TAU LEPTON DECAYS AND PERIPHERAL PROCESSES IN THE FRAMEWORK OF THE NJL MODEL

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The known results about unitarity violation in QED are revised. Taking into account the peripheral processes in high energy ions collisions with production of arbitrary number of pairs of charged leptons and pions, we confirm the known result about violation of unitarity in frames of QED. Namely the total cross section of production of an arbitrary number of pairs of charged particles grow more rapidly than any power of logarithm of the center-of-mass energy. Contribution from production of a single neutral meson (scalar and pseudo-scalar) is discussed.

An interpretation of this fact is expected to have the similar base as the QED problem — the absence of the total cross section in scattering of a charged particle on the Coulomb centrum. The relevant conditions cannot be realized in the Nature.

Explicit expressions for production of a finite number of pairs cross sections are presented.

The decays $\tau \to \pi\pi(\pi')\nu$ and $\tau \to \omega\pi\nu$ were calculated in the framework of the extended NJL model. The intermediate vector $\rho(770)$ and $\rho'(1450)$ mesons are taken into account. The radial excited states of mesons are described by the second-order polynomial form factor.

The decays $\tau \to \eta(\eta')\pi\nu$ allowed due to quark mass differences are also described. In these decays the scalar mesons $a_0(980)$ and $a_0(1450)$ are also taken into account.

In contrast to other works the calculations of τ lepton decays in the framework of the NJL model do not require attraction of any arbitrary parameters.

1. INTRODUCTION

We consider cross sections of production of pairs of charged fermions and pions in high energy collisions of two charged fermions. The problem of violation of unitarity arises for an object which is the sum of arbitrary number of pairs. Namely it turns out that this “generalize” cross section grows with center-of-mass energy $\sqrt{s}$ faster than any power of the logarithm of energy $L = \log(s/m^2)$.

Dynamics of inelastic processes in peripheral collisions consisted in creation of set of hadrons with small invariant mass, separated by rapidity gaps. Description of subprocess can be performed in terms of realistic models of strong interactions such us bag model, CHPT, Nambu–Jona-Lasinio model.

The NJL model allows us to describe the processes of the τ lepton decays without introduction of any arbitrary parameters. Firstly, such calculations were fulfilled in the works [11] where decays $\tau \to 3\pi\nu$ and $\tau \to \pi\gamma\nu$ were described. Here we consider in the framework of the extended NJL decays $\tau \to \pi\pi(\pi')\nu$ and $\tau \to \pi\eta(\eta')\nu$ taking into account intermediate vector mesons $\rho(770), \rho(1450)$ and scalar mesons $a_0(980), a_0(1450)$. For description of first radial-excited state the polynomial form-factor of second order over transversal momentum is used.

2. GENERAL FORMALISM

Consider first the two-photon mechanism of fermion pair creation at charged ions collision

$$Y_1(Z_1, P_1) + Y_2(Z_2, P_2) \to Y_1(Z_1, P_1') + Y_2(Z_2, P_2') + \mu_+(q_+) + \mu_-(q_-),$$
$$s = (P_1 + P_2)^2 >> M_{1,2}^2, \qquad (1)$$

where $M_{1,2}$ are the ions masses.

Phase volume in the final state (one pair is created):

$$d\Gamma_1 = \frac{(2\pi)^4}{(2\pi)^{12}} \frac{d^3p_1'}{2E_1'} \frac{d^3p_2'}{2E_2'} \frac{d^3q_-}{2E_-} \frac{d^3q_+}{2E_+} \delta^4(p_1 + p_2 - p_1' - p_2' - q_+ - q_-), \quad (2)$$

with Sudakov parametrization of 4-momenta [1]

$$q_{1,2} = \alpha_{1,2}p_2 + \beta_{1,2}p_1 + q_{1,2}^{\perp}, q_{\pm} = \alpha_{\pm}p_2 + \beta_{\pm}p_1 + q_{\pm\perp}, \qquad (3)$$

takes the form

$$d\Gamma_1 = \frac{1}{(2\pi)^8} \frac{1}{4s} \frac{d\beta_1}{\beta_1} d^2\mathbf{q}_1 d^2\mathbf{q}_2 \frac{dx}{x\bar{x}} d^2\mathbf{q}_-, \qquad (4)$$

with $x = \beta_-/\beta_1$. Here and further we imply p_1, p_2 to be light-like 4-vectors constructed from the 4-momenta P_1, P_2 of the initial ions.

The matrix element is

$$\mathcal{M} = \frac{(4\pi\alpha)^2}{q_1^2 q_2^2} \left(\frac{2}{s}\right)^2 (sN_1)(sN_2)(s\phi), \qquad (5)$$

where $q_1 = p_1 - p_1'$, $q_2 = p_2 - p_2'$ and

$$N_1 = \frac{1}{s}\bar{u}(p_1')\widehat{p}_2 u(p_1),$$
$$N_2 = \frac{1}{s}\bar{u}(p_2')\widehat{p}_1 u(p_2),$$
$$\phi = \frac{1}{s}\bar{u}(q_-) O_{\mu\nu} v(q_+) p_1^{\mu} p_2^{\nu},$$

$$\mathcal{O} \equiv O_{\mu\nu} p_1^\mu p_2^\nu = \widehat{p}_1 \frac{\widehat{q}_- - \widehat{q}_1 + m}{d_1} \widehat{p}_2 + \widehat{p}_2 \frac{\widehat{q}_- - \widehat{q}_2 + m}{d_2} \widehat{p}_1, \tag{6}$$

$$d_{1,2} = (q_- - q_{1,2})^2 - m^2. \tag{7}$$

Quantities $N_{1,2}$ describe the subprocesses $Y_{1,2} \to Y_{1,2}\gamma^*$. Quantity Φ describe the subprocess $\gamma^* + \gamma^* \to \mu\bar{\mu}$.

