Field Theory On Cosmological Spacetime: Some Results from AdS/CFT

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Soumyabrata Chatterjee

To My Family

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List of Publications Arising From The Thesis

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Contents

SYNOPSIS

While the conventional perturbative approaches are well suited for the weakly coupled field theories, these methods fail when the theories become strongly coupled. One therefore needs to search for other avenues to explore the strongly coupled phases of field theories. Fortunately, the discovery of AdS/CFT correspondence of Maldacena in 1997 [1] has opened up one such direction. This correspondence relates a d spacetime dimensional gauge theory to a gravitational theory in $d + 1$ spacetime dimensions [2, 3]. One of the striking features of this approach is that it is inherently non-perturbative. This is in the sense that when one theory is weakly coupled, the other is in its strongly coupled phase. Consequently, while a direct proof of this conjecture still eludes us, it has passed through several non-trivial checks by now. Among the various forms of this correspondence, particularly well studied one, and mainly of interest to us in this thesis, is known as the weak form of the duality. In this form, the conjecture states that the type IIB supergravity on $AdS_5 \times S_5$ is dual to the $\mathcal{N} = 4$ large N, $SU(N)$ gauge theory, at zero temperature, formulated on a four-dimensional flat spacetime. The gauge theory under consideration is strongly coupled in the sense that the 't Hooft coupling is large. The gauge theory at finite temperature can also be studied within this framework by introducing a black hole in the gravity side in such a way that the asymptotic symmetries of AdS_5 remain unchanged.

In this thesis, our primary aim is to explore some features of this gauge theory (and some cousins of it), formulated on *time dependent* backgrounds. Time dependent geometries typically suffer from space-like singularities. Consequently, one expects to encounter strong gravitational fluctuations close to these singularities and, therefore, invalidating any semiclassical approach to study field theory on generic time dependent backgrounds. Even if one is probing away from these strongly fluctuating regions, because of the lack of the time-translational symmetry, defining a vacuum in a field theory on such a geometry becomes ambiguous. This appears to be even more serious when the spacetime possesses no static in and out regions [9]. It may therefore seem that there is very limited scope of learning anything about strongly coupled theories on time dependent backgrounds. Though, in general, it is true, some progress can indeed be made for geometries which possess AdS duals [14,28,29]. In this thesis we construct several four dimensional time dependent spacetimes with five dimensional AdS duals (besides those which are already known) and make a general attempt to study gauge theories on these geometries. In the next few paragraphs we summarize our works and the findings.

From the evidences coming from the thermal background radiation, we know that our universe is isotropic on large scales. In order to uncover the reasons for such a high degree of isotropy, physicists, even as early as in 1970, asked the following question: Could it be that the particle creation in an initial anisotropic universe may back-react on the geometry, making it isotropic [30]? One such anisotropic background, belonging to the Bianchi I class, is the Kasner metric [31][1]. It turns out that this is the simplest geometry possessing an AdS dual. In the following, we call this five dimensional dual as the AdS-Kasner geometry. It generically has a spacelike singularity in the past which run all the way to the boundary. Since the Kasner metric appears as a near-brane limit of a D3 brane (with time dependent world volume) of type IIB supergravity, from the AdS/CFT correspondence, we know the gauge theory that lives on the boundary is the $\mathcal{N} = 4$, $SU(N)$ SYM at large N. If one includes extra fluid

[1]Unlike Friedmann-Robertson-Walker (FRW) metric, this one parameter family of solution of Einstein equations (without matter) are not conformally flat. For earlier studies of perturbative field theories on the Kasner background, see [32] and references therein.

matter in to the five-dimensional Einstein action[2], it is possible to construct generalizations of four dimensional Kasner solutions which possess AdS duals [33, 34]. In the first part of the thesis, we construct these AdS-Kasner like geometries. One of the unpleasant features of the AdS-Kasner metric is that it generally suffers from horizon singularity deep inside the bulk. This can be cured, as we discuss, by adding a periodic coordinate which essentially caps off the horizon singularity. These are known as AdS-solitons, generalized appropriately in the time dependent setting. We provide a way to achieve this from the brane geometries. We further construct AdS-FRW spacetimes where now the boundaries have the FRW metrics. Motivated by the fact that the embedding of geometries in higher dimensional *flat* space often illustrates many geometrical features [66], we also provide explicit constructions of flat space embeddings of all the time dependent AdS duals that we construct. Though we do not use these constructions in the later part of the thesis, we believe that the results might be useful in the future.

The simplest of the Kasner class of metrics is the four dimensional Milne metric. Here, one of the space direction expands linearly with time and the other directions remain static. In fact this metric, upon appropriate coordinate transformation, can be brought to a Minkowski space. However, the new coordinates only cover a patch of the Minkowski namely the past and the future lightcones of the flat spacetime. Features of the gauge theory on Milne space can be studied using the bulk AdS-Milne space[3]. We first compute the two point correlator of two scalar operators by studying appropriate massive bulk scalars. We find that the correlator is in a thermal state and compare our results with the flat vacuum. In the large N limit, leading contribution to the two point space-like correlator of higher dimensional operators of $SU(N)$ SYM is expected to come from the UV-regulated length of bulk geodesic(s) connecting two boundary points. We calculate this correlator for Milne in the *geodesic approximation* and

[2] which may or may not follow from ten dimensional type II supergravity upon dimensional reductions

[3] It has translational symmetries along four directions, a $SO(1,2)$ from rotation and boost plus a scaling symmetry.

find its matching with our previous exact computation [34]. It turns out that for the other Kasner and Kasner-like geometries, one can only compute the correlator with in the geodesic approximation. Owing to the anisotropic scaling symmetries of these metrics, we find that the space-like geodesic equations can be solved exactly and the correlators can be computed. A somewhat surprising result that arises from our computation is the following. For the cases where the two points in the boundary are separated along a spatial direction associated with the positive Kasner coefficient, no divergences appear in the correlator even if we take the space-like surface close to the cosmological singularity. This may mean that, at least in these cases, the boundary selects a perfectly normalizable state [33] and no signatures of such spacetime singularities show up on the boundary correlations. This should be contrasted with the results of [35, 36]. In these references, the correlators are computed for the space-like separated operators along the contracting directions of the Kasner geometries. It is then found that the correlators become singular as the geodesics approach the past singularity. This blowing up of the space-like correlator is subsequently argued to be a consequence of boundary conformal field theory picking up a non-normalizable state.

As we mentioned before, AdS-Kasner metric has a horizon singularity in the UV. This singularity can be removed by lifting the solutions to AdS-Kasner soliton geometry. We compute the correlators and find that the qualitative features remain unchanged.

The method that we develop can also be used to compute the geodesics and the correlators for certain other geometries with *timelike* singularities [37]. These are the space dependent analogue of the Kasner metrics and, as we show, can be obtained from certain near-brane limit of the D3 brane in type IIB. For the correlators we find that, regardless of the choice of the Kasner direction, the correlators are non-singular near the timelike singularity.

Study of field theory on time dependent backgrounds is marked with various difficulties. Within the AdS/CFT correspondence, we could only initiate a preliminary investigation. We therefore end the thesis with a discussion on some possible directions of further explorations.

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Chapter 1

Introduction

One of the most exciting developments in theoretical physics over the past couple of decades is perhaps the AdS/CFT correspondence [1–3]. On one hand, this conjectured correspondence provides us with a non-perturbative definition of string theory and, on the other hand, it renders a concrete realization of the holographic principle proposed earlier by 't Hooft and Susskind [4], [5].

According to the AdS/CFT conjecture, there exists a correspondence between certain strongly coupled gauge theory and string theory. One thoroughly studied realization comes from the type IIB string theory. The correspondence suggests that the $\mathcal{N} = 4$, four dimensional $SU(N)$ super Yang-Mills (SYM) theory at zero temperature is dual to the type IIB string theory formulated on $AdS_5 \times S^5$. Here, AdS_5 represents the five dimensional anti-de Sitter spacetime and S^5 is the metric on the five-sphere. In the limit when both N and the gauge coupling are infinitely large, the bulk reduces to the IIB supergravity on AdS_5. The correspondence at finite temperature requires the introduction of a black hole in AdS space. Because of the dual nature of the correspondence, a direct proof of this conjecture is still lacking. However, various non-trivial checks led one to believe the correctness of the conjecture. In this thesis, we will attempt to exploit this conjecture to explore some features of strongly coupled gauge theories formulated on *time dependent backgrounds*.

The study of time dependent geometries and that of the field theories formulated on such backgrounds continue to be an active direction of research. It has its most prominent relevance in cosmology. In this context, it is of prime importance to know the origin, the large scale structure and the ultimate fate of our universe. Our present understanding of the evolution of the universe is based on the Friedmann-Robertson-Walker (FRW) cosmological model where one analyzes the time dependent Einstein equations arising from the general theory of relativity in the presence of appropriate matter. Recent results from the Cosmic Microwave Background Radiation tell us that our universe is homogeneous and isotropic on large scale. It has been speculated that even if we start from an anisotropic phase of the universe at early time, the creation of particles in time dependent background will back react and drive the universe to an isotropic phase. The simplest time dependent anisotropic geometry that has been studied in this context is the Kasner metric [31]. It belongs to the Bianchi type I class in cosmology.

Within the general theory of relativity, the time dependent geometries typically suffer from spacelike singularities. Consequently, one expects to encounter strong gravitational fluctuations close to these singularities and, therefore, it invalidates any semi-classical approach to study field theory on generic time dependent backgrounds. Even if one probes away from these strongly fluctuating regions, because of the lack of the time translational symmetry, defining a vacuum in a field theory on such a geometry suffers from ambiguities. This appears to be even more serious when the spacetime possesses no static in and out regions [9]. It may therefore seem that there is very limited scope of learning anything about strongly coupled theories on the time dependent backgrounds. Though, in general, it is true, some progress can still be made for those geometries which possess AdS duals [14, 28, 29].

The Kasner geometry, for one, is a simple example of the time dependent spacetime possessing an AdS dual. In this thesis, we call this the AdS-Kasner metric. It has been known for some time that AdS-Kasner metric appears from the near brane limit of a D3 brane (with time dependent world volume metric) of the type IIB supergravity [13, 14]. Therefore, from the AdS/CFT correspondence, we know that the gauge theory that lives on the boundary is the $\mathcal{N} = 4$, four dimensional $SU(N)$ super Yang-Mills at large N. In this thesis, besides Kasner, we find many geometries, including the FRW, which possess AdS duals. These geometries require the support of extra matter fields that may or may not follow from the type IIB supergravity. We make a broad attempt to study strongly coupled gauge theory on this backgrounds using AdS/CFT correspondence as a guide. In particular, we study the two point correlations of primary scalar operators of the gauge theory on these time dependent backgrounds. In some cases, like in Milne, it is possible to compute the correlators exactly. However, in most of the cases, owing to several ambiguities that appear in the time dependent backgrounds (and also for technical complexities), we resort to the geodesic approximation for the spacelike correlators. These results allow us to unearth some generic features of the correlators. The method that we develop can also be used to compute the geodesics and the correlators for certain other geometries with timelike singularities [37]. These are the space dependent analogues of the Kasner metrics and, as we show, can be obtained from certain near-brane limit of the D3 brane in type IIB.

Before we go on to present our results in the later chapters, we briefly review the D branes and the AdS/CFT correspondence. In the later sections, we elaborate upon some selected aspects of the weakly coupled quantum field theory on curved spacetime. The choices are made here to facilitate a comparison with the strong coupling results that we get in the later chapters.

1.1 D-branes and supergravity solutions

1.1.1 D-brane

The string theory [22–25] is based on the idea that all the particles that we observe are excitations of an one dimensional string. Different boundary conditions allow us to have two different kind of strings – one is called the open string and the other one is a closed string. Internal consistency demands that the strings are supersymmetric and they live in ten dimensional spacetime. The strings are generally characterized by their tensions (T), a length scale l_s known as string length. They are related by

$$T = \frac{1}{2\pi\alpha'}, \tag{1.1}$$

with $\alpha' = l_s^2$.

The spectrum of a closed string gives rise to a massless spin-2 particle state, known as graviton, the mediator of the gravitational interaction. Similarly the spectrum of an open string gives rise to the gauge fields. The interactions between strings are governed by a dimensionless parameter g_s which turns out to be the vacuum expectation value of a scalar field called the dilaton, which is itself a part of the string spectrum. Consistency further requires that there are only five different string theories in 10-dimensions. They are known as type I, type IIA, type IIB, SO(32) heterotic and $E_8 \times E_8$ heterotic string theory. However, all of these different string theories are connected through different dualities transformations. Besides the one dimensional objects, string theory also contains higher dimensional solitonic configurations known as the Dirichlet branes (D-branes) [26]. A Dp-brane is a $p + 1$ dimensional hypersurface in 10 dimensions on which the open strings can end. Therefore, they obey the Dirichlet boundary conditions along $10 - p - 1$ directions. Though both the type IIA string and the type IIB string contain the D-branes, in this thesis, our focus will be on the type IIB string. The Dp-branes are charged under a $p+2$ form field and have a tension proportional to $\frac{1}{g_s}$. These Dp-branes are normally the BPS objects which means that they preserve a certain number of spacetime supersymmetries. If we consider putting N number of such branes on top of each other, then for large N, the stack gets heavy and consequently it curves the spacetime geometry. On the other hand, as we mentioned earlier, since a D-brane acts as the end points of open strings, it describes a gauge theory on its world volume. The gauge theory in consideration here is a maximally supersymmetric $U(N)$ gauge theory. Therefore, we see that two different descriptions of the D-branes – one in terms of their world volume gauge fields and the other is in terms of gravity. In his celebrated work in 1997, Maldacena proposed that, in the decoupling limit (will be described later), these two sectors decouple from each other and the gauge/gravity duality or AdS/CFT correspondence follows. In the next few subsections we present a very brief review of this correspondence. There are several excellent reviews which present different aspects of this conjecture [6–8].

1.1.2 D-Branes as Solitonic Solutions of Supergravity

In the low energy limit, the type IIB string theory reduces to the type IIB supergravity. The bosonic part of the type II supergravity action contains a metric $g_{\mu\nu}$, a dilaton ϕ, axion C and a $p+2$ form field strength $F_{(p+2)}$. In the Einstein frame, the type II action can be written as [8]

$$S_E = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{|g|} \left(R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \sum_p \frac{1}{(p+2)!} e^{a_p \phi} F^2_{p+2} + ... \right). \tag{1.2}$$

where $a_p = -\frac{1}{2}(p-3)$ and the dots represents fermionic terms as well as the NS-NS 3-form field strength term. G_{10} here is the ten dimensional gravitational constant. The resulting equation of motions for the metric, the dilaton and the form field are

$$\begin{aligned}
R^{\mu}{}_{\nu} &= \frac{1}{2}\partial^{\mu}\phi\partial_{\nu}\phi + \frac{1}{2(p+2)!}e^{a_p\phi}\left((p+2)F^{\mu\xi_2\ldots\xi_{p+2}}F_{\nu\xi_2\ldots\xi_{p+2}} - \frac{p+1}{8}\delta^{\mu}_{\nu}F^2_{p+2}\right),\\
\nabla^2\phi &= \frac{1}{\sqrt{g}}\partial_{\mu}(\sqrt{g}\partial_{\nu}\phi g^{\mu\nu}) = \frac{a_p}{2(p+2)!}F^2_{p+2},\\
\partial_{\mu}&(\sqrt{g}e^{a\phi}F^{\mu\nu_2\ldots\nu_{p+2}}) = 0.
\end{aligned} \tag{1.3}$$

The $p+2$ form field strength also satisfies the Bianchi identity:

$$\partial_{[\mu_1}F_{\mu_2\ldots\mu_{p+3}]} = 0. \tag{1.4}$$

This is known as the electric $p+2$ form. The magnetic field strength is defined as

$$\tilde{F}_{10-p-2} = e^{a_p\phi} * F_{p+2}. \tag{1.5}$$

It is straightforward to verify that the equations (1.3) are invariant under the following transformations

$$a_p\phi \to -a_p\phi, \quad p+2 \to 10-p-2, \quad F_n \to \tilde{F}_{10-p-2}. \tag{1.6}$$

Simplest realization of AdS/CFT comes from the $D3$ brane. These are the solutions of type IIB supergravity and it has 4-dimensional worldvolume. It is a half BPS solution, which preserves half of the total spacetime supersymmetry. For the $D3$ brane the axion and dilaton fields are constant and the five form field strength is self dual. The solution of (1.3) for the $D3$ brane can be written as,

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}\left(-f dt^2 + \sum_{i=1}^{3}(dx^i)^2\right) + H^{\frac{1}{2}}\left(f^{-1}dr^2 + r^2 d\Omega_5^2\right),\\
H &= 1 + (\frac{h}{r})^4,\\
f &= 1 - (\frac{r_0}{r})^4,\\
h^8 + r_0^4 h^4 &= \frac{Q^2}{16},\\
\phi &= \text{ constant.}
\end{aligned}
\tag{1.7}
$$

where r_0 and Q are the integration constants and Q is related to the electric and magnetic charges of the $D3$ brane. The five form field strength is given by,

$$
F_{ti_1i_2i_3r} = \epsilon_{i_1i_2i_3} H^{-2}\frac{Q}{r^5}. \tag{1.8}
$$

The electric and magnetic charge densities of the brane are obtained by computing the flux of the five form field strength through S^5

$$
\begin{aligned}
\mu_3 &= \frac{1}{\sqrt{16\pi G_{10}}}\int_{S^5} *F_5 = \frac{\Omega_5}{\sqrt{16\pi G_{10}}}Q,\\
g_3 &= \frac{1}{\sqrt{16\pi G_{10}}}\int_{S^5} F_5 = \frac{\Omega_5}{\sqrt{16\pi G_{10}}}Q.
\end{aligned}
\tag{1.9}
$$

For the $D3$ brane both charge densities are equal due to the self duality of the five form field strength. The string coupling constant g_s for this geometry can be expressed in terms of the dilaton ϕ by $g_s = e^{\phi}$ and this is essentially a constant. In the above equations, $r = r_0$ is the horizon of this geometry and it has a singularity ar $r = 0$. For generic r_0, the solution breaks all the spacetime supersymmetry. Half of it is recovered when one sends r_0 to zero.

