On the Eikonal approximation in AdS space

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ABSTRACT: We explore the eikonal approximation to graviton exchange in AdS_5 space, as relevant to scattering in gauge theories. We restrict ourselves to the regime where conformal invariance of the dual gauge theory holds, and to large 't Hooft coupling where the computation involves pure gravity. We give a heuristic argument, a direct loop computation, and a shock wave derivation. The scalar propagator in AdS_3 plays a key role, indicating that even at strong coupling, two-dimensional conformal invariance controls high-energy four-dimensional gauge-theory scattering.

Contents

1 Introduction

There has been much interest in high-energy scattering, and the eikonal approximation in particular, in the contexts of gauge theories, string theories, and the duality which relates them. Relevant recent papers include [1–6]. In this note, we obtain the eikonal approximation to scattering in AdS_5. This is relevant for high-energy limits of Green functions in four-dimensional conformal field theory, and plays a partial role in dual descriptions of high-energy hadron scattering and small-x deep-inelastic scattering in nonconformal quantum field theories. Applications will be addressed elsewhere; here we present some basic results.[1]

The string-dual description of scattering at very high energies and low momentum transfer in large-N, large-λ gauge theories (here $\lambda \propto g^2 N$ is the 't Hooft coupling) was considered in [2], building on work of [8–10]. The dual string theory has strings propagating on a space which is asymptotically $AdS_5 \times X$ (X a compact five-dimensional space of little consequence here) with metric approaching

$$ds^2 \approx \frac{R^2}{z^2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right) + ds_X^2 \tag{1.1}$$

as $z \to 0$. The metric is strongly corrected for z near $z_{max} \sim 1/\Lambda$, where Λ is the confinement scale.

The form of a two-to-two hadron scattering amplitude in the gauge theory, at leading order in $1/N$ and s large compared to the confinement scale and to t, was shown in [2],

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[1]Work with overlapping results appeared [7] as this paper was in preparation.

following [9], to be proportional to a Pomeron propagator or "kernel" $\mathcal{K}_0$.

$$\text{Im}\left[\mathcal{M}_{2\to2}(s,t)\right] \propto \int dz \sqrt{g(z)}\, dz' \sqrt{g(z')}\, \Phi_3(z)\Phi_4(z)\left[\tilde{s}^2 \left(\frac{zz'}{R^2}\right)^2 \mathcal{K}_0(s,t,z,z')\right]\Phi_1(z')\Phi_2(z').$$
(1.2)

Here the Φ_i are wave functions for states dual to gauge-theory hadrons, and the variable $\tilde{s}$ represents the square of the proper[2] center-of-mass energy.

$$\tilde{s} \equiv \frac{s}{\sqrt{g_{+-}(z)g_{-+}(z')}} = \frac{zz's}{R^2}\ .$$
(1.3)

(Note $\tilde{s}$ is slightly ambiguous, though generally the ambiguity is subleading in the Regge limit; we have taken $\tilde{s}$ to have a symmetric dependence on z and z', as was done in [2].) An analogous formula for deep-inelastic scattering at small-x was obtained earlier in [9]. One may obtain similar formulas from four-point functions, as in [11], by taking high-energy limits of their (suitably regulated) Fourier transforms. In each of these cases, the amplitude resembles (1.2); there are four external functions $\Phi_i(z)$ which are normalizable or nonnormalizable modes of fields in the bulk, combined with a kernel $\mathcal{K}_0$.

In a conformal field theory, or in a confining theory with $t \ll -\Lambda^2$, the kernel takes the form

$$\mathcal{K}_0\left(s,t,z,z'\right) = \frac{2}{\pi^2} s^{J_0-2} \int_0^\infty d\nu\, \nu\, \sinh \pi\nu\, K_{i\nu}\left(z|t|^{1/2}\right) K_{-i\nu}\left(z'|t|^{1/2}\right) e^{-\nu^2\tau}\ .$$
(1.4)

where $\tau = \frac{1}{2\sqrt{\lambda}} \ln(s/s_0)$. This is similar in form to the BFKL kernel [12–14], matching its form precisely (but not its coefficients, which are λ-dependent) for $t = 0$, and having similar form for large t. Like the BFKL kernel, its analytic structure involves a cut[3] in the J-plane extending along the negative real axis starting from $J = J_0$,

$$J_0 = 2 - \frac{2}{\sqrt{\lambda}} + O\left(\lambda^{-1}\right).$$
(1.5)

As $\lambda \to \infty$, s fixed, the scattering amplitude is described by a pure graviton exchange, with $J_0 \to 2$. Define the amputated four-point amplitude $\mathcal{A}^{(1)}$ for a single t-channel Pomeron exchange by

$$\mathcal{A}^{(1)}\left(s,x^\perp,z;y^\perp,w;x'^\perp,z';y^\perp,w\right) = \mathcal{M}^{(1)}\left(s,x^\perp,z;x'^\perp,z'\right)\left[g(z)g(z')\right]^{-1/2}$$
(1.6)
$$\times\delta^2\left(x^\perp - y^\perp\right)\delta\left(z - w\right)\delta^2\left(x'^\perp - y'^\perp\right)\delta\left(z' - w'\right)$$

In the limit λ large, $s^2\mathcal{K}_0$, up to some metric factors and delta functions, is just the imaginary part of $\mathcal{M}^{(1)}$,

$$\frac{1}{2i}\text{Disc}\left[\mathcal{M}^{(1)}\left(s,x^\perp,z;x'^\perp,z'\right)\right] \sim \tilde{s}^2 \int d^2x^\perp e^{iq_\perp \cdot x^\perp} \left(\frac{zz'}{R^2}\right)^2 \mathcal{K}_0\left(s,q_\perp^2,z,z'\right)$$
(1.7)

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[2]i.e., measured by a local observer in the bulk,

[3]In both cases, the cut becomes a discrete and dense set of poles when the coupling runs slowly.

The full amplitude can be constructed from the discontinuity and crossing-symmetry as usual through methods of analyticity. A short calculation using eqs. (1.2) and (1.4) reveals, as $\lambda \to \infty$,

$$\mathcal{M}^{(1)}\left(s, x^\perp, z; x'^\perp, z'\right) = \frac{\kappa_5^2}{R} \tilde{s}^2 \left(\frac{zz'}{R^2}\right) G_3\left(x^\perp, z; x'^\perp, z'\right) \tag{1.8}$$

where κ_5 is the gravitational coupling in AdS_5, G_3 is the dimensionless scalar propagator for a particle of mass $\sqrt{3}/R$ in an Euclidean AdS_3 space of curvature radius R, i.e., a propagator over a three dimensional hyperboloid,

$$G_3\left(x^\perp, z; x'^\perp, z'\right) = G_3(u) = \frac{1}{4\pi} \frac{1}{\left[1 + u + \sqrt{u(2+u)}\right]^2 \sqrt{u(2+u)}} \tag{1.9}$$

and

$$u = \frac{\delta_{ij}\left(x^\perp - x'^\perp\right)^i \left(x^\perp - x'^\perp\right)^j + (z - z')^2}{2zz'} \tag{1.10}$$

(with $i, j = 1, 2$) is the chordal distance on the AdS_3 transverse to the momentum direction of the scattering particles. We will explain in the next section why this result should be expected.