The sums on spin states of module squared of these quantities are

$$\sum |N_1|^2 = \sum |N_2|^2 = 2, \quad \sum |\phi|^2 = 2x\bar{x}Z = \Phi^{\mu\mu}, \tag{8}$$

with

$$\begin{aligned} Z = {} & \frac{1}{D_-^2}[m^2 + (\vec{q}_2 - \vec{q}_+)^2][m^2 + \vec{q}_-^{\,2}] + \\ & + \frac{1}{D_+^2}[m^2 + (\vec{q}_1 - \vec{q}_+)^2][m^2 + \vec{q}_+^{\,2}] + \frac{2}{D_- D_+}[-(m^2 + \vec{q}_-^{\,2})(m^2 + \vec{q}_+^{\,2}) + \\ & + m^2(\vec{q}_-\vec{q}_+) + (\vec{q}_2\vec{q}_+)(\vec{q}_1\vec{q}_-) + (\vec{q}_1\vec{q}_+)(\vec{q}_2\vec{q}_-) - (\vec{q}_-\vec{q}_+)(\vec{q}_1\vec{q}_2)]. \end{aligned} \tag{9}$$

Here we use $d_1 = -\frac{1}{x}D_-$, $d_2 = -\frac{1}{\bar{x}}D_+$ with

$$\begin{aligned} D_- &= x[(\vec{q}_2 - \vec{q}_+)^2 + m^2] + \bar{x}[\vec{q}_-^{\,2} + m^2], \\ D_+ &= \bar{x}[(\vec{q}_2 - \vec{q}_-)^2 + m^2] + x[\vec{q}_+^{\,2} + m^2]. \end{aligned} \tag{10}$$

It is useful to note that the quantity Z can be written in a more elegant equivalent form

$$\begin{aligned} Z &= \frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{D_- D_+} - \frac{x\bar{x}}{(D_- D_+)^2} R^2, \\ R(\vec{q}_1, \vec{q}_2; \vec{q}_-, \vec{q}_+) &= (\vec{q}_1^{\,2} - 2\vec{q}_1\vec{q}_-)(\vec{q}_2^{\,2} - 2\vec{q}_2\vec{q}_-) - 2(\vec{q}_1\vec{q}_2)[m^2 + \vec{q}_-^{\,2}]. \end{aligned} \tag{11}$$

The quantity R possesses the Bose symmetry:

$$R = R(\vec{q}_1, \vec{q}_2; \vec{q}_-, \vec{q}_+) = R(\vec{q}_2, \vec{q}_1; \vec{q}_-, \vec{q}_+) = R(\vec{q}_1, \vec{q}_2; \vec{q}_+, \vec{q}_-). \tag{12}$$

Consider now the subprocess of charged pions pair creation $\gamma^* + \gamma^* \to \pi^+\pi^-$. Corresponding quantity ϕ^π has a form

$$\phi^\pi = \frac{1}{s}\left[\frac{s^2\alpha_-\beta_+ x}{D_-} + \frac{s^2\alpha_+\beta_-\bar{x}}{D_+} - s\right]. \tag{13}$$

After some algebra we find

$$\begin{aligned} \phi^\pi &= -\frac{x\bar{x}R}{D_+ D_-}, \\ \phi^\pi &= \frac{(x\bar{x})^2 R^2}{(D_+ D_-)^2}. \end{aligned} \tag{14}$$

3. ONE PAIR CREATION CROSS SECTION

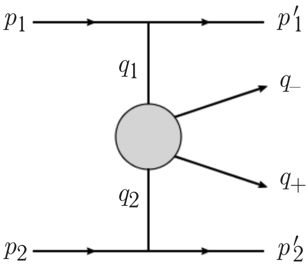

Fig. 1.

Differential cross section has a form (Fig. 1)

$$d\sigma = \frac{1}{8s}\sum |M|^2 d\Gamma_4. \tag{15}$$

Using the expressions for the phase volume of the final state and matrix element we obtain

$$d\sigma^a = \frac{\alpha^4(Z_1Z_2)^2}{\pi}\frac{d\beta_1}{\beta_1}\frac{d\vec{q}_1^{\,2}\,d\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}F^a, \tag{16}$$

with

$$F^a = \int\limits_0^1 \frac{dx}{x\bar{x}}\frac{d^2\vec{q}_-}{\pi}\Phi^a, \quad a = \mu\bar{\mu}, \pi,$$
$$\Phi^{\mu\bar{\mu}} = 2x\bar{x}[\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{D_-D_+} - \frac{x\bar{x}}{(D_-D_+)^2}R^2], \tag{17}$$
$$\Phi^{\pi} = \frac{(x\bar{x})^2R^2}{(D_+D_-)^2}.$$

For the case of very small values of $\vec{q}_{1,2}^{\,2}$ the replacement [1]

$$\vec{q}_1^{\,2} \to \vec{q}_1^{\,2} + M_1^2\beta_1^2, \quad \vec{q}_2^{\,2} \to \vec{q}_2^{\,2} + M_2^2\alpha_2^2 \tag{18}$$

must be done. Further we restrict ourselves to the case $\vec{q}_{1,2}^{\,2} \sim M_{1,2}^2$.