1.1.3 $AdS_5 \times S^5$ from the Near Horizon Limit

Now we consider a set of N coincident $D3$ branes. First, let us notice that if we start with a single $D3$-brane, then we can write the flux (1.9) through S^5 as [8]

$$\mu_3 = T_3 \sqrt{16\pi G_{10}}, \tag{1.10}$$

where the $D3$ brane tension T_3 and Newton's constant G_{10} are given by

$$\begin{aligned} T_3 &= \frac{2\pi}{(2\pi l_s)^4 g_s}, \\ 16\pi G_{10} &= \frac{(2\pi l_s)^8}{2\pi} g_s^2. \end{aligned} \tag{1.11}$$

Now, for the N coincident $D3$ branes, we write $\mu \to N\mu$ in the equation (1.9) with fixed Q. Using the equations (1.10), (1.11), we can re-express Q in terms of N, l_s and g_s as,

$$Q = 16N\pi l_s^4 g_s. \tag{1.12}$$

If we want to restore supersymmetry, we need to chose $r_0 = 0$. Then from (1.7) we obtain,

$$H = 1 + \frac{4N\pi l_s^4 g_s}{r^4} = 1 + \frac{l^4}{r^4}, \tag{1.13}$$

with $l^4 = 4N\pi l_s^4 g_s$. In the near horizon limit, $r \to 0$, the solution approximates to

$$\begin{aligned} H &\sim \frac{l^4}{r^4}, \\ ds^2 &= \frac{r^2}{l^2}\left[-dt^2 + \sum_{i=1}^{3}(dx^i)^2\right] + \frac{l^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right). \end{aligned} \tag{1.14}$$

This is the metric of $AdS_5 \times S^5$. In these coordinates the AdS boundary is at $r \to \infty$.

1.1.4 Matching of Symmetries: Towards a Duality

The AdS space, that we just obtained, is a negatively curved, maximally symmetric solution of the Einstein equation in the presence of a negative cosmological constant. The geometry of $d+1$ dimensional AdS space can be represented by a hyperboloid as

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2, \tag{1.15}$$

with constant R, in flat $d+2$ dimensional space with metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2. \tag{1.16}$$

From this construction, we see that the AdS space has the isometry group of $SO(2,d)$. This isometry group is also the conformal group in d dimensions. The S^5 part that appear in (1.14) has an associated $SO(6)$ symmetry. This group has the same Lie algebra as $SU(4)$. This turns out to be the global internal symmetry group of $\mathcal{N}=4$ SYM (called the R symmetry group). The full symmetry group of $\mathcal{N}=4$ SYM theory is the superconformal group $SU(2,2|4)$. It has maximal bosonic subgroup $SU(2,2)\times SU(4)$. The Lie algebra of $SU(2,2)$ and $SO(2,4)$ are equivalent and, as we just discussed, the $SO(2,4)$ is the isometry group of AdS_5. These observations immediately leads to the fact that the isometry groups of the SYM and the $AdS_5\times S^5$ spacetime are identical - providing a hint towards the AdS/CFT correspondence. Further support comes from the fact that there is a direct mapping of the bulk supergravity states and the single trace operators and their descendants of the above CFT living on the flat boundary of the AdS [6].

These observations along with several others and the general prescription of holography then suggests that the two apparently disjoint theories namely the supergravity theory on AdS_5 and the $\mathcal{N}=4$, $SU(N)$ SYM theory actually carry the same information. This leads to the

AdS/CFT conjecture of Maldacena. In its strong form, it says that the type IIB superstring theory on $AdS_5 \times S^5$ (where both AdS_5 and S^5 have the same radius l, and the 5-form field F_5 has integer flux N through S^5) is dual to the $\mathcal{N} = 4$ super-Yang-Mills theory in 4-dimensions, with gauge group $SU(N)$ once the following identifications between the parameters of both the theories are made:

$$g_s = g_{YM}^2 \qquad\qquad l^4 = 4\pi g_s N l_s^4 \tag{1.17}$$

This equivalence then provides a precise map between the states (and fields) on the superstring side and the local gauge invariant operators on the $\mathcal{N} = 4$ SYM side, as well as a correspondence between the correlators in both theories.

1.1.5 The 't Hooft Limit

The 't Hooft coupling, which works as an effective coupling constant for the $SU(N)$ SYM theory, is defined by $\lambda = g_{YM}^2 N = g_s N$. The 't Hooft limit is defined as $N \to \infty$ keeping λ fixed. In this limit, the string coupling constant $g_s = \frac{\lambda}{N}$ is small and the string theory is weakly coupled. Therefore, on one hand, it allows us to neglect the non-planar diagrams in the gauge theory and, on the other, the supergravity limit of the type IIB string theory becomes trustworthy. This provides us with a correspondence between the supergravity and the large N limit of the gauge theory. However, since large λ limit corresponds to the strongly coupled phase of the gauge theory, this correspondence is dual in nature, namely, between the strongly coupled $SU(N)$ $\mathcal{N} = 4$ SYM theory and the weakly coupled IIB string theory. It is in this limit that controlled computations can be carried out in the bulk. This, in turn, opens up a possibility to explore the properties of the strongly coupled gauge theory. In the next two subsections, for the illustrative purpose, we provide a few standard computations. More details can be found in [7].

1.1.6 The AdS/CFT Dictionary and Recipe For Calculating Boundary Correlation Functions

The gauge/gravity duality that we have just discussed, associates a gauge invariant boundary operator O_ϕ for every bulk field ϕ. The bulk here is the AdS or could be the asymptotically $AdS\,(AAdS)$ spacetime. For the asymptotically AdS spacetime the AdS isometries are broken. However, asymptotically one can recover all the isometries. According to the proposal of [2,3], the bulk fields in $AAdS$ can be identified as the sources coupled to the boundary operator. The relation between these two is that the on-shell bulk partition function is equal to the boundary QFT generating functional:

$$Z_{sugra}\left[\phi_{(0)}\right] = \int_{\phi\sim\phi_{(0)}} D\Phi e^{-S[\phi]} = \langle exp\left(-\int_{\partial AAdS}\phi_{(0)}O\right)\rangle_{QFT} \tag{1.18}$$

where $S\,[\phi]$ is the action of the bulk field ϕ and $\partial AAdS$ is the boundary of the $AAdS$. Here, $\phi_{(0)}$ is the boundary value of the field ϕ. The correlation functions for the boundary theory can be obtained by taking functional derivative of the on-shell bulk action with respect to the field $\phi_{(0)}$ as

$$\begin{aligned}
\langle O(x)\rangle &= \frac{\delta S_{on-shell}}{\delta\phi_{(0)}(x)}\,|_{\phi_{(0)}=0}\\
\langle O(x_1)O(x_2)\rangle &= \frac{\delta^2 S_{on-shell}}{\delta\phi_{(0)}(x_1)\delta\phi_{(0)}(x_2)}\,|_{\phi_{(0)}=0}\\
\langle O(x_1)....O(x_n)\rangle &= \frac{\delta^n S_{on-shell}}{\delta\phi_{(0)}(x_1)....\delta\phi_{(0)}(x_n)}\,|_{\phi_{(0)}=0}
\end{aligned} \tag{1.19}$$

Normally, the QFT correlation functions suffer from divergences and one needs a renormalization procedure to obtain finite correlators. A systematic procedure has been developed to obtain finite correlators [27], known as holographic renormalization.

If the classical action S could depend upon a number of fields, for example, scalar field

ϕ, a vector field A_μ or a tensor field $g_{\mu\nu}$. Each field acts as a source for different boundary operators and symmetry currents. The vector field A_μ couples to the boundary symmetry current J_i, the metric field $g_{\mu\nu}$ couples to the boundary stress-tensor T_{ij}. The corresponding one point function for the boundary field theory can be obtained by these formulas:

$$\begin{aligned}\phi \to \langle O(x)\rangle_s &= \frac{1}{\sqrt{g_{(0)}(x)}}\frac{\delta S_{ren}}{\delta\phi_{(0)}(x)}\\ A_\mu \to \langle J_i(x)\rangle &= \frac{1}{\sqrt{g_{(0)}(x)}}\frac{\delta S_{ren}}{\delta A_{i(0)}(x)}\\ G_{\mu\nu} \to \langle T_{ij}(x)\rangle &= \frac{1}{\sqrt{g_{(0)}(x)}}\frac{\delta S_{ren}}{\delta g_{(0)ij}(x)}\end{aligned} \tag{1.20}$$

where S_{ren} is the renormalized action, $g_{(0)ij}$ is the boundary metric. The above set of formulas provide a proper set up to calculate the boundary field theoretic quantities from its gravity dual in the bulk. The simplest perhaps is the computation of the two point correlators.

1.1.7 Calculation of the Two Point Function

Using the previous prescription, we can calculate the two point correlation function for the boundary gauge theory using its bulk dual. To start with, let us consider the AdS_5 in Euclidean signature. The metric in this signature is:[1]

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \sum_{i=1}^{4} dx_i^2\right). \tag{1.21}$$

The action of a bulk scalar field of mass m is given by

$$S_E = K\int dz d^4x\sqrt{g}\left[g^{zz}(\partial_z\phi)^2 + g^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2\right], \tag{1.22}$$

[1]This AdS metric is related to our previous metric (1.14) by a coordinate transformation $z = \frac{1}{r}$ and the boundary in this case is located at $z = 0$.

where K is a normalization constant. Since the metric (1.21) has translational symmetries along x_i, we can take the following Fourier ansatz for the solution of the bulk field ϕ

$$\phi(z,x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} f_k(z)\phi_0(k). \tag{1.23}$$

The equation of motion for $f_k(z)$ follows from the above action is

$$f_k''(z) - \frac{3}{2} f_k'(z) - \left(k^2 + \frac{m^2R^2}{z^2}\right) f_k(z) = 0. \tag{1.24}$$

Using equation (1.23) and performing the integration with respect to x, the above action (1.22) can be written in the following form:

$$S_E = KR^3 \int_\epsilon^\infty dz \int \frac{d^4kd^4k'}{(2\pi)^4} \frac{\delta(k+k')}{z^3} \left[\partial_z f_k \partial_z f_{k'} - kk' f_k f_k' + \frac{m^2R^2}{z^2} f_k f_k'\right] \phi_0(k)\phi_0(k'). \tag{1.25}$$

Here, ϵ is an ultraviolet cutoff. The equation (1.24) has two linearly independent solutions in terms of modified Bessel functions:

$$\phi_k(z) = Az^2 K_\nu(kz) + Bz^2 I_\nu(kz), \tag{1.26}$$

where $\nu = \sqrt{4 + m^2R^2}$. We introduce $\Delta = \nu + 2$, which is the conformal weight of the operator O. We also demand that the solution is regular at $z = \infty$ and equal to 1 at $z = \epsilon$. These boundary conditions fix the solution:

$$f_k(z) = \frac{z^2 K_\nu(kz)}{\epsilon^2 K_\nu(k\epsilon)}. \tag{1.27}$$

After substituting the equation of motion (1.24) into the action (1.25), the on shell action reduces to the boundary term,

$$S_E = KR^3 \int \frac{d^4k d^4k'}{(2\pi)^8}(2\pi)^4\delta^4(k+k')\phi_0(k)\phi_0(k')\frac{f_k'(z)\partial_z f_k(z)}{z^3}\bigg|_{\epsilon}^{\infty}. \quad (1.28)$$

Functional derivatives with respect to ϕ_0 now gives the two point correlation function in the momentum space:

$$\langle O(k)O(k')\rangle = -KR^3(2\pi)^4\delta^4(k+k')\frac{f_k'(z)\partial_z f_k(z)}{z^3}\bigg|_{\epsilon}^{\infty}. \quad (1.29)$$

Using the explicit solution (1.27) and after some straightforward manipulation, one arrives at the following form of two point correlation function

$$\begin{aligned}\langle O(k)O(k')\rangle &= -KR^3\epsilon^{2(\Delta-4)}(2\pi)^4\delta^4(k+k')k^{2\nu}2^{1-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(\nu)} + ... \\ &= KR^3\epsilon^{2(\Delta-4)}(2\pi)^4\delta^4(k+k')k^{2\Delta-4}\frac{\Gamma(3-\Delta)}{2^{2\Delta-5}\Gamma(\Delta-2)} + ... \end{aligned} \quad (1.30)$$

Here the dots represent the polynomials in $k^2\epsilon^2$ (or vanishing in the limit $\epsilon \to 0$). The position space correlation function can be obtained by taking a Fourier transform of the above momentum space correlation function, which gives:

$$\langle O(x)O(x')\rangle \sim \frac{1}{|x-x'|^{2\Delta}} \quad (1.31)$$

From the above expression we note that the form of the correlation function is same as CFT two point correlation function for the primary operator O of scaling dimension Δ. The translational symmetry, which is a part of the conformal symmetry, restricts the correlator to be a function of $|x-x'|$. The scaling symmetry then dictates the power that appears on $|x-x'|$. Finally, the special conformal transformation tells us that the correlator vanishes between two primary operators of dimension Δ and Δ' unless they are equal.

In this thesis, our focus will be primarily on the boundary geometries which break the conformal symmetry. Consider, for example, $\mathcal{N} = 4$, $SU(N)$ SYM on Kasner metric,

$$ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2p_i} dx_i^2, \ \text{ with } \sum_i p_i = \sum_i p_i^2 = 1. \tag{1.32}$$

The metric has the spatial translational symmetries. It also enjoys an anisotropic scaling symmetry

$$t \to \lambda t, \quad x_i \to \lambda^{1-p_i} x_i, \tag{1.33}$$

in the sense that the metric only gets scaled as $ds^2 \to \lambda^2 ds^2$. If we now, for example, consider a two point equal time correlator of operator with scaling dimension Δ separated along x_1, it will satisfy a Ward identity of the form

$$\Big[2\Delta + t\frac{\partial}{\partial t} + (1-p_1)x_1\frac{\partial}{\partial x_1}\Big]\langle O(t,x_1)O(t,0)\rangle = 0. \tag{1.34}$$

The general solution of this equation can be written in terms of a scaling function as

$$\langle O(t,x_1)O(t,0)\rangle = t^{-2\Delta} f(x_1 t^{p_1-1}). \tag{1.35}$$

Due to the lack of the conformal symmetry, f is in general difficult to determine. In this thesis, among others things, we will make an attempt to compute such scaling functions using the AdS/CFT conjecture.

Time dependent geometries, as above, often bring in many non-trivialities. Typically they contain spacelike singularities. Near these singularities, gravitational fluctuations are expected to become important and hence the metric is expected ot receive corrections. At present, in general, there is no way to obtain these corrections - the Einstein action for gravity or the perturbative corrections around that may no longer be the right framework to work with. Even if we neglect the issue of the quantum fluctuations, studying perturbative field theory theory it self on these backgrounds is plagued with many conceptual difficulties. In the next section, we provide a very brief account of some of these difficulties. It is a well studied subject and has vast literature. We refer the book [9] for the interested readers.

1.2 Some Aspects of Quantum Field Theory on Time Dependent Spacetime

Quantum field theory on a time dependent spacetime is a generalization of the quantum field theory on the Minkowski space. In such curved geometries the field equations, the local entities and the commutation relations are governed by the principle of general covariance and the equivalence principle. Often one encounters situations where some global quantities lose their unique meaning in such spacetimes. For example, the notion of vacuum state, which is Poincare invariant in the Minkowski spacetime, may turn out to be non-unique. In fact, defining a vacuum state itself is troublesome when the metric does not asymptote to flat spacetime in the past or in the future. Further, the notion of particle also becomes ambiguous and observer dependent in curved spacetime. Here, and in the next subsection, we elaborate some of these issues through (i) a toy model of a massless scalar on an isotropically expanding universe and (ii) a massless scalar on the Milne space [11, 32].

We start with the case of the metric describing an expanding universe. The metric has an FRW form

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \tag{1.36}$$

where $a(t)$ is the cosmological scale factor. In the subsequent analysis, we do not assume any particular form of $a(t)$ except that it asymptotically approaches constant values at early and late time t as

$$
\begin{aligned}
a(t) &\to a_1 \quad \text{as } t \to -\infty \\
a(t) &\to a_2 \quad \text{as } t \to \infty.
\end{aligned}
\tag{1.37}
$$

We now consider a minimally coupled, massless scalar field propagating in this background. The equation of motion is:

$$\Box\phi = 0. \tag{1.38}$$

It reduces to the following form in the above background

$$a^{-3}\partial_t(a^3\partial_t\phi) - a^{-2}\sum_i \partial_i^2\phi = 0. \tag{1.39}$$

We now expand the field operator ϕ in the following form:

$$\phi = \sum_{\vec{k}} \{A_{\vec{k}} f_{\vec{k}}(x) + A^{\dagger}_{\vec{k}} f^{*}_{\vec{k}}(x)\}, \tag{1.40}$$

where $f_{\vec{k}} = V^{-\frac{1}{2}} e^{i\vec{k}.\vec{x}} \psi_k(\tau)$. We have discretized the space into lattice points and V is the volume of a cubic lattice. The new time variable τ defined by

$$\tau = \int^t a^{-3}(t')dt'. \tag{1.41}$$

We further normalize the modes $f_{\vec{k}}$ and $f^{*}_{\vec{k}}$ in such a way that they satisfy the following scalar products

$$
\begin{aligned}
(f_{\vec{k}}, f_{\vec{k}'}) &= \delta_{\vec{k},\vec{k}'} \\
(f^*_{\vec{k}}, f^*_{\vec{k}'}) &= -\delta_{\vec{k},\vec{k}'} \\
(f_{\vec{k}}, f^*_{\vec{k}}) &= 0,
\end{aligned}
\tag{1.42}
$$

where

$$
(f_1, f_2) \equiv i \int_{\Sigma_t} d^3x \, \sqrt{g} g^{0\nu} \left[f_1^*(x,t)\partial_\nu f_2(x,t) - \partial_\nu f_1^*(x,t) f_2(x,t) \right]. \tag{1.43}
$$

Here Σ_t is a constant t hypersurface.

