simply k (mod n), give us a good quantum number, the so-called n-ality, for the many-body energy eigenstates.

Because the (k,Φ)-dependence of the eigenvalues of ρ_i is through the combination $\Phi + k\Phi_o^*/n$, the energy or any physical property satisfies

$$E(k,\Phi,\{\alpha\})=f(\Phi + k\Phi_o^*/n, \{\alpha\}), \quad (8)$$

where $\{\alpha\}$ is a set of usual quantum numbers. Thus the energy spectrum is actually a collection of n sectors, each corresponding to a one-component system with a central flux, differing from each other by Φ_o^*/n and therefore admits a smaller period Φ_o^*/n.

Moreover, (8) implies the existence of level crossings or spectral flow. Because the topological n-ality is a good quantum number, a gap can never be open at the level crossing points unless the two levels involved have the same n-ality. A numerical result showing the pattern of level crossings for a 3-anyon system on a 3 × 3 cylindrical lattice with $\theta = \pi/5$ and $e^* = e/3$ is given by Fig. 1. (The details

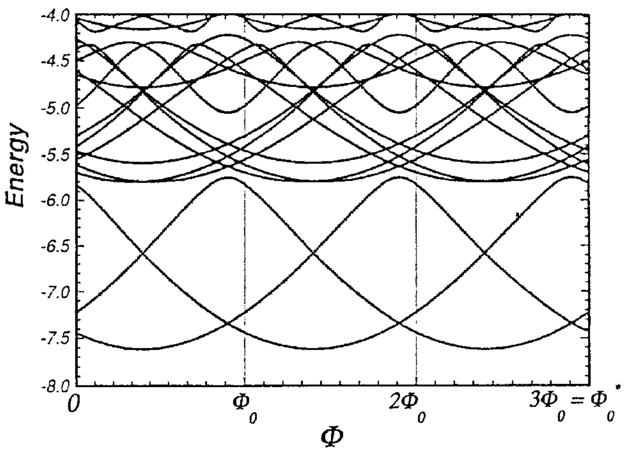

Figure 1. Spectral flow on a cylinder

will be presented elsewhere[7].) Note in particular that the three lowest levels flow into each other with a period 1/3, but for each fixed level the period is three times larger. Though this kind of pattern is not typical for an anyon system, but there are good reasons to believe that the ground states of a cylindrical FQH system have such a pattern of level crossings. Such a scenario is essentially what proposed by Tao and Wu[8] six years ago and recently refined by Thouless[9] This has been shown by Niu, Thouless and Wu[10] to be sufficient to give the fractional quantization of the Hall conductance in the topological approach.

A consequence of our results is that the $\nu = 1/q$ FQH <u>edge</u> states on a cylinder must, like the bulk states, carry a Z_q-like quantum number.

When put on a torus, the Laughlin states for $\nu = 1/q$ with q odd correspond to an irreducible BGR. So from gauge invariance alone we can infer that the fractional charge of quasiparticles must be an integral multiple of e/q. In the torus case, we have two non-commuting winding operators W_1 and W_2, similar to (7), respectively corresponding to moving an anyon along different fundamental loops: $W_1W_2 = W_2W_1e^{2i\theta}$. They are symmetries in the thermodynamic limit and the spectrum has q-fold exact degeneracy for each level of anyons.

Our topological discussion is quite general and model-independent, but does not tell what underlying dynamical mechanism will give rise to the spectrum required by gauge invariance. It would be interesting to speculate on the possible relevance to quarks, which are also fractionally charged and carry a triality.

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