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New results on $g-2$ calculation¹

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Abstract. In this paper I summarize the current situation of the theoretical calculation of the electron $g-2$, including the result from the recent calculation of the QED mass-independent 4-loop contribution with a precision of 1100 digits.

A particle of mass m and spin s possesses a magnetic moment μ

$$\mu = g \frac{eh}{4\pi mc} s, \quad (1)$$

where g is the gyromagnetic ratio. According to the Dirac's theory [1], an electron has $g = 2$. This agreed with the experimental measurements until Kusch and Foley [2] measured a value of the *anomaly* a_e slightly different from zero:

$$a_e = \frac{g-2}{2} = 0.001\,15(4). \quad (2)$$

The deviation is due to the interaction of the electron with photons; using Q.E.D. Schwinger [3,4] was able to calculate at the first order that

$$a_e = \frac{\alpha}{2\pi} = 0.001\,161\dots, \quad (3)$$

where α is the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}. \quad (4)$$

The current measurements of a_e are based on the Penning trap method, developed by the group at the University of Washington. This trap uses an axial magnetic field and a quadrupole electric field; the anomaly is expressed as the ratio of two frequencies, which can be measured to a very high precision. For the development of this technique, the Nobel prize in Physics 1989 was awarded to H. Dehmelt. Their final results were [5]:

$$a_{e^-}^{exp} = 1\,159\,652\,188.4(4.3) \times 10^{-12} \quad (4.3 \text{ ppb}), \quad (5)$$

$$a_{e^+}^{exp} = 1\,159\,652\,187.9(4.3) \times 10^{-12} \quad (4.3 \text{ ppb}). \quad (6)$$

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² since 1 September 2017



$A_1^{(4)}$	$A_1^{(6)}$	$A_1^{(8)}$	$A_1^{(10)}$
-2.973... [12]	1.49(20) [31]	-1.434(138) [24]	7.795(336) [29]
-0.328 478 965 579 ... [8, 9]	1.195(26) [32]	-1.5098(384) [35]	6.599(223) [29]
	1.17611(42) [33]	-1.7283(35) [25]	
	1.181259(40) [34]	-1.9144(35) [26]	
	1.181241456587200006... [10]	-1.9106(20) [28]	
		-1.91298(84) [29]	
		-1.912245764926445... [11]	

Table 1. Numerical results of the evaluations of $A_1^{(4)}$, $A_1^{(6)}$, $A_1^{(8)}$ and $A_1^{(10)}$.

The most recent value obtained with the same technique by the Harvard group is [6, 7]:

$$a_e^{exp} = 1\,159\,652\,180.73(.28) \times 10^{-12} \quad (0.24 \text{ ppb}) . \quad (7)$$

In the standard model

$$a_e^{SM} = a_e^{QED} + a_e^{\text{weak}} + a_e^{\text{hadr}} . \quad (8)$$

The QED contribution can be split up in mass-independent and mass-dependent parts:

$$a_e^{QED} = A_1 + A_2 \left(\frac{m_e}{m_\mu} \right) + A_2 \left(\frac{m_e}{m_\tau} \right) + A_3 \left(\frac{m_e}{m_\mu}, \frac{m_e}{m_\tau} \right) . \quad (9)$$

The functions $A^{(i)}$ can be expanded in power series

$$A_i = A_i^{(2)} \left(\frac{\alpha}{\pi} \right) + A_i^{(4)} \left(\frac{\alpha}{\pi} \right)^2 + A_i^{(6)} \left(\frac{\alpha}{\pi} \right)^3 + A_i^{(8)} \left(\frac{\alpha}{\pi} \right)^4 + A_i^{(10)} \left(\frac{\alpha}{\pi} \right)^5 + \dots . \quad (10)$$

