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CLOSED-ORBIT DISTORTIONS OF PERIODIC FODO LATTICES DUE TO PLANE GROUND WAVES

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An analytical investigation of the closed-orbit distortions due to ground waves has been made for a model periodic FODO lattice. The response of the closed orbit to a single plane wave and the rms response to an ensemble of uncorrelated waves are expressed as a function of C/λ (ring circumference/wavelength). It is found that transverse focusing (represented by the tune Q) and superperiodic lattice geometry (represented by the number of FODO cells N) can resonantly amplify the closed-orbit response to certain frequencies. As a consequence, as C/λ increases through some values $mN \pm Q$ (m positive integer), the response function is expected to show a steplike behavior followed by a slow fall off. The results are compared with models that have both continuous focusing and uncorrelated quadrupole motion. Numerical examples are presented.

1. INTRODUCTION

In an ideal storage ring the periodic closed orbit is a smooth curve that is close to the design orbit and that therefore passes through the centers of the guide-field magnets and quadrupoles. This maximizes the aperture for the stored particle beam and guarantees simple and predictable beam optics for particles close to the closed orbit.

In an actual storage ring, however, the closed orbit differs considerably (typically several millimeters) from the design orbit. This is due to additional dipole kicks arising from field errors in the bending magnets and misalignments of quadrupole magnets. The dipole components of field errors in the bending magnets cause closed-orbit kicks that are typically smaller than 0.1% of the design deflection angle ϑ_0. In practice, the horizontal (Δx) and vertical (Δz) misalignments of quadrupole magnets can be easily controlled within a few tenths of a millimeter. If f is the focal length of a quadrupole, this results in error kicks $\Delta x/f$ and $\Delta z/f$, respectively. Taking into account that with increasing design energy ϑ_0 falls much faster than $1/f$, one expects that for small storage rings the dipole errors of bending magnets are the dominant source of closed-orbit distortions, whereas for large storage rings the misalignments of quadrupoles dominate. These closed-orbit errors can be detected and corrected with the aid of many beam-position monitors and correction dipole magnets. Unfortunately, it takes some time to evaluate the closed–orbit measurement and to apply the appropriate set of dipole corrections. Therefore, it is of especial interest to investigate those types of errors that might vary quickly on the time scale needed to make corrections.

This paper deals with the effect of ground vibrations on the closed orbit in a storage ring. Estimates of ground-vibration amplitudes[1] and measurements[1,2] show that in the frequency domain above 1 Hz, most of the disturbances are man made (e.g. motor traffic, heavy machinery) with typical amplitudes of 10^{-7} m.†

At lower frequencies, i.e., with wavelengths larger than about one km, the natural "microseismic noise" clearly dominates. Both classes of noise vary considerably with respect to site and time and depend on traffic conditions, soil properties, weather, seismic activity, and so on.

Although, with such small amplitudes, the quadrupole movement due to ground vibration is small compared to the alignment tolerances, the distortion of the closed orbit might still be troublesome or even intolerable, depending on beam-stability requirements and the closed-orbit sensitivity. This problem is of particular interest to

(a) very large colliders using two separate rings (HERA, LHC, SSC), because of the large number of quadrupoles and strong nonlinearity of the beam-beam interaction
(b) dedicated synchrotron radiation sources, because of the ambitious beam-stability requirements of the users.

The effect of magnet imperfections on beam dynamics has been an important field of investigation since the very introduction of alternating-gradient rings. For uncorrelated misalignment of quadrupoles in a periodic lattice, the rms closed-orbit distortion $\sigma_{co}(s)$ at an azimuthal position s has been shown[3] to be

$$\sigma_{co}(s) = \frac{\sqrt{\beta(s)}\sqrt{\langle\beta\rangle}}{2\sin\pi Q}\frac{\sigma_q}{|f|}\sqrt{N} \tag{1}$$

This formula is equally valid for horizontal and vertical misalignments once the appropriate horizontal or vertical beta function β, betatron tune Q, focal length of quadrupoles f, and transverse rms quadrupole misalignment σ_q are inserted. N is the number of identical FODO sections, whence the number of quadrupoles is $2N$.‡ The average $\langle\beta\rangle$ is taken at the quadrupoles. If correlated quadrupole misalignments are involved, it is probably sufficient in many realistic cases to assume that plane ground waves§ are traversing the storage ring. In a model with continuous focusing [i.e. $\beta(s) = \text{const} = R/Q$], with the design orbit taken as a pure circle of radius R, it has been shown[4] that the response of the closed orbit

† Normally vertical amplitudes are larger than the horizontal amplitudes, and they are more harmful since the vertical emittance of the beam is smaller than the horizontal one.

