DOUBLE PARTON DISTRIBUTION FUNCTIONS AND THEIR QCD EVOLUTION EQUATIONS*

Emilia Lewandowska

The Henryk Niewodniczański Institute of Nuclear Physics PAN
Radzikowskiego 152, 31-342 Kraków, Poland

Double parton distribution functions (DPDFs) are used in the description of double parton scattering. Their QCD evolution equations, known in the leading logarithmic approximation, obey non-trivial momentum and valence quark number sum rules. In this paper, we discuss the QCD evolution of the DPDFs, in particular, the specification of initial conditions for the evolution equations which obey the new momentum sum rules. We also present results on factorization of DPDFs on the product of two single parton distributions.

1. Introduction

Double parton distribution functions are used in the description of double parton scattering, which is the simplest process in the analysis of multiparton interactions, studied for many years from both the theoretical [1–13] and phenomenological side [14–19]. The experimental evidence of the double parton scattering has been presented in [20–25].

Let us recall the main facts about the double parton distribution functions

$$D_{f_1 f_2}(x_1, x_2, Q_1, Q_2) \,. \tag{1.1}$$

They depend on parton flavours f_1, f_2 (including gluons) and parton momentum fractions, x_1, x_2, obeying the condition

$$x_1 + x_2 \leq 1 \,, \tag{1.2}$$

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which says that the sum of two parton momenta cannot exceed the total nucleon momentum. The DPDFs also depend on two hard scales, Q_1 and Q_2, of two hard processes in which the partons take part.

In the most naive approach, one can assume that DPDFs are products of two single parton distribution functions (SPDFs):

$$D_{f_1 f_2}(x_1, x_2, Q_1, Q_2) = D_{f_1}(x_1, Q_1) \, D_{f_2}(x_2, Q_2) \, \Theta(1 - x_1 - x_2). \quad (1.3)$$

However, this is not generally true because of correlations between partons.

2. Evolution equations for double parton distributions

In the leading logarithmic approximation (LLA), the QCD evolution of the DPDFs is a two-step process. In the first step, DPDFs with equal scales are evolved from an initial scale Q_0 up to the smaller scale $Q_1 < Q_2$ (which relation we assume from now on), treating both momentum fractions symmetric. In the second step, the single parton distribution evolution from Q_1 to Q_2 is preformed keeping the momentum fraction x_1 fixed. Schematically,

$$D_{f_1 f_2}(x_1, x_2, Q_0, Q_0) \to D_{f_1 f_2}(x_1, x_2, Q_1, Q_1) \to D_{f_1 f_2}\left(x_1^{\text{fix}}, x_2, Q_1, Q_2\right). \quad (2.1)$$

The evolution equations for the first step are written for the DPDFs with equal evolution parameters: $D_{f_1 f_2}(x_1, x_2, t) \equiv D_{f_1 f_2}(x_1, x_2, Q, Q)$,

$$\partial_t D_{f_1 f_2}(x_1, x_2, t) = \sum_{f'} \int_0^{1-x_2} du \, \mathcal{K}_{f_1 f'}(x_1, u, t) \, D_{f' f_2}(u, x_2, t)$$

$$+ \sum_{f'} \int_0^{1-x_1} du \, \mathcal{K}_{f_2 f'}(x_2, u, t) \, D_{f_1 f'}(x_1, u, t)$$

$$+ \sum_{f'} \mathcal{K}_{f' \to f_1 f_2}(x_1, x_2, t) \, D_{f'}(x_1 + x_2, t), \quad (2.2)$$

where the evolution parameter $t = \ln(Q^2/Q_0^2)$. The upper integration limits reflect condition (1.2), written for the momentum fractions (u, x_2) and (x_1, u). The limits can be formally set to 1 by extending the DPDFs into the non-physical domain through the condition

$$D_{f_1 f_2}(x_1, x_2, t) = 0 \qquad \text{for} \qquad x_1 + x_2 > 1. \quad (2.3)$$