Since the amplitude grows faster than s, it violates unitarity at large s, and a resummation of higher-loop amplitudes is required to obtain sensible physics. In certain restricted regions of z and z', this resummation can be done via the eikonal approximation. Note however that a complete field-theory computation, which requires integrating over z and z', typically is not possible within the bulk eikonal approximation. With this caveat, we proceed to consider the eikonal approximation to the amplitude in a very limited regime. We keep only leading-λ effects (pure gravity), $x^\perp \ll 1/\Lambda$ (no effects from confinement), $z \ll z_{max}$ (where the hadrons are small compared to the confinement scale, and the metric is pure AdS_5), but with z and z' large enough that the proper energy $\sqrt{s}$ is large compared to the Planck mass. In addition, the proper distance between the points $(x^\perp, z)$ and $(x'^\perp, z')$ must not be too small, so that the scattering involves only linear gravity. We also ignore the space X, assuming there is no angular-momentum transfer in the compact directions.

In the regime where the eikonal approximation is appropriate, it is easy to adapt flat-space methods to write the eikonal result. This is because AdS spaces have Minkowski slices with ordinary boost invariance, and because the derivation of the eikonal approximation involves separating the light-cone directions from the transverse directions, which need not be translationally invariant. (This will be most clear in our perturbative derivation.) The result for the amputated 5-dimensional amplitude in the eikonal approximation is

$$\mathcal{A}_{eik}\left(s, x^\perp, z; y_\perp, w; x'^\perp, z'; y_\perp, w\right) = \mathcal{M}_{eik}\left(s, x^\perp, z; x'^\perp, z'\right) \left[g(z)g(z')\right]^{-1/2} \tag{1.11}$$

$$\times \delta^2\left(x^\perp - y^\perp\right) \delta(z - w) \delta^2\left(x'^\perp - y'^\perp\right) \delta(z' - w')$$

showing the classic eikonal reduction of a function of four positions to a function of two positions in the transverse dimensions, and with

$$\mathcal{M}_{eik}\left(s, x^\perp, z; x'^\perp, z'\right) = -2i \left(\frac{zz'}{R^2}\right)^2 s \left[\exp\left\{i\chi\left(s, x^\perp, z; x'^\perp, z'\right)\right\} - 1\right] \tag{1.12}$$

where[4]

$$\chi(s, x^\perp, z; x'^\perp, z') = \frac{1}{2} \frac{\kappa_5^2}{R^3} \, zz's \, G_3(u) \tag{1.13}$$

We emphasize again that this form for the amplitude is valid only in the restricted regime described above.

Our result can now be combined with external wave functions, normalizable or non-normalizable, and integrated over z and z', to allow partial computation of unamputated high-energy scattering amplitudes, operator matrix elements or Green functions in a dual gauge theory. However, since the result above holds only in limited regions of the coordinates z, z', no complete physical amplitude can be obtained from this result alone, at least not without additional arguments showing that all other regions give small contributions to the full unamputated amplitude. In the remaining sections we give multiple lines of argument that support the result (1.11)–(1.13).

2 A consistency argument

First, we check the form of (1.11)–(1.13) by matching it to previous work. The amputated form for the scattering amplitude in *transverse representation* (longitudinal momentum space and transverse position space) must agree at transverse proper distances short compared to R with known results on the eikonal approximation obtained by other methods, including direct computation of multi-loop amplitudes at high energy [15–18] and the shock-wave approach [19, 20].

In transverse representation, flat-space results for eikonal scattering take simple forms. We know that if the *proper* transverse distance $\tilde{b}$ between two particles is sufficiently small compared to R, as they scatter at high *proper* energy $\sqrt{\tilde{s}}$, then standard results must apply.[5] In $D = 5$ (bulk) spacetime dimensions, the leading order amputated amplitude in transverse representation (with transverse delta functions removed) will be

$$\mathcal{M}^{(1)}\left(\tilde{s}, \tilde{b}\right) = \frac{1}{\sqrt{g_{+-}g_{-+}}} \kappa_5^2 \frac{\tilde{s}^2}{4\pi\tilde{b}} \tag{2.1}$$

The functional dependence on $\tilde{s}$ and $\tilde{b}$ is as in flat space; the metric factor in front is due[6] to our use of a transverse position basis. The flat-space eikonal approximation then implies

$$\mathcal{M}_{eik}\left(\tilde{s}, \tilde{b}\right) \propto -2i\frac{\tilde{s}}{\sqrt{g_{+-}g_{-+}}} \left[\exp\left\{\frac{i}{2}\kappa_5^2 \frac{\tilde{s}}{4\pi\tilde{b}}\right\} - 1\right] . \tag{2.2}$$

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[4]Since $\kappa_5^2/R^3 = 1/(M_P R)^3$, where M_P is the five-dimensional Planck constant, we see that χ depends only on the number of colors N in the gauge theory, $\chi \sim N^{-2}$, where $N \propto (M_P R)^{3/2}$.

[5]Note that this argument requires that $\tilde{s}$, though large compared to the momentum transfer, must not be so large that the eikonal approximation is nowhere valid for $\tilde{b} < R$. For fixed $\tilde{s}$ we can always consider taking R sufficiently large that the argument applies; as we will see this is enough to fix the answer. Similarly $\tilde{b}$ must not be so small as to probe the nonlinear gravity near the scattering objects.

[6]More precisely, though natural, it is actually conventional, since one could absorb it into the amputation prescription.

Here we have used the fact that every order of the eikonal approximation must transform in the same way under diffeomorphisms, so the phase shift in the exponent must be an invariant.

On the other hand, we also know that for $\lambda \to \infty$, the leading-order high-energy scattering amplitude in AdS_5 is due simply to t-channel graviton exchange, so the gravitational propagator $\mathcal{G}_{MN,M'N'}$ must appear in the tree amplitude. At high energy, only the term in $\mathcal{G}$ proportional to $g_{++}g_{--}$ survives, and this term is just the massless scalar propagator G_5 in AdS_5, a function by AdS isometries (the conformal invariance of the dual gauge theory) of the chordal distance

$$u_5 = \frac{\eta_{\mu\nu}\,(x - x')^\mu\,(x - x')^\nu + (z - z')^2}{2zz'} \tag{2.3}$$

Explicitly [11, 21]

$$\mathcal{G}_{++,--}\left(x, z; x', z'\right) = 2\left(\frac{R^2}{zz'}\right)^2 G_5(u_5). \tag{2.4}$$

with

$$G_5(u_5) = \frac{1}{8\pi^2}\left[2 + \frac{1 + u_5}{[u_5(2 + u_5)]^{3/2}} - \frac{2(1 + u_5)}{[u_5(2 + u_5)]^{1/2}}\right] \tag{2.5}$$

For near-forward scattering at high energy, the total momentum transfer is limited. In a collinear frame where the light-cone momentum components become large, the total momentum transfer is nearly transverse and the longitudinal components of momentum transfer must go to zero asymptotically. In such a frame, the AdS_5 massless scalar propagator G_5 should be evaluated at zero longitudinal momentum transfer, i.e.

$$G_5(q_\pm = 0, x^\perp, z; x'^\perp, z') = \int dx^+ dx^- G_5(u_5) = zz' G_3(u) \tag{2.6}$$

since $u_5 = u - x^+ x^-/zz'$. Here we used the definitions in (1.9) and (1.10).[7]

Now we may simply note that a scalar propagator in 5 dimensions approaches $1/\tilde{b}$ as $\tilde{b} \to 0$, since the propagator satisfies a flat transverse Laplacian at short distances. Observing that $\tilde{b} \to R\sqrt{2u}$ as $u \to 0$ and matching to (2.1) and (2.2) fixes the results

$$\mathcal{M}^{(1)}\left(\tilde{s}, x^\perp, z; x'^\perp, z'\right) = \frac{1}{R}\left(\frac{zz's}{R^2}\right)^2\left(\frac{zz'}{R^2}\right) G_3(x^\perp, z; x'^\perp, z') \tag{2.7}$$

and

$$\begin{aligned}
\mathcal{M}_{eik} &= -2i\frac{1}{\sqrt{g_{+-}g_{-+}}}\tilde{s}\left[\exp\left\{\frac{i}{2}\kappa_5^2 R^{-1}\tilde{s}\, G_3(u)\right\} - 1\right] \\
&= -2is\left(\frac{zz'}{R^2}\right)^2\left[\exp\left\{\frac{i}{2}\frac{\kappa_5^2}{R^3}zz's\, G_3(u)\right\} - 1\right],
\end{aligned} \tag{2.8}$$

in agreement with our earlier claim.