‡ Equation (1) contains a factor $1/\sqrt{2}$ as compared to the respective formula in Ref. 3, because here we are discussing the rms closed-orbit deviation and not the rms emittance of the closed orbit. This difference should be borne in mind in all that follows, but the reader is free to multiply by $\sqrt{2}$ to get the results for the closed-orbit emittance if this is more appropriate to a particular problem.

§ To be precise, one should talk about straight waves being damped with increasing depth below ground surface (e.g., Rayleigh waves), but for the movement of the quadrupoles mounted on the ground surface there is no difference.

exhibits resonancelike properties at $\lambda \simeq \lambda_\beta = 2\pi R/Q$. If the ground vibration wavelength λ is not much smaller than the betatron wavelength λ_β, this model, although crude, seems to be reasonable. There are, however, practical cases where vibration contributions with $\lambda < \lambda_\beta$ are not at all negligible. Then it might be instructive to use a more general analytical model that remains valid quantitatively if λ becomes equal or even smaller than the distance between quadrupoles. We don't consider here any amplification or damping of the magnet motion with respect to the ground that might result from resonances in the supports.

2. CLOSED-ORBIT DISTORTION DUE TO A SINGLE PLANE GROUND WAVE

2.1. *The Model*

We use a right-handed curvilinear coordinate system x, z, s attached to the design orbit; s is the path length along this trajectory, and x is in the horizontal plane of the design orbit, perpendicular to s. Let us consider the following model. The storage ring consists of N identical FODO cells. Each FODO cell is composed of one horizontally and one vertically focusing quadrupole and two identical bending magnets located symmetrically between them (see Fig. 1). The focal lengths f of both quadrupoles are assumed to be equal. This is not essential for the model but simplifies the notation somewhat. The betatron phase advance per full FODO cell is $\Delta\phi$; i.e., the betatron tune of the whole ring is $Q = N \cdot \Delta\phi/2\pi$. Quadrupoles are treated as thin lenses; i.e., the variations of β and ϕ within each quadrupole are neglected. Within this model all quadrupole magnets lie on a circle with radius R, which is (in close approximation) $1/2\pi$ of the design orbit circumference C, if $N > 2$. In general, the direction of wave propagation and the point where the orbit distortion is observed are arbitrary parameters. Without severely restricting generality, however, we can assume that the observation point is in the center of a focusing quadrupole located at $s = 0$, per definition. The beta value at the observation point β_0 can be regarded as variable, nevertheless. The azimuthal angle θ and the betatron phase advance ϕ refer to the point $s = 0$ (see Figure 2). Then the azimuthal position θ_n of the nth quadrupole magnet and betatron phase ϕ_n within this magnet are related by

$$\phi_n = n\frac{\Delta\phi}{2} = Q_z\theta_n \tag{2}$$

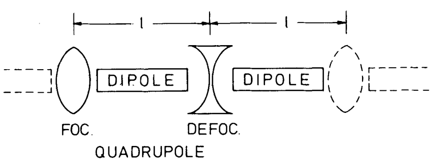

FIGURE 1 Sketch of a simple FODO cell as used in the model.

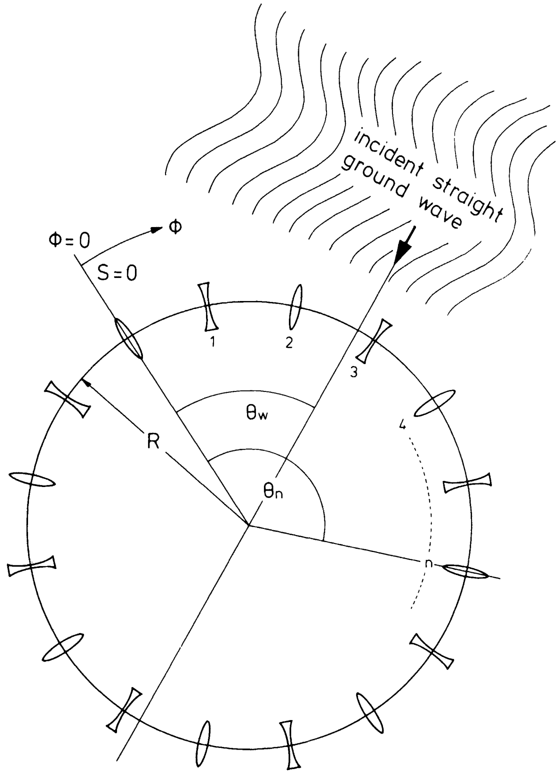

FIGURE 2 Sketch of geometry as described by the model and nomenclature.