The third term in Eq. (2.2), called from now on the *splitting term*, describes the parton splitting

$$\left(f', x_1 + x_2\right) \to (f_1, x_1) + (f_2, x_2). \quad (2.4)$$

Note that this term contains the SPDF, $D_{f'}(x, t)$, which obey the DGALP evolution equations in the LLA

$$\partial_t D_f(x, t) = \sum_{f'} \int_0^1 du \, \mathcal{K}_{ff'}(x, u, t) D_{f'}(u, t) \,. \tag{2.5}$$

Thus, Eqs. (2.2) and (2.5) form a closed set of equations to be solved simultaneously. The integral kernels $\mathcal{K}_{ff'}$ in Eqs. (2.2) and (2.5) are given by the relation

$$\mathcal{K}_{ff'}(x, u, t) = \mathcal{K}_{ff'}^{\mathrm{R}}(x, u, t) - \delta(u - x) \, \delta_{ff'} \, \mathcal{K}_f^{\mathrm{V}}(x, t) \tag{2.6}$$

with the real part equal to

$$\mathcal{K}_{ff'}^{\mathrm{R}}(x, u, t) = \frac{\alpha_{\mathrm{s}}(t)}{2\pi} \frac{1}{u} P_{ff'}^{(0)} \left(\frac{x}{u} \right) \theta(u - x) \,, \tag{2.7}$$

where the splitting functions, $P_{ff'}^{(0)}$, are given in the LLA. The virtual part of the kernel (2.6) is computed from the momentum sum rule for SPDFs,

$$\sum_f \int_0^1 dx \, x D_f(x, t) = 1 \,, \tag{2.8}$$

and has the following form

$$x \, \mathcal{K}_f^{\mathrm{V}}(x, t) = \sum_{f'} \int_0^1 du \, u \, \mathcal{K}_{f'f}^{\mathrm{R}}(u, x, t) \,. \tag{2.9}$$

The functions $\mathcal{K}_{f' \to f_1 f_2}$ in the splitting term in Eq. (2.2) are related to the real emission kernels (2.7). Let us notice that in the LLA there exists only one flavour f' which allows splitting into given flavours, f_1 and f_2. Thus, the sum over f' contains only the term with this flavour

$$\mathcal{K}_{f' \to f_1 f_2}(x_1, x_2, t) = \mathcal{K}_{f_1 f'}^{\mathrm{R}}(x_1, x_1 + x_2, t) = \mathcal{K}_{f_2 f'}^{\mathrm{R}}(x_2, x_1 + x_2, t) \,. \tag{2.10}$$

The last equality results from the properties of the splitting functions. For example, for the splitting $q \to gq$, we have

$$P_{gq}^{(0)}(z) = P_{qq}^{(0)}(1 - z) \tag{2.11}$$

which leads to this equality for $z = x_1/(x_1 + x_2)$ and $1 - z = x_2/(x_1 + x_2)$. Similarly, for the splittings $g \to qq$ and $g \to gg$, the following relations hold

$$P_{qg}^{(0)}(z) = P_{qg}^{(0)}(1 - z) \,, \qquad P_{gg}^{(0)}(z) = P_{gg}^{(0)}(1 - z) \,. \tag{2.12}$$

Thus, the splitting term, as well as the sum of the first two terms in Eq. (2.2), are symmetric with respect to the parton interchange: $(f_1, x_1) \leftrightarrow (f_2, x_2)$. In consequence, evolution equations (2.2) are invariant under parton interchange provided the initial conditions obey the same symmetry.