Substituting the form of $f_{\vec{k}}$ into equation (1.39), we get the following equation for $\psi(\tau)$

$$
\frac{d^2\psi_k}{d\tau^2} + k^2 a^4 \psi_k(\tau) = 0. \tag{1.44}
$$

In order to solve this equation, we need to impose the boundary condition that at early time $t \to -\infty$, the solution is plane wave with positive frequency. Since at $t \to -\infty$ we have $a(t) \to a_1$, the solution at early time is:

$$
f_{\vec{k}} \sim (Va_1^3)^{-\frac{1}{2}} (2\omega_1)^{-\frac{1}{2}} e^{i(\vec{k}.\vec{x} - \omega_1 t)}. \tag{1.45}
$$

where $\omega_1 = \frac{k}{a_1}$. In τ coordinate therefore

$$
\psi(\tau) \sim (2a_1^3\omega_1)^{-\frac{1}{2}} e^{-i\omega_1 a_1^3 \tau}. \tag{1.46}
$$

The annihilation and creation operators $A_{\vec{k}}$ and $A^\dagger_{\vec{k}}$ of (1.40) satisfy the commutation relations

$$
\begin{aligned}
\left[A_{\vec{k}}, A^\dagger_{\vec{k}}\right] &= \delta_{\vec{k},\vec{k}'}, \\
\left[A_{\vec{k}}, A_{\vec{k}'}\right] &= 0.
\end{aligned}
\tag{1.47}
$$

The operator $A_{\vec{k}}$ is time independent and the above commutation relations are valid for all times. Using this annihilation operator we can define the early time vacuum state

$$A_{\vec{k}}|0\rangle = 0. \tag{1.48}$$

In the far future ($t \to \infty$) the equations (1.39) and (1.44) similarly admit two linearly independent plane wave solutions:

$$\begin{aligned} g_{\vec{k}} &\sim (Va_2^3)^{-\frac{1}{2}}(2\omega_2)^{-\frac{1}{2}}e^{i(\vec{k}.\vec{x}-\omega_2 t)}, \\ g^*_{\vec{k}} &\sim (Va_2^3)^{-\frac{1}{2}}(2\omega_2)^{-\frac{1}{2}}e^{-i(\vec{k}.\vec{x}-\omega_2 t)}, \end{aligned} \tag{1.49}$$

where $\omega_2 = \frac{k}{a_2}$. With respect to these new modes, we can define new creation and annihilation operators

$$\phi = \sum_{\vec{k'}} \left[a_{\vec{k'}} g_{\vec{k'}} + a^\dagger_{\vec{k'}} g^*_{\vec{k'}} \right]. \tag{1.50}$$

The new annihilation operator $a_{\vec{k}}$ defines a new vacuum state $|\bar{0}\rangle$ in the far future

$$a_{\vec{k}}|\bar{0}\rangle = 0. \tag{1.51}$$

Now, as both the sets $\{f_{\vec{k}}, f^*_{\vec{k}}\}$ and $\{g_{\vec{k}}, g^*_{\vec{k}}\}$ are linearly independent and complete, we can write

$$\phi(x) = \sum_{\vec{k}} \{A_{\vec{k}} f_{\vec{k}}(x) + A^\dagger_{\vec{k}} f^*_{\vec{k}}(x)\} = \sum_{\vec{k'}} \{a_{\vec{k'}} g_{\vec{k'}}(x) + a^\dagger_{\vec{k'}} g^*_{\vec{k'}}(x)\}. \tag{1.52}$$

Taking the scalar product with respect to $g_{\vec{l}}$ and $f_{\vec{l}}$, and using the relations (1.42) and (1.43), we obtain the following equations

$$a_{\vec{l}} = \sum_{\vec{k}} \left[\alpha^*_{kl} A_{\vec{k}} - \beta^*_{kl} A^\dagger_{\vec{k}}\right],$$
$$A_{\vec{l}} = \sum_{\vec{k}} \left[\alpha_{kl} a_{\vec{k}} + \beta^*_{kl} a^\dagger_{\vec{k}}\right], \tag{1.53}$$

where $\alpha_{kl} = (g_{\vec{k}}, f_{\vec{l}})$ and $\beta_{kl} = -(g_{\vec{k}}, f^*_{\vec{l}})$. The above relations are known as Bogolubov transformations and it relates the early time creation and annihilation operators with the late time creation and annihilation operators. The above equations also impose constraints on the Bogolubov coefficients α_{kl} and β_{kl}

$$\sum_m (\alpha_{lm}\alpha^*_{km} - \beta_{lm}\beta^*_{km}) = \delta_{lk}$$
$$\sum_m (\alpha_{lm}\beta_{km} - \beta_{lm}\alpha_{km}) = 0. \tag{1.54}$$

Using the above relations one can show that the late time annihilation and creation operators $a_{\vec{l}}$ and $a^\dagger_{\vec{l}}$ respectively satisfy the same commutation relation as eq.(1.47).

The number operator for particle is $N = a^\dagger_{\vec{k}} a_{\vec{k}}$. If initially there are no particles, i.e, if we start with the vacuum state $|0\rangle$, then at late time we have:

$$\langle N_{\vec{k}} \rangle_{t\to\infty} = \langle 0| a^\dagger_{\vec{k}} a_{\vec{k}} |0\rangle = \sum_l |\beta_{kl}|^2. \tag{1.55}$$

To derive this we have used equations(1.47) and (1.53). The above equation implies that there is a possibility of creation of particles at late time due to the expansion of the universe. Here we note that, in obtaining the above formulas, we have assumed that the spacetime has asymptotic flat regions at far past and at far future so that the mode decomposition of the field had the usual Minkowski field theory interpretation and the notion of particles are therefore well defined in these regions. During the period of expansion, the simple plane wave decomposition of the field operator is not possible and the notion of particle is not operationally well defined. However, in some cases, especially for slowly varying $a(t)$ an approximate vacuum can be defined. The adiabatic approximation is one such approximation scheme. For a detailed discussion we refer [9] to the readers. We note, in passing, that unlike this example, if we do not have asymptotically flat regions, even defining the "in" and the "out" vacua becomes difficult.

Non-trivialities also appear when the spacetime (on which one quantizes a field) covers only a part of the Minkowski spacetime. Two simple examples of such spacetimes are provided by the Rindler space and the Milne space. In the next subsection, we discuss the quantization of a massless scalar on a four dimensional Milne spacetime. This selection is made considering its relevance on what would follow in later chapters.

1.2.1 Field Theory on Milne Spacetime

The Milne universe in four dimensions is a special case of Kasner spacetime (1.32) for which the Kasner exponents p_is are $(1, 0, 0)$. It is a flat hyperbolic spacetime in an unusual coordinate system. In this spacetime, the free scalar wave equation can be solved exactly and the corresponding field theory can be studied without any approximation. Here we briefly discuss this field theory and for simplicity we consider massless scalar field in this spacetime. The line element of the four dimensional Milne universe is

$$ds^2 = -dt^2 + t^2(dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{1.56}$$

After the following coordinate transformations,

$$y^0 = t \cosh x^1, \quad y^1 = t \sinh x^1, \tag{1.57}$$

the above metric becomes Minkowski spacetime

$$ds^2 = -(dy^0)^2 + (dy^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{1.58}$$

The range of the coordinates y^0 and y^1 are however $0 < y^0 < \infty$ and $-\infty < y^1 < \infty$. It shows that the Milne universe covers only a patch of the Minkowski spacetime and is very similar to the Rindler spacetime. Now we consider a free scalar field in this background which satisfies the following equation

$$\Box\phi = 0.$$

To solve this equation we take

$$\phi(t, \vec{x}) = \int d^3k \left[a_k f_{\vec{k}} + a_k^\dagger f_{\vec{k}}^*\right], \tag{1.59}$$

with $f_{\vec{k}} = (2\pi)^{-\frac{3}{2}} e^{i\vec{k}.\vec{x}} \psi_k(t)$. Substituting the above form into the wave equation, we obtain a differential equation for ψ as

$$t^2\psi_k''(t) + t\psi_k'(t) + \left(k_1^2 + (k_2^2 + k_3^2)t^2\right)\psi_k(t) = 0. \tag{1.60}$$

This differential equation has two sets of linearly independent solutions, (ψ_k, ψ_k^*) and $(\bar{\psi}_k, \bar{\psi}_k{}^*)$

$$\begin{aligned}\psi_k &= \frac{\sqrt{\pi}}{2} e^{\frac{\pi k_1}{2}} H_{ik_1}^{(2)}(\omega t)\,,\\ \bar{\psi}_k &= \left(\frac{2}{\pi}\sinh(\pi k_1)\right)^{-\frac{1}{2}} J_{-ik_1}(\omega t)\,,\end{aligned} \tag{1.61}$$

where $\omega = \sqrt{k_2^2 + k_3^2}$.

Here we have determined the normalization factors for the above solutions from the following condition of the Wronskian:

$$tW = t\left(\psi_k^* \partial_t \psi_k - \psi_k \partial_t \psi_k^*\right) = i = t\left(\bar{\psi_k}^* \partial_t \bar{\psi_k} - \bar{\psi_k} \partial_t \bar{\psi_k}^*\right),$$

which follows from eqns.(1.42)and (1.43).

The above set of modes are related by Bogolubov transformation:

$$\bar{\psi_k} = \alpha_k \psi_k + \beta_k \psi_k^*. \tag{1.62}$$

From the properties of Bessel and Hankel functions one can find the Bogolubov coefficients α_k and β_k:

$$\begin{aligned} \alpha_k &= \left[\frac{e^{\pi k_1}}{2\sinh(\pi k_1)}\right]^{\frac{1}{2}}, \\ \beta_k &= \left[\frac{e^{-\pi k_1}}{2\sinh(\pi k_1)}\right]^{\frac{1}{2}}. \end{aligned} \tag{1.63}$$

Here the two modes ψ_k and $\bar{\psi_k}$ defines two inequivalent vacua $|0\rangle$ and $|\bar{0}\rangle$ respectively. The first vacuum is the usual Minkowski vacuum and to see this, we use the following integral representation of the Hankel function [38]

$$H^{(2)}_{ik_1}(\omega t) = i\frac{e^{-\frac{k_1\pi}{2}}}{\pi}\int_{-\infty}^{\infty} e^{-i\omega t\cosh\rho' - ik_1\rho'} d\rho'. \tag{1.64}$$

Changing the integration variable to $\rho = \rho' - x^1$ and using the eq.(1.57) we can write the above integral as

$$H^{(2)}_{ik_1}(\omega t) = i\frac{e^{-\frac{k_1\pi}{2}}}{\pi}\int_{-\infty}^{\infty} d\rho e^{-i\omega\left[y^0\cosh\rho + y^1\sinh\rho\right]} e^{-ik_1\rho} e^{-ik_1 x^1}. \tag{1.65}$$

Now, defining $p(\rho) = -\omega\sinh\rho$ and $\omega_p(\rho) = \omega\cosh\rho = \sqrt{\omega^2 + p^2}$, the above equation can be written in the following form

$$H^{(2)}_{ik_1}(\omega t) = i\frac{e^{-\frac{k_1\pi}{2}}}{\pi}\int_{-\infty}^{\infty} d\rho e^{-i\left[y^0\omega_p - y^1 p\right]} e^{-ik_1\rho} e^{-ik_1 x^1}. \quad (1.66)$$

Using this form, we can write the mode function $f_{\vec{k}} = (2\pi)^{-\frac{3}{2}} e^{i\vec{k}.\vec{x}}\psi_k(t)$ associated with this mode as

$$f_{\vec{k}} = i\frac{e^{-\frac{k_1\pi}{2}}}{\pi}\int_{-\infty}^{\infty} d\rho e^{i\left(k_2 x^2 + k_3 x^3 + y^1 p(\rho)\right)} e^{-i\omega_p(\rho) y^0} e^{-ik_1\rho}. \quad (1.67)$$

From the above expression we see that $f_{\vec{k}}$ is superposition of the positive frequency modes with respect to the Minkowski time y^0 and the corresponding vacuum is the Minkowski vacuum. It was shown in [11] that the other vacuum $|\bar{0}\rangle$ corresponds to the Rindler vacuum in the right Rindler wedge, which is inequivalent to Minkowski vacuum. Further, the non vanishing Bogolubov coefficients (1.63) imply that if an inertial detector registers no particles in the $|0\rangle$ vacuum then it must register particles in the $|\bar{0}\rangle$ vacuum.

Later in the thesis, we will study the strongly coupled gauge theory on this Milne metric using the AdS/CFT techniques. Owing to the simplicity of this geometry, some computations can be performed exactly.

1.3 Plan of The Thesis

After this brief review, we now present the plan of the rest of the thesis. In the next chapter we discuss about a class of time dependent geometries which possesses AdS duals. These include the AdS-Kasner spacetime, the AdS-Kasner Soliton and the AdS-de Sitter. These are the solutions of the Einstein equations with a negative cosmological constant. In order to identify the gauge theory on the boundary of some of these geometries, we focus on their supergravity origin from the ten and the eleven dimensional perspective. Indeed we find that the AdS-Kasner and AdS-Kasner soliton arise from the near-brane limit of the $D3$ and the

$M5$ branes respectively. Subsequently, we show that the vacuum AdS-Kasner spacetime has a non-vacuum extension. These are, as we will discuss, sourced by extra matter represented by a perfect fluid stress tensor. This additional matter modifies one of Kasner conditions and allows us even to construct geometries where all the directions expanding anisotropically. This chapter ends with a discussion on higher dimensional flat space embedding of all the above time dependent backgrounds.

In chapter 3, we aim to compute the gauge theory correlators, holographically, on the Kasner class of geometries. To illustrate the conceptual complications with the time dependent geometries, we first start with the Milne spacetime where exact computations can be performed. Subsequently, we emphasis the difficulties in performing exact computations for the other Kasner class of geometries. We then resort to the available approximation methods. The one that turned out to be particularly useful in this context is known as the *geodesic approximation*. It allows us to compute two point spacelike separated correlators involving scalar operators of large scaling dimensions in the gauge theory. As these geometries contain curvature singularities at $t = 0$, we then proceed to examine the signatures of these singularites on the correlators. Regardless of any particular background within this class, a generic result that appears is the following: if two points are separated along an expanding direction, the correlators are smooth even if we take the spacelike surface close to the singularity. An analysis of this behaviour along with a comparison with the other existing results in the literature is provided at the end of this chapter.

The next chapter deals with a space dependent version of the AdS-Kasner geometry. The space dependent Kasner geometry is both inhomogeneous and anisotropic. Like the time dependent AdS-Kasner, this spacetime can also be obtained from the near horizon limit of a $D3$ brane with a space dependent five-form field. This geometry has a timelike singularity. Our primary aim here is to study the dual gauge theory near this singularity in the same spirit as in the previous chapter. We find that, in contrast to the time dependent AdS-Kasner, here the boundary correlators are smooth near the singularity for any (positive or negative) Kasner exponents. The chapter ends with an analysis of our results.

Finally, we conclude this thesis with a discussion on the possible avenues in which our investigations could be further extended.

Chapter 2

Time Dependent Geometries and Their Duals

Time dependent spacetimes typically contain naked singularities and, since these singularities are marked by strong gravitational fluctuations, available classical frameworks to study gravitational physics are expected to be inadequate near these singularities. Though we are yet to have a completely satisfying theory of quantum gravity, AdS/CFT correspondence offers us with an indirect way to address such situations. Following our discussions in the previous chapter, we now know that it relates a gravitational theory to a gauge theory in one lower dimension. Furthermore, we know that this correspondence is dual in nature. Namely, it relates a strongly coupled gravitational theory to a weakly coupled gauge theory. Could we then have a possibility to study time dependent backgrounds in terms of, much controlled and weakly coupled, dual degrees of freedom? Indeed, in the past in several works, this question was addressed with different degrees of success. See for example [14], [20], [28], [29], [33], [34], [35], [36], [52]-[63] for an incomplete set of references.

In order to proceed, one first requires to construct the cosmological backgrounds which permit *AdS duals*. Next, to identify the gauge theory in question, it is important to find ways to embed these dual time dependent AdS backgrounds in ten or eleven dimensional supergravity theories. This is our primary concern in the present chapter.

One of the simplest backgrounds that offers AdS dual is the four dimensional Kasner spacetime. It is an anisotropic geometry in which two of the directions expand and the other shrinks with time. It further has a naked singularity in the past. Its five-dimensional dual, which we call AdS-Kasner, arises from a near-brane limit of a time dependent $D3$ brane of ten dimensional IIB supergravity. Therefore, the gauge theory that lives on the Kasner boundary is the usual $\mathcal{N} = 4$, $SU(N)$ SYM – however in a state that breaks all the supersymmetries. In this context, we also describe another time dependent background. This is called the time dependent AdS-Kasner soliton which arises from the $M5$ brane of the eleven dimensional supergravity. The advantage of constructing the solitonic generalization is that it contains a compact circle. Tuning the periodicity of this circle, it is possible to avoid the horizon singularity of the AdS-Kasner. We will discuss this in detail later in the chapter. We also present the five-dimensional duals of a large class of time dependent backgrounds by turning on appropriate matter fields. We refer them as the non-vacuum solutions. Unfortunately, except for a few cases, we do not have the brane descriptions of these geometries. Consequently the gauge theories still remain illusive. Finally, in the last part of this chapter, we embed all the above backgrounds in appropriate higher-dimensional *flat* geometries. Though we do not use these constructions in later chapters, we believe that the constructions will be useful in the future.

2.1 Kasner, AdS-Kasner, AdS-Kasner Soliton Geometries

The Kasner spacetime is an exact solution of the vacuum Einstein equation

$$R_{\mu\nu} = 0, \tag{2.1}$$

with the following metric

$$ds^2 = -dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2p_3}dx_3^2. \tag{2.2}$$

The three exponents p_1, p_2, p_3 are known as Kasner exponents and they satisfy the following relations:

$$\begin{aligned} p_1 + p_2 + p_3 &= 1, \\ p_1^2 + p_2^2 + p_3^2 &= 1. \end{aligned} \tag{2.3}$$

These are known as the Kasner relations. The exponents p_1, p_2, p_3 can be parametrized by a real variable $u \geq 1$:

$$\begin{aligned} p_1 &= \frac{-u}{1+u+u^2}, \\ p_2 &= \frac{1+u}{1+u+u^2}, \\ p_3 &= \frac{u(1+u)}{1+u+u^2}, \end{aligned} \tag{2.4}$$

which is called the Lifshitz-Khalatnikov parametrization. The above relations imply that except for the two sets of Kasner exponents $(1, 0, 0)$ and $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, all other sets of exponents contain one negative number and two positive numbers which are all different. The Kasner metric belongs to the homogeneous Bianchi type-I classification of the metric. For this anisotropic geometry, the linear distances along two of the axes increase, while the distance decreases along the third. At $t = 0$, this metric has a curvature singularity.