The above arguments can all be easily generalized to other dimensions. The only work is to obtain the correct normalization.

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[7]Note this is a special case of a general relation, which states that the propagator of a bulk scalar in D dimensions of mass $(mR)^2 = \Delta(\Delta - D + 1)$, at zero longitudinal momentum transfer, is zz' times the propagator for a bulk scalar in AdS_{D-2} of mass $(mR)^2 = (\Delta - 1)(\Delta - D + 2)$. This is dual to the following simple statement. The two-point function for an operator of dimension Δ in $D-1$ dimensions on the boundary of AdS_D, at zero longitudinal momentum transfer, reduces to the two-point function of an operator of dimension $\Delta - 1$ in $D - 3$ boundary dimensions.
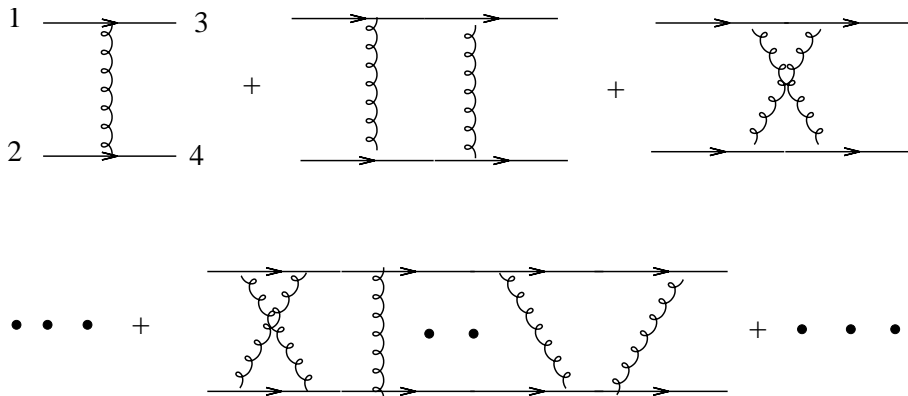
Figure 1. Ladder and crossed ladder diagrams contributing to the eikonal approximation in the high energy limit.

3 A diagrammatic derivation

We may also derive the eikonal result as a sum of the high-energy contribution of perturbative diagrams illustrated in figure 1, along the lines of Cheng and Wu [15, 22–26] or the methods pioneered by many other authors in the gravitational case; see for example [16–20, 27, 28]. For Anti de Sitter space this consists of summing a class of Witten diagrams, where we choose scalar fields for the external lines and gravitons for the exchanged rungs between these two sides. The sum includes all orders of the coupling to the sources giving rise to all ladder and crossed ladder diagrams. The mechanism leading to eikonalization at high energies in flat background has long been understood. From the perspective of perturbative summation, the key simplification necessary is the separation of dependence on the *longitudinal* light-cone momentum coordinates, $q_i^{\pm}$ and the *transverse* impact $x_i^{\perp}$ co-ordinates. This feature can already be appreciated by analyzing the high-energy behavior for the sum of box and crossed-box diagrams in flat space. Therefore, we begin by providing a brief description in ϕ^3 theory, paying particular attention to the dependence of amplitudes on transverse coordinates, before treating the case of graviton exchange in AdS_5.

3.1 Flat background

We begin by considering the 4-particle amputated Green's function, $\mathcal{A}(p_i)$, for elastic scattering $p_1 + p_2 \to (-p_3) + (-p_4)$, by exchange of a massless field of spin J. We use the "all-incoming" convention, with $s = -(p_1 + p_2)^2$, $t = -q^2 = -(p_1 + p_3)^2$. The reasons for starting with the Green's function rather than an on-shell scattering amplitude are twofold. First, in the conformal case there is no on-shell S matrix, and second, we choose to transform all transverse momenta $p_i^{\perp}$ to co-ordinate space, which is not possible on the mass shell ($p_i^2 = -m^2$). This procedure greatly simplifies the analysis of the high-energy dependence of the ladder graphs. After this transformation to the "transverse representation", we also choose to drop the overall conservation delta function for longitudinal momentum:

$(2\pi)^2\delta(p_1^{\pm} + p_2^{\pm} + p_3^{\pm} + p_4^{\pm})$. In the high-energy limit where $p_1^+ \simeq -p_3^+$ and $p_2^- \simeq -p_4^-$ are large, we will verify that, order by order, the scattering is local in transverse coordinates,

$$\mathcal{A}\left(p_i^{\pm}, x_i^{\perp}\right) = \sum_{n=1} \mathcal{A}^{(n)} \rightarrow \left(\sum_n \mathcal{M}^{(n)}\left(p_1^+, p_2^-, x_1^{\perp} - x_2^{\perp}\right)\right) \delta^2\left(x_1^{\perp} - x_3^{\perp}\right) \delta^2\left(x_2^{\perp} - x_4^{\perp}\right)$$

(3.1)

and that the resulting sum,

$$\mathcal{M}_{eik}\left(p_1^+, p_2^-, x^{\perp} - x'^{\perp}\right) = \sum_{n=1} \mathcal{M}^{(n)}\left(p_1^+, p_2^-, x^{\perp} - x'^{\perp}\right) = -2is\left[e^{i\chi\left(s, x^{\perp} - x'^{\perp}\right)} - 1\right] ,$$

(3.2)

takes the eikonal form.

3.1.1 Tree and one-loop scattering in a flat background

The tree level amputed amplitude in transverse representation,

$$\mathcal{A}^{(1)}(p_i^{\pm}, x_i^{\perp}) = \mathcal{M}^{(1)}(p_i^{\pm}, x_1^{\perp} - x_2^{\perp})\delta^2(x_1^{\perp} - x_3^{\perp})\delta^2(x_2^{\perp} - x_4^{\perp}) \tag{3.3}$$

$$\mathcal{M}^{(1)}(p_i^{\pm}, x_1^{\perp} - x_2^{\perp}) = g_0^2 s^J G(q^{\pm}, x_1^{\perp} - x_2^{\perp}) \tag{3.4}$$

is given in terms of the t-channel massless propagator,

$$G(q^{\pm}, x^{\perp}) = \int \frac{d^2 q^{\perp}}{(2\pi)^2} \frac{e^{iq^{\perp} x^{\perp}}}{q^{\perp 2} - 2q^+ q^- - i\epsilon} , \tag{3.5}$$

where $q^{\pm} = -(p_1^{\pm} + p_3^{\pm})$. We have introduced a factor s^J for each t-channel exchange in anticipation of the case of a graviton exchange, where $J = 2$; of course our scalar model has $J = 0$. At high energies, $q^{\pm} = O(1/\sqrt{s})$, we have

$$\mathcal{M}^{(1)}(p_1^+, p_2^-, x^{\perp} - x'^{\perp}) \simeq g_0^2 s^J G(q^{\pm} = 0, x^{\perp} - x'^{\perp}) . \tag{3.6}$$

This is to be compared with eq. (1.8) for the one-graviton exchange contribution in AdS_5.