The explicit form of Eqs. (2.2) can be given by

$$\partial_t D_{f_1 f_2}(x_1, x_2, t) =$$

$$\sum_{f'} \left\{ \int_{\frac{x_1}{1-x_2}}^{1} \frac{dz}{z} P_{f_1 f'}^{(0)}(z) \, D_{f' f_2}\left(\frac{x_1}{z}, x_2, t\right) - D_{f_1 f_2}(x_1, x_2, t) \int_0^1 dz z P_{f' f_1}^{(0)}(z) \right\}$$

$$+ \sum_{f'} \left\{ \int_{\frac{x_2}{1-x_1}}^{1} \frac{dz}{z} P_{f_2 f'}^{(0)}(z) \, D_{f_1 f'}\left(x_1, \frac{x_2}{z}, t\right) - D_{f_1 f_2}(x_1, x_2, t) \int_0^1 dz z P_{f' f_2}^{(0)}(z) \right\}$$

$$+ \sum_{f'} \frac{1}{x_1 + x_2} P_{f_1 f'}^{(0)}\left(\frac{x_1}{x_1 + x_2}\right) D_{f'}(x_1 + x_2, t), \tag{2.13}$$

where, for simplicity, we keep the number of active flavours fixed. This allows to absorb the running coupling constant, $\alpha_{\rm s}$, into a new definition of the evolution parameter

$$t = \frac{6}{33 - 2n_f} \ln \frac{\ln\left(Q^2/\Lambda_{\rm QCD}^2\right)}{\ln\left(Q_0^2/\Lambda_{\rm QCD}^2\right)}. \tag{2.14}$$

According to relation (2.10), the splitting term in (2.13) can also be written with the splitting function $P_{f_2 f'}^{(0)}(x_2/x_1 + x_2)$.

The second step in the evolution chain (2.1) is realized by the DGLAP evolution equation (2.5) with respect to the second parton, while keeping the first parton fixed

$$\partial_{t_2} D_{f_1 f_2}(x_1, x_2, t, t_2) = \sum_{f'} \int_0^{1-x_1} du \, \mathcal{K}_{f_2 f'}(x_2, u, t) \, D_{f_1 f'}(x_1, u, t, t_2). \tag{2.15}$$

This is why the upper limit in the integral above equals $(1 - x_1)$.

3. Sum rules for DPDFs

Evolution equations (2.2) and (2.5) preserve the following relation valid for any flavour f_2

$$\sum_{f_1} \int_0^{1-x_2} dx_1 x_1 D_{f_1 f_2}(x_1, x_2, t) = (1 - x_2) D_{f_2}(x_2, t) \qquad (3.1)$$

once it is imposed on initial conditions for the evolution. This formula can be better understood after dividing both sides by $D_{f_2}(x_2, t)$. The quantity which appears under the integral,

$$D_{f_1}(x_1 | x_2; t) = \frac{D_{f_1 f_2}(x_1, x_2, t)}{D_{f_2}(x_2, t)}, \qquad (3.2)$$

is the conditional probability to find the parton (f_1, x_1) in the nucleon once the second parton is fixed. Thus, formula (3.1) says that active partons carry the fraction $(1 - x_2)$ of the total nucleon momentum. Strictly speaking, both the double and single parton distributions cannot be interpreted as probability distributions since they are not always normalizable. Nevertheless, condition (3.1) is preserved during evolution.

From Eq. (3.1), the following relation is found after integrating and summing over the second parton characteristics

$$\sum_{f_1, f_2} \int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{x_1 x_2 D_{f_1 f_2}(x_1, x_2, t)}{(1 - x_2)} = \int_0^1 dx_2 \, x_2 \sum_{f_2} D_{f_2}(x_2, t). \qquad (3.3)$$

The integral on the r.h.s. equals 1 from the momentum sum rule for SPDFs (2.8). Thus, the momentum sum rule for the DPDFs can also be written as

$$\sum_{f_1, f_2} \int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{x_1 x_2}{(1 - x_2)} D_{f_1 f_2}(x_1, x_2, t) = 1. \qquad (3.4)$$

It is important to notice that after imposing the parton exchange symmetry,

$$D_{f_1 f_2}(x_1, x_2) = D_{f_2 f_1}(x_2, x_1), \qquad (3.5)$$

relation (3.1) can be rewritten for the situation when the parton (f_1, x_1) is fixed

$$\sum_{f_2} \int_0^{1-x_1} dx_2 x_2 D_{f_1 f_2}(x_1, x_2, t) = (1 - x_1) D_{f_1}(x_1, t). \qquad (3.6)$$