The AdS space is a maximally symmetric solution, and as we discussed previously, it solves the Einstein equations with negative cosmological constant Λ, namely,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2.5}$$

In the Poincare patch, the metric of a five dimensional AdS space can be written as

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right), \tag{2.6}$$

where $\eta_{\mu\nu}$ is the four dimensional Minkowski metric. Equations (2.5) and (2.6) possess a property that if we replace the Minkowski metric by any Ricci flat metric then the resultant metric is still a solution of (2.5). This allows us to construct different non equivalent asymptotically locally AdS spacetimes. As Kasner spacetime is Ricci flat, the AdS-Kasner metric,

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2p_3}dx_3^2 + dz^2\right), \tag{2.7}$$

also satisfies (2.5). The above equation also implies that the boundary of the AdS ($z = 0$) is the four dimensional Kasner spacetime.

The metric enjoys the translational symmetries along x^is and an anisotropic scaling symmetry of the coordinates,

$$z \to \lambda z, \quad t \to \lambda t, \quad x_i \to \lambda^{(1-p_i)}x_i \; (i = 1, 2, 3), \tag{2.8}$$

for some constant λ.

Besides the curvature singularity at $t = 0$, the AdS-Kasner geometry also suffers from the horizon singularity at $z = \infty$. This singularity can be removed from the geometry by adding an extra compact dimension to the spacetime, giving us the AdS-Kasner soliton. Before we present the AdS-Kasner soliton metric, we first briefly review the static AdS soliton geometry.

The AdS soliton geometry can be obtained by a double analytic continuation of the near horizon geometry of a p brane [40, 41]. The near horizon geometry of a p brane is

$$ds^2 = \frac{r^2}{l^2}\left[-\left(1 - \frac{r_0^{p+1}}{r^{p+1}}\right)dt^2 + \sum_{i=1}^{p-1}(dx^i)^2 + (dx^p)^2\right] + \left(1 - \frac{r_0^{p+1}}{r^{p+1}}\right)^{-1}\frac{l^2}{r^2}dr^2. \tag{2.9}$$

If we analytically continue this metric with both $t \to i\tau$ and $x^p \to it$ then we obtain,

$$ds^2 = \frac{r^2}{l^2}\left[\left(1 - \frac{r_0^{p+1}}{r^{p+1}}\right)d\tau^2 + \sum_{i=1}^{p-1}(dx^i)^2 - dt^2\right] + \left(1 - \frac{r_0^{p+1}}{r^{p+1}}\right)^{-1}\frac{l^2}{r^2}dr^2. \tag{2.10}$$

We refer this metric as AdS soliton. The τ coordinate is periodically identified with period $\beta = \frac{4\pi l^2}{(p+1)r_0}$ to avoid the conical singularity at $r = r_0$. This metric satisfies (2.5) and it is the lowest energy(negative) solution for asymptotically locally AdS spacetimes. This metric is used in AdS/CFT to study the confinement in dual gauge theory [40]. The boundary gauge theory for the AdS soliton is nonsupersymmetric $SU(N)$ Yang-Mills theory. The breaking of supersymmetry occurs here due to the antiperiodic boundary conditions for the fermions around the τ circle. Due to the total negative energy of the supergravity solution, the dual gauge theory of the AdS soliton has a negative Casimir energy [41] which can be verified from the gauge theory calculation.[1].

The AdS-Kasner soliton is obtained by replacing the Minkowski part of the world volume of the above AdS soliton metric by Kasner like metric. In Poincare coordinate $z = \frac{1}{r}$ the AdS-Kasner soliton takes the following form

$$ds^2 = \frac{1}{z^2}\left[-dt^2 + \frac{1}{f(z)}dz^2 + \sum_{i=1}^{n-1} t^{2p_i}dx_i^2 + f(z)d\theta^2\right]. \tag{2.11}$$

Here p_is satisfy Kasner conditions (2.3) and $f(z) = 1 - \frac{z^{n+1}}{z_0^{n+1}}$. This metric still satisfies (2.5) and it smoothly ends at $z = z_0$, before the singular Poincare horizon, because of the periodic identification of θ. As the time translation symmetry is broken, it is not possible to define the

[1]The standard definition of the energy for a spacetime which has a time translation symmetry is $E = -\frac{1}{8\pi G}\int N(K - K_0)$, where N is the norm of the timelike killing vector, K is the trace of the extrinsic curvature of a spacelike surface(const t) for which $r = const$ and large, and K_0 is the trace of the extrinsic curvature of a surface with the same intrinsic geometry in the background(for soliton it is simply $r_0 = 0$). This calculation yields a negative energy density of the gauge theory from bulk supergravity. The weak coupling calculation of the gauge theory yields a result which mismatches with the supergravity calculation by a factor of $\frac{3}{4}$. Perhaps this mismatch occurs due to the fact that the boundary gauge theory under consideration is a strongly coupled theory [41].

total energy of the AdS-Kasner soliton.

The asymptotically locally AdS solutions that we discussed above can be realized in ten and eleven dimensional supergravities. In the next section we review how to obtain the above solutions from the near horizon geometry of $D3$ and $M5$ branes.

2.2 Supergravity Origin Of AdS-Kasner and AdS-Kasner Soliton Geometry

2.2.1 AdS-Kasner From $D3$ Brane

We first review the time dependent solutions from 10D type IIB supergravity. We primarily concentrate on the Bosonic part of the theory, since for generic time dependent solutions supersymmetry is explicitly broken. The equations of motion following from the relevant part of standard IIB supergravity action[2]

$$S_{IIB} = -\frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{1}{2\times 5!}F_5^2 \right) . \tag{2.12}$$

has the forms:

$$\begin{aligned} R^{\mu}_{\nu} &= \frac{1}{2}\partial^{\mu}\phi\partial_{\nu}\phi + \frac{1}{2\times 5!}(5F^{\mu\xi_2\ldots\xi_5}F_{\nu\xi_2\ldots\xi_5} - \frac{1}{2}\delta^{\mu}_{\nu}F_5^2), \\ \partial_{\mu}(\sqrt{g}F^{\mu\xi_2\ldots\xi_5}) &= 0, \\ \nabla^2\phi &= 0. \end{aligned} \tag{2.13}$$

It was shown in [13] that these equations are solved by the following metric and gauge

[2]We impose the self-duality condition of the 5-form field strength at the level of equation of motion.

field configuration:

$$
\begin{aligned}
ds^2 &= \Big(1+\frac{l^4}{r^4}\Big)^{-\frac{1}{2}}\Big[-dt^2+\sum_{i=1}^{3} t^{2p_i}dx_i^2\Big]+\Big(1+\frac{l^4}{r^4}\Big)^{\frac{1}{2}}\Big[dr^2+r^2 d\Omega_5^2\Big],\\
F_{tx_1x_2x_3r} &= \frac{4l^4 t^{p_1+p_2+p_3} r^3}{(l^4+r^4)^2}, \qquad F_{ijklm} = \sqrt{-g}\,\epsilon_{tx_1x_2x_3rijklm}F^{tx_1x_2x_3r}\\
\phi &= 0,
\end{aligned}
\tag{2.14}
$$

provided the exponents, p_i, satisfy Kasner conditions (2.3).

In the near-brane limit, $r \to 0$, the metric reduces to

$$
ds^2 = -\frac{r^2}{l^2}dt^2+\frac{l^2}{r^2}dr^2+r^2(t^{2\alpha}dx^2+t^{2\beta}dy^2+t^{2\gamma}dz^2)+l^2 d\Omega_5^2, \tag{2.15}
$$

with

$$
F_{txyzr} = \frac{4tr^3}{l^4}, \ \ \text{giving potential } C_{txyz} = \frac{tr^4}{l^4}. \tag{2.16}
$$

Besides the $d\Omega_5^2$ part, this is the AdS-Kasner solution[3].

Kasner solutions sourced by scalar fields can also be realized, likewise, from the same supergravity set-up [14]. The scalar field profile, however, in this case gets an interpretation of the stiff matter on the brane configuration in question.

2.2.2 AdS-Kasner Soliton From $M5$ Brane

AdS-Kasener soliton similarly follows from eleven dimensional supergravity. To see this, let us note that the $M5$ brane with time dependent worldvolume a solution of the (bosonic sector

[3]This form is related to the form given in (2.7) by a coordinate transformation, $r = \frac{1}{z}$

of) $D = 11$ supergravity with time dependent gauge fields as

$$ds^2 = \left(1+\frac{l^3}{r^3}\right)^{-\frac{1}{3}}\left[-dt^2+\sum_{i=1}^{5} t^{2p_i}dx_i^2\right] + \left(1+\frac{l^3}{r^3}\right)^{\frac{2}{3}}\left[dr^2+r^2 d\Omega_4^2\right], \tag{2.17}$$

along with

$$F_{tx_1x_2x_3x_4x_5r} = \frac{3\ l^3\ t\ r^2}{(l^3+r^3)^2}, \tag{2.18}$$

where the exponents, p_is, again satisfy similar Kasner conditions (2.3), namely

$$\sum_{i=1}^{5} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{5} p_i^2 = 1. \tag{2.19}$$

In the near-brane limit, i.e. $r \to 0$, the metric and the non-zero component of the form field reduce to the forms :

$$\begin{aligned} ds^2 &= \frac{r}{l}\left[-dt^2+t^{2\alpha_1}dx_1^2+t^{2\alpha_2}dx_2^2+t^{2\alpha_3}dx_3^2+t^{2\alpha_4}dx_4^2+t^{2\alpha_5}dx_5^2\right] \\ &+ \frac{l^2}{r^2}\left[dr^2+r^2d\Omega_4^2\right], \\ F_{tx_1x_2x_3x_4x_5r} &= \frac{3\ t\ r^2}{l^3}, \end{aligned} \tag{2.20}$$

Through a change of coordinate,

$$w^2 = \frac{r}{l^3}. \tag{2.21}$$

the metric in (2.20) further takes the form :

$$ds^2 = \frac{w^2}{4l^2}\left(-d\bar{t}^2+\bar{t}^{2\alpha_1}d\bar{x}_1^2+\bar{t}^{2\alpha_2}d\bar{x}_2^2+\bar{t}^{2\alpha_3}d\bar{x}_3^2+\bar{t}^{2\alpha_4}d\bar{x}_4^2+\bar{t}^{2\alpha_5}d\bar{x}_5^2\right)+4\ l^2\frac{dw^2}{w^2}+l^2d\Omega_4^2, \tag{2.22}$$

where $\bar{x}_i$ and $\bar{t}$ are suitably scaled versions of the coordinates, x_i and t respectively. This space we call $KAdS_7 \times S^4$.

There also exists another negative energy solution of the same supergravity. At the level of solutions, such negative energy solutions are obtained through a double analytic continuation of the time and the p-th worldvolume coordinate of a non-extremal p-brane solution (which we have already discussed in the previous section)

$$t \to i\theta, \qquad x^p \to it. \tag{2.23}$$

In the near horizon limit, one gets the so called AdS solitons which are energetically favoured and hence a more suitable candidate to study the boundary gauge theory. Now, t being the time coordinate and θ, a periodic angular coordinate, this double analytic continuation amounts to changing the asymptotic topology R^p of the parent p-brane configuration to $R^{p-1} \times S^1$. Next we look in detail the case in 11-D supergravity when the AdS solitons have time dependent world-volume.

The generic action for the Bosonic part of $d = 11$ supergravity is

$$S_{11d} = -\frac{1}{2\,\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{48}\, F_4^2 \right) , \tag{2.24}$$

The equations of motion arising from (2.24) admits the solitonic solution:

$$\begin{aligned} ds^2 &= \left(1 + \frac{l^3}{r^3}\right)^{-\frac{1}{3}} \left[-dt^2 + \sum_{i=1}^{4} t^{2p_i} dx_i^2 + \gamma(r)\, d\theta^2 \right] \\ &+ \left(1 + \frac{l^3}{r^3}\right)^{\frac{2}{3}} \left[\frac{1}{\gamma(r)} dr^2 + r^2 d\Omega_4^2 \right], \end{aligned} \tag{2.25}$$

where $\gamma(r) = 1 - \frac{r_0^3}{r^3}$, and the gauge field is given by

$$F_{\theta x_1 x_2 x_3 x_4 t r} = \frac{3\sqrt{l^3 + r_0^3}\, l^{\frac{3}{2}}\ t\ r^2}{(l^3 + r^3)^2}, \tag{2.26}$$

iff the exponents, p_i's satisfy Kasner condition, namely

$$\sum_{i=1}^{4} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{4} p_i^2 = 1. \tag{2.27}$$

We call this solution a M5-soliton.

In near horizon limit, the M5-Kasner soliton solution takes the form

$$\begin{aligned} ds^2 &= \frac{r}{l}\left[-dt^2 + \sum_{i=1}^{4} t^{2p_i} dx_i^2 + \gamma(r) d\theta^2\right] + \frac{l^2}{r^2}\left[\frac{1}{\gamma(r)} dr^2 + r^2 d\Omega_4^2\right], \\ & F_{t x_1 x_2 x_3 x_4 x_5 r} = \frac{3\ t\ r^2}{l^3}. \end{aligned} \tag{2.28}$$

In the coordinates defined in (2.21) metric takes the form

$$ds^2 = \frac{w^2}{4l^2}\left[-d\bar{t}^2 + \sum_{i=1}^{4} \bar{t}^{2p_i} d\bar{x}_i^2 + \left(1 - \frac{w_0^6}{w^6}\right) d\bar{\theta}^2\right] + 4\, l^2 \left(1 - \frac{w_0^6}{w^6}\right)^{-1} \frac{dw^2}{w^2} + l^2 d\Omega_4^2, \tag{2.29}$$

where $w_0 = \frac{r_0}{l^3}$. $\bar{x}_i$ and $\bar{t}$ are suitably scaled versions of the coordinates, x_i and t respectively. Note here, additionally, θ is also rescaled to $\bar{\theta}$ and hence the period of $\bar{\theta}$ has to be adjusted accordingly.

Employing the coordinate transformation $\omega = \frac{4l^2}{z}$, the AdS_7 part of the metric reduces to the familiar form

$$ds^2 = \frac{1}{z^2}\left[-d\bar{t}^2 + \sum_{p_i=1}^{4} \bar{t}^{2\alpha_1} d\bar{x}_i^2 + \left(1 - \frac{z^6}{z_0^6}\right) d\bar{\theta}^2 + \left(1 - \frac{z^6}{z_0^6}\right)^{-1} dz^2\right], \tag{2.30}$$

which is a seven dimensional AdS-Kasner soliton. In the next chapter we will require this background for our study.

2.3 Other Time Dependent Duals in the Presence of Matter

If we turn on other matter fields in the five-dimensional spacetime, it is possible to construct cosmological metric with perfect fluids which possess time dependent bulk duals. For such construction, we take help from [20]. We start with the Poincare AdS (2.6) in D dimensions (where $\mu,\nu = 0,1,..D-2$) which is a solution of the vacuum Einstein equation (2.5) with negative cosmological constant $\Lambda = \frac{-(D-1)(D-2)}{2}$. We now replace the Minkowski part of this metric by a $(D-1)$-dimensional non-vacuum metric $\gamma_{\mu\nu}$ which is sourced by a matter stress tensor $\hat{T}_{\mu\nu}$, namely

$$\hat{G}_{\mu\nu} = 8\pi\hat{T}_{\mu\nu}, \tag{2.31}$$

where, $\hat{G}_{\mu\nu}$ is the $(D-1)$-dimensional Einstein tensor for the metric $\gamma_{\mu\nu}$. The authors of [20] showed that the full D-dimensional non-vacuum metric is a solution of the following equation:

$$R_{ab} - \frac{1}{2}g_{ab}R = -\Lambda g_{ab} + T_{ab}, \tag{2.32}$$

provided the followings are satisfied

$$\begin{aligned} T_{\mu\nu} &= \hat{T}_{\mu\nu}, \\ T_{zz} &= \frac{1}{z^2}\frac{T}{2}, \text{ with } T = z^2\gamma_{\mu\nu}T^{\mu\nu}. \end{aligned} \tag{2.33}$$

The above equations of motion restrict the types of matter stress tensors. While passing, we note that a massless scalar field or a perfect fluid type stress tensor can be used as a source for the above construction. We can, for example, construct AdS-FRW metric by taking the source as the massless scalar field ϕ, for which

$$T_{ab} = \frac{1}{8\pi}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}g^{cd}\nabla_c\phi\nabla_d\phi\right), \qquad \nabla^2\phi = 0, \tag{2.34}$$

Choosing for $D = 5$, $\phi = \phi(t)$, and $\gamma_{\mu\nu}$ as

$$d\hat{s}^2 = \gamma_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)((dx^1)^2 + (dx^2)^2 + (dx^3)^2), \tag{2.35}$$

we get from the equations of motion,

$$a^2(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \qquad \phi = -\sqrt{\frac{2}{3}}\ln\left(\frac{t}{t_0}\right), \tag{2.36}$$

and the AdS-FRW metric takes the form

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + \left(\frac{t}{t_0}\right)^{\frac{2}{3}}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right) + dz^2\right). \tag{2.37}$$

Here, we note that the above stress tensor for the massless scalar field is a perfect fluid obeying the stiff matter equation of state $p = \rho$, where p is the pressure and ρ is the energy density.

We next discuss the non-vacuum extension of the five dimensional AdS-Kasner spacetime (2.7). The matter stress tensor for this case is the perfect fluid stress tensor

$$\hat{T}_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\gamma_{\mu\nu}. \tag{2.38}$$

U_μ is the four velocity of the fluid and p, ρ are respectively the pressure and energy density of the fluid with the equation of state $p = w\rho$. For such a system, the conservation of stress tensor and the Einstein equations give

$$\rho = \frac{\rho_0}{t^2}, \tag{2.39}$$

(for some constant ρ_0; the weak energy condition would hold if $\rho_0 > 0$.) and

$$\begin{aligned}
&p_1 p_2 + p_1 p_3 + p_2 p_3 = \rho_0, \\
&p_1 - p_1^2 + p_2 - p_1 p_2 - p_2^2 = \omega\rho_0, \\
&p_2 - p_2^2 + p_3 - p_2 p_3 - p_3^2 = \omega\rho_0, \\
&p_3 - p_3^2 + p_1 - p_1 p_3 - p_1^2 = \omega\rho_0,
\end{aligned} \tag{2.40}$$

respectively. To analyze the solutions, we introduce two parameters S and Q such that

$$\begin{aligned}
&p_1 + p_2 + p_3 = S, \\
&p_1^2 + p_2^2 + p_3^2 = Q.
\end{aligned} \tag{2.41}$$

In what follows, we review few possibilities that would be of our interest later.