2.2. Vertical

We are now able to describe the vertical closed-orbit distortion in response to a single vertical plane wave (there is no horizontal response to a vertical magnet displacement since there are no sextupoles or skew quadrupoles in our model). Let $\hat{z}$ be the amplitude of the ground wave and φ_0 its phase with respect to the center of the ring; $\omega/2\pi$ is the frequency of the wave, v its velocity of propagation, and θ_w its direction of incidence. Then, the vertical motion of the nth quadrupole magnet is given by

$$\Delta z_n(t) = \hat{z}\,\mathrm{Re}\exp\left\{i\left[\omega t + \frac{\omega R}{v}\cos(\theta_n - \theta_w) + \varphi_0\right]\right\}. \tag{3}$$

$\omega R/v$ may be replaced by C/λ. If we now consider the distortion z_c of the closed orbit at $s = \phi_z = 0$, we may use the fact that the frequency of the ground wave is much smaller than the transverse betatron frequencies, so that the distortion is adiabatic. Therefore, we may treat Δz_n as a constant misalignment with t as a parameter. The closed-orbit distortion due to Δz_n is[3]

$$z_c = \frac{\sqrt{\beta_0}}{2\sin\pi Q_z}\sqrt{\beta_{z,n}}\frac{\Delta z_n}{f_n}\cos(\phi_n - \pi Q_z),$$

with β_0 and ϕ_n in the vertical plane. Since our model system is linear we may sum up contributions from all misaligned quadrupoles. If we use

$$\frac{\sqrt{\beta_{z,n}}}{f_n} = \frac{\sqrt{\beta}}{f} \text{ in focusing and } \frac{\sqrt{\beta_{z,n}}}{f_n} = -\frac{\sqrt{\breve{\beta}}}{f} \text{ in defocusing}$$

quadrupole magnets and use Eqs. (2) and (3), we get

$$z_c = \mathrm{Re}\left\{\frac{\sqrt{\beta_0}}{2\sin\pi Q_z}\frac{\hat{z}}{f}e^{i(\omega t+\varphi_0)}\left[\sum_{n=2}^{2N}\sqrt{\beta}\,e^{i(C/\lambda)\cos(\phi_n/Q_z-\theta_w)}\cos(\phi_n-\pi Q_z)\right.\right.$$
$$\left.\left.-\sum_{n=1}^{2N-1}\sqrt{\breve{\beta}}\,e^{i(C/\lambda)\cos(\phi_n/Q_z-\theta_w)}\cos(\phi_n-\pi Q_z)\right]\right\}. \tag{4}$$

The sums over n can be executed analytically, if we use

$$\exp\left[i\frac{C}{\lambda}\cos\left(\frac{\phi_n}{Q_z}-\theta_w\right)\right] = \sum_{p=-\infty}^{+\infty} i^p J_p\left(\frac{C}{\lambda}\right)\cos\left[p\left(\frac{\phi_n}{Q_z}-\theta_w\right)\right],$$

because the Bessel function of the first kind $J_p\left(\frac{C}{\lambda}\right)$ no longer depends on n. Thus, with the help of Eq. (2), we have to deal with

$$z_c = \mathrm{Re}\left\{\frac{A}{\sin\pi Q_z}\left[\sqrt{\beta}\sum_{p=-\infty}^{+\infty} i^p J_p\sum_{n=1}^{N}\cos p\left(\frac{n\,\Delta\phi}{Q_z}-\theta_w\right)\cos(n\,\Delta\phi-\pi Q_z)\right.\right.$$
$$\left.\left.-\sqrt{\breve{\beta}}\sum_{p=-\infty}^{+\infty} i^p J_p\sum_{n=1}^{N}\cos p\left(\frac{n\,\Delta\phi-\Delta\phi/2}{Q_z}-\theta_w\right)\cos(n\,\Delta\phi-\pi Q_z-\Delta\phi/2)\right]\right\}.$$