As a result, we obtain an equivalent relation for the momentum sum (3.4) by interchanging $x_1 \leftrightarrow x_2$ in the integration measure

$$\sum_{f_1,f_2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \frac{x_1 x_2}{1-x_1} D_{f_1 f_2}(x_1, x_2, t) = 1 \,. \qquad (3.7)$$

The DPDFs also obey valence quark number sum rule which is more complicated than that for the SPDFs

$$\int_0^1 dx \left\{ D_{q_i}(x,t) - D_{\bar{q}_i}(x,t) \right\} = N_i \,. \qquad (3.8)$$

For the DPDFs, the analogous sum rule depends on the second parton flavour f_2 (see [4, 26] for more details)

$$\int_0^{1-x_2} dx_1 \left\{ D_{q_i f_2}(x_1, x_2, t) - D_{\bar{q}_i f_2}(x_1, x_2, t) \right\}$$

$$= \begin{cases} N_i \, D_{f_2}(x_2, t) & \text{for} \quad f_2 \neq q_i, \bar{q}_i \,, \\[2mm] (N_i - 1) \, D_{f_2}(x_2, t) & \text{for} \quad f_2 = q_i \,, \\[2mm] (N_i + 1) \, D_{f_2}(x_2, t) & \text{for} \quad f_2 = \bar{q}_i \,. \end{cases} \qquad (3.9)$$

Let us stress again that the momentum and valence quark number sum rules are conserved by the evolution equations (2.2) and (2.5) only after imposing them on initial conditions.

4. Initial conditions

In order to solve the evolution equations (2.2) and (2.5), the initial distributions $D_{f_1 f_2}(x_1, x_2, t_0)$ and $D_f(x, t_0)$ have to be specified. In the literature, the following *symmetric* ansatz is often discussed [3, 4]

$$D_{f_1 f_2}(x_1, x_2, t_0) = D_{f_1}(x_1, t_0) \, D_{f_2}(x_2, t_0) \, \frac{(1 - x_1 - x_2)^2}{(1-x_1)^{2+n_1}(1-x_2)^{2+n_2}} \qquad (4.1)$$

with $n_{1,2} = 0.5$ for valence parton distributions. However, the valence number sum rule (3.9) is strongly violated for such an ansatz. This is shown in Fig. 1 as the deviation from 1. Thus, the natural question arises how to specify initial conditions which exactly obey the new sum rules. One of

the solution would be to specify the initial DPDFs and then generate SPDF from Eqs. (3.1) and (3.9). However, this is not practical since the double parton distributions are not constrained at present by experimental data, in contrast to the SPDFs which are very well known. Therefore, we reverse the logic and will try to build the initial DPDFs from the known SPDFs.
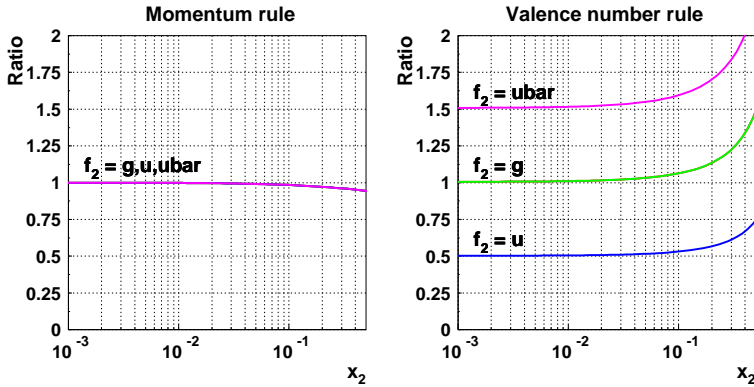

Fig. 1. The ratio of the r.h.s. to the l.h.s. of Eq. (3.1) (left) and Eq. (3.9) (right) for the input distributions (4.1) as a function x_2 for $f_2 = g, d, u, \bar{u}$. The violation of the sum rules is indicated by the departure of the ratios from 1.