When $S = Q = 1$ or equivalently, when $p = \rho = 0$, we have the anisotropic Kasner metric with p_is given in (2.4). With time, the metric expands in two directions and contracts along one.

For $\omega = 1$, pressure and energy density satisfy $p = \rho$. This leads to

$$S = 1, \quad Q = 1 - 2\rho_0. \tag{2.42}$$

This implies that one of the Kasner condition gets modified due to the non-vacuum effect. Now, for a range of ρ_0, universe can have anisotropic non-contracting behaviour in all the directions. To see this, we first express p_2 and p_3 in terms of p_1 are therefore,

$$p_2 = \frac{1}{2}(1 - p_1 - \sqrt{1 + 2p_1 - 3p_1^2 - 4\rho_0})$$
$$p_3 = \frac{1}{2}(1 - p_1 + \sqrt{1 + 2p_1 - 3p_1^2 - 4\rho_0}). \tag{2.43}$$

Now, for given $p_1 < 1$, for p_2, p_3 to be real

$$1 + 2p_1 - 3p_1^2 - 4\rho_0 \geq 0. \tag{2.44}$$

Or, in turn,

$$\rho_0 \leq \frac{1}{4}(1 + 2p_1 - 3p_1^2). \tag{2.45}$$

For $0 < p_1 < 1$, the expression on the right varies from $1/2$ to 0 picking at $p_1 = 1/3$ giving $\rho_0 = 1/3$. So, it is always possible to find a positive p_1 if $\rho_0 \leq 1/3$. Now, for p_2 to be positive, we need

$$1 - p_1 > \sqrt{1 + 2p_1 - 3p_1^2 - 4\rho_0}. \tag{2.46}$$

Or, in other words,

$$\rho_0 > p_1(1 - p_1). \tag{2.47}$$

The expression on the right varies between 0 and $1/2$ peaking at $p_1 = 1/2$. Therefore, we see that if $\rho_0 < 1/3$, it is always possible to find p_1, such that p_2 is positive. Now, since $p_3 > p_2$, we see that, for ρ_0 below $1/3$, we will always find all p_is to be positive. Consequently, for the metric it means, all the directions are anisotropically non-contracting starting from the initial big bang singularity. We note, in particular, a special case $p_1 = p_2 = 1/2, p_3 = 0$ with $\rho_0 = 1/4$ satisfy (2.41) which we will consider later.

Now, to embed this non-vacuum Kasner metric into AdS, we need to introduce an extra component of stress tensor T_{zz} as given by (33). The complete five-dimensional energy-momentum tensor describes an anisotropic fluid. In this context it is also interesting to check whether this fluid satisfies the null/weak/strong energy condition(s). For an arbitrary null-vector k_a in five-dimensions, the null energy condition is given by $\mathcal{T}_{ab}k^a k^b \geq 0$, where $\mathcal{T}_{ab} = T_{ab} - \Lambda g_{ab}$ is the total five-dimensional stress tensor (matter + background). The null energy condition is satisfied by $\mathcal{T}_{ab}$ provided the parameter ω and the energy density ρ appearing in the four-dimensional equation of state, satisfy the conditions $\omega \geq -\frac{1}{3}$ and $\rho \geq 0$ respectively. We recognize these as the conditions for the strong energy condition to be satisfied by the four-dimensional fluid. For the weak energy case, we first need to recognize that the weak energy condition may be violated by $\mathcal{T}_{ab}$ (Note that empty AdS also violates the weak energy condition). However, for the five-dimensional matter stress tensor T_{ab} in (2.32), both the weak and the strong energy conditions [4] are satisfied provided ω and ρ satisfy the same conditions i.e $\omega \geq -\frac{1}{3}$ and $\rho \geq 0$.

Before ending this section we finally review the construction of the AdS de Sitter geometry as a vacuum time dependent AdS spacetime. It is very well known fact that our universe is very close to the de-Sitter spacetime, which is a time dependent expanding spacetime with positive cosmological constant. This is a maximally symmetric spacetime which has an isometry group $SO(1,3)$. One can study free field theory on this background which allows a family of privileged vacuum states, known as α vacua, that respect the isometry group of the de Sitter space [9]. The embedding of this spacetime into AdS allows one to study various aspects of strongly coupled field theory on de Sitter spacetime holographically [58, 60–63]. Here we follow [58] for the construction of AdS de Sitter which can be used in holography to study the

[4]Recall that the weak energy condition for the five-dimensional matter stress tensor is given by

$$\widehat{T}_{\mu\nu}\xi^\mu\xi^\nu + T_{zz}(\xi^z)^2 \geq 0, \tag{2.48}$$

where ξ^a is a five dimensional timelike vector and u_μ is the four-velocity of the boundary fluid. The strong energy condition for the same is given by

$$\left(T_{ab} - \frac{T'}{2}g_{ab}\right)v^a v^b \geq 0, \tag{2.49}$$

for an arbitrary future directed timelike vector field v^a, where $T' = \frac{3}{2}T$.

strongly coupled field theory on this background.

The metric of a four dimensional de Sitter space in static patch is given by

$$ds^2 = -(1 - H^2 r^2)dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\Omega_2^2. \tag{2.50}$$

It covers the causal diamond associated with a single geodesic observer and it has a timelike killing vector ∂_t[5]. Now, our task is to embed this static de Sitter metric into AdS so that the boundary of the AdS becomes de Sitter. To do this we work in the Fefferman-Graham form, which states that any asymptotically AdS metric can be written in the following form [39],

$$ds^2 = \frac{1}{z^2}\left(g_{\mu\nu}(z, x)dx^\mu dx^\nu + dz^2\right), \tag{2.51}$$

From this form one can identify that the boundary($z = 0$) metric of the AdS is $g_{\mu\nu}(x) = g_{\mu\nu}(0, x)$. Near the boundary the metric $g_{\mu\nu}(z, x)$ has an expansion[6]

$$g_{\mu\nu}(z, x) = g_{\mu\nu}(x) + z^2 g^{(2)}_{\mu\nu}(x) + \ldots + z^d g^{(d)}_{\mu\nu}(x) + z^d \log(z^2) h^{(d)}_{\mu\nu}(x) + \ldots, \tag{2.52}$$

To obtain a bulk dual to a field theory on the above de Sitter space we write the bulk metric in the above Fefferman-Graham form and due to the static boundary, we assume that the bulk metric is time independent, i.e,

$$ds^2 = \frac{1}{z^2}\left(-f(r, z)dt^2 + j(r, z)dr^2 + h(r, z)d\Omega_2^2 + dz^2\right). \tag{2.53}$$

The next step is to write the functions $f(r, z)$, $j(r, z)$ and $h(r, z)$ as a series expansion in z and

[5]Here we note that there is no globally defined timelike killing vector for the de Sitter spacetime.

[6]In this expansion only even powers of z survive upto order d which can be seen by analyzing the Einstein equation(2.5) near the boundary.

solve order by order the Einstein equations (2.5).

$$\begin{aligned} f(r,z) &= f_0(r) + f_2(r)z^2 + f_4(r)z^4 + \cdots , \\ j(r,z) &= j_0(r) + j_2(r)z^2 + j_4(r)z^4 + \cdots , \\ h(r,z) &= h_0(r) + h_2(r)z^2 + h_4(r)z^4 + \cdots . \end{aligned} \tag{2.54}$$

The boundary condition is that at $z \to 0$ the metric should be given by (2.50). This gives

$$f_0(r) = 1 - H^2r^2, \qquad j_0(r) = \frac{1}{1 - H^2r^2}, \quad \text{and} \quad h_0(r) = r^2. \tag{2.55}$$

Using the above forms (2.55) we substitute (2.54) into the Einstein equation (2.5) and we obtain a set of algebraic equations containing the above unknown functions and solving these upto z^2 term we obtain

$$\begin{aligned} f_2(r) &= -\frac{(1 - H^2r^2)H^2}{2}, \\ j_2(r) &= -\frac{H^2}{2(1 - H^2r^2)}, \\ h_2(r) &= -\frac{r^2H^2}{2}. \end{aligned} \tag{2.56}$$

By similar iterations one can find that the solutions are truncated at order $O(z^4)$ and the full solution to eqn.(2.54) is

$$\begin{aligned} f(r,z) &= (1 - H^2r^2)\left(1 - \frac{H^2z^2}{4}\right)^2 , \\ j(r,z) &= \frac{1}{1 - H^2r^2}\left(1 - \frac{H^2z^2}{4}\right)^2 , \\ h(r,z) &= r^2\left(1 - \frac{H^2z^2}{4}\right)^2 . \end{aligned} \tag{2.57}$$

This completes the construction of AdS de Sitter spacetime. From the above solution we can see that in the Fefferman-Graham form the bulk metric factorizes as $g_{\mu\nu}(z, x) = f(z) g^{\rm dS}_{\mu\nu}(x)$, with

$$f(z) = \left(1 - \frac{H^2 z^2}{4}\right)^2 . \tag{2.58}$$

One can calculate the curvature invariants for this geometry which show that the spacetime is completely smooth and free from singularity. This spacetime has a Killing horizon(with respect to the Killing vector ∂_t) at $z = 2/H$ and there is a temperature $T_{\rm dS} = H/2\pi$ associated with this horizon.

There are many different coordinates(which correspond to different patches) by which the de Sitter metric can be written. One such coordinate system is known as planar coordinates in which the de Sitter space is conformal to the Minkowski space:

$$ds^2 = \frac{1}{H^2\eta^2}\left(-d\eta^2 + d\vec{x}^2\right). \tag{2.59}$$

This metric covers half of global de Sitter and the time η is known as conformal time. For this coordinate system, the corresponding bulk metric reads

$$ds^2 = \frac{1}{z^2}\left(\frac{f(z)}{H^2\eta^2}\left(-d\eta^2 + d\vec{x}^2\right) + dz^2\right), \tag{2.60}$$

where the function $f(z)$ is given in (2.58). For this metric the time translation symmetry is broken and the spacetime is an expanding one obeying the Hubble's law.

2.4 Flat Space Embedding of The Dual Geometries

All of the geometries that we constructed in the previous sections turn out to have higher dimensional flat-space embeddings. In this section, we construct these explicitly.

Embeddings of geometries in higher dimensional flat space illustrate many geometrical features of these spacetimes. For example, one can obtain useful information about the isometry structure of that spacetime. Further, it helps in studying various thermal properties of the the spacetime, see for example [65]- [67]. For time dependent Friedmann-Robertson-Walker geometries, embedding approach was used to exhibit three dimensional conformal symmetry on future space-like hypersurfaces in [68]. Though the results of this section will not be explicitly used in the later chapters, we believe our results will be useful in the future.

2.4.1 AdS-FRW embedding

We first consider the AdS-FRW metric as it is simpler. We write it as

$$dS^2 = \frac{1}{z^2}\Big[dz^2 - dt^2 + a^2(t)\sum_{i=1}^{3} dx_i dx^i\Big]. \tag{2.61}$$

The dependence of the scale factor $a(t)$ on t follows from Einstein equations and generally goes as t^p for some constant p. Let us now consider all the terms inside the square brackets except the dz^2 part. Introducing new coordinates [69],

$$\begin{aligned} y^i &= a(t)x^i, \quad i = 1, 2, 3, \\ y^0 &= ca(t)\sum_{i=1}^{3} x^i x^i + c\int \frac{dt}{\dot{a}(t)} + \frac{a(t)}{4c}, \\ y^4 &= ca(t)\sum_{i=1}^{3} x^i x^i + c\int \frac{dt}{\dot{a}(t)} - \frac{a(t)}{4c}. \end{aligned} \tag{2.62}$$

with c being an arbitrary constant, we get

$$-dt^2 + a^2(t)\sum_{i=1}^{3}(dx^i)^2 = -(dy^0)^2 + \sum_{i=1}^{4}(dy^i)^2. \tag{2.63}$$

Consequently, the metric (2.61) reduces to AdS in Poincare coordinates,

$$dS^2 = \frac{1}{z^2}\Big[dz^2 - (dy^0)^2 + \sum_{i=1}^{4}(dy^i)^2\Big]. \tag{2.64}$$

This can now easily be embedded in flat space using the standard map

$$\begin{aligned}
X_0 &= \frac{1}{2z}\Big(z^2 + 1 + \sum_{i=1}^{4}(y^i)^2 - (y^0)^2\Big),\\
X_i &= \frac{y^i}{z}, \quad i = 1,2,3,4,\\
X_5 &= \frac{1}{2z}\Big(z^2 - 1 + \sum_{i=1}^{4}(y^i)^2 - (y^0)^2\Big),\\
X_6 &= \frac{y^0}{z}.
\end{aligned} \tag{2.65}$$

To summarize, the five dimensional AdS-FRW metric can be embedded in seven dimensional flat space

$$dS^2 = -dX_0^2 - dX_6^2 + \sum_{i=1}^{5} dX_i^2, \tag{2.66}$$

with

$$\begin{aligned}
X_0 &= \frac{1}{2z}\Big[z^2 + 1 - a(t)\int \frac{dt}{\dot{a}(t)}\Big],\\
X_j &= \frac{a(t)x^j}{z}, \quad j = 1,2,3,\\
X_4 &= \frac{1}{z}\Big[ca(t)\sum_{j=1}^{3} x_j^2 + c\int \frac{dt}{\dot{a}(t)} - \frac{a(t)}{4c}\Big],\\
X_5 &= \frac{1}{2z}\Big[z^2 - 1 - a(t)\int \frac{dt}{\dot{a}(t)}\Big],\\
X_6 &= \frac{1}{z}\Big[ca(t)\sum_{j=1}^{3} x_j^2 + c\int \frac{dt}{\dot{a}(t)} + \frac{a(t)}{4c}\Big].
\end{aligned} \tag{2.67}$$

From the higher dimensional perspective, the AdS-FRW metric sits on the intersection of the hypersurfaces

$$-X_0^2 - X_6^2 + \sum_1^5 X_i^2 = -1, \tag{2.68}$$

and

$$\frac{1 - X_0^2 + X_5^2}{(X_0 - X_5)^2} = (a(t)\int \frac{dt}{\dot{a}(t)})\Big|_{a(t)=2c\frac{X_6-X_4}{X_0-X_5}}. \tag{2.69}$$

Particularizing to the case $a(t) = t^p$, we get

$$2c(X_6 - X_4)(X_0 - X_5)^{(p-1)} = \Big((2-p)p(1 - X_0^2 + X_5^2)\Big)^{\frac{p}{2}}. \tag{2.70}$$

2.4.2 Embedding AdS-Kasner

For the flat embedding of AdS-Kasner, it requires a bit more work. Consider the metric in (2.7). We first take the part

$$-dt^2 + t^{2p_1}(dx^1)^2. \tag{2.71}$$

This is in the FRW form. Therefore, using our previous results, we can embed this part to flat space. We introduce

$$\begin{aligned} y^0 &= (c(x^1)^2 + \frac{1}{4c})t^{p_1} + \frac{ct^{2-p_1}}{p_1(2-p_1)}, \\ y^1 &= t^{p_1}x^1, \\ \tilde{y} &= (c(x^1)^2 - \frac{1}{4c})t^{p_1} + \frac{ct^{2-p_1}}{p_1(2-p_1)}. \end{aligned} \tag{2.72}$$

Then (2.7) takes the form

$$dS^2 = \frac{1}{z^2}\Big[dz^2 - (dy^0)^2 + (dy^1)^2 + (d\tilde{y})^2 + t^{2p_2}dx^2dx^2 + t^{2p_3}dx^3dx^3\Big]. \tag{2.73}$$

Let us now define light-cone coordinates $u = y^0 - \tilde{y}, v = y^0 + \tilde{y}$ to write (2.7) as

$$dS^2 = \frac{1}{z^2}\Big[dz^2 - dudv + (dy^1)^2 + (2cu)^{\frac{2p_2}{p_1}}(dx^2)^2 + (2cu)^{\frac{2p_3}{p_1}}(dx^3)^2\Big]. \tag{2.74}$$

So Kasner can be embedded as a hypersurface

$$uv - \tilde{y}^2 = \frac{1}{p_1(2-p_1)}(2cu)^{\frac{2}{p_1}}, \tag{2.75}$$

in the plane wave metric. It is useful now to convert the above plane wave metric into Brinkmann form by defining

$$\begin{aligned} &y^2 = (2cu)^{\frac{p_2}{p_1}}x^2, \ \ y^3 = (2cu)^{\frac{p_3}{p_1}}x^3, \ \ U = u, \\ &V = v + \frac{2cp_2}{p_1}(2cu)^{\frac{2p_2-p_1}{p_1}}(x^2)^2 + \frac{2cp_3}{p_1}(2cu)^{\frac{2p_3-p_1}{p_1}}(x^3)^2. \end{aligned} \tag{2.76}$$

The metric becomes

$$dS^2 = \frac{1}{z^2}\Big[dz^2 - dUdV + \sum_{i=1}^{3} dy^i dy^i - HdU^2\Big], \tag{2.77}$$

where

$$H = \frac{(2cU)^{\frac{2p_2}{p_1}}p_2(p_1-p_2)(x^2)^2 + (2cU)^{\frac{2p_3}{p_1}}p_3(p_1-p_3)(x^3)^2}{p_1^2U^2} \tag{2.78}$$

$$= \frac{(p_2(p_1-p_2)(y^2)^2 + p_3(p_1-p_3)(y^3)^2}{p_1^2U^2}. \tag{2.79}$$

In the Brinkmann form, hypersurface equation can be written as

$$UV - \frac{p_2}{p_1}(y^2)^2 - \frac{p_3}{p_1}(y^3)^2 - (y^1)^2 = \frac{1}{p_1(2-p_1)}(2cU)^{\frac{2}{p_1}}. \tag{2.80}$$

Further, introducing

$$Z_0 = \frac{V+U+HU}{2},\; Z_4 = \frac{V-U+HU}{2},\; Z_5 = \frac{H+\frac{U^2}{2}}{2},$$
$$Z_6 = \frac{H-\frac{U^2}{2}}{2},\; Z_i = y^i \; (i=1,2,3), \tag{2.81}$$

we get flat space embedding of plane wave. So 4-dimensional Kasner can be embedded in 7 dimensional flat space. Using this, we get

$$dS^2 = \frac{1}{z^2}\Big[dz^2 - dZ_0^2 - dZ_6^2 + \sum_{i=1}^{5} dZ_i^2\Big]. \tag{2.82}$$

This is now AdS in Poincare form and can be easily embedded in flat space. So to summarize, the five dimensional AdS-Kasner metric can be embedded in flat space

$$dS^2 = -dX_0^2 - dX_7^2 - dX_8^2 + \sum_{i=1}^{6} dX_i^2, \tag{2.83}$$

and to write it in a compact way, we first define

$$Z_0 = \frac{t^{p_1}}{4c} + c\Big[\frac{t^{2-p_1}}{(2-p_1)p_1} + \frac{1}{p_1^2}(p_1^2 t^{p_1} x_1^2 + (2p_1-p_2)p_2 t^{2p_2-p_1} x_2^2 + (2p_1-p_3)p_3 t^{2p_3-p_1} x_3^2)\Big],$$
$$Z_1 = x_1 t^{p_1},\;\; Z_2 = x_2 t^{p_2},\;\; Z_3 = x_3 t^{p_3},$$
$$Z_4 = -\frac{t^{p_1}}{4c} + c\Big[\frac{t^{2-p_1}}{(2-p_1)p_1} + \frac{1}{p_1^2}(p_1^2 t^{p_1} x_1^2 + (2p_1-p_2)p_2 t^{2p_2-p_1} x_2^2 + (2p_1-p_3)p_3 t^{2p_3-p_1} x_3^2)\Big],$$
$$Z_5 = \frac{t^{2p_1}}{16c^2} + \frac{2c^2}{p_1^2}\Big[(p_1-p_2)p_2 t^{2(p_2-p_1)} x_2^2 + (p_1-p_3)p_3 t^{2(p_3-p_1)} x_3^2\Big]$$
$$Z_6 = -\frac{t^{2p_1}}{16c^2} + \frac{2c^2}{p_1^2}\Big[(p_1-p_2)p_2 t^{2(p_2-p_1)} x_2^2 + (p_1-p_3)p_3 t^{2(p_3-p_1)} x_3^2\Big]. \tag{2.84}$$

The flat coordinates are then

$$
\begin{aligned}
X_0 &= \frac{1}{2z}\Big[z^2 + 1 + \sum_{i=1}^{5} Z_i^2 - Z_0^2 - Z_6^2\Big],\\
X_i &= \frac{Z_i}{z}, \quad i = 1, ..., 5,\\
X_6 &= \frac{1}{2z}\Big[z^2 - 1 + \sum_{i=1}^{5} Z_i^2 - Z_0^2 - Z_6^2\Big],\\
X_7 &= \frac{Z_0}{z},\\
X_8 &= \frac{Z_6}{z}.
\end{aligned}
\tag{2.85}
$$

Here c is an arbitrary constant.