Here the abbreviation $A = \frac{\sqrt{\beta_0}}{2}\frac{\hat{z}}{f}e^{i(\omega t+\varphi_0)}$ has been introduced. The execution of the sums over n is quite tedious but straightforward and results in

$$z_c = \mathrm{Re}\left[A\sum_{p=-\infty}^{+\infty} i^p J_p\left(\frac{C}{\lambda}\right)C_p\right] \tag{5}$$

with

$$C_p = \frac{(-1)^{p+1}}{\sin\left(\frac{\pi p}{N}-\frac{\Delta\phi}{2}\right)}\left\{\sqrt{\breve{\beta}}\cos\left[p\left(\pi\frac{N+1}{N}-\theta_w\right)-\frac{\Delta\phi}{2}\right]-\sqrt{\breve{\beta}}\cos p\,(\pi-\theta_w)\right\}. \tag{6}$$

The real part yields

$$z_c(t) = \sum_{p=-\infty}^{+\infty}\frac{J_pC_p}{2}[i^pA+(-i)^pA^*]$$
$$= \frac{\sqrt{\beta_0}}{2}\frac{\hat{z}}{f}\left[\cos(\omega t+\varphi_0)\sum_{p=-\infty}^{+\infty}(J_{4p}C_{4p}-J_{4p-2}C_{4p-2})\right.$$
$$\left.+\sin(\omega t+\varphi_0)\sum_{p=-\infty}^{+\infty}(J_{4p-1}C_{4p-1}-J_{4p-3}C_{4p-3})\right]. \tag{7}$$

If we define the response R as the amplitude of the closed-orbit distortion $\hat{z}_c$ divided by the amplitude of the ground wave $\hat{z}$, we get the final result

$$R_z = \frac{\hat{z}_c}{\hat{z}} = \frac{\sqrt{\beta_0}}{2f}\left\{\left[\sum_{p=-\infty}^{+\infty} J_{4p}\left(\frac{C}{\lambda}\right)C_{4p} - J_{4p-2}\left(\frac{C}{\lambda}\right)C_{4p-2}\right]^2 + \left[\sum_{p=-\infty}^{+\infty} J_{4p-1}\left(\frac{C}{\lambda}\right)C_{4p-1} - J_{4p-3}\left(\frac{C}{\lambda}\right)C_{4p-3}\right]^2\right\}^{1/2}. \tag{8}$$

With this result, we have succeeded in replacing the finite sum over the error kicks [see Eq. (4)] by an infinite sum. What do we win? The advantage becomes clear if we realize that C_p becomes resonant for

$$p = mN + Q_z \quad \text{with } p,\ m \text{ integers,}$$

that is to say, recalling that $N > Q_z$, for

$$|p| = |m|\, N \pm Q_z. \tag{9}$$

For arbitrary Q_z and m there is always some p such that

$$\left|\sin\left(\frac{\pi p}{N} - \frac{\Delta\phi}{2}\right)\right|^{-1} \tag{10}$$

becomes very large. If we denote the distance of Q_z from the closest integer $[Q_z]$ with δQ_z, we may express the maxima of Eq. (10) as

$$\frac{1}{\sin\frac{\pi}{N}\delta Q_z}, \text{ occurring whenever } p_{\text{res}} = mN + [Q_z]. \tag{11}$$

Due to this factor, all contributions with $p \neq p_{\text{res}}$ in Eq. (8) are very effectively suppressed. As a result, ground waves with C/λ above $|p_{\text{res}}|$ contribute much more to the response function R_z than those with $C/\lambda < |p_{\text{res}}|$. According to Eqs. (9) and (11), the lowest-order closed-orbit resonances occur for

$$\begin{aligned} |p_{\text{res}}|_1 &= [Q_z], \\ |p_{\text{res}}|_2 &= N - [Q_z], \\ |p_{\text{res}}|_3 &= N + [Q_z], \text{ etc.} \end{aligned} \tag{12}$$

Consequently, with increasing C/λ, the response function is expected to increase in a steplike manner whenever C/λ exceeds some resonance parameter $|p_{\text{res}}|_i$, as illustrated in Figs. 3 and 4. This behavior has been expected qualitatively.[4] Since the continuous-focusing model predicts this behavior only for the case when $p_{\text{res}} = [Q_z]$, the quantitative agreement is good for $\nu \leqslant Q \cdot v/C$ only. Eqs. (6–8) now give an analytic framework for a quantitative understanding of the response of a periodic FODO lattice for ground-wave frequencies up to

$$\nu < v/l_{\text{q}} \quad (l_{\text{q}} = \text{length of quadrupole magnets}).$$

The lower parts of Figs. 3 and 4 show that R depends markedly on the difference between the angle of observation and the angle of wave propagation θ_{w}.