It is easy to check that the following ansatz satisfies Eq. (3.1) and the first relation in Eq. (3.9)

$$D_{f_1 f_2}(x_1, x_2, t_0) = \frac{1}{1 - x_2} D_{f_1}\left(\frac{x_1}{1 - x_2}, t_0\right) D_{f_2}(x_2, t_0), \qquad (4.2)$$

where the condition $x_1 + x_2 \leq 1$, imposed by the theta function, is implicit. However, the last two valence quark relations (3.9) are not satisfied. In order to fulfil them, we have to correct ansatz (4.2) for the same flavours or anti-flavours, $f_1, f_2 \in \{q_i, \bar{q}_i\}$:

$$D_{f_1 f_1}(x_1, x_2, t_0) = \frac{1}{1 - x_2}\left\{D_{f_1}\left(\frac{x_1}{1 - x_2}, t_0\right) - \frac{1}{2}\right\} D_{f_1}(x_2, t_0), \quad (4.3)$$

$$D_{f_1 \bar{f}_1}(x_1, x_2, t_0) = \frac{1}{1 - x_2}\left\{D_{f_1}\left(\frac{x_1}{1 - x_2}, t_0\right) + \frac{1}{2}\right\} D_{\bar{f}_1}(x_2, t_0). \quad (4.4)$$

The additional factors with $\pm 1/2$ account for the factors with ± 1 on the r.h.s. of Eq. (3.9) and do not spoil the momentum sum rule relation (3.1). However, the proposed ansatz is not symmetric with respect to the exchange of partons

$$D_{f_1 f_2}(x_1, x_2, t_0) \neq D_{f_2 f_1}(x_2, x_1, t_0). \qquad (4.5)$$

In addition, the quark distributions $D_{q_i q_i}(x_1, x_2)$ and $D_{\bar{q}_i \bar{q}_i}(x_1, x_2)$ are not positive definite in the whole physical domain: $x_1 + x_2 \leq 1$. In Fig. 2 we show the graphical comparison of the two inputs as functions of x_1 with fixed $x_2 = 10^{-3}$, using the MSTW08 LO parametrization of the SPDFs [27] at $Q_0^2 = 2\,\mathrm{GeV}^2$. For such a small x_2, both ansatze give practically the same distributions with gluons, D_{gg} and D_{gu}. However, this is not the case for the pure quark distributions, D_{uu} and $D_{u\bar{u}}$, which are significantly different in the large x_1 region. As expected, D_{uu} from Eq. (4.3) is negative for $x_1 > 0.6$. Notice also that the modifications given by Eqs. (4.3) and (4.4) lead to non-zero values of the distributions $D_{f_i f_i}$ and $D_{f_i \bar{f}_i}$ at the kinematic boundary: $x_1 + x_2 = 1$.
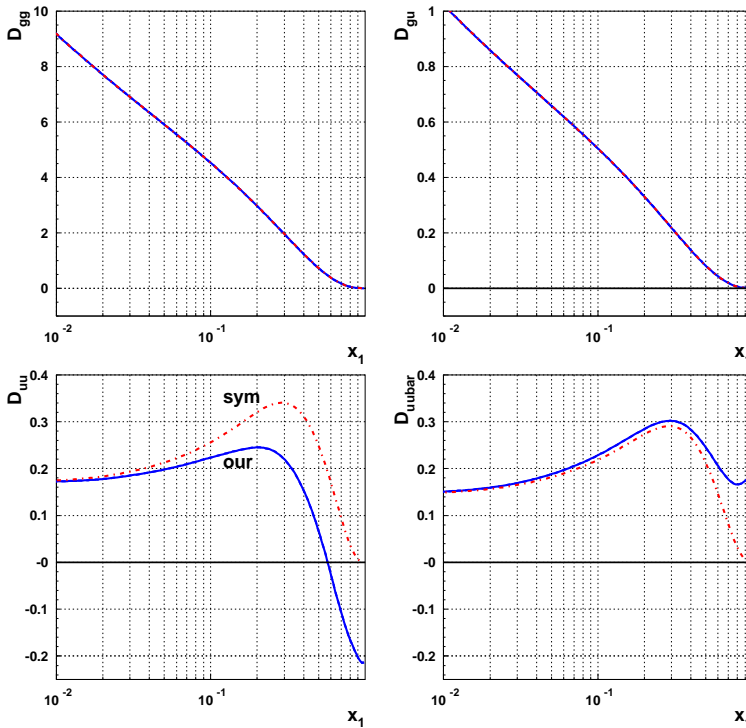