The amputated amplitude at one-loop order involves a box diagram, and a crossed box diagram, obtained from the box by interchanging $(p_1^{\pm}, x_1^{\perp})$ with $(p_3^{\pm}, x_3^{\perp})$ or equivalently $(p_2^{\pm}, x_2^{\perp})$ and $(p_4^{\pm}, x_4^{\perp})$. The sum of the two diagrams, in transverse representation, can be written in a compact form (see figure 2):

$$\mathcal{A}^{(2)}\left(p_i^{\pm}, x_1^{\perp}, x_3^{\perp}, x_2^{\perp}, x_4^{\perp}\right) = \frac{i(2\pi)^2}{2!} \int \frac{d^2 q_1^{\pm}}{(2\pi)^2} \frac{d^2 q_2^{\pm}}{(2\pi)^2} \delta^2\left(q^{\pm} - q_1^{\pm} - q_2^{\pm}\right) A_{13}\left(p_1^{\pm}, q_1^{\pm}, x_1^{\perp}, x_3^{\perp}\right)$$

$$\times \left[\left(-ig_0^2 s^J\right) G\left(q_1^{\pm}, x_1^{\perp} - x_2^{\perp}\right)\right]$$

$$\times \left[\left(-ig_0^2 s^J\right) G\left(q_2^{\pm}, x_3^{\perp} - x_4^{\perp}\right)\right] A_{24}\left(p_2^{\pm}, -q_1^{\pm}, x_2^{\perp}, x_4^{\perp}\right) \quad (3.7)$$

$$A_{13}(p_1^{\pm}, q_1^{\pm}, x_1^{\perp}, x_3^{\perp}) = S(p_1^{\pm} + q_1^{\pm}, x_3^{\perp}, x_1^{\perp}) + S(p_1^{\pm} + q_2^{\pm}, x_1^{\perp}, x_3^{\perp})$$

$$A_{24}(p_2^{\pm}, q_1^{\pm}, x_2^{\perp}, x_4^{\perp}) = S(p_2^{\pm} + q_1^{\pm}, x_4^{\perp}, x_2^{\perp}) + S(p_2^{\pm} + q_2^{\pm}, x_2^{\perp}, x_4^{\perp})$$
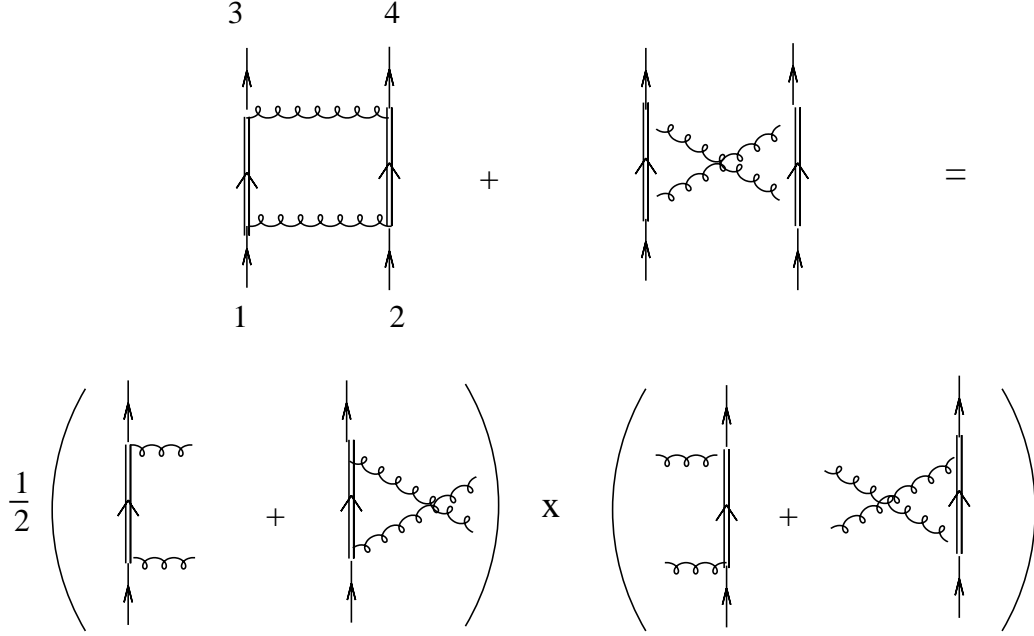
(3.8)

Figure 2. Sum of box and cross box diagram is factorized with combinatoric weight 1/2!.

Here S is the propagator for the scattered particles, and for a scalar of mass μ, it is

$$S\left(p^{\pm}, x^{\perp}\right) = \int \frac{d^2 p^{\perp}}{(2\pi)^2} \frac{e^{ip^{\perp} x^{\perp}}}{p^{\perp 2} - 2p^+ p^- + \mu^2 - i\epsilon} \ . \tag{3.9}$$

The two terms on the right-hand side of each equation in (3.8) represent a sum over all possible orderings of the rungs of the ladder as they connect to one of the ladder's sides. The product of A_{13} and A_{24} in eq. (3.7) then leads to four terms. Since the rungs are indistinguishable, these four terms represent a double-counting of each Feynman diagram. Consequently a factor of 1/2! has been added to compensate for this over-counting.

Extracting the high-energy behavior of eq. (3.7) can be done in the following two steps. We first note that, for near-forward scattering at high energies, the limit s large is characterized by $p_1^+ \simeq -p_3^+$ and $p_2^- \simeq -p_4^-$ both large, and q, q_1, q_2 asymptotically space-like, with $q_i^{\pm} = O(1/\sqrt{s})$. In this limit, $G(q_1^{\pm}, x_1^{\perp} - x_2^{\perp}) \simeq G(q_1^{\pm} = 0, x_1^{\perp} - x_2^{\perp})$ and $G(q_2^{\pm}, x_3^{\perp} - x_4^{\perp}) \simeq G(q_2^{\pm} = 0, x_3^{\perp} - x_4^{\perp})$ can be taken out of the $\int d^2 q_1^{\pm} d^2 q_2^{\pm} \delta^2(q^{\pm} - q_1^{\pm} - q_2^{\pm})$ integrals.