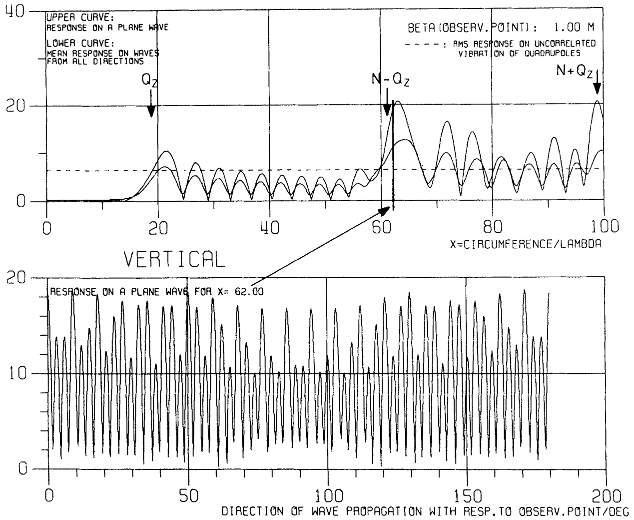

FIGURE 3 As an example for the vertical response to ground waves, parameters of the European Synchrotron Radiation Facility ESRF have been adopted: $C = 850$ m, $Q_z = 19.2$, $N = 80$. The upper plot shows R_z and $R_z^{\rm rms}$ as a function of C/λ for $\theta_w = 0$. It shows that the response increases whenever C/λ increases through the values Q_z, $N - Q_z$, $N + Q_z, \ldots$, see Eq. (12). The lower plot shows that, for C/λ fixed, R_z depends on θ_w in a complicated way. It is mainly characterized by the superposition of all $\cos(|p_{\rm res}|_i\, \theta_w)$ terms, with $|p_{\rm res}|_i = mN \pm [Q_z]$ smaller than C/λ considered [see Eqs. (6) and (12)]. For this example, $C/\lambda = 62$ has been chosen arbitrarily. Due to averaging, $R_z^{\rm rms}$ does not depend on θ_w, of course. For ease of scaling, $\beta_0 = 1$ m has been chosen for both plots.

2.3. Horizontal

Unlike vertical waves, horizontal waves also move quadrupole magnets longitudally (i.e. in the direction of the design orbit) by an amount which depends on the difference between the azimuthal position θ_n of the magnet and the direction of incidence θ_w. For example, if plane (again, more precisely, straight) compressional waves are considered,† the magnet motion is purely transversal only for $\theta_n = \theta_w$ and $\theta_n = \theta_w + \pi$. The horizontal motion of the nth quadrupole magnet is given in this case by

$$\Delta x_n(t) = \hat{x} \cos(\theta - \theta_w) \operatorname{Re} \exp\left\{ i \left[\omega t + \frac{C}{\lambda} \cos(\theta - \theta_w) - \varphi_0 \right] \right\}. \tag{13}$$

If we now use

$$i \cos\theta e^{ix\cos\theta} = \sum_{p=-\infty}^{+\infty} i^p \frac{dJ_p(x)}{dx} \cos p\theta,$$

we can express the horizontal closed orbit distortion $x_c(t)$ in analogy to Eq. (5)

† Treatment of horizontal shear waves requires evaluation of sums of the type $\sum_{n=1}^{N} \sin(\theta_n - \theta_w) \cos p(\theta_n - \theta_w) \cos(\phi_n - \pi Q_x)$ and yields resonances at $p = mN + Q_x \pm 1$ (p, m integer).