To perform the integration over the transverse component of pair momenta $d^2\vec{q}_-$ we apply the Feynman trick of joining the denominators

$$\int \frac{d^2\vec{q}_-}{\pi D_+D_-} = \int\limits_0^1 \frac{dy d^2\vec{q}_-}{\pi[D_-y + D_+\bar{y}]^2} = \int\limits_0^1 \frac{dy}{D},$$
$$\int \frac{d^2\vec{q}_-}{\pi(D_+D_-)^2} = 2\int\limits_0^1 \frac{y\bar{y}dy}{D^3}, \tag{19}$$
$$D = m^2 + \vec{q}_1^{\,2}x\bar{x} + \vec{q}_2^{\,2}y\bar{y}.$$

Here we use $D_-y + D_+\bar{y} = \vec{q}^{\,2} + D$ with $\vec{q} = \vec{q}_- - x\vec{q}_1 - y\vec{q}_2$. Performing the shift of variables and averaging on the azimuthal angles we obtain

$$F^{\mu\bar{\mu}} = \vec{q}_1^{\,2}\vec{q}_2^{\,2}\phi^{\mu},$$
$$F^{\pi} = \vec{q}_1^{\,2}\vec{q}_2^{\,2}\phi^{\pi}, \tag{20}$$

with

$$\phi^{\mu} = 2\int_0^1 dx \int_0^1 dy \left[\frac{1}{D} - \frac{XY}{D^3}K\right],$$
$$\phi^{\pi} = \int_0^1 dx \int_0^1 dy \frac{XY}{D^3}K;$$
$$K = 8D^2 + 2D[\vec{q}_2^{\,2}(1-4Y) + \vec{q}_1^{\,2}(1-4X)] + 2\vec{q}_1^{\,2}\vec{q}_2^{\,2}(1-4X)(1-4Y), \tag{21}$$
$$X = x(1-x), \quad Y = y(1-y).$$

For the cross section of one pair production we obtain

$$\vec{q}_1^{\,2}\vec{q}_2^{\,2}\frac{d\sigma_1^a}{d\vec{q}_1^{\,2}d\vec{q}_2^{\,2}} = \frac{(Z_1Z_2)^2\alpha^4}{\pi}L\phi^a(\vec{q}_1^{\,2}, \vec{q}_2^{\,2}), \tag{22}$$

where we insert the boost integral $L = \ln\frac{s}{m^2} = \int d\beta_1/\beta_1$. Here with the logarithmic accuracy we can put the value of m of order of mass of the particle of some pair.

4. MANY PAIRS CREATION CROSS SECTION

By analogy to the case of one pair production we have for phase volume of 2 pairs production (Fig. 2, a)

$$d\Gamma_2 = \frac{\pi^5}{s(2\pi)^{14}2^5}\frac{d\beta_1}{\beta_1}\frac{d\beta'}{\beta'}\frac{d^2\vec{q}_1}{\pi}\frac{d^2\vec{q}_2}{\pi}\frac{d^2\vec{q}\,'}{\pi}d\gamma_1 d\gamma_2,$$
$$d\gamma_i = \frac{d^2\vec{r}_i}{\pi}\frac{dx_i}{x_i(1-x_i)}, \quad \frac{m^2}{s} \ll \beta' \ll \beta_1 \ll 1. \tag{23}$$

Here x_i are the energy fractions of the negative charged particle from the created pairs and $\vec{r}_i$ are the corresponding transverse momenta.

Matrix element of two pair production process is

$$M_2 = \frac{8s(4\pi\alpha)^3}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}(\vec{q}\,')^2}\Phi^{(1)}\Phi^{(2)}. \tag{24}$$

The relevant cross section

$$\vec{q}_1^{\,2}\vec{q}_2^{\,2}\frac{d\sigma_2}{d\vec{q}_1^{\,2}d\vec{q}_2^{\,2}} = \frac{(Z_1Z_2)^2\alpha^6}{4\pi^3}\left(\frac{L^2}{2}\right)\phi^{(1)} \times \phi^{(2)}(\vec{q}_1^{\,2}, \vec{q}_2^{\,2}), \tag{25}$$

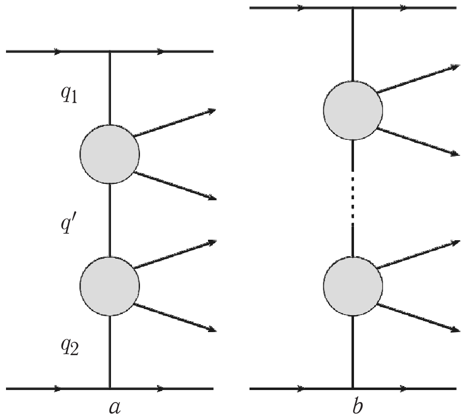

Fig. 2.

with

$$\phi^{(1)} \times \phi^{(2)}(\vec{q}_1^{\,2}, \vec{q}_2^{\,2}) = \int\limits_0^\infty d(\vec{q}\,')^2 \phi(\vec{q}_1^{\,2}, (\vec{q}\,')^2)\phi((\vec{q}\,')^2, \vec{q}_2^{\,2}). \tag{26}$$

For n pairs production (Fig. 2, *b*) we will obtain the enhancement factor $L^n/n!$. Let us write it in the form

$$\frac{L^n}{n!} = \int\limits_C \frac{dj}{2\pi i j^{n+1}} \left(\frac{s}{m^2}\right)^j, \tag{27}$$

where contour C is the line in the j plane $Re\, j = \sigma$, $0 < \sigma < 1$, $-\infty < Im\, j < \infty$. We obtain for the cross section of n pair production

$$\vec{q}_1^{\,2}\vec{q}_2^{\,2}\frac{d\sigma_n}{d\vec{q}_1^{\,2} d\vec{q}_2^{\,2}} = \frac{(Z_1 Z_2 \alpha^2)^2}{\pi} \int\limits_C \frac{dj}{2\pi i j^2} \left(\frac{s}{m^2}\right)^j \left(\frac{\alpha^2}{4\pi^2 j}\right)^n \times$$
$$\times\, \phi(\vec{q}_1^{\,2}, (\vec{q}\,'_1)^2) d(\vec{q}\,'_1)^2 \ldots d(\vec{q}\,'_{n-1})^2 \phi((\vec{q}\,'_{n-1})^2, \vec{q}_2^{\,2}). \tag{28}$$