AdS Kasner is then an intersection of following hypersurfaces in flat space.

$$
\begin{aligned}
&-X_0^2 - X_7^2 - X_8^2 + X_6^2 + \sum_{i=1}^{5} X_i^2 = -1,\\
&(X_7 - X_4)^2 = 2(X_5 - X_8)(X_0 - X_6),\\
&X_5^2 - X_8^2 = \frac{p_2(p_1 - p_2)}{2p_1^2} X_2^2 + \frac{p_3(p_1 - p_3)}{2p_1^2} X_3^2,\\
&X_7^2 - X_4^2 - 2X_5^2 + 2X_8^2 - \frac{p_2}{p_1} X_2^2 - \frac{p_3}{p_1} X_3^2\\
&= (X_0 - X_6)^2 \left(\frac{1}{(2 - p_1)p_1} \left(2c \frac{X_7 - X_4}{X_0 - X_6} \right)^{\frac{2}{p_1}} + \left(\frac{X_1}{X_0 - X_6} \right)^2 \right).
\end{aligned}
\tag{2.86}
$$

2.4.3 AdS-Kasner soliton

Finally, we embed the Kasner-AdS soliton (2.11)in 6-dimensions in a higher dimensional flat spacetime. First, we embed the Kasner part of the metric as was done in equation (2.82). We get

$$
dS^2 = \frac{1}{z^2}\Big[\Big(1 - \frac{z^5}{z_0^5}\Big)^{-1} dz^2 + \Big(1 - \frac{z^5}{z_0^5}\Big) d\theta^2 - dZ_0^2 - dZ_6^2 + \sum_{i=1}^{5} dZ_i^2\Big]. \tag{2.87}
$$

Now, with the following choice of coordinates

$$
\begin{aligned}
&X_0 = \frac{Z_0}{z},\ \ X_1 = \frac{Z_6}{z},\ \ X_2 = \frac{2z_0}{5}\sqrt{g(z)}\,\mathrm{Sin}\left(\frac{5\theta}{2z_0}\right),\ \ X_3 = \frac{2z_0}{5}\sqrt{g(z)}\,\mathrm{Cos}\left(\frac{5\theta}{2z_0}\right),\\
&X_4 = \frac{Z_1}{z},\ \ X_5 = \frac{Z_2}{z},\ \ X_6 = \frac{Z_3}{z},\ \ X_7 = \frac{Z_4}{z},\ \ X_8 = \frac{Z_5}{z}\\
&X_9 = \frac{1}{2z}\left[Z_0^2 + Z_6^2 - \sum_{i=1}^{5} Z_i^2 + zD(z) + 1\right],\\
&X_{10} = \frac{1}{2z}\left[Z_0^2 + Z_6^2 - \sum_{i=1}^{5} Z_i^2 + zD(z) - 1\right],
\end{aligned}
\tag{2.88}
$$

the metric reduces to

$$
dS^2 = -dX_0^2 - dX_{10}^2 - dX_1^2 + \sum_{i=2}^{9} dX_i^2 \tag{2.89}
$$

In the above

$$
g(z) = \frac{1}{z^2}\left(1 - \frac{z^5}{z_0^5}\right),\ \ D(z) = -\int \frac{dz}{z^2 g(z)}\left(1 - \frac{z_0^2 z^4 g'(z)^2}{25}\right). \tag{2.90}
$$

The soliton metric therefore sits at the intersection of six hypersurfaces.

2.5 Summary

In this chapter our primary aim was to construct some cosmological geometries possess AdS duals [34]. We started with the anisotropic AdS-Kasner spacetime and its solitonic generalization. By relating these solutions with the near-brane limits of the $D3$ and the $M5$ branes, we could identify the boundary gauge theories. We then proceeded to construct some non-vacuum generalizations, for example, the AdS-FRW and the non-vacuum extension of the AdS-Kasner by introducing a class of perfect fluid stress tensors on the boundary. Although, for most of solutions in the later categories, we do not yet have ways to arrive from the higher dimensional brane configurations, in parts of the later chapters, we will *assume* that a version of AdS/CFT correspondence still exists in these cases as well. Finally, we concluded the chapter with the explicit construction of higher dimensional flat space embeddings of our constructed cosmological backgrounds.

Chapter 3

Gauge Theory Correlators On Time Dependent Spacetimes

In the previous chapter, we have constructed a large number of cosmological backgrounds which possess AdS duals. In particular, we now know that the strongly coupled $\mathcal{N} = 4$, $SU(N)$ SYM at large coupling formulated on Kasner background is dual to a weakly coupled gravitational theory on the AdS-Kasner geometry. In this chapter, we make use of the correspondence to compute spacelike separated two point correlators of scalar operators of the SYM. We further extend these results for those cosmological backgrounds where the gauge theory is not explicitly known. Our aim in this section is to search for imprints of the cosmological singularities on the correlators.

We start with the Kasner class. It is parametrized by the constants p_is with $i = 1, 2, 3$ satisfying two constraints. The simplest among this class is the Milne metric where one of these constants takes the value *one* and the rest are *zero*. It turns out that this is the only geometry for which a two point correlator can be exactly computed. The reason for this stems from the fact that the metric is in fact a part of the flat Minkowski space and is therefore devoid of any real cosmological singularities. For the rest, we need to employ some approximation scheme. The one that is particularly useful in this context is known as the *geodesic approximation* [15]- [17]. We illustrate this approximation with some simple examples later in the chapter. Subsequently, within this framework, we compute spacelike separated correlators for a large class of cosmological backgrounds that include the Kasner, the Kasner solitons and finally the *non-vacuum* Kasner. One result that comes out from our analysis is the following. Regardless of the specific background we choose to work with, a large class of spacelike correlators remain *non-singular* even if we take the spacelike surface close to the cosmological singularities. Signature of singularity may however show up in higher point correlators. Computations of the higher point correlators on time dependent backgrounds are difficult and is beyond the scope of this thesis.

3.1 AdS-Milne spacetime: An exactly solvable Model

We start with the Milne spacetime. The dual of Milne is the AdS-Milne background is given by (we have chosen $p_1 = 1, p_2 = p_3 = 0$)

$$ds^2 = \frac{1}{z^2}\left[-dt^2 + t^2 dx_1^2 + dx_2^2 + dx_3^2 + dz^2\right]. \tag{3.1}$$

For this case the scalar field equations and, consequently, finding the boundary correlator becomes simple. As discussed in the previous chapter, in order to compute the correlator for the SYM scalar operators, we first need to solve the scalar field equation on the above background and impose the right set of boundary conditions.

To proceed further, it is useful to first analytically continue the coordinates to $t = i\tau$ and $x_1 = i\psi$. The geometry has a horizon at $\tau = 0$ and the boundary at $z = 0$. The equation of motion of a massive scalar field is given by

$$(\Box - m^2)\phi = 0. \tag{3.2}$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\phi)$.

If we insist the field to vanish at $\tau = 0$ and $z = \infty$, the solution can be written as

$$\phi = \sum_{n=-\infty}^{\infty} \epsilon^{\Delta_-} \int_0^{\infty} \int_0^{\infty} d\omega dk_2 dk_3 C_{\omega,n,k_2,k_3} \sqrt{\tilde{\omega}} e^{in\psi} e^{ik_2x_2} e^{ik_3x_3} \frac{z^2 K_\alpha(\tilde{\omega}z)}{\epsilon^2 K_\alpha(\tilde{\omega}\epsilon)} J_n(\omega\tau). \tag{3.3}$$

Here, J_n is the Bessel function, K_α is a modified Bessel function, ϵ is a small cutoff near $z = 0$ and C_{ω,n,k_1,k_2} are constants. The other constants are defined as $\tilde{\omega} = \sqrt{\omega^2 + k_1^2 + k_2^2}$, $\alpha = \sqrt{4 + m^2}$ and $\Delta_- = 2 - \alpha$. Now, following the standard prescription, we evaluate the scalar field action on the solution. This gives

$$S = \sum_{n_1,n_2} \int d\tau dx_2 dx_3 d\omega d\omega' dk_2 dk_3 dk_2' dk_3' \delta_{m+n} e^{i(k_2+k_2')x_2} e^{i(k_3+k_3')x_3}$$
$$C_{\omega,n,k_2,k_3} C_{\omega',m,k_2',k_3'} \tau J_n(\tau\omega) J_m(\tau\omega') \sqrt{\omega\omega'}\tilde{\omega}^{2\alpha}. \tag{3.4}$$

Further, differentiating the above action twice with respect to C, we reach at the following two point correlator in momentum space:

$$\langle O(k_i,\omega,n) O(k_i',\omega',m)\rangle \sim (-)^m \delta_{n+m} \delta(k_2 + k_2') \delta(k_3 + k_3') \delta(\omega - \omega') \tilde{\omega}^{2\alpha}. \tag{3.5}$$

Here, we have used

$$\int_0^{\infty} d\tau \tau J_n(\omega\tau) J_n(\omega'\tau) = \frac{1}{\omega}\delta(\omega - \omega'). \tag{3.6}$$

To obtain the correlator in position space, we use the identity

$$\sum_{n=-\infty}^{\infty} J_n(\omega\tau) J_n(\omega\tau') e^{in(\psi-\psi')} = J_0(\omega\tilde{\tau}), \tag{3.7}$$

where

$$\tilde{\tau} = \sqrt{\tau^2 + \tau'^2 - 2\tau\tau' \cos(\psi - \psi')}, \tag{3.8}$$

and get

$$\langle O(\tau, \psi, x_2, x_3) O(\tau', \psi', x_2', x_3') \rangle = \int_0^\infty d\omega \int_{-\infty}^\infty dk_2 \int_{-\infty}^\infty dk_3\, \omega\tilde{\omega}^{2\alpha} J_0(\omega\tilde{\tau}) e^{ik_2(x_2 - x_2')} e^{ik_3(x_3 - x_3')}. \tag{3.9}$$

Going to the polar coordinates the integrals can be easily performed and we arrive at

$$\langle O(\tau, \psi, x_2, x_3) O(\tau', \psi', x_2', x_3') \rangle = \frac{1}{[\tau^2 + \tau'^2 - 2\tau\tau' \cos(\psi - \psi') + (x_2 - x_2')^2 + (x_3 - x_3')^2]^{2+\alpha}}, \tag{3.10}$$

up to some α dependent prefactor. Now, analytically continuing back to the earlier coordinates, we reach at

$$\langle O(t, x_1, x_2, x_3) O(t', x_1', x_2', x_3') \rangle = \frac{1}{[-t^2 - t'^2 + 2tt' \cosh(x_1 - x_1') + (x_2 - x_2')^2 + (x_3 - x_3')^2]^{2+\alpha}}. \tag{3.11}$$

For space-like separated points, say by a distance $2x$ along x_1, the formula further simplifies to

$$\langle O(t, x, x_2, x_3) O(t, -x, x_2, x_3) \rangle = \frac{1}{(2t \sinh x)^{2\Delta}}, \tag{3.12}$$

where $\Delta = 2 + \alpha$.

3.2 Problems in Calculating Correlators For The Others Geometries

For the other cosmological backgrounds, exact computation as above is difficult to carry out. The reasons are the following. For one, the scalar wave equation on a generic time dependent background is hard to solve analytically. Secondly, even if it is possible in some cases, as these spacetimes contain naked curvature singularities it is not clear what boundary conditions to be imposed. Thirdly, if the backgrounds do not have the asymptotically past or future flat regions, there could be unambiguoity in identifying the positive and the negative energy modes. In this section, we try to elaborate upon these issues.

We begin with the AdS-Kasner background

$$ds^2 = \frac{1}{z^2}\left[-dt^2 + \sum_{i=1}^{3} t^{2p_i} dx_i^2 + dz^2\right]. \tag{3.13}$$

Using the fact that for the above metric,

$$\sqrt{-g} = \frac{t^{p_1+p_2+p_3}}{z^5} = \frac{t}{z^5}, \tag{3.14}$$

we get the following form of the wave equation,

$$z^2\partial_z^2\phi - 3z\partial_z\phi - z^2\partial_t^2\phi - \frac{z^2}{t}\partial_t\phi + \sum_{i=1}^{3}\frac{z^2}{t^{2p_i}}\partial_i{}^2\phi - m^2\phi = 0. \tag{3.15}$$

Substituting the ansatz,

$$\phi = \sum_{i=1}^{3}\xi(t)\rho(z)e^{ik_i x^i}, \tag{3.16}$$

we reach at

$$\begin{aligned} &z^2\partial_z^2\rho - 3z\partial_z\rho - (m^2 + \omega^2 z^2)\rho = 0, \\ &t\partial_t^2\xi + \partial_t\xi + \sum_{i=1}^{3}\frac{k_i^2}{t^{2p_i-1}}\xi - \omega^2 t\xi = 0. \end{aligned} \tag{3.17}$$

Here ω is an arbitrary constant. Now, first is the usual radial equation of AdS. As we know, the solutions can be written in terms of Bessel functions. In order to solve the time dependent part, it is instructive to introduce a new variable $\chi(t)$ as

$$\xi = t^{-\frac{1}{2}}\chi(t), \tag{3.18}$$

and re-write the second equation in (3.17) as

$$\partial_t^2\chi + \Big(\frac{1}{4t^2} + \sum_{i=1}^{3}\frac{k_i^2}{t^{2p_i}} - \omega^2\Big)\chi = 0. \tag{3.19}$$

Normalization can be fixed by setting the the Wronskian to a constant, for example, by choosing

$$\chi\partial_t\chi^* - \chi^*\partial_t\chi = i. \tag{3.20}$$

If we treat (3.19) as a Schrodinger-like equation, the effective potential has a form

$$V(t) = -\frac{1}{4t^2} - \sum_i \frac{k_i^2}{t^{2p_i}}. \tag{3.21}$$

Near $t = 0$, for generic p_is, the first term dominates making the potential singular and scale invariant.

For generic values of p_is, the $\xi(t)$ in (3.17) can not be solved analytically. However, for specific cases, some progress can be made. To illustrate this further, let us consider the case $\{p_1 = \frac{1}{2}, p_2 = \frac{1}{2}, p_3 = 0\}$. As discussed in the previous chapter, for $\rho_0 = \frac{1}{4}$, the metric solves the Einstein equation. Here one gets,

$$\begin{aligned} \chi(t) \quad &= C_1 M[-i\beta, 0, 2i\alpha t] + C_2 W[-i\beta, 0, 2i\alpha t], \text{ for } k_3^2 > \omega^2 \\ &= C_3 M[\beta, 0, 2\alpha t] + C_4 W[\beta, 0, 2\alpha t], \text{ for } k_3^2 < \omega^2, \end{aligned} \tag{3.22}$$

where,

$$\alpha = \sqrt{|k_3^2 - \omega^2|},\ \beta = \frac{k_1^2 + k_2^2}{2\alpha}. \tag{3.23}$$

M, W are the Whittaker functions and C_is are integration constants. Owing to the properties of M and W, we note that, at $t = 0$, the solutions identically vanish. Further, because of time dependence of the metric, there is no natural way to identify the positive and negative energy modes. Therefore we find it difficult to put appropriate boundary conditions to isolate the relevant one from the general solution. In the case of Milne, as we saw in chapter one or previously in this chapter, such problems do not occur.

3.3 Geodesic Approximation

In the above circumstance, one needs to use some approximation scheme and it turns out, for our purpose, the geodesic approximation [15]- [17] is the suitable one. It follows from the Feynman path integral representation of the two point correlation function. Here one represents the inverse of the operator $\Box - m^2$ as a path integral. In the large mass limit and in the saddle point approximation, the path integral is then expected to reduce to finding the geodesic. This method works well for the Euclidean time. However, for Lorentzian metric (as in the case here), there are often subtleties due to the appearance of oscillating phases. In our discussion, we will *assume* such uncertainties [17] - [19] do not occur.