For definiteness, let us next use $q_1^{\pm}$ as independent integration variables. At high energies, the dependence of A_{13} on q_1^+ drops out and it becomes a function of q_1^- only. Conversely, A_{24} is independent of q_1^- and is a function of q_1^+. This factorizable dependence, $\int d^2 q_1^- \int dq_1^+ A_{13} A_{24} \simeq \int d^2 q_1^- A_{13} \int dq_1^+ A_{24}$, allows us to carry out the $q_1^{\pm}$ integrations explicitly. Focus first on the integral over A_{13},

$$\int \frac{dq_1^-}{2\pi} A_{13}\left(p_1^+, q_1^-, x_1^{\perp}, x_3^{\perp}\right) \tag{3.10}$$

which involves a sum of two pole terms coming from the S propagators. At high energies, the s-channel pole occurs at $0 = \mu^2 + (p_1 + q_1)^2 - i\epsilon$, i.e., $q_1^- \simeq O(1/p_1^+) - i\epsilon$, and the u-channel pole occurs at $q_1^- \simeq O(1/p_1^+) + i\epsilon$. Although each pole term vanishes only as $O(1/q_1^-)$ for q_1^- large, the sum goes as $O(1/(q_1^-)^2)$. Closing the contour leads simply to

$$(-i)\, \text{Residue}\, [\, S \,]_{q^- \simeq 0 - i\epsilon} = (i/2p_1^+) \int \frac{d^2 p^\perp e^{ip^\perp (x_1^\perp - x_3^\perp)}}{(2\pi)^2} = (i/2p_1^+)\, \delta^2\left(x_1^\perp - x_3^\perp\right) \tag{3.11}$$

Similarly for A_{24}, closing the contour in q_1^+ leads to $(i/2p_2^-)\delta^2(x_2^\perp - x_4^\perp)$. Putting these together, we wind up with

$$\mathcal{A}^{(2)}\left(p_1^+, p_2^-, x_1^\perp, x_2^\perp, x_3^\perp, x_4^\perp\right) \simeq \mathcal{M}^{(2)}\left(p_1^+, p_2^-, x_1^\perp - x_2^\perp\right)\delta^2\left(x_1^\perp - x_3^\perp\right)\delta^2\left(x_2^\perp - x_4^\perp\right) \tag{3.12}$$

$$\mathcal{M}^{(2)}\left(p_1^+, p_2^-, x^\perp - x'^\perp\right) = -2is\, \frac{1}{2!}\, \left[ig_0^2 s^{J-1} G\left(q^\pm = 0, x^\perp - x'^\perp\right)/2\right]^2. \tag{3.13}$$

Note that all the dependences on $x_1^\perp - x_3^\perp$ and $x_2^\perp - x_4^\perp$ reduce to delta-functions, i.e., the effective interaction remains local, with "zero transverse deflection". This is a key feature common to all eikonal results, and we will see in a moment that it generalizes to the case of AdS space.[8]

3.1.2 Eikonal exponentiation

We can now re-sum the infinite series of loop graphs to obtain the eikonal approximation, as in [15]. Consider a ladder with n rungs, where arbitrary crossings of rungs are allowed. Denote transverse coordinates for these $2n$ vertices by $\{x_i\}$ and $\{x'_j\}$, $i, j = 1, \cdots, n$. The sum of all n^{th} order diagrams can be obtained by evaluating

$$\frac{i(2\pi)^2}{n!} \int \prod_{k=1}^n \frac{dq_k^+ dq_k^-}{(2\pi)^2} \delta^2\left(\sum q_i^\pm\right) A_{13}\left(p_1^\pm, p_3^\pm, q_i^\pm, x_j^\perp\right)$$

$$\times \left[\prod_k \left(-ig_0^2 s^J\right) G\left(q_k^\pm \simeq 0, x_k^\perp, x_k'^\perp\right)\right] A_{24}\left(p_2^\pm, p_4^\pm, -q_l^\pm, x_p'^\perp\right) \tag{3.14}$$

where we have ordered the light-cone momenta of exchanged rungs, $q_k^\pm$, $k = 1, \cdots, n$, and have also dropped the dependence of exchanged propagators on $\{q_k^\pm\}$, as is valid at high energies, and as was done earlier for the one-loop contribution. Here A_{13} is given by a sum of $n!$ terms, each a product of $n - 1$ propagators, corresponding to all possible ways of attaching n exchanged propagators to one side of the eikonal ladder. A_{24} is defined

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[8]It is worth providing a more intuitive interpretation of the result just obtained. The q_1^- integral over A_{13} can be written more symmetrically as

$$\int dq_1^- dq_2^- \delta(q_1^- + q_2^-) A_{13} \sim \frac{1}{-2p_1^+} \int dq_1^- dq_2^- \delta(q_1^- + q_2^-) \times \left[\frac{1}{q_1^- + i\epsilon} + \frac{1}{q_2^- + i\epsilon}\right]$$

The integrand can be shown simply to correspond to the Fourier transform of $\theta(x_3^+ - x_1^+) + \theta(x_1^+ - x_3^+)$. That is, the different permutations in eq. (3.8) simply correspond to scattering in different "time-orderings". For each ordering, the scattering amplitude is constant and local, proportional to $(1/p_1^+)\delta^2(x_1^\perp - x_3^\perp)$. This physical picture generalizes to the case of multiple exchanges.

similarly. Both are generalization of eq. (3.8) from $n = 2$ to $n > 2$. As for the $n = 2$ case, a factor of $1/n!$ is supplied to account for over-counting.

To extract the high-energy behavior, we again take advantage of $\{q_i^+\}$ and $\{q_i^-\}$ factorization, which allows us to carry out the integrations

$$\prod_i \int \frac{dq_i^-}{2\pi} \delta \left(\sum_i q_i^- \right) A_{13} \left(p_1^+, q_i^-, x_j^\perp \right) \int \frac{dq_i^+}{2\pi} \delta \left(\sum_i q_i^+ \right) A_{24} \left(p_2^-, -q_k^+, x_l'^\perp \right) \quad (3.15)$$

Following the analysis of Cheng and Wu [15], one finds that the net result of these integrations is to produce

$$[(i)^2/2s]^{n-1} \prod_{i=1}^{n-1} \left[\delta^2 \left(x_i^\perp - x_{i+1}^\perp \right) \delta^2 \left(x_i'^\perp - x_{i+1}'^\perp \right) \right] . \quad (3.16)$$

That is, we can set $x_i^\perp = x^\perp$ and $x_j'^\perp = x'^\perp$, and the feature of "zero transverse deflection" persists in each order. Integrating over all but 2 of these transverse coordinates, we are led to

$$\mathcal{A}^{(n)} \left(p_1^+, p_2^-, x_1^\perp, x_3^\perp, x_2^\perp, x_4^\perp \right) \simeq \mathcal{M}^{(n)} \left(p_1^+, p_2^-, x_1^\perp, x_2^\perp \right) \delta^2 \left(x_1^\perp - x_3^\perp \right) \delta^2 \left(x_2^\perp - x_4^\perp \right) \quad (3.17)$$

$$\mathcal{M}^{(n)} \left(p_1^+, p_2^-, x^\perp - x'^\perp \right) = -2is \frac{1}{n!} \left[ig_0^2 s^{J-1} G \left(q^\pm = 0, x^\perp - x'^\perp \right)/2 \right]^n \quad (3.18)$$

After summing over n, we arrive at the eikonal amplitude

$$\mathcal{M}_{eik} \left(p_1^+, p_2^-, x^\perp - x'^\perp \right) = \sum_{n=1} \mathcal{M}^{(n)} \left(p_1^+, p_2^-, x^\perp - x'^\perp \right) = (-2is) \left[e^{i\chi(s, x^\perp - x'^\perp)} - 1 \right] \quad (3.19)$$

where the eikonal is

$$\chi \left(s, x^\perp - x'^\perp \right) = \frac{1}{2} g_0^2 s^{J-1} G \left(q^\pm = 0, x^\perp - x'^\perp \right) \quad (3.20)$$

Upon taking a 2-d Fourier transform, we arrive at the on-shell amplitude

$$T(s,t) = \int d^2 x^\perp e^{-ix^\perp \cdot q^\perp} \mathcal{M} \left(p_1^+, p_2^-, x^\perp \right) \simeq -2is \int d^2 x^\perp e^{-ix^\perp \cdot q^\perp} \left[e^{i\chi(s, x^\perp - x'^\perp)} - 1 \right] \quad (3.21)$$

3.2 Eikonal expansion for AdS_5 gravity

Let us return to the problem of summing eikonal graphs in AdS_5, which can be carried out in close analogy with the flat background. As described earlier, we begin by considering a gauge theory scattering amplitude (or correlation function), truncated by dropping the external hadron wave functions (or external boundary-to-bulk AdS_5 propagators) on each external leg, and then written in the transverse representation $(p^\pm, x_i^\perp, z_i)$. We work only in the regime where the amputated amplitude can be evaluated using propagators in AdS_5 space, which are conformal Green's function in AdS_5 with 3 transverse dimensions in an AdS_3 submanifold.