5. SUM ON PAIRS NUMBER

Let us introduce the function $\Psi_j(1,2) = \Psi_j(\vec{q}_1^{\,2}, \vec{q}_2^{\,2})$ defined as

$$\Psi_j(1,2) = \phi(1,2) + \left(\frac{\alpha^2}{4\pi^2 j}\right)\phi(1,1') \times \phi(1',2) + \left(\frac{\alpha^2}{4\pi^2 j}\right)^2 \phi(1,1')\times$$
$$\times\, \phi(1',2') \times \phi(2',2) + \ldots, \tag{29}$$

where we imply $\phi(1,1') \times \phi(1',2) = \int\limits_0^\infty \phi(\vec{q}_1^{\,2}, (\vec{q}'_1)^2)\phi((\vec{q}'_1)^2, \vec{q}_2^{\,2}) d(\vec{q}\,')^2$. This function obeys the integral equation

$$\Psi_j(1,2) = \phi(1,2) + \frac{\alpha^2}{4\pi^2 j}\phi(1,1') \times \Psi_j(1',2). \tag{30}$$

It was shown ([2, 3]) that this function has a cut in j-plane which starts in some point j_0 on the real axis. For the case $j_0 > 0$ it results in violation of unitarity in QED. Really the "generalized" cross section $\Sigma(1,2)$ — sum of cross sections of production of any number of pairs:

$$\Sigma(1,2) = \frac{(Z_1 Z_2 \alpha^2)^2}{\pi} \int\limits_C \frac{dj}{2\pi i j^2} \left(\frac{s}{m^2}\right)^j \Psi_j(1,2), \tag{31}$$

will grow with center-of-mass energy $\sqrt{s}$ as

$$\Sigma(1,2)(s) \sim \Sigma_0(1,2) \left(\frac{s}{s_0}\right)^{j_0}. \tag{32}$$

The cross section obtained grows with energy $2E = \sqrt{s}$ more rapidly than any power of $L = \ln(s/m^2)$ and is the consequence of long-range behavior of the Coulomb field. We remind that in the theory of strong interaction the total cross section growth is restricted by Froissart boundary $\sigma(s) \sim \sigma_0 L^2$ which is the consequence of the short like character of the strong interaction forces [5]. The "remedy" from this "disease" of QED is in fact the absence in the nature of the kinematical situations with the arbitrary large impact parameters d and the related orbital quantum number l related by $lh = dE$.

To find the start point of a cut in the j-plane we may consider the homogeneous integral equation for the discontinuity of function Ψ_j i.e. its difference from the value on the opposite branch of the cut $\Delta\Psi_j$

$$\Delta\Psi_j(x_1,x_2) = \frac{1}{h}\phi(x_1,x) \times \Delta\Psi_j(x,x_2), \frac{1}{h} = \frac{\alpha^2}{4\pi^2 j}, \quad x_i = \vec{q}_i^{\,2}. \tag{33}$$

This equation can be solved for the case $x_i \gg m^2$. Really [3] one can look for the solution in the form

$$\Delta\Psi_j(x_1,x_2) = c(h)\frac{1}{\sqrt{x_1 x_2}}\left(\frac{x_1}{x_2}\right)^\lambda \tag{34}$$

with pure imaginary value of λ. Using this anzatz and the explicit form of ϕ^μ, ϕ^π we obtain after some algebra

$$h^i = \frac{\pi^2 \sin(\pi\lambda)}{32\cos^2(\pi\lambda)} \frac{P^i(\lambda)}{\lambda(1-\lambda^2)}, \quad i = \pi, \mu,$$

$$P^{\pi} = \frac{1}{2}(5 - 4\lambda^2), P^{\mu} = 11 - 12\lambda^2. \tag{35}$$

For completeness we give a rough estimate of the order of total cross sections of some processes in $\gamma\gamma$ collisions with bound state production by photon exchange mechanism:

$$\sigma^{\gamma\gamma\to PsPs} = \frac{\pi\alpha^8}{96} r_e^2 (1 + 2\cos^2\phi_0), \quad \sigma^{\gamma\gamma\to A_\pi A_\pi} = (\frac{r_e}{4r_\pi})^2 \sigma^{\gamma\gamma\to PsPs},$$
$$\sigma^{\gamma\gamma\to PsA_\pi} = \frac{\pi\alpha^8}{64} r_\pi^2 (3 - 2\cos^2\phi_0), \quad \sigma^{\gamma\gamma\to\pi^0 Ps} = \frac{\alpha^7}{32\pi^2 f_\pi^2}(1 + 2\cos^2\phi_0), \tag{36}$$
$$r_e = \frac{\alpha}{m_e}, \quad r_\pi = \frac{\alpha}{m_\pi}.$$

They are really very small quantities of order 10^{-8} nb.

6. DISCUSSION

The quantities $h = 2\alpha^2/\pi^2$, $\phi(\vec{q}_1^{\,2}, \vec{q}_2^{\,2})$ obtained in [2, 3]

$$\phi(\vec{q}_1^{\,2}, \vec{q}_2^{\,2}) = \int_0^1 dx \int_0^1 dy \frac{X + Y - 5XY}{X\vec{q}_1^{\,2} + Y\vec{q}_2^{\,2} + m^2}, \tag{37}$$

differ from our results $h = \alpha^2/(4\pi^2)$ and $\phi(\vec{q}_1^{\,2}, \vec{q}_2^{\,2})$ given in (21).

In paper [3] the short explanations of manipulations with one-loop expression of light-light scattering tensor was done to arrive at the form cited above. We underline, nevertheless, that the result (21) can be used in the problem of fitting the experimental data directly.