The mass-independent coefficients at 1, 2 and 3 loop are known in analytical form [3, 4, 8–10]:

$$A_1^{(2)} = \frac{1}{2} , \quad (11)$$

$$A_1^{(4)} = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) = -0.328\,478\,965\,579 \dots , \quad (12)$$

$$A_1^{(6)} = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3} \left[\left(a_4 + \frac{1}{24}\ln^4 2 \right) - \frac{1}{24}\pi^2\ln^2 2 \right] \\ - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} = 1.181\,241\,456 \dots ,$$

where $\zeta(n) = \sum_{i=0}^{\infty} i^{-n}$, $a_n = \sum_{i=0}^{\infty} 2^{-i} i^{-n}$. In table 1 we list some older theoretical evaluations of the two, three and four loop coefficients. In Ref. [11] I have evaluated up to 1100 digits of precision the 4-loop contribution $A_1^{(8)}$, finalizing a twenty-year effort [13–19] begun after the completion of the calculation of $A_1^{(6)}$ [10]. The first digits of the result are

$$A_1^{(8)} = -1.912245764926445574152647167439830054060873390658725345171329848 \dots . \quad (13)$$

The full-precision result is shown in table 3. The result (13) is in excellent agreement (0.9σ) with the numerical value

$$A_1^{(8)}(\text{Ref. [29]}) = -1.91298(84) , \quad (14)$$

latest result of a really impressive pluridecennial effort [20–29], and with the independent value

$$A_1^{(8)}(\text{Ref. [30]}) = -1.87(12) . \quad (15)$$

contribution	value in units of 10^{-12}
$A_1^{(2)}(\alpha/\pi)$	1 161 409 733.631(720)
$A_1^{(4)}(\alpha/\pi)^2$	- 1 772 305.065(3)
$A_1^{(6)}(\alpha/\pi)^3$	14 804.203
$A_1^{(8)}(\alpha/\pi)^4$	- 55.667
$A_1^{(10)}(\alpha/\pi)^5$	0.446(15)
$A_2^{(4)}(m_e/m_\mu)(\alpha/\pi)^2$	2.804
$A_2^{(6)}(m_e/m_\mu)(\alpha/\pi)^3$	- 0.092
$A_2^{(8)}(m_e/m_\mu)(\alpha/\pi)^4$	0.026
$A_2^{(10)}(m_e/m_\mu)(\alpha/\pi)^5$	- .0002
$A_2^{(4)}(m_e/m_\tau)(\alpha/\pi)^2$	0.010
$A_2^{(6)}(m_e/m_\tau)(\alpha/\pi)^3$	- 0.0008
$a_e(\text{hadronic v.p.})$	1.866(11)
$a_e(\text{hadronic v.p.,NLO})$	- 0.223(1)
$a_e(\text{hadronic v.p.,NNLO})$	0.028(1)
$a_e(\text{hadronic l-l})$	0.035(10)
$a_e(\text{weak})$	0.0297(5)

Table 2. Contributions to a_e .

At 5-loop level there is only the numerical evaluation by the Kinoshita's group

$$A_1^{(10)}(\text{Ref. [29]}) = 6.599(223) . \quad (16)$$

Concerning the mass-dependent part $A_2(r)$, $A_2^{(4)}(r)$ is known in analytical form [36], as well as $A_2^{(6)}(r)$ [37–41]; the first terms of the expansion for small r of the 4-loop coefficient $A_2^{(8)}(r)$ are known analytically [42, 43]. $A_2^{(10)}(m_e/m_\mu)$ and $A_2^{(10)}(m_e/m_\tau)$ have been calculated numerically [29]; the first terms of the expansion for small mass ratios of $A_3^{(6)}(m_e/m_\mu, m_e/m_\tau)$ and $A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)$ are known analytically [43]. The hadronic and weak contribution are

$$a_e(\text{hadronic v.p.}) = 1.866(11) \times 10^{-12} \quad (\text{see Ref. [44]}) , \quad (17)$$

$$a_e(\text{hadronic v.p.,NLO}) = -0.223(1) \times 10^{-12} \quad (\text{see Ref. [45]}) , \quad (18)$$

$$a_e(\text{hadronic v.p.,NNLO}) = 0.028(1) \times 10^{-12} \quad (\text{see Ref. [45]}) , \quad (19)$$

$$a_e(\text{hadronic l-l}) = 0.035(10) \times 10^{-12} \quad (\text{see Ref. [46]}) , \quad (20)$$

$$a_e(\text{weak}) = 0.0297(5) \times 10^{-12} \quad (\text{see Ref. [47]}) . \quad (21)$$