Within the semi-classical limit, this approximation suggests that the boundary correlator of two operators can be written as

$$\langle\psi|O(x)O(x')|\psi\rangle \sim e^{-m\mathcal{L}_{reg}(x,x')}, \tag{3.24}$$

where $|\psi\rangle$ is some state of the strongly coupled Yang-Mills theory residing on the boundary and O is an operator whose dimension Δ is large and close to m, the mass of a heavy bulk field. For our purpose O will be a scalar operator and, consequently, we will consider only the scalar field. The quantity $\mathcal{L}_{reg}$ is the length of the regularized bulk geodesic connecting points x and x'. We will not go into the details of the approximation method; interested readers can consult the original literature suggested in the previous paragraph. Here we will rather provide an example as an illustration.

Let us consider the Milne universe. In order to compute the space-like two point correlator, we need to find the geodesic whose end points are stuck at the boundary, say at $t = t_0$. We take the points along x_1 direction at $x_1 = \pm x$. The effective metric for the geodesic is

$$ds^2 = \frac{1}{z^2}\left[-dt^2 + t^2 dx_1^2 + dz^2\right]. \tag{3.25}$$

Using t as a parameter, the geodesic equations are

$$\begin{aligned} &\ddot{x}_1 + 2\frac{\dot{x}_1}{t} - t\dot{x}_1^3 = 0, \\ &z\ddot{z} + \dot{z}^2 + t^2\dot{x}_1^2 - tz\dot{z}\dot{x}_1^2 - 1 = 0. \end{aligned} \tag{3.26}$$

Here, the dots are derivatives with respect to t. The solutions of the above equations are

$$x_1 = \pm \log\left[\frac{\sqrt{c}t}{1+\sqrt{1-ct^2}}\right], \quad z = \sqrt{t^2 - t_0^2}. \tag{3.27}$$

The constant c is related to the turning point of the geodesic in the bulk. Since at this point, $x_1 = 0, dt/dx = 0$, and therefore $t = 1/\sqrt{c}$. Further, since at $t = t_0$, $x_1 = \pm x$, we get

$$c = \frac{4e^{2x}}{(1+e^{2x})^2 t_0^2}. \tag{3.28}$$

For calculational simplicity, without loss of generality, we take $t_0 = 1$ henceforth. The length of the geodesic can be easily computed. It is

$$\mathcal{L} = \int_{t=\frac{1}{\sqrt{c}}}^{1+t(z=\epsilon)} \frac{2dt}{z}\sqrt{-1 + t^2\dot{x}^2 + \dot{z}^2}. \tag{3.29}$$

The lower integration limit is the turning point of the geodesic and the upper limit is the value of t at which one approaches the boundary $z = \epsilon$ with $\epsilon \to 0$. The result diverges as we take ϵ to zero. The regularization is done by subtracting the equivalent piece coming from AdS. This gives

$$\mathcal{L}_{reg} = \lim_{\epsilon\to 0}\left(\log\left[\frac{4\sinh^2 x}{\epsilon}\right] - 2\log\left[\frac{1}{\epsilon}\right]\right) = \log[4\sinh^2 x]. \tag{3.30}$$

Consequently,

$$\langle O(1,x,0,0)O(1,-x,0,0)\rangle = \frac{1}{(2\sinh x)^{2m}}. \tag{3.31}$$

This is same as (3.12) for the operators of large dimensions. This shows that the geodesic approximation reproduces the correct two point correlation function in the large mass limit for the AdS-Milne geometry.

3.4 Correlators in the Geodesic Approximation

Due to the lack of direct solvability, in this section we study the spacelike two point correlators of the boundary gauge theory for AdS-Kasner and AdS-Kasner soliton geometries using the geodesic approximation. As the boundary of these geometries are time dependent, the equal time correlation function depends on the time slicing of the boundary. Here we calculate this correlation function between two points separated along an expanding direction with time. Further, we will see how some generic features of the result emerge as a consequence of the underlying scaling symmetry of those solutions. For AdS-Kasner soliton, owing to the complication due to the background geometry, we compute the correlator numerically and compare the result with that of AdS-Kasner.

3.4.1 AdS - Kasner

We first would like to compute $\langle O(x_1', t_0)O(x_1'', t_0)\rangle$ where the bulk geometry is given by (3.13) and we write $t^{p_i} = a_i$ This is a correlator along x_1 direction with two boundary points at

x_1', x_1'' computed at a fixed time $t = t_0$. Corresponding space-like geodesic must then have two fixed end points x_1', x''_1 on the boundary $z = 0$ at time $t = t_0$. For this particular calculation, therefore, the other boundary directions $x_i, i \neq 1$ are irrelevant. For the moment, we work with a general scale factor $a_1(t)$ along x_1. Later, we will use the explicit form $a_1 = t^{2p_1}$ for $p_1 > 0$.

Calling x_1 as x and a_1 as a for notational simplicity, the geodesic equations for (3.13) are given by

$$\ddot{x} + 2\frac{\dot{a}}{a}\dot{x} - a\dot{a}\dot{x}^3 = 0,$$
$$z\ddot{z} + \dot{z}^2 + \dot{x}^2 a^2 - a\dot{a}z\dot{z}\dot{x}^2 - 1 = 0. \tag{3.32}$$

Here, we have taken time as a parameter and derivatives are with respect to time.

General solutions of these equations can be written as

$$x(t) = \pm \int \frac{a(t^*)dt}{a(t)\sqrt{a^2(t^*) - a^2(t)}}, \tag{3.33}$$

and

$$z(t) = +\sqrt{-2\int dt[\frac{a(t)}{\sqrt{a^2(t^*) - a^2(t)}}\Big(\int^t dt' \frac{a(t')}{\sqrt{a^2(t'^*) - a^2(t')}}\Big)\Big]}. \tag{3.34}$$

To write (3.33) in this form, we have used the fact that there is a turning point of the geodesic in the bulk and at that point dx/dt diverges. For the solution above, we have taken the point to be $t = t^*$.

Given a functional form for $a(t)$, one would then try to integrate the left hand sides of the above equations. In this process, three integration constants would appear. However, all of these can be fixed by boundary conditions. The constant appearing from (3.33) can be set to zero by using $x \to x +$ constant symmetry of the metric. Other two constants arise from the two integrations in (3.34). Both of them can be fixed - (1) by demanding $dz/dx = 0$ at the turning point of the geodesic in $z - x$ plane and (2) by requiring $z = 0$ for $t = t_0$.

We now argue that since the AdS-Kasner metric has anisotropic scaling symmetry, the constant $a(t^*)$ can be scaled away. Taking $a(t) = t^p$ and defining new coordinates $\bar{z}, \bar{t}$ and $\bar{x}$ as

$$z = t^*\bar{z}, \quad t = t^*\bar{t}, \quad x = {t^*}^{1-p}\bar{x} \tag{3.35}$$

we can re-write (3.33) and (3.34) as

$$\bar{x}(\bar{t}) = \pm \int \Big[\frac{d\bar{t}}{\bar{t}^p\sqrt{1-\bar{t}^{2p}}}\Big], \tag{3.36}$$

and

$$\bar{z}(\bar{t}) = +\sqrt{-2\int d\bar{t}\Big[\frac{\bar{t}^p}{\sqrt{1-\bar{t}^{2p}}}\Big(\int^{\bar{t}} d\bar{t}' \frac{\bar{t}'^p}{\sqrt{1-\bar{t}'^{2p}}}\Big)\Big]}. \tag{3.37}$$

With this, (3.36) and (3.37) can be easily integrated. This gives, for *generic* p,

$$\bar{x}(\bar{t}) = \frac{\bar{t}^{1-p}}{1-p}\, {}_2F_1\Big(\frac{1}{2}, \frac{1-p}{2p}, \frac{1+p}{2p}, \bar{t}^{2p}\Big) - \frac{\sqrt{\pi}\Gamma(\frac{1-p}{2p})}{(1-2p)\Gamma(\frac{1-2p}{2p})}, \tag{3.38}$$

and

$$\begin{aligned}\bar{z}(\bar{t}) = \quad & \Big[\bar{t}^2\Big[1 - {}_3F_2\Big(\{1, \frac{1}{2p}, \frac{1}{p}\}, \{\frac{1}{2} + \frac{1}{2p}, 1 + \frac{1}{p}\}, \bar{t}^{2p}\Big)\Big] \\ & + \frac{4\sqrt{\pi}p\Gamma(\frac{1+p}{2p})\bar{t}^{1-p}}{(1-2p)\Gamma(\frac{1-2p}{2p})}\Big[{}_2F_1\Big(\frac{1}{2}, \frac{1-p}{2p}, \frac{1+p}{2p}, \bar{t}^{2p}\Big) - \sqrt{1-\bar{t}^{2p}}\Big] + c\Big]^{\frac{1}{2}}.\end{aligned} \tag{3.39}$$

Here, c is a constant which can fixed using $\bar{z} = 0$ for $\bar{t} = \bar{t}_0$. ${}_2F_1$ and ${}_3F_2$ are the hypergeometric function and the generalized hypergeometric function respectively. For some specific values of p, the solutions however simplify. In the appendix A, we provide a way to solve (3.32) and get to these results.

Having reached this far, we proceed to find the geodesic length. For the correlator,

$\langle O(x,t_0)O(-x,t_0)\rangle$, we first need to calculate the integral (3.24)

$$\mathcal{L} = \int \frac{2dt}{z}\Big[\sqrt{-1+\Big(\frac{dz}{dt}\Big)^2 + t^{2p}\Big(\frac{dx}{dt}\Big)^2}\Big], \tag{3.40}$$

with appropriate limits. Now, as for the lower limit, the turning point of the geodesic is at $t = t^*$. In terms of scaled time, it is at $\bar{t} = 1$. For the upper limit, we note that the correlator is being calculated at a constant $t = t_0$ slice. This, in terms of scaled variable, is $\bar{t} = \bar{t}_0$. We further need to UV-regulate the integral by introducing a cut-off, $\bar{\delta}$. The geodesic length is therefore

$$\mathcal{L} = \int_{\bar{t}=1}^{\bar{t}=\bar{t}_0-\bar{\delta}} \frac{2d\bar{t}}{\bar{z}}\Big[\sqrt{-1+\Big(\frac{d\bar{z}}{d\bar{t}}\Big)^2 + \bar{t}^{2p}\Big(\frac{d\bar{x}}{d\bar{t}}\Big)^2}\Big]. \tag{3.41}$$

In general, $\mathcal{L}$ is infinite. In order to render it finite, we need to subtract, from $\mathcal{L}$, the equivalent AdS part. This removes the $\bar{\delta} \to 0$ singularity in the geodesic length. Consequently, the regulated $\mathcal{L}$ will only depend on $\bar{t}_0$.

Above observation, in turn, means that the gauge theory correlator has the form

$$\langle O(x,t_0)O(-x,t_0)\rangle = e^{-m\mathcal{L}_{reg}} = f(\bar{t}_0), \tag{3.42}$$

for some function f. For general p, we are unable to evaluate this function analytically. Nevertheless, numerically it can be calculated. We provide our results later. However, for some values of p, expressions simplify and analytic computations can be done. As an illustrative example, we do it for $p = 1/3$. Results are given below.

$$\begin{aligned} \bar{z}(\bar{t}) &= \sqrt{3(\bar{t}^{\frac{4}{3}} - \bar{t}_0^{\frac{4}{3}}) + (\bar{t}^2 - \bar{t}_0^2)} \\ \bar{x}(\bar{t}) &= \pm 3\sqrt{1 - \bar{t}^{\frac{2}{3}}}. \end{aligned} \tag{3.43}$$

The geodesic length turn out to be

$$\begin{aligned}
\mathcal{L} &= \int_{\bar{t}=1}^{\bar{t}_0-\bar{\delta}} d\bar{t} \frac{2\bar{t}^{\frac{1}{3}}\sqrt{1-\bar{t}_0^{\frac{2}{3}}}(2+\bar{t}_0^{\frac{2}{3}})}{\sqrt{1-\bar{t}^{\frac{2}{3}}}(3\bar{t}^{\frac{4}{3}}+\bar{t}^{2}-3\bar{t}_0^{\frac{4}{3}}-\bar{t}_0^{2})} \\
&= 2\tanh^{-1}\left[\frac{\sqrt{1-(\bar{t}_0-\bar{\delta})^{\frac{2}{3}}}\left(2+(\bar{t}_0-\bar{\delta})^{\frac{2}{3}}\right)}{\sqrt{1-\bar{t}_0^{\frac{2}{3}}}\left(2+\bar{t}_0^{\frac{2}{3}}\right)}\right].
\end{aligned} \tag{3.44}$$

Finally, to obtain the regularized length, we need to subtract appropriate AdS contribution. Therefore,

$$\begin{aligned}
\mathcal{L}_{reg} &= \mathcal{L} - 2\log\left[\frac{\bar{t}_0^{\frac{1}{3}}}{\bar{z}(\bar{t}_0-\bar{\delta})}\right] \\
&= \log\left[\frac{4(4-\bar{t}_0^{2}-3\bar{t}_0^{\frac{4}{3}})}{\bar{t}_0^{\frac{2}{3}}}\right].
\end{aligned} \tag{3.45}$$

Therefore, we find that $f(\bar{t}_0)$ goes to zero as we take $\bar{t}_0 \to 0$. As we mentioned previously, for arbitrary p, it is not possible to evaluate the correlator analytically. However, it is straightforward to carry out a numerical computation. The result is shown in figure (3.1). Indeed, we find the correlators do not pick up singularities as we take $\bar{t}_0 \to 0$.

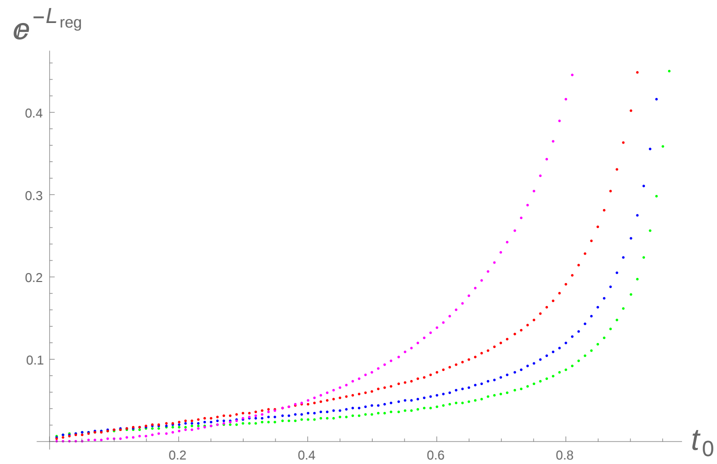

Figure 3.1: Plot of $e^{-L_{reg}}$ vs t_0 for different values of p. Magenta, blue and green are for $p = 9/10, 1/5$ and $1/7$ respectively. For $p = 1/3$, the numerical and analytical results coincide. The behaviour is shown above in red.

Having computed the correlator for $p > 0$, we proceed to make some general remarks about the correlator. First, let us notice that we can re-write (3.42) as

$$\langle O(t^{*(1-p)}\bar{x}, t^*\bar{t}_0)O(-t^{*(1-p)}\bar{x}, t^*\bar{t}_0)\rangle = f(\bar{t}_0). \tag{3.46}$$

However, since t^* is a free parameter, we are free to choose it. Let us take $t^* = \bar{t}_0^{-1}$. Then the correlator takes the form

$$\langle O(\bar{t}_0^{(p-1)}\bar{x}, 1)O(-\bar{t}_0^{(p-1)}\bar{x}, 1)\rangle = f(\bar{t}_0). \tag{3.47}$$

Calling $O(\bar{t}_0^{(p-1)}\bar{x}, 1) = \tilde{O}(\bar{t}_0^{(p-1)}\bar{x})$ and so on we get,

$$\langle \tilde{O}(\bar{t}_0^{(p-1)}\bar{x})\tilde{O}(-\bar{t}_0^{(p-1)}\bar{x})\rangle = f(\bar{t}_0). \tag{3.48}$$

Dependence on the arguments of correlator in this fashion is indeed expected in a scale invariant theory. Note that for $p > 0$, as we push the space-like surface close to $\bar{t}_0 = 0$, the separation between the two points in the correlator increases. So we capture the large separation behaviour of the correlator.

General spacelike geodesics

We now focus on to space-like geodesics in more general Kasner-AdS spacetime. Our computation will equally go through for non-vacuum solution. Here, specifically, we look for the geodesics for which the end points are situated at (x_1, x_2, x_3) and $(-x_1, -x_2, -x_3)$ at time $t = t_0$. Taking t as a parameter, the geodesic equations are

$$\ddot{x}^i + 2p_i \frac{\dot{x}^i}{t} - \dot{x}^i \sum_{j=1}^{3} p_i t^{2p_j - 1} (\dot{x}^j)^2 = 0.$$
$$z\ddot{z} + \dot{z}^2 + \sum_{j=1}^{3} t^{2p_j} (\dot{x}^j)^2 - z\dot{z} \sum_{j=1}^{3} p_j t^{2p_j - 1} (\dot{x}^j)^2 - 1 = 0. \tag{3.49}$$

Though we have complicated coupled set of equations, it turns out that much can be said about the solutions without assuming explicit values of p_is. We first define

$$y^i = \dot{x}^i, \quad K = z\dot{z}, \tag{3.50}$$

and rewrite (3.49) as

$$\dot{y}^i + \frac{2p_i}{t} y^i - y^i \sum_{j=1}^{3} p_i t^{2p_j - 1} (y^j)^2 = 0.$$
$$\dot{K} - K \sum_{j=1}^{3} p_j t^{2p_j - 1} (y^j)^2 + \sum_{j=1}^{3} t^{2p_j} (y^j)^2 - 1 = 0. \tag{3.51}$$

The first set of equations integrate to

$$y^i = C_{ij} t^{-2(p_i - p_j)} y^j, \tag{3.52}$$

where C_{ij}s are the integration constants satisfying $C_{ji} = C_{ij}^{-1}$. We now assume it there is a turning point of the geodesic and we take that to be $t = t^*$. Using (3.52) in the first equation of (3.51) and solving for y^i, we get

$$y^i = \pm \frac{t^{-2p_i}}{\sqrt{t^{-2p_i} - 3t^{*-2p_i} + \sum_{j\neq i} t^{*-2(p_i-p_j)} t^{-2p_j}}}. \tag{3.53}$$

To fix the integration constants, we have used the boundary condition $dt/dx^i = 0$ at $t = t^*$ and also the fact that the equations in (3.51) have certain scaling symmetry. To elaborate further, we note that under the transformation

$$z = \lambda\bar{z}, \quad t = \lambda\bar{t}, \quad x^i = \lambda^{1-p_i}\bar{x}^i, \tag{3.54}$$

with λ constant, the forms of the equations do not change. This alone can be used to fix all the C_{ij}s. We get

$$C_{ij} = t^{*p_i - p_j}. \tag{3.55}$$

Hence,

$$x^i = \pm \int dt \ \frac{t^{-2p_i}}{\sqrt{t^{-2p_i} - 3t^{*-2p_i} + \sum_{j\neq i} t^{*-2(p_i-p_j)} t^{-2p_j}}}. \tag{3.56}$$

The constants arising from these integrals are to be fixed using the boundary conditions $x^i = 0$ at $t = t^*$. We can now use this to rewrite K (and therefore z) of the second equation in (3.51) in the form

$$z(t) = \sqrt{\int \frac{2}{g_1(t)} \Big[\int_1^t d\tau \ \Big(1 - g_2(\tau)\Big) g_1(\tau) + C_1 \Big] dt + C_2}, \tag{3.57}$$

with

$$g_1(t) = e^{-\int^t (\sum_{j=1}^3 p_j \tau^{2p_j - 1} y^{j2}) d\tau}, \quad g_2(t) = \sum_{j=1}^{3} t^{2p_j} y^{j2}. \tag{3.58}$$

In (3.57), $C_{1,2}$ are the integration constants which are to be fixed using $z = 0$ at $t = t_0$ and z has a maximum at $t = t^*$.