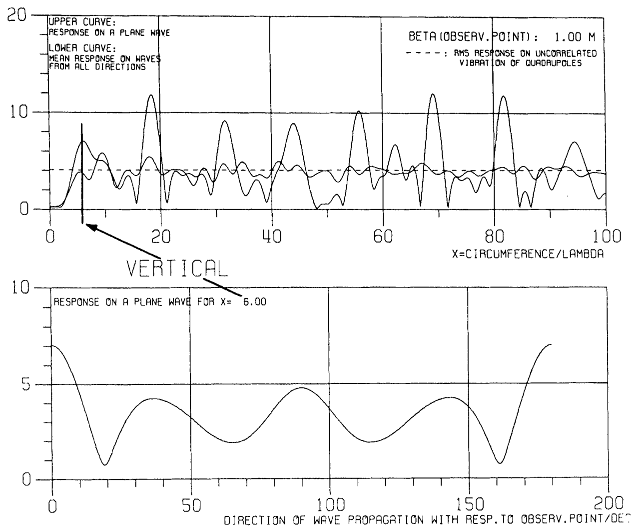

FIGURE 4 Since in a small storage ring the tune Q_z and the number of cells N are quite small ($Q_z = 4.2$, $N = 12$, $C = 100\,\text{m}$ in this example), the contributions of Bessel functions of resonant orders $p = mN \pm [Q_z]$ cannot be distinguished. It is clearly seen from the upper plot that the average of R_z^{rms} equals the rms response to uncorrelated quadrupole motion (broken line). Note that the maximum of $C/\lambda = 100$ corresponds to a ground wavelength of $\lambda = 1\,\text{m}$ in this plot. Since the length of realistic quadrupoles would not be much shorter, this marks the limit of the thin-lens model. However, amplitudes of those high-frequency waves will be negligible in most realistic cases. The lower plot shows the angular dependence of $R_z(\theta_w)$ at $C/\lambda = 6 = \text{const.}$

as[4]

$$x_c(t) = \operatorname{Re}\left[A_x \sum_{p=-\infty}^{+\infty} i^p J_p'(x) C_p\right], \tag{14}$$

with

$$A_x = \frac{\sqrt{\beta_0}}{2}\frac{\hat{x}}{f} e^{i}\left(\omega t + \varphi_0 - \frac{\pi}{2}\right),$$

where β_0 and all optics parameters in C_p (see Eq. (6)] refer to the horizontal plane. For the response function we get

$$R_x = \frac{\hat{x}_c}{\hat{x}} = \frac{\sqrt{\beta_0}}{2f}\left\{\left[\sum_{p=-\infty}^{+\infty} J_{4p}'\left(\frac{C}{\lambda}\right)C_{4p} - J_{4p-2}'\left(\frac{c}{\lambda}\right)C_{4p-2}\right]^2 + \left[\sum_{p=-\infty}^{+\infty} J_{4p-1}'\left(\frac{C}{\lambda}\right)C_{4p-1} - J_{4p-3}'\left(\frac{C}{\lambda}\right)C_{4p-3}\right]^2\right\}^{1/2}. \tag{15}$$

In analogy with the vertical case, resonances occur for

$$p = mN + Q_x \quad \text{with } p,\ m \text{ integers.}$$

I.e., with increasing C/λ the response R_x behaves in a similar way to R_z, see Fig. 5.

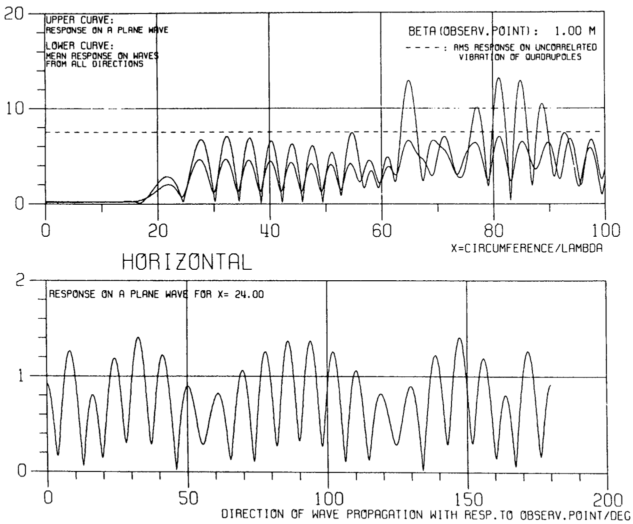

FIGURE 5 The parameters for this example of horizontal response are similar to those of Fig. 3: $C = 850\,\mathrm{m}$, $Q_x = 22.19$, and $N = 80$, and the structure of the response is similar. Note that, with the present definition of R_x and R_x^{rms}, the average R_x^{rms} differs from the rms response to uncorrelated motion by a factor of $\sqrt{2}$, since each horizontal wave contains components that are longitudinal with respect to the beam line, see Eq. (13).

3. CLOSED-ORBIT DISTORTION BY AN ENSEMBLE OF PLANE GROUND WAVES

3.1. Vertical

As mentioned earlier, the response to a single wave depends sensitively on the difference between the angle of observation and the angle of wave propagation. In practice, however, it might be more appropriate to describe the ground-surface motion by a large number M of plane ground waves coming from arbitrary directions θ_m, with phases φ_m, and amplitudes $\hat{z}_m$. In this case, we are still far from having uncorrelated motion of the magnets (see, e.g., Ref. 5 and the considerations below). Let us assume that all waves of our ensemble have equal (but variable, of course) frequency $\omega/2\pi$ and wavelength λ. Then, the vertical motion of the nth quadrupole magnet is given by