In the high-energy limit we only need to keep the $++, --$ component of the graviton propagator, which simplifies the analysis greatly. In what follows, we generalize this to n-graviton exchanges and observe how eikonalization arises for scattering in AdS_5.

Consider the case of the one-graviton-exchange Witten diagram for scalar sources on the boundary of AdS_5. The amplitude for this diagram is [11]

$$\kappa_5^2 \int dz \sqrt{g} \int dz' \sqrt{g'} \; \tilde{T}^{MN}(p_1, p_3, z) \tilde{G}_{MNM'N'}(q, z, z') \tilde{T}^{M'N'}(p_2, p_4, z') \qquad (3.22)$$

in momentum representation, where $\tilde{T}^{MN}$ is the energy-momentum tensor for the scalar source in the bulk and $\tilde{G}_{MNM'N'}$ is the graviton propagator, both in momentum representation. At high energies, keeping the leading $++, --$ component, we find for the amputated amplitude in transverse representation

$$\mathcal{M}^{(1)}(s, x^\perp, z; x'^\perp, z') = \frac{\kappa_5^2}{R} \, \tilde{s}^2 \, \left(\frac{zz'}{R^2} \right) G_3(x^\perp, z, x'^\perp, z') \qquad (3.23)$$

which has previously been given in eq. (1.8). Here R is the AdS radius and G_3 is the AdS_3 scalar propagator, eq. (1.9), which can be expressed in terms of the chordal distance, eq. (1.10).

Let us turn next to the one-loop contribution, which involves a box diagram and a crossed box. The total contribution at high energies, generalizing eq. (3.7) by keeping only graviton exchanges of helicity structure $(++, --)$, can be expressed as

$$\mathcal{A}^{(2)} \left(p_i^\pm, x_i^\perp, z_i \right) = i \frac{1}{2!(2\pi)^2} \int d^2 q_1^\pm d^2 q_2^\pm \delta^2 \left(q^\pm - q_1^\pm - q_2^\pm \right)$$
$$\times [A_{13}] \left[-i \left(\kappa_5^2/R^3 \right) \left(z_1 z_2 s/R^2 \right)^2 \left(z_1 z_2 \right) G_3 \left(u[1,2] \right) \right]$$
$$\times \left[-i \left(\kappa_5^2/R^3 \right) \left(z_3 z_4 s/R^2 \right)^2 \left(z_3 z_4 \right) G_3 \left(u[3,4] \right) \right] [A_{24}] \qquad (3.24)$$

where $u[1,2]$ and $u[3,4]$ are chordal distances in an obvious notation. Again, similar to eq. (3.8), we have

$$A_{13}(p_1^\pm, q_1^\pm, q_2^\pm, x_1^\perp, z_1, x_3^\perp, z_3) = \frac{1}{R^3} \left[\; G_5(p_1^\pm + q_1^\pm, x_3^\perp, z_3, x_1^\perp, z_1) \right.$$
$$\left. + G_5(p_1^\pm + q_2^\pm, x_1^\perp, z_1, x_3^\perp, z_3) \right]$$
$$A_{24}(p_2^\pm, -q_1^\pm, -q_2^\pm, x_2^\perp, z_2, x_4^\perp, z_4) = \frac{1}{R^3} \left[\; G_5(p_1^\pm - q_1^\pm, x_4^\perp, z_4, x_2^\perp, z_2) \right.$$
$$\left. + G_5(p_1^\pm - q_2^\pm, x_2^\perp, z_2, x_4^\perp, z_4) \right] \qquad (3.25)$$

which account for both the "box" and the "cross-box" diagrams. Here $G_5(p^\pm, x^\perp, z; x'^\perp, z')$ is the AdS_5 scalar propagator in a transverse representation, which has previously been introduced, eq. (2.6). It can be expressed as

$$G_5(p^\pm, x^\perp, z; x'^\perp, z') = (zz')^2 \int \frac{d^2 p^\perp}{(2\pi)^2} e^{ip^\perp (x^\perp - x'^\perp)} \int k dk \frac{J_2(zk) J_2(z'k)}{k^2 + p^{\perp^2} - 2p^+ p^-} \; . \qquad (3.26)$$

Let us concentrate on extracting the high-energy behavior for $\mathcal{A}^{(2)}$. The situation is nearly identical to that for a flat background, leading to factorization in $q_1^\pm$, and the need

to evaluate $(1/2\pi) \int dq_1^- A_{13}$ and $(1/2\pi) \int dq^+ A_{24}$ separately. Focus on the A_{13} integral, which again involves two terms, each an integral over a propagator G_5. Using eq. (3.26), integration over q_1^- leads to

$$\frac{-i}{R^3} \text{Residue} \left[G_5 \right]_{q^- \simeq 0 - i\epsilon} = \left(iR^{-3}/2p_1^+ \right) \int \frac{d^2 p^\perp}{(2\pi)^2} e^{ip^\perp (x_1^\perp - x_3^\perp)} (z_1 z_3)^2 \int k dk J_2 (z_1 k) J_2 (z_3 k)$$

$$= \left(iR^2/2p_1^+ z_1^2 \right) \delta^2 \left(x_1^\perp - x_3^\perp \right) \delta (z_1 - z_3) / \sqrt{g_1} \qquad (3.27)$$

using the Bessel function completeness relation; here $g_1 \equiv \det g(z_1)$. Similarly, we obtain

$$(1/2\pi) \int dq_1^+ A_{24} = \left(iR^2/2p_2^- z_2^2 \right) \delta^2 \left(x_2^\perp - x_4^\perp \right) \delta (z_2 - z_4) / \sqrt{g_2} \, . \qquad (3.28)$$

Putting these together, we again verify "zero transverse deflection", and

$$\mathcal{M}^{(2)} \left(p_1^+, p_2^-, x^\perp, z; x'^\perp, z' \right) = -2i \left(zz'/R^2 \right)^2 s \, \frac{1}{2!} \left[i \left(\kappa_5^2 / 2R^3 \right) (zz's) G_3(u) \right]^2 \qquad (3.29)$$

This represents a direct generalization of the flat-space result, eq. (3.13).