Inserting Eq.s(11-16,17-21), the known $A_2^{(j)}$ and the measurement of the fine structure constant [48, 49]

$$\alpha^{-1} = 137.035\,998\,996(85) \quad (0.62 \text{ ppb}) ,$$

into Eq.s(8-10) one finds

$$a_e^{\text{th}} = 1\,159\,652\,182.031(15)(15)(720) \times 10^{-12} , \quad (22)$$

where the first error comes from $A_1^{(10)}$, the second one from the hadronic and electroweak corrections, the last one from α , respectively. The values of the single contributions to a_e are listed in table 2. Conversely, assuming the validity of the theory and using the experimental measurement (7) one finds

$$\alpha^{-1}(a_e) = 137.035\,999\,1500(18)(18)(330)(0.25 \text{ ppb}) ,$$

where the errors come from $A_1^{(10)}$, hadronic and electroweak corrections, and a_e , respectively.

-1.91224576492644557415264716743983005406087339065872534517132984800603844398065170614276089270000363158375584153314732700563785149128545391
9028043270502738223043455789570455627293099412966997602778221157847203390641519081665270979708674381150121551479722743221642734319279759586
07405005783738496070187432831402483802519224946074229855893046350614049225266343109442400023563568812806206454940132249775943004292888367617
48899236915180878086989705263578533753776964117024536196013497574494361268486175162606832387186747303831505962741878015305514879400536977798
36946427868432691843117588958115974356695043304834907361342658649953116387811743475385423488364085584441882237217456706871041823307430517443
05573945961171550858961148995261266061246994073118403927472340023464969531735482584817998224097373710773657404645135211230912425281111372153
02154453721014811121159848970884223279879720484201445122828451516585236561786594592600991733031721302865467212345340500349104700728924487200
6160442613254496900043191519823004748818149431103849537829940629675867875385249781946989793132162197975750676701142904897962085050785592...

Table 3. First 1100 digits of $A_1^{(8)}$.

Methods of calculation of $A_1^{(8)}$

I sketch here the methods of calculation used in the literature, only for comparison. For further information on the technical aspects of my calculation of the 4-loop coefficient $A_1^{(8)}$, see Ref. [51] in these proceedings.

In QED the contributions to $g-2$ at n loops can be expressed as combinations of n -loop 4-dimensional Feynman integrals, belonging to a variety of Feynman diagrams.

- In Ref. [29], the n -loop 4-dimensional integrals are transformed in $(3n - 2)$ -dimensional integrals of (huge) rational functions of Feynman parameters. The integrals are computed using the MonteCarlo adaptive routine VEGAS [50]; an enormous amount of computing time is needed to sample adequately the integrands; for more information, see Ref. [20–29].
- My method, used in [11], consists in:
 - (i) reduction of contributions from each Feynman diagram to a small number (334 for $A_1^{(8)}$) of n -loop D -dimensional master integrals by using a suitable algorithm [10, 13];
 - (ii) determination of systems of difference or differential equations satisfied by the master integrals [13];
 - (iii) high precision calculation of these integrals by solving these systems of equations by means of rapidly convergent series expansions [13, 14];

This method allowed to obtain 1100 digits of $A_1^{(8)}$ (and up to 9800 digits for some selected important integrals). See Ref. [51] for further details.

- In Ref. [30] the contributions of the various diagrams are reduced to combinations of a small number of master integrals. Most of these master integrals are computed with MonteCarlo methods.
- I note the alternative approach recently introduced by S.Volkov in Ref. [52, 53]. It is also based on MonteCarlo integration. It seems promising at 5-loop level.

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