$$\Delta z_n(t) = \sum_{m=1}^{M} \hat{z}_m \,\mathrm{Re} \exp\left\{ i\left[\omega t + \frac{C}{\lambda}\cos(\theta_n - \theta_m) + \varphi_m \right]\right\}, \tag{16}$$

and the closed-orbit distortion is [see Eq. (5)]

$$z_c(t) = \mathrm{Re} \sum_{m=1}^{M} A_m \sum_{p=-\infty}^{+\infty} i^p J_p\left(\frac{C}{\lambda}\right) C_p^m \tag{17}$$

with

$$A_m = \frac{\sqrt{\beta_0}}{2}\frac{\hat{z}_m}{f} e^{i(\omega t + \varphi_m)}$$

and

$$C_p^m = \frac{(-1)^{p+1}}{\sin\left(\frac{\pi p}{N} - \frac{\Delta\phi}{2}\right)} \left\{\sqrt{\beta}\cos\left[p\left(\pi\frac{N+1}{N} - \theta_m\right) - \frac{\Delta\phi}{2}\right] - \sqrt{\beta}\cos p(\pi - \theta_m)\right\}. \tag{18}$$

In reality, we don't know all $\hat{z}_m$, θ_m, and φ_m, but we may assume some statistical properties if many different ensembles of waves are considered. Amplitudes, directions, and phases of different partial waves are uncorrelated. All directions and phases are random. If $\langle\ \rangle$ denotes averaging over all possible ensembles, we find with these assumptions that $\langle z_c(t)\rangle = 0$. More relevant, however, is the rms value of $z_c(t)$: rms $- z_c = (\langle z_c^2\rangle)^{1/2}$. It is determined by

$$(\text{rms} - z_c)^2 = \sum_{m=1}^{M}\sum_{n=1}^{M}\sum_{p=-\infty}^{+\infty}\sum_{q=-\infty}^{+\infty} J_p J_q \langle C_p^m C_q^n\rangle \langle \text{Re}\,(i^p A_m)\cdot \text{Re}\,(i^q A_N)\rangle. \tag{19}$$

In this formula we have been able to factorize the averages, since phases and directions are uncorrelated. The most important step is now to realize from Eq. (18) that $\langle C_p^m C_q^n\rangle$ vanishes if $|p| \neq |q|$ and if $m \neq n$. It vanishes for $m \neq n$ since directions θ_m, θ_n are random, leaving the self-correlation term $m = n$ only if ensemble averaging is performed. It vanishes for $|p| \neq |q|$ due to the orthogonality of $\cos|p|\,\theta_m \cdot \cos|q|\,\theta_m$ and of $\sin|p|\,\theta_m \cdot \sin|q|\,\theta_m$ in the interval $0 < \theta_m < 2\pi$. As a result, Eq. (19) becomes

$$(\text{rms} - z_c)^2 = \sum_{m=1}^{M} J_0^2\langle(\text{Re}\,A_m)^2\rangle\langle(C_0^m)^2\rangle + \sum_{m=1}^{M}\sum_{p=1}^{\infty} J_p^2\langle[\text{Re}(i^p A_m)]^2\rangle\langle(C_p^m + C_{-p}^m)^2\rangle. \tag{20}$$

Here, $i^{-p}J_{-p} = i^p J_p$ has been used, and it has been convenient to split the sum over p, since the term with $p = 0$ behaves singularly with respect to averaging. Since all phases are random, we may write

$$\langle[\text{Re}\,(i^p A_m)]^2\rangle = \frac{\beta_0}{4f^2}\frac{\langle\hat{z}_m^2\rangle}{2}. \tag{21}$$

The rms ground motion (with frequency $\omega/2\pi$) is, from Eq. (16),

$$\text{rms} - \Delta z = \langle[\Delta z_n(t)]^2\rangle^{1/2} = \left(\frac{1}{2}\sum_m \langle\hat{z}_m^2\rangle\right)^{1/2}. \tag{22}$$