The generalization to higher loops can similarly carried out as for flat space. We obtain

$$\mathcal{A}^{(n)} \left(p_i^\pm, x_i^\perp, z_i \right) \simeq \mathcal{M}^{(n)} \left(s, x_1^\perp, z_1, x_2^\perp, z_2 \right)$$

$$\times \left[\delta^2 \left(x_1^\perp - x_3^\perp \right) \delta (z_1 - z_3) / \sqrt{g_1} \right] \left[\delta^2 \left(x_2^\perp - x_4^\perp \right) \delta (z_2 - z_4) / \sqrt{g_2} \right] \qquad (3.30)$$

$$\mathcal{M}^{(n)} \left(p_1^+, p_2^-, x^\perp, z; x'^\perp, z' \right) = -2i \left(zz'/R^2 \right)^2 s \, \frac{1}{n!} \left[i \left(\kappa_5^2 / 2R^3 \right) (zz's) G_3(u) \right]^n \qquad (3.31)$$

Summing over n, we have

$$\mathcal{M}_{eik}(p_1^+, p_2^-, x^\perp, z; x'^\perp, z') = -2is \left(\frac{zz'}{R^2} \right)^2 \left[e^{i\chi(s, x^\perp - x'^\perp, z, z')} - 1 \right] \qquad (3.32)$$

where

$$\chi(s, x^\perp - x'^\perp, z, z') = \frac{1}{2} \left(\frac{\kappa_5^2}{R^3} \right) (zz's) G_3(u) = \frac{1}{2} (RM_P)^{-3} (zz's) G_3(u) \qquad (3.33)$$

as promised.

Lets up summarize the essential feature of the eikonal approximation. The dependences on $x_1^\perp - x_3^\perp$, $z_1 - z_3$ and $x_2^\perp - x_4^\perp$, $z_1 - z_3$ reduce to delta-functions so that there is "zero transverse deflection" of the incoming states during the interaction. As we now will see in the shock wave derivation, this "freezing" of transverse motion is a consequence of the instantaneous interaction in light-cone time x^+.

4 Shock wave derivation

An alternative approach to the eikonal approximation for gravity is to study the semi-classical limit of one particle scattering in the presence of a shock wave created by the other.

In particular 't Hooft computed the eikonal amplitude for high-energy scalar particles in flat space gravity [19, 20]. The shock is given by the Aichelburg-Sexl metric [29], which is the Schwarzschild metric for a particle with mass $m_i \ll M_P$ boosted to the light-cone and approximated for impact parameters far outside the Schwarzschild radius. Here we will show that by generalizing this metric to a shock wave in the bulk 5-d AdS space [30–32], we are able to derive the eikonal amplitude without recourse to perturbation theory used in section 3. This has the advantage that it provides greater insight and a complementary way to understand the source of corrections to this approximation.

Consider the shock wave created by particle 2 with a very large longitudinal light-cone "energy" p_2^-, on the light-cone $x^+ = (x^0+x^3)/\sqrt{2} = 0$ at fixed transverse position $x^\perp = x'^\perp$ and $z = z'$. The energy momentum tensor for this particle in the *bulk* is

$$T^{--}(x^+, x^\perp, z; x'^\perp, z') = (z^2/R^2)\, p_2^-\, \delta\left(x^+\right) \delta^2\left(x^\perp - x'^\perp\right) \delta\left(z - z'\right)/\sqrt{g}. \tag{4.1}$$

Although tensor indices are raised and lowered by the AdS background metric $g_{MN} = \eta_{MN} R^2/z^2$, we choose to treat the momentum components, $p^\mu = \eta^{\mu\nu} p_\nu$, as flat space 4-vectors, to match with the Noether currents on the boundary Yang-Mills theory. Note that extra factors of z in T^{--} ensure that $T_{++} = g_{+-} g_{+-} T^{--}$ and h_{++} both scale like z^{-2} under $z \to \gamma z, x^\mu \to \gamma x^\mu, p_\mu \to \gamma^{-1} p_\mu$, as they must for a conformal dual gauge theory. With this as the source to the Einstein equation, one arrives at the modified metric,

$$ds^2 = (g_{MN} + h_{NM})dx^M dx^N = R^2\, \frac{-2dx^+ dx^- + (dx^\perp)^2 + dz^2}{z^2} + h_{++}(x^+, x^\perp, z)dx^+ dx^+, \tag{4.2}$$

where $\sqrt{g} = R^5/z^5$ and $g_{+-} = g_{-+} = -R^2/z^2$.

The eikonal approximation requires solving gravity in the Gaussian approximation for fluctuations h_{MN} relative to the fixed AdS_5 background metric, $g_{MN}(z)$. Expanding the Einstein Hilbert action to quadratic order for the relevant terms we have,

$$S_{EH}[g+h] \simeq S_{EH}[g] + \frac{1}{2\kappa_5^2} \int d^4x dz \sqrt{g}\, \left[-\frac{1}{2}\, \partial_N h^+{}_- \, \partial^N h_+{}^- - \frac{1}{2}\, \partial_N h^-{}_+ \, \partial^N h_-{}^+ \right]$$
$$+ \int d^4x dz \sqrt{g}\, \left[h_{++} T^{++} + h_{--} T^{--} \right] + O(h^3) , \tag{4.3}$$

where for convenience we have introduced the dependent metric functions: $h^-{}_+ = h_+{}^- = g^{-+}h_{++}$ and $h^+{}_- = h_-{}^+ = g^{+-}h_{--}$. This leads to the linearized Einstein equation,[9]

$$-\Delta_2\, h_{++}(x^+, x^\perp, z) = 2\kappa_5^2 T_{++}(x^+, x^\perp, z; x'^\perp, z') , \tag{4.4}$$

where $\kappa_5^2 = 1/M_P^3$ and

$$\Delta_j = z^{-j} \frac{1}{\sqrt{g}} \partial_M \sqrt{g} g^{MN} \partial_N z^j = \frac{1}{R^2}\left[z^2 \partial_z^2 + (2j-3)z\partial_z + j(j-4) + z^2 \nabla_\perp^2 \right] , \tag{4.5}$$

---

[9] Note that in flat space, we would need to solve for the transverse Greens function,

$$-\nabla_\perp^2 h_{++} = 2\kappa_D^2 p_2^- \delta(x^+)\delta^{D-2}(x^\perp) ,$$

which for $D = 4$ agrees with the Aichelburg-Sexl metric: $h_{++} = -p_2^- \kappa_4^2 \log(|x^\perp|/C)\delta(x^+)/\pi$.

is the general tensor Laplacian operator for AdS_5 defined in ref. [2]. The solution to the Einstein equation (4.4) is proportional to the bulk-to-bulk scalar propagator in AdS_3:

$$h_{++}(x^+, x^\perp, z; x'^\perp, z') = 2 \left(z'/z \right) \left(\kappa_5^2/R \right) p_2^- G_3(x^\perp - x'^\perp, z, z') \delta(x^+) , \qquad (4.6)$$

where we have reinserted the explicit dependence on the location, $(x'^\perp, z')$, of the source (4.1). Note that the factor of z'/z is uniquely determined at this point resulting below in a scattering phase symmetric in z, z'. From this solution we also obtain by the raising operator $g^{-+} = -z^2/R^2$,

$$h^{--}(x^+, x^\perp, z; x'^\perp, z') = 2zz' \left(\kappa_5^2/R^3 \right) \left(z^2/R^2 \right) p_2^- G_3(x^\perp - x'^\perp, z, z') \delta(x^+) . \qquad (4.7)$$

Since the AdS_3 propagator, $G_3(u)$ defined in eq. (1.9), is a function of the scale-invariant variable u, h^{--} has scaling dimension -2 as it should.