Similar phenomena take place as well in process of production of an arbitrary number of gluon jets in high energy hadron collisions. It was shown in frames of Quantum Chromodynamics (QCD) [6] that the violation of unitarity result is

$$\Sigma(s) \sim \left(\frac{s}{s_0}\right)^{j_{\mathrm{BFKL}}}, \quad j_{\mathrm{BFKL}} = N_c \frac{\alpha_s}{\pi} 4\ln 2 \approx 0.5. \tag{38}$$

For the case of production of a single pseudoscalar particle by two virtual photons the eigenvalue equation has a form

$$\Delta\Psi_j^{\pi}(x_1, x_2) = \frac{1}{h^{\pi}} F(x_1, x) \times \Delta\Psi_j^{\pi}(x, x_2), \tag{39}$$

with $x_i = \vec{q}_i^{\,2}/m_q^2$, m_q is the constituent quark mass in the triangle fermion loop, describing the decay $\pi_0 \to \gamma^*\gamma^*$ calculated in NJL frame and

$$F(x_1, x) = \frac{1}{x_1 - x}[\ln^2 x_1 - \ln^2 x]. \tag{40}$$

Unfortunately, the conform-invariance induced anzatz used above cannot be applied here. A similar problem arises in the case of production of a single scalar meson, such as σ, a_0, f_0 in frames of NJL model. We hope to return to this problem in future. We note in conclusion that eigenvalue problem can be solved in the separable approximation for function $F(x,y) = \phi(x)\psi(y)$. In any case the unitarity violation problem remains as well in this case.

7. THE PROCESSES $\tau \to \pi\pi\nu$ AND $\tau \to \omega\pi\nu$

Lagrangians

For the description of the processes with intermediate W^- and ρ, and ω mesons in the ground state we need the part of the standard NJL Lagrangian [7] which describes interactions of mesons and gauge bosons with quarks

$$\Delta\mathcal{L}_1 = \bar{q}\left[i\widehat{\partial} - m + \frac{g_{\mathrm{EW}}}{2}\tau_\pm \widehat{W}^\pm + g_\sigma I a_0^\pm + i g_\pi \gamma_5 \tau_\pm \pi^\pm + \frac{g_\rho}{2}\tau_\pm \widehat{\rho}^\pm\right] q, \tag{41}$$

where $\bar{q} = (\bar{u}, \bar{d})$ with u and d quark fields; $m = \mathrm{diag}\,(m_u, m_d)$, $m_u = 280$ MeV is the constituent quark mass, $m_d - m_u \approx 3.7$ MeV [1]; g_{EW} is the electroweak constant; W, $a_0^\pm$, $\pi^\pm$, and $\rho^\pm$ are the electroweak, scalamp, pion, and ρ meson fields, respectively; g_π is the pion coupling constant, $g_\pi = m_u/f_\pi$, where $f_\pi = 93$ MeV is the pion decay constant; g_ρ is the vector meson coupling constant, $g_\rho \approx 6.14$ corresponds to the standard relation $g_\rho^2/(4\pi) \approx 3$; $g_\sigma = g_\rho/\sqrt{6}$ is the scalar meson coupling constant; $\tau_\pm = (\tau_1 \mp i\tau_2)/\sqrt{2}$, $I = \mathrm{diag}\,(1,1)$ and $\tau_{1,2,3}$ are Pauli matrices. The integrals through the quark loops take the form

$$I_1(m) = -i\frac{N_c}{(2\pi)^4}\int^{\Lambda_4} \frac{d^4k}{(m^2 - k^2)} = \frac{N_c}{(4\pi)^2}\left[\Lambda_4^2 - m^2 \log\left(\frac{\Lambda_4^2}{m^2} + 1\right)\right], \tag{42}$$

$$I_2(m) = -i\frac{N_c}{(2\pi)^4}\int^{\Lambda_4} \frac{d^4k}{(m^2 - k^2)^2} = \frac{N_c}{(4\pi)^2}\left[\log\left(\frac{\Lambda_4^2}{m^2} + 1\right) - \left(1 + \frac{m^2}{\Lambda_4^2}\right)^{-1}\right], \tag{43}$$

where $N_c = 3$ is a number of quark colors and $\Lambda_4 \approx 1250$ MeV is a 4-dimensional cut-off parameter in the standard NJL model [7].

[1] We take into account the quark mass difference only in calculation of $\tau \to \eta(\eta')\pi\nu$ decays.

For description of the radial excited mesons interactions we use the extended version of the NJL Lagrangian [8, 9]

$$\Delta\mathcal{L}_2^{\rm int} = \bar{q}(k')\Big\{A_\pi\tau_\pm\gamma_5\pi(p) - A_{\pi'}\gamma_5\tau_\pm\pi'(p)+ \\ + A_\rho\tau_\pm\widehat{\rho}^{\pm}(p) - A_{\rho'}\tau_\pm\widehat{\rho}'^{\pm}(p)\Big\}q(k), \quad p = k - k', \quad (44)$$

$$A_\pi = g_{\pi_1}\frac{\sin(\alpha+\alpha_0)}{\sin(2\alpha_0)} + g_{\pi_2} f\left(k^{\perp 2}\right)\frac{\sin(\alpha-\alpha_0)}{\sin(2\alpha_0)},$$
$$A_{\pi'} = g_{\pi_1}\frac{\cos(\alpha+\alpha_0)}{\sin(2\alpha_0)} + g_{\pi_2} f\left(k^{\perp 2}\right)\frac{\cos(\alpha-\alpha_0)}{\sin(2\alpha_0)},$$
$$A_\rho = g_{\rho_1}\frac{\sin(\beta+\beta_0)}{\sin(2\beta_0)} + g_{\rho_2} f\left(k^{\perp 2}\right)\frac{\sin(\beta-\beta_0)}{\sin(2\beta_0)},$$
$$A_{\rho'} = g_{\rho_1}\frac{\cos(\beta+\beta_0)}{\sin(2\beta_0)} + g_{\rho_2} f\left(k^{\perp 2}\right)\frac{\cos(\beta-\beta_0)}{\sin(2\beta_0)}.$$