Fig. 2. The initial DPDFs, $x_1 x_2 D_{f_1 f_2}(x_1, x_2)$, as functions of x_1 for fixed $x_2 = 10^{-3}$ and the input scale $Q_0^2 = 2\,\mathrm{GeV}^2$. The two input distributions (4.2) (our) and (4.1) (sym.) are plotted.

Unfortunately, we could not find the positive definite initial conditions which are built from the SPDF and fulfil relations (3.1) and (3.9). We summarize the features of the two discussed initial conditions in Table I.

TABLE I

Properties of the discussed initial conditions.

Properties	Symmetric ansatz	Our ansatz
Parton symmetry	yes	no
Positivity	yes	no
Sum rules	no	yes

5. Effects of the evolution

In order to study the evolution of the DPDFs we have constructed a numerical program based on Chebyshev polynomial expansion which simultaneously solves Eqs. (2.5) and (2.2). In Fig. 3 we show the DPDFs evolved up to $Q^2 = 100 \text{ GeV}^2$ from the initial conditions (4.1) (sym.) and (4.2) (our). We see that the evolved distributions D_{gg} and D_{gu} give exactly the
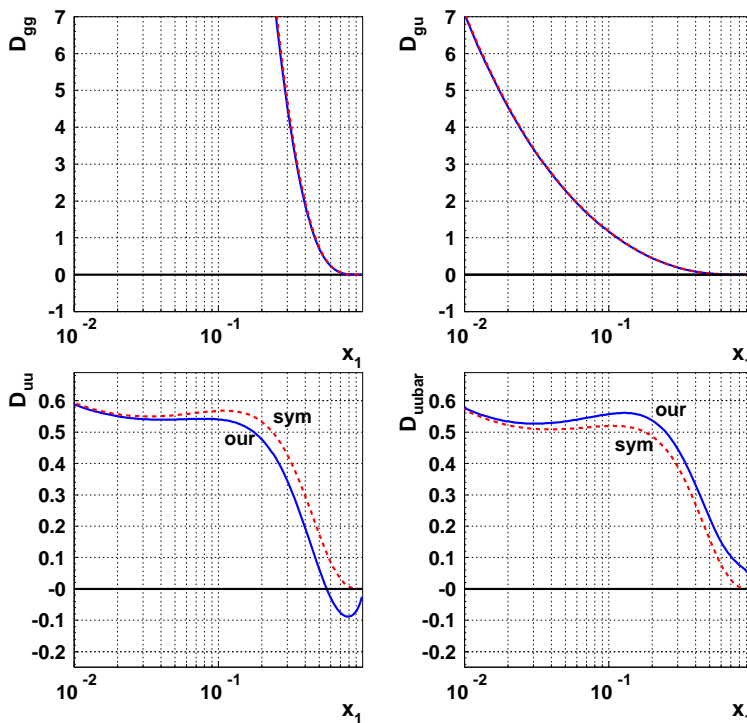

Fig. 3. The DPDFs from Fig. 2 for the symmetric (dashed curves) and our (solid curves) inputs evolved to $Q^2 = 100 \text{ GeV}^2$ as functions of x_1 for fixed $x_2 = 10^{-3}$.

same results in the whole domain of x_1, while the distributions D_{uu} and $D_{u\bar{u}}$ are different for large values of x_1. As expected, D_{uu} from our input stays negative for $x_1 > 0.6$. On the other hand, for small values of parton momentum fractions, $x_1, x_2 \ll 1$, both distributions tend to the factorized form

$$D_{f_1 f_2}(x_1, x_2, Q) \approx D_{f_1}(x_1, Q)\, D_{f_2}(x_2, Q). \tag{5.1}$$