Finally, averaging with respect to directions requires some trigonometrical gymnastics. Using Eqs. (20)–(22), we may express the rms response $R_z^{\rm rms}$ by

$$
\begin{aligned}
R_z^{\rm rms} &= (\text{rms closed-orbit distortion})/(\text{rms ground motion}) \\
&= \frac{\sqrt{\beta_0}}{2f}\left\{ J_0^2\!\left(\frac{C}{\lambda}\right)\left(\frac{\sqrt{\hat\beta}\cos\frac{\Delta\phi}{2}-\sqrt{\breve\beta}}{\sin^2\frac{\Delta\phi}{2}}\right)^2 \right. \\
&\quad + \sum_{p=1}^{\infty} J_p^2\!\left(\frac{C}{\lambda}\right)\left[\frac{\hat\beta+\breve\beta-2\sqrt{\hat\beta\breve\beta}\cos\left(\frac{p\pi}{N}-\frac{\Delta\phi}{2}\right)}{2\sin^2\left(\frac{\pi p}{N}-\frac{\Delta\phi}{2}\right)}\right. \\
&\quad + \frac{\hat\beta+\breve\beta-2\sqrt{\hat\beta\breve\beta}\cos\left(\frac{p\pi}{N}+\frac{\Delta\phi}{2}\right)}{2\sin^2\left(\frac{\pi p}{N}+\frac{\Delta\phi}{2}\right)} \\
&\quad \left.\left. + 2\,\frac{\hat\beta\cos\Delta\phi+\breve\beta-2\sqrt{\hat\beta\breve\beta}\cos\frac{\Delta\phi}{2}\cos\frac{p\pi}{N}}{\cos\frac{2\pi p}{N}-\cos\Delta\phi}\right]\right\}^{1/2}.
\end{aligned}
\tag{23}
$$

This result is exact within the model of a (linear) periodic FODO lattice with thin lenses. The resonance behaviour is obviously the same as in the single-wave case [see Eq. (9)]. Due to averaging, however, the rms response no longer depends on angles, and the dependence on C/λ is smoother than for single waves, as seen from Figs. 3 and 4. In the average over any frequency range $\Delta(\omega R/v) = \Delta(C/\lambda) \gg N$ one expects the rms response to equal the response to uncorrelated quadrupole motion [see Eq. (1)]. The latter value is indicated in Figs. 3–5 to show that this is indeed so.

3.2. *Horizontal*

It has been shown in section 2.3 that the response to a horizontal compressional wave can be described by the same equations that are valid in the vertical case if one just replaces J_p by J_p' and uses the appropriate horizontal values for the optics parameters. Since J_p doesn't enter the averaging considerations in the previous section, it follows that the same rule holds in the case of an ensemble of horizontal compressional waves with amplitudes, phases, and directions randomly distributed, so that Eq. (23) may be used correspondingly for the horizontal rms response. In doing so, one must not forget to consider the additional factor $\sqrt{2}$

[as compared to Eqs. (22) and (23)] arising from averaging $\cos^2(\theta - \theta_m)$ in

$$\text{rms} - \Delta x = \langle [\Delta x_n(t)]^2 \rangle^{1/2} = \left(\frac{1}{4} \sum_m \langle \hat{x}_m^2 \rangle \right)^{1/2}.$$

Note that R_x for a *single* wave [Eq. (15)] refers the closed-orbit amplitude to the ground-wave amplitude and not to the mean amplitude of the transverse quadrupole motion which is smaller by $\sqrt{2}$ due to the factor $\cos(\theta - \theta_0)$ in Eq. (13). Therefore, for the sake of coherency with the definition of R_x for a single wave, this factor has been omitted in the example of R_x^{rms} in Fig. 5.

4. CONCLUSIONS

If the sensitivity of a storage ring to ground motion is considered, an exact analytic analysis of a simple but realistic model is a valuable tool in the lattice design and in the interpretation of both measurements and computer simulations. In the model of a periodic FODO lattice with thin lenses, quite simple analytic formulae can be given that describe the spectral response of the closed orbit to horizontal and vertical plane waves and uncorrelated wave groups as a function of C/λ (ring circumference/wavelength). The final results, given for symmetric FODO cells, may easily be generalized to asymmetric FODO cells: one simply omits f and replaces $\sqrt{\hat{\beta}}$ by $\sqrt{\hat{\beta}}/f_{\text{foc}}$ and $\sqrt{\check{\beta}}$ by $\sqrt{\check{\beta}}/f_{\text{defoc}}$.

It is seen that transverse focusing (represented by the tune Q) and lattice geometry (represented by the number of FODO cells N) resonantly amplify the closed-orbit response to certain frequencies. The lowest lattice-driven resonance frequency ($C/\lambda = N - Q$) is somewhat higher than the lowest frequency that is resonantly driven by the betatron tune ($C/\lambda = Q$). However, since the lattice-driven resonances are typically the stronger ones, the spectrum of potentially disturbing sources of ground motion is quite large and must be specified in each case individually.

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