The radially excited states were introduced in the NJL model with the help of the form factor in the quark-meson interaction:

$$f\left(k^{\perp 2}\right) = \left(1 - d|k^{\perp 2}|\right)\Theta\left(\Lambda^2 - |k^{\perp 2}|\right), \\ k^\perp = k - \frac{(kp)p}{p^2}, \quad d = 1.788\ \text{GeV}^{-2}, \quad (45)$$

where k and p are the quark and meson momenta, respectively. The cut-off parameter $\Lambda_3 = 1.03$ GeV. Thus, the divergent integrals have the form

$$I_m^{f\dots f} = -iN_c\int^{\Lambda_3}\frac{d^4k}{(2\pi)^4}\frac{\left(f(k^{\perp 2})\right)^n}{(m^2-k^2)^m}, \quad (46)$$

where n is a number of vertexes with form-factor, $d = 1.788$ is the slope parameter.

The coupling constants $g_{\rho_1} = g_\rho$ and $g_{\pi_1} = g_\pi$ are the same as in the standard NJL version. The constants $g_{\rho_2} = 10.56$ and $g_{\pi_2} = g_{\rho_2}/\sqrt{6}$, and the mixing angles $\alpha_0 = 58.39°$, $\alpha = 58.70°$, $\beta_0 = 61.44°$, and $\beta = 79.85°$ were defined in [10].

8. THE PROCESSES $\tau \to \pi\pi\nu$ AND $\tau \to \omega\pi\nu$

The amplitude of the $\tau \to \pi^-\pi^0\nu_\tau$ decay is described in the NJL model by the Feynman diagrams given in Fig. 3

$$T = G_F|V_{ud}|f_{a_1}(p^2)m_\rho^2\left(\frac{1 - i\sqrt{q^2}\,\Gamma_\rho(p^2)/m_\rho^2}{m_\rho^2 - p^2 - i\sqrt{p^2}\,\Gamma_\rho(p^2)} + \right. \\ \left. + \frac{e^{i\pi}C_{W\rho'}C_{\rho'\pi\pi}(1/g_\rho)p^2/m_\rho^2}{m_{\rho'}^2 - p^2 - i\sqrt{p^2}\,\Gamma_{\rho'}(p^2)}\right)(p^\mu_{\pi^-} - p^\mu_{\pi^0})l_\mu\pi^-\pi^0, \quad (47)$$

where

$$f_{a_1}(p^2) = Z + (1 - Z) + \left(\frac{p^2 - m_\pi^2}{(g_\rho F_\pi)^2}\right)\left(1 - \frac{1}{Z}\right) = \\ = 1 + \left(\frac{p^2 - m_\pi^2}{(g_\rho F_\pi)^2}\right)\left(1 - \frac{1}{Z}\right), \quad (48)$$

where $Z = (1 - 6m_u^2/m_{a_1}^2)^{-1}$ is the additional renormalizing factor pion fields that appeared after the inclusion of a_1–π transitions. This function describes the creation of pions at the ends of the triangle quark diagram with taking into account the possibility of creation of these pions through the intermediate axial-vector $a_1(1260)$ meson. The first term of this amplitude corresponds to the triangle diagram without a_1–π transitions, the second term corresponds to diagram with a_1–π transition on the one of the pion lines and the third term corresponds to the diagram with transitions on both pion lines.

The NJL model allows us to describe the processes of the τ lepton decays without introduction of arbitrary parameters. Firstly, such calculations were fulfilled in the work [11, 12] where decays $\tau \to 3\pi\nu$ and $\tau \to \pi\gamma\nu$ were described. Here we consider in the framework of the extended NJL decays $\tau \to \pi\pi(\pi')\nu$, $\tau \to \pi\omega\nu$, $\tau \to \pi\eta(\eta')\nu$ taking into account intermediate vector mesons $\rho(770)$, $\rho(1450)$ and scalar mesons $a_0(980)$, $a_0(1450)$. For description of first radial-excited state the polynomial form-factor of second order over transversal momentum is used.

The $W^- \to \rho^-$ transition was defined in [13]

$$C_{W\rho'} \frac{G_F |V_{ud}|}{g_\rho} (g^{\mu\nu} p^2 - p^\mu p^\nu), \quad (49)$$

$$C_{W\rho'} = -\left(\frac{\cos(\beta + \beta_0)}{\sin(2\beta_0)} + \Gamma \frac{\cos(\beta - \beta_0)}{\sin(2\beta_0)}\right), \quad (50)$$

where $\Gamma = 0.54$. The vertex $\rho'\pi\pi$ is described by the term

$$C_{\rho'\pi\pi} = -\left(\frac{\cos(\beta + \beta_0)}{\sin(2\beta_0)} g_{\rho_1} + \frac{\cos(\beta - \beta_0)}{\sin(2\beta_0)} \frac{I_2^f}{I_2} g_{\rho_2}\right) = 1.68. \quad (51)$$

After using the expression for the decay width we get $\mathcal{B}(\tau^- \to \to \pi^-\pi^0\nu_\tau) = 24.76\%$, when the world average found in PDG [14] is $(25.51 \pm 0.09)\%$. With the help of the method used here we can obtain also a qualitative prediction for branching of the process $\tau \to \pi\pi'(1300)\nu$. This value approximately equals 0.2%, which does not contradict modern experimental data regarding the decays $\tau \to 4\pi\nu$. This prediction can be useful result for future experimental measurement.