Let us address the question to what extent the double parton distributions are different from the naive product of two single PDFs which are evolved by the standard DGLAP evolution equations. For this purpose, in Fig. 4 we plot the ratio

$$R_{f_1 f_2} = \frac{D_{f_1 f_2}\left(x_1, x_2, Q^2\right)}{D_{f_1}\left(x_1, Q^2\right) D_{f_2}\left(x_2, Q^2\right)} \tag{5.2}$$

as a function of x_1 for $x_2 = 10^{-3}$ and $Q^2 = 100 \text{ GeV}^2$. As we can see, the effect of the violation of factorization for small values of x_1 and x_2 is only
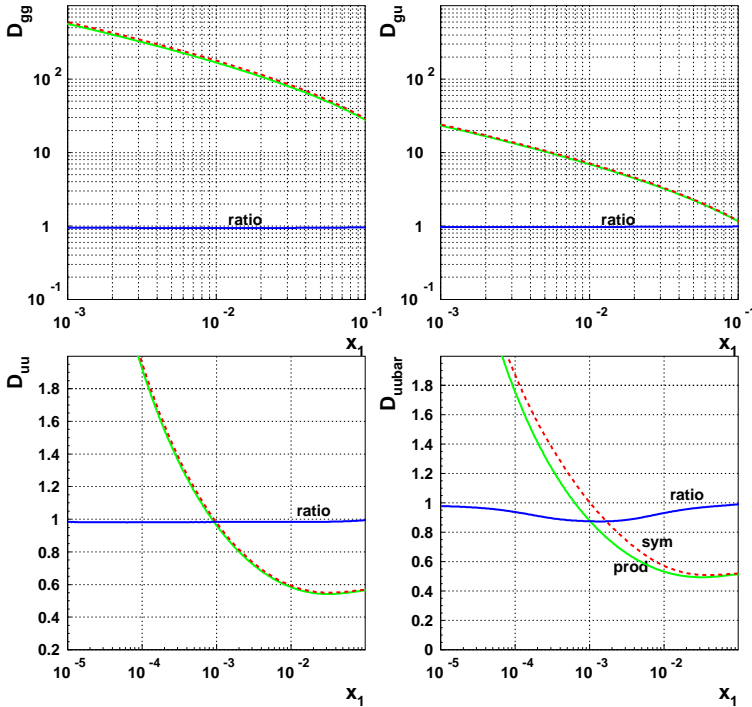

Fig. 4. The ratio (5.2) for the symmetric (4.1) (dashed line) and our (4.2) (solid line) inputs evolved to $Q^2 = 100 \text{ GeV}^2$ as functions of x_1 for fixed $x_2 = 10^{-3}$.

seen for the distribution $D_{u\bar{u}}$. This is due to the third term in the evolution equations (2.2) which characterise parton splitting. In particular, the violation is significant only for the splitting $g \to q\bar{q}$, due to a large value of the gluon distribution $g(x)$ at small x. For the distributions D_{gg}, D_{gu}, D_{uu}, the factorization holds very well. The same conclusions are valid for the other quark flavours.

6. Summary

The double parton distributions and their QCD evolution have been studied in detail. We presented the evolution equations for DPDFs in the leading logarithmic approximation and the sum rules which they should obey — the momentum and valence quark number.

Since a little is known about the DPDFs from experiments, we are bound to construct the initial conditions for the QCD evolution using the existing well known parametrizations of the SPDFs. We considered two forms of such inputs: the parton exchange symmetric (4.1), which do not fulfil the valence number sum rule, and (4.2)–(4.4) which exactly obeys this rule.

We also shown that the factorized form (5.1) for small values of x_1 and x_2 is conserved during the QCD evolution except for the distribution $D_{q\bar{q}}$. In this case, the splitting contribution $g \to q\bar{q}$ in the evolution equations (2.2) violates the factorization because of a large gluon distribution $g(x)$ at small x.

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