Next we find the amplitude for particle 1 to propagate in this background metric. This is just the bulk-to-bulk propagator $G(x, z; x', z')$ for particle 1 in the presence of the shock wave at $x^+ = 0$ introduced by particle 2. Its equation is the same as for $G_5(u)$, except for an additional term[10] for the contribution of h^{--}:

$$\left[2z^2 \partial_+ \partial_- - z^2 \partial_z^2 + 3z \partial_z - z^2 \nabla_\perp^2 + R^2 h^{--} \partial_-^2 \right] G(x, z; x', z') = R^5 \delta^4 \left(x - x' \right) \delta \left(z - z' \right) / \sqrt{g}. \qquad (4.8)$$

The metric $h^{--}(x^\perp, x^+, z)$ preserves translational invariance in x^-, so it is natural to transform to fixed p_1^+,

$$\widetilde{G}_{p_1^+}(x^+ - x'^+, x^\perp - x'^\perp, z, z') = \int dx^- e^{ip_1^+(x^- - x'^-)} G(x, z; x', z') . \qquad (4.9)$$

The resultant equation is just the light-cone Schroedinger equation with "time" $\tau = x^+$ and conjugate "Hamiltonian" $H = P^-$:

$$[-i\partial_\tau + H] \widetilde{G}_{p_1^+}(x^+ - x'^+, x^\perp - x'^\perp, z, z') = \left(z^3/2p_1^+ \right) \delta^2 \left(x^\perp - x'^\perp \right) \delta(x^+ - x'^+)\delta(z - z_0) \qquad (4.10)$$

where

$$2p_1^+ H = -\partial_z^2 + 3z^{-1}\partial_z - \nabla_\perp^2 - (p_1^+)^2 R^2 z^{-2} h^{--}(x^\perp, x^+, z) . \qquad (4.11)$$

The solution is given by the time-ordered product

$$\widetilde{G}_{p_1^+}(x_3^+ - x_1^+, x_3^\perp - x_1^\perp, z_3, z_1) = \langle x_3^\perp, z_3 | \, T_\tau \left[\exp\left(-i \int_{x_1^+}^{x_3^+} d\tau H \right) \right] |x_1^\perp, z_1\rangle. \qquad (4.12)$$

The Hamiltonian operator has states enumerated by $|x^\perp, z\rangle$. We can factorize this into the product of three segments $\tau < 0, \tau = 0, \tau > 0$.

$$T_\tau \left[\exp\left(-i \int_\epsilon^{x_3^+} d\tau \hat{H} \right) \right] \exp\left(-i \int_{-\epsilon}^\epsilon d\tau H \right) T_\tau \left[\exp\left(-i \int_{x_1^+}^{-\epsilon} d\tau \hat{H} \right) \right] \qquad (4.13)$$

---

[10]Note that to linear order the inverse of $g_{MN} + h_{MN}$ is $g^{MN} - h^{MN}$, which implies that the shock potential in the propagator (4.8) is $-h^{--}(x^+, x^\perp, z; x'^\perp, z')$ with the correct sign for gravitational attraction.

The first and the third factors above would contribute to the bulk-to-boundary propagators, which are dropped when amputating the Green's function. The integral over H for the middle term receives its only contribution from the delta function in $h^{--}(x^+) \sim \chi \, \delta(x^+)$, giving rise to the eikonal phase shift,

$$\chi(s, x^\perp - x'^\perp, z, z') = (\kappa_5^2/R^3)szz'G_3(x^\perp - x'^\perp, z, z')/2 \,, \qquad (4.14)$$

for a (diagonal) unitary S matrix:

$$S(s, x^\perp - x'^\perp, z, z') = e^{i\chi(s, x^\perp - x'^\perp, z, z')} \,. \qquad (4.15)$$

This phase is in agreement with our earlier result for the truncated bulk scattering amplitude,

$$\mathcal{M}(s, x^\perp - x'^\perp, z, z') = -2is \left(\frac{zz'}{R^2}\right)^2 \left[e^{i\chi(s, x^\perp - x'^\perp, z, z')} - 1 \right] \,. \qquad (4.16)$$

In principle, one could derive the prefactor in this equation within the context of the shock wave calculation; here we have merely matched to the known tree-level amplitude.

5 Conclusion

We have considered the eikonal approximation to high-energy scattering in the bulk of AdS space, as might be relevant for a portion of a calculation of high-energy scattering in gauge theory, as well as other physical processes. We gave three approaches to the eikonal amplitude: a heuristic picture for the AdS scaling form, an explicit resummation of Witten diagrams, and a shock wave derivation. All have their advantages for further generalizations and clearer physical intuition.

However, our results for the eikonal phase are valid only for linearized semiclassical gravity. For most physically important applications, the restrictions on our results must be relaxed. There are a number of technical and conceptual advances that are needed, some of which are well within reach.

- Generalization to finite λ. There is no obstruction to extending our results to the case of Regge behavior at finite λ, using [2]. Interesting comparisons can be made with eikonal studies of flat-space string theory, such as [18, 24, 27, 33].

- Generalization to non-conformal settings. This is necessary for a study of how the string theory realizes the dual gauge theory's Froissart-Martin bound. In [2] we studied effects of confinement and running couplings, and again the obstructions to extending our results to this case are purely technical.

- Corrections to the eikonal approximation. Full gauge-theory computations require integrals over bulk coordinates z and z', but the eikonal approximation is typically valid only in part of the bulk; for instance, it may fail as $z \to z'$. While reasonable approximations will allow some gauge-theory computations to be carried out reliably,

a stronger understanding of scattering in all regions of bulk coordinates is clearly desirable. A minimal consistency requirement is that of "local small angle" scattering, e.g., local momentum transfer should be less than the local energy. This has been commented upon briefly in the final section of [2] and more extensively in [34]. In addition, one must also take into account other nonlinear effects.

- Accounting for nonlinear corrections. At small bulk impact parameter, the gravitational fields of the scattering particles become sufficiently large to require nonlinear gravity to be incorporated. In some regimes, one must incorporate string interactions, such as triple-Pomeron vertices; a role for effective Reggeon field theories may be expected. In other regimes one must include nonperturbative effects, such as black holes [1, 35, 36]. Only in these contexts can one begin to address questions of how field-theory unitarity is restored at strong coupling, as relevant to high-energy cross-sections, saturation phenomena, and heavy-ion collisions. It is clear that quantum string corrections must be addressed as well for some relevant processes, if contact is to be made with QCD itself.

Despite the limited region of validity of our results, we see signs of what we expect are general features that go well beyond this regime. The eikonal phase is proportional to the Euclidean transverse AdS_3 Green's function, a strong-coupling manifestation of conformal symmetry of the gauge theory in the transverse plane, which is known to arise for the weak-coupling BFKL kernel [37, 38]. The conventional picture in 4-d flat space, where the scattering particle picks up a phase at a fixed position in the transverse impact parameter space $x^\perp = (x_1, x_2)$, is generalized here to a phase at a fixed *bulk* transverse position $x^\perp, z$. In both the perturbative and shock wave pictures, the exchange of an arbitrary number of rungs in a ladder graph becomes effectively local, thus freezing all transverse motion. We expect these and other features will survive, or be naturally extended, as other regimes are explored, and a deeper understanding of high-energy scattering in gauge and string theory emerges.

Acknowledgments

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