9. THE DECAY $\tau \to \pi\pi\nu$

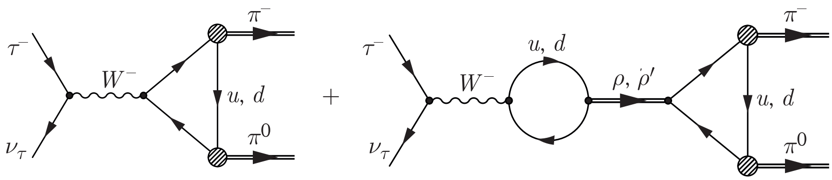

Fig. 3. The diagrams describing the decay $\tau \to \pi\pi\nu$

Table 1

	CLEO	ALEPH	BELLE	NJL
$\mathcal{B}$, %	25.32 ± 0.15	$25.471 \pm 0.097 \pm 0.085$	$25.24 \pm 0.01 \pm 0.39$	24.76

10. THE DECAY $\tau \to \omega\pi\nu$

The expression for the amplitude describing the decay $\tau \to \pi\omega\nu$ (see Fig. 4):

$$T = G_F|V_{ud}|\bar{\nu}(1-\gamma^5)\gamma^\mu\tau(T_{W\rho} + T_{\rho'})\epsilon_{\mu\nu\rho\sigma}p^\rho_\omega p^\sigma_\pi \omega^\nu \pi. \quad (52)$$

The $T_{W\rho}$ term corresponds to the contribution given by the contact diagram and the diagram with an intermediate $\rho(770)$ meson. Using the factor for $W-\rho$ transition, we can get the expression that coincides with one given by the vector meson dominance model:

$$T_{W\rho} = \frac{C_\rho}{g_{\rho_1}} \frac{1 - i\Gamma_\rho/m_\rho}{m^2_\rho - p^2 - im_\rho\Gamma_\rho} m^2_\rho. \quad (53)$$

The contribution of the amplitude with an intermediate $\rho(1450)$ meson reads

$$T_{\rho'} = C_{\rho'}C_{W\rho'} \frac{p^2}{m^2_{\rho'} - p^2 - i\sqrt{p^2}\,\Gamma_{\rho'}(p^2)}. \quad (54)$$

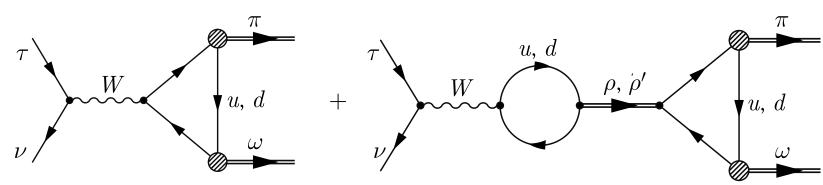

Fig. 4. The diagrams describing the decay $\tau \to \omega\pi\nu$

The vertex constants C_ρ and $C_{\rho'}$ are defined from the extended NJL model Lagrangian

$$\frac{C_\rho}{g_{\pi_1}} = \left(g_{\rho_1}\frac{\sin(\beta+\beta_0)}{\sin(2\beta_0)}\right)^2 I_3 + \left(g_{\rho_2}\frac{\sin(\beta-\beta_0)}{\sin(2\beta_0)}\right)^2 I_3^{ff} + \\ + 2g_{\rho_1}g_{\rho_2}\frac{\sin(\beta+\beta_0)}{\sin(2\beta_0)}\frac{\sin(\beta-\beta_0)}{\sin(\beta_0)}I_3^f, \quad (55)$$

$$-\frac{C_{\rho'}}{g_{\pi_1}} = g_{\rho_1}\frac{\sin(\beta+\beta_0)}{\sin(2\beta_0)}g_{\rho_1}\frac{\cos(\beta+\beta_0)}{\sin(2\beta_0)}I_3 + \\ + g_{\rho_2}\frac{\sin(\beta-\beta_0)}{\sin(2\beta_0)}g_{\rho_2}\frac{\cos(\beta-\beta_0)}{\sin(2\beta_0)}I_3^{ff} + \\ + g_{\rho_1}\frac{\sin(\beta+\beta_0)}{\sin(2\beta_0)}g_{\rho_2}\frac{\cos(\beta-\beta_0)}{\sin(2\beta_0)}I_3^f + \\ + g_{\rho_2}\frac{\cos(\beta+\beta_0)}{\sin(2\beta_0)}g_{\rho_1}\frac{\sin(\beta-\beta_0)}{\sin(2\beta_0)}I_3^f, \quad (56)$$

Using these formulas we get values for the branching $\mathcal{B}_{\rm NJL}(\tau \to \pi\omega\nu) = 1.85\%$. The CLEO [15] measurement equals $(1.95 \pm 0.08)\%$ and the ALEPH [16] is $(1.91 \pm 0.13)\%$.

11. THE DECAY $\tau^- \to \eta(\eta')\pi^-\nu$

At present, the decays $\tau^- \to \eta(\eta')\pi^-\nu$ are not well studied in experiments [14]. The experiment gives us only upper limit. The processes $\tau^- \to \eta(\eta')\pi^-\nu$ are the second class decays. These decays are going due to quark mass differences. For calculation of these decays we should first calculate two non-diagonal transitions $\pi^0 \to \eta$ and $W^- \to a_0^-$ within the

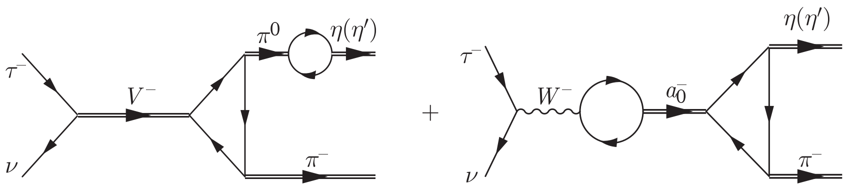

Fig. 5. The diagrams describing the decay $\tau^- \to \eta(\eta')\pi^-\nu$

Table 2

	CLEO	ALEPH	NJL
$\mathcal{B}$, %	1.95±0.08	1.91±0.13	1.85

Table 3

	$\mathcal{B}_V^{\pi\eta}\cdot 10^{-6}$	$\mathcal{B}_S^{\pi\eta}\cdot 10^{-6}$	$\mathcal{B}_{\rm tot}^{\pi\eta}\cdot 10^{-6}$
NJL	4.35	0.38	4.72
PR	1.58–5.70	10.7–65.9	$\simeq 26$
NS	3.6	~ 10	3–10
EXP	—	—	< 99

Table 4

	$\mathcal{B}_V^{\pi\eta'}\cdot 10^{-8}$	$\mathcal{B}_S^{\pi\eta'}\cdot 10^{-8}$	$\mathcal{B}_{\rm tot}^{\pi\eta'}\cdot 10^{-8}$
NJL	1.11	1.98	3.09
PR	0.14–3.4	6–18	—
NS	$< 2+8$	$< 10 + (20–120)$	< 140
EXP	—	—	< 72

NJL model. These transitions go through quark loops containing u and d quarks (see Fig. 5).

We use the amplitude for $\tau\to\pi\pi\nu$ with the $\pi^0\to\eta(\eta')$

$$T_V = \epsilon_{\pi\eta(\eta')} m_\rho^2 \left(\left(1 - \frac{i\sqrt{q^2}\,\Gamma_\rho(p^2)}{m_\rho^2}\right) BW_\rho(p^2) + \beta_\rho \frac{p^2}{m_\rho^2} BW_{\rho'}(p^2)\right)\times \\ \times (p^\mu_{\pi^-} - p^\mu_{\eta(\eta')}) l_\mu \pi^- \eta(\eta'), \quad (57)$$

For the processes with the intermediate vector meson we get contributions to branching fractions

$$\mathcal{B}_V(\tau\to\eta\pi\nu) = 4.35\cdot 10^{-6}, \quad (58)$$

$$\mathcal{B}_V(\tau\to\eta'\pi\nu) = 1.11\cdot 10^{-8}. \quad (59)$$

The $W^-\to a_0^-$ transition takes the form

$$\frac{\sqrt{3}}{4g_\rho} g_{\rm EW}|V_{ud}|(m_d - m_u)p^\mu W^-_\mu a_0^-\,, \quad (60)$$

The amplitude with the intermediate scalar meson (see Fig. 5) takes the form

$$T_S = 2Zm_u(m_d - m_u)\epsilon_{\eta(\eta')}(BW_{a_0}(p^2) + \beta_{a_0\eta(\eta')\pi} BW_{a_0'}(p^2))p^\mu l_\mu \pi^-\eta(\eta'), \quad (61)$$

where $BW_{a_0(a_0')}(p^2)$ is the Breit–Wigner formula for the $a_0(a_0')$ meson with $m_{a_0} = 980$ MeV, $m_{a_0'} = 1474$ MeV, $\Gamma_{a_0'}(m_{a_0'}) = 265$ MeV taken from PDG [14] and $\Gamma_{a_0}(m_{a_0}) = 100$ MeV calculated in the NJL model which coincides with the upper PDG limit [14]. For the estimation of the contribution of the radial-excited $a_0^-(1450)$ to the τ decays we should use the extended NJL model [8–10]. The amplitudes $A_{a_0'\to\eta(\eta')\pi}$ can be found in [17, 18]. The transition $W^-\to a^-(1450)$ takes the form

$$C_{Wa_0'} = \frac{\sqrt{3}}{4g_\rho} g_{\mathrm{EW}}|V_{ud}|(m_d - m_u)\times \\ \times\left(\frac{\cos(\phi+\phi_0)}{\sin(2\phi_0)} + \Gamma\frac{\cos(\phi-\phi_0)}{\sin(2\phi_0)}\right) p^\mu W_\mu^- a_0^-, \quad (62)$$

where $\phi_0 = 65.5^\circ$ and $\phi = 72.0^\circ$ are the mixing angles.

Thus, we get the $\beta_{a_0\eta(\eta')\pi}$ parameter:

$$\beta_{a_0\eta(\eta')\pi} = e^{i\pi} C_{Wa_0'} \frac{\sqrt{6}}{4Z}\frac{A_{a_0'\to\eta(\eta')\pi}}{m_u}. \quad (63)$$

The values $\beta_{a_0\eta\pi} = -0.24$ and $\beta_{a_0\eta'\pi} = -0.26$ do not contradict the ones given in [19, 20]. The contributions to the branching fractions from the amplitude (61) are

$$\mathcal{B}_S(\tau\to\eta\pi\nu) = 0.37\cdot 10^{-6}, \quad (64)$$

$$\mathcal{B}_S(\tau\to\eta'\pi\nu) = 2.63\cdot 10^{-8}. \quad (65)$$

The expression for the total width is

$$\Gamma = \frac{G_f^2|V_{ud}|^2}{384\pi m_\tau^2} \int\limits_{m^2_{\eta(\eta')}+m_\pi^2}^{m_\tau^2} \frac{\widehat{s}}{s^3}\lambda^{1/2}(s, m^2_{\eta(\eta')}, m_\pi^2)(m_\tau^2 - s)^2\times \\ \times\left(|T_V|^2(2s+m_\tau^2)\lambda(s, m^2_{\eta(\eta')}, m_\pi^2) + |T_S|^2 3m_\tau^2(m^2_{\eta(\eta')} - m_\pi^2)^2\right). \quad (66)$$

Note that there is no interference between the vector and scalar intermediate state contributions. Thus, for branchings we get

$$\mathcal{B}(\tau^-\to\eta\pi^-\nu) = 4.72\cdot 10^{-6}, \quad (67)$$

$$\mathcal{B}(\tau^-\to\eta'\pi^-\nu) = 3.74\cdot 10^{-8}. \quad (68)$$

Let us note that our estimations for scalar contributions are much less than ones in previous works. One can see comparison in Tables 1, 2 and 3, 4.

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