To proceed further, one needs to use explicit values of p_is. For some set of values, the integrations can be analytically performed and for the rest, numerical means are required. For AdS-Milne, it is particularly easy to solve the equations, calculate the regulated length and obtain the correlator. The result matches with (3.11) in the large Δ limit for the space-like separated points. We do not reproduce the calculation here.

As a nontrivial example, let us next look at the case of $p_1 = p_2 = 1/2$ and $p_3 = 0$. Metric now has a true singularity in the past. Here (3.56) and (3.57) can be explicitly evaluated. The solutions are

$$\begin{aligned}
\bar{x}^1(\bar{t}) &= \pm\sqrt{2}\Big(\sin^{-1}\sqrt{\bar{t}} - \frac{\pi}{2}\Big),\\
\bar{x}^2(\bar{t}) &= \pm\sqrt{2}\Big(\sin^{-1}\sqrt{\bar{t}} - \frac{\pi}{2}\Big),\\
\bar{x}^3(\bar{t}) &= \pm\frac{1}{\sqrt{2}}(\frac{\pi}{2} - \sqrt{\bar{t}-\bar{t}^2}\sin^{-1}\sqrt{\bar{t}}),
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
\bar{z}(\bar{t}) = \frac{1}{\sqrt{2}}\Big[&(\bar{t}-\bar{t}_0)(\bar{t}+\bar{t}_0-1) - 2\sqrt{(1-\bar{t})\bar{t}}\sin^{-1}\sqrt{1-\bar{t}} - (\sin^{-1}\sqrt{1-\bar{t}})^2\\
&+ (\sin^{-1}\sqrt{1-\bar{t}_0})^2 + 2\sqrt{(1-\bar{t}_0)\bar{t}_0}\sin^{-1}\sqrt{1-\bar{t}_0}\Big]^{\frac{1}{2}}.
\end{aligned} \tag{3.60}$$

Here we have used the scaling symmetry of (3.54) with $\lambda = t^*$ to set the turning point to one. The parameter t_0 appearing in the above equation is the space-like surface on which the correlator would be calculated.

These solutions can now be used to compute the regularized geodesic length by subtracting the equivalent AdS part from

$$\mathcal{L} = -\int_{\bar{t}=1}^{\bar{t}=\bar{t}_0+\bar{\delta}} \frac{2d\bar{t}}{\bar{z}}\Big[\sqrt{-1+\Big(\frac{d\bar{z}}{d\bar{t}}\Big)^2 + \sum_{j=1}^{3}\bar{t}^{2p_j}\Big(\frac{d\bar{x}^j}{d\bar{t}}\Big)^2}\Big] \tag{3.61}$$

and then to use it in (3.24) to evaluate the correlator. It turns out that, for the situation we are considering, the integral can not be done analytically. However, we obtain it numerically and the result is shown in the figure. What we see is that the correlator goes to zero as we take the space-like surface t_0 close to the initial singularity.

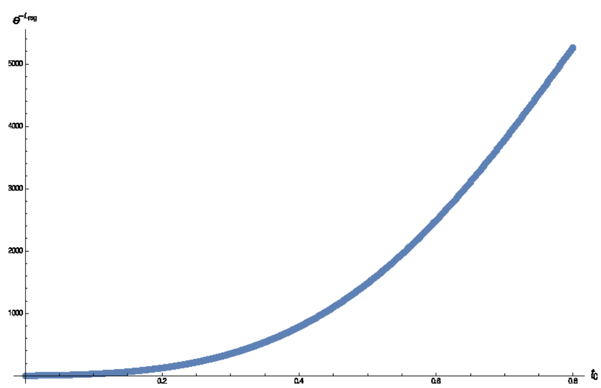

Figure 3.2: Plot of $e^{-L_{reg}}$ as a function of t_0 for $p_1 = p_2 = \frac{1}{2}, p_3 = 0$ non-vacuum AdS-Kasner solution. As the space-like surface at t_0 is pushed towards $t = 0$, the correlator vanishes.

3.4.2 AdS-Kasner Soliton

As obvious from the metric, AdS-Kasner geometry has a horizon singularity at $z = \infty$. We now ask as to how the correlator would change if we could somehow cap this singularity. This brings us to the AdS-Kasner soliton metric. Details of the metric and its appearance from the near-brane limit of eleven dimensional supergravity have already been discussed in the previous chapter. We find that the behaviour of the correlator qualitatively remains the same.

The AdS_7 Kasner soliton is given by

$$ds^2 = \frac{1}{z^2}\left[-d\bar{t}^2 + \sum_{i=1}^{4} \bar{t}^{2\alpha_i} d\bar{x}_i^2 + \left(1 - \frac{z^6}{z_0^6}\right) d\bar{\theta}^2 + \left(1 - \frac{z^6}{z_0^6}\right)^{-1} dz^2\right]. \tag{3.62}$$

As before, without any loss of generality, we can consider equal time correlators where the boundary points are separated only in x_1 direction. The geodesic equations are:

$$-\frac{2\dot{z}}{z} + \alpha t^{-1+2\alpha}\dot{x}^2 = f(t), \tag{3.63}$$

$$\ddot{x} + \frac{2\alpha}{t}\dot{x} - \frac{2}{z}\dot{x}\dot{z} = f(t)\dot{x}, \tag{3.64}$$

$$\frac{\left(z^6 - z_0^6\right)\left(-\dot{x}^2\left(z^6 - z_0^6\right)t^{2\alpha} + z^6 + zz_0^6\ddot{z} - z_0^6\right) + \dot{z}^2\left(z_0^6 - 4z^6\right)z_0^6}{z_0^6\left(z^7 - zz_0^6\right)} = f(t)\dot{z}, \tag{3.65}$$

where for notational simplicity we have denoted x_1 as x, avoided the bars from the variables and also called α_1 as α. In the above equations, $f(t)$ is a function of t.

Now substituting $f(t)$ from (3.63) into (3.64) and (3.65) we obtain:

$$\ddot{x}t = \alpha\dot{x}\left(-2 + t^{2\alpha}\dot{x}^2\right), \tag{3.66}$$

$$\left(z^6 - z_0^6\right)\left(z^6 + z_0^6 z\ddot{z} - z_0^6\right) - \dot{z}^2 z_0^6\left(2z^6 + z_0^6\right) - \dot{x}^2\left(z^6 - z_0^6\right)t^{2\alpha-1}\left(t\left(z^6 - z_0^6\right) + \alpha z\dot{z}z_0^6\right) = 0. \tag{3.67}$$

We further concentrate on the case $\alpha = \frac{1}{3}$ to see a parallel with the case of the Kasner example we considered in the previous section. For other positive α, qualitative behaviour of the correlator remains same. Equation (3.66) can be solved analytically. We substitute the solution in (3.67) and re-express the z-equation as a differential equation in x.

$$\begin{aligned} &\left(x^2-9\right)^3 z^{12} + z_0^{12}\left[81\left\{\left(x^2-9\right)zz'' + \left(x^2-9\right)z'^2 - 2xzz'\right\} + \left(x^2-9\right)^3\right] \\ + \quad &z_0^6 z^6\left[-81\left(x^2-9\right)zz'' + 162z'\left\{\left(x^2-9\right)z' + xz\right\} - 2\left(x^2-9\right)^3\right] = 0\,. \end{aligned} \tag{3.68}$$

Unlike its counterpart in Kasner-AdS, this equation cannot be solved analytically. However we do find numerical solutions implementing the boundary conditions, namely,

- $\frac{dz}{dx} = 0$ at the turning point, $t = t^*$ of the geodesic.

- $z = 0$ at $t = t_0$.

Further, the geodesic length can be written as

$$L = \frac{2}{9}\int_0^{x_0-\delta} \frac{dx}{z(x)} \left(x\sqrt{9-x^2}\sqrt{\frac{81 z_0^6 z'(x)^2}{x^2\left(9-x^2\right)\left(z_0^6 - z(x)^6\right)} + \frac{9}{x^2} - 1} \right). \tag{3.69}$$

Here x_0 is related to the fixed time-slice t_0 at the boundary through the solution of (3.66).

$$x_0 = \pm 3\sqrt{1 - t_0^{\frac{2}{3}}}. \tag{3.70}$$

Coordinates in (3.69) and (3.70) are all scaled coordinates as per (3.35) so that the turning point is now at $t = 1$. δ is a sharp cut-off in x and signifies the UV cut-off near AdS boundary. The singularity $\delta = 0$ can however be taken care of by subtracting from it the corresponding length in AdS with the same UV cut-off, δ, namely

$$L_{AdS} = 2\,\log\left[\frac{t_0^{\frac{1}{3}}}{z(x_0-\delta)}\right]. \tag{3.71}$$

In figure (3.3), we plot $e^{-L_{reg}}$ as a function of t_0 where

$$L_{reg} = L - L_{AdS}. \tag{3.72}$$

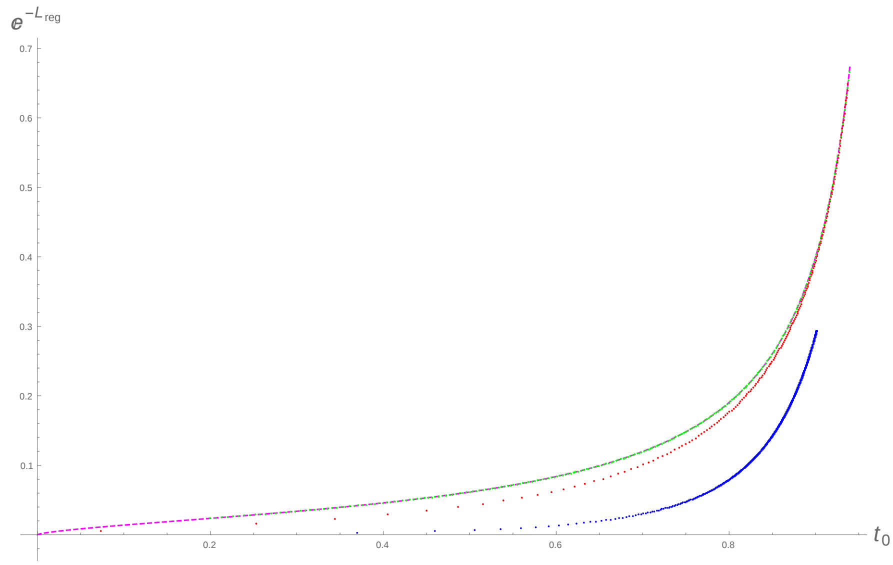

Figure 3.3: Plot of $e^{-L_{reg}}$ vs t_0 for AdS_7 Kasner soliton. Blue, red and green curves are for $z_0 = .5, 1$ and 10 respectively. For comparison, we also showed, in magenta, $e^{-L_{reg}}$ for pure AdS_7 Kasner solution.

The correlator qualitatively shows a behaviour similar to the one we argued for the Kasner geometry. We see here that the function $e^{-L_{reg}}$ goes to zero smoothly as we tune $t_0 \to 0$. We further see from the figure that, as z_0 increases, the plots more and more resemble that of Kasner-AdS. This is expected. As the point z_0 moves away from the boundary, the correlator sense less of the bulk solitonic geometry.

Following the similar analysis done in case of AdS-Kasner, one can analyse the general geodesic equations and hence the correlators for the AdS-Kasner soliton. The qualitative results remain unchanged. For the illustrative purpose, we do it here for $p_1 = p_2 = 1/2, p_3 = 0$. While the solutions for the geodesic equations for x^is remain same as (3.59), the one for the z changes. It is given by

$$\ddot{z} + \Gamma^z_{tt} + \sum_{i=1}^{3} \Gamma^z_{x_i x_i} \dot{x}_i^2 + (\Gamma^z_{zz} + \frac{2}{z})\dot{z}^2 - \frac{1}{2}\dot{x}_1^2 \dot{z} - \frac{1}{2}\dot{x}_2^2 \dot{z} = 0, \tag{3.73}$$

where

$$\Gamma^z_{tt} = -\Gamma^z_{x_3x_3} = \frac{(z-z_0)(z^4+z^3z_0+z^2z_0^2+zz_0^3+z_0^4)}{zz_0^5},$$
$$\Gamma^z_{x_1x_1} = \Gamma^z_{x_2x_2} = t\Gamma^z_{x_3x_3},$$
$$\Gamma^z_{zz} = -\frac{7z^5-2z_0^5}{2z(z-z_0)(z^4+z^3z_0+z^2z_0^2+zz_0^3+z_0^4)}. \quad (3.74)$$

The length of the geodesic now reads

$$\mathcal{L} = -2\int \frac{dx}{z(x)}\frac{\sin(\sqrt{2}x)}{\sqrt{2}} \left[\frac{2}{f(z)\sin^2(\sqrt{2}x)}z'(x)^2 - 1 + 4\cos^2(\frac{x}{\sqrt{2}})\,\mathrm{cosec}^2(\sqrt{2}x) + \cot^2(\frac{x}{\sqrt{2}})\right]^{\frac{1}{2}}, \quad (3.75)$$

where $f(z) = 1 - z^5/z_0^5$ and the prime denotes derivative with respect to the argument. Equation (3.73) and the integral in (3.75) can be solved numerically. The result of the correlator as a function of t_0 is shown in figure (3.4). We clearly notice that the qualitative behaviour of the correlator is *insensitive* to the presence or the absence of the horizon deep in the AdS bulk.

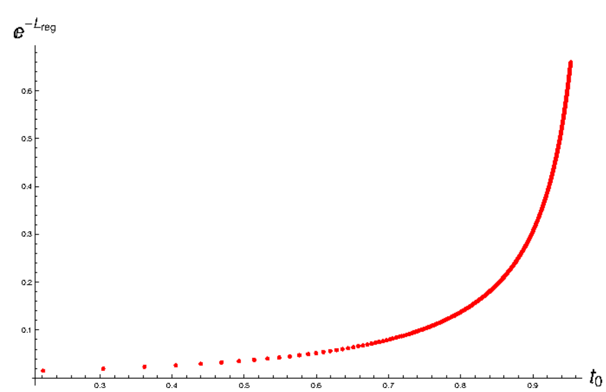

Figure 3.4: Plot of $e^{-L_{reg}}$ as a function t_0 for the soliton solution.

3.5 Non Singular Behaviour And Comparison With Other Results In The Literature

The study of spacelike two point correlator shows that it is non singular along an expanding direction for the above AdS-Kasner and AdS-Kasner soliton geometries. From the previous plots we clearly see that the correlator decreases to zero near the singularity ($t = 0$). This feature is not entirely expected from the beginning. Although we do not have a fully satisfactory explanation for this behaviour but some light can still be shed.

The correlators we have just computed is the expectation value of the product of the scalar operator at different positions with respect to some state in the boundary field theory. From our study in the previous chapter, we found from the flat space embedding that a five dimensional AdS-Kasner spacetime can be embedded in a nine dimensional flat space as a hyperboloid. This implies that there are three constraint surfaces more than the Poincare AdS and these constraints break the full conformal symmetry $SO(2,4)$ of the AdS. In general, this breaking of conformal symmetry implies that the boundary CFT is not in the ground state but in some excited state. The geodesic approximation picks a state in the gauge theory – presumably the right state – and provides us with an answer. However, to identify the state uniquely, one needs to solve the massive scalar field equation in the bulk and implement the right boundary condition. At the moment, this is still lacking. This is due to the inherent ambiguities associated with the cosmological backgrounds.

It is perhaps prudent at this point to compare our results with the existing ones. In [35], [36], the correlators are computed for space-like separated operators along the direction with $p_i < 0$. It is found that the correlators become singular as the geodesics approach the past singularity. This blowing up of the space-like correlator is subsequently argued to be a consequence of boundary conformal field theory picking up a non-normalizable state. In the present context, for $p_i > 0$, we see that the geodesic approximation selects a perfectly normalizable CFT state with the usual behaviour of space-like correlator. Furthermore, we note that the AdS-Kasner metric can be written as a conformal to anisotropic de Sitter metric by using a coordinate transformation

$$ds^2 = \frac{1}{z^2}\left[-dt^2 + \sum_{i=1}^{3} t^{2p_i} dx_i^2 + dz^2\right] \tag{3.76}$$
$$= \frac{e^{2\tau}}{z^2}\left[-d\tau^2 + \sum_{i=1}^{3} e^{2\tau(p_i-1)} dx_i^2 + dz^2\right],$$

where $t = e^{\tau}$. The second metric in the above expression is conformal to anisotropic de Sitter spacetime. Now, at any fixed time t_0 the proper boundary separation along x_i in anisotropic de Sitter space is $\mathcal{L}_{bdy} = e^{\tau(p_i-1)}\Delta x_i = t_0^{p_i-1}\Delta x_i$. Using this fact and several other observations, the authors of [35] proposed that for general $p < 1$, the power law fall off of the correlator satisfies

$$\langle O(x, t_0) O(-x, t_0)\rangle \propto \mathcal{L}_{bdy}^{-\frac{2\Delta}{1-p}}. \tag{3.77}$$

From this expression we see that as $t_0 \to 0$ the proper boundary separation in anisotropic de Sitter diverges and the correlator decreases to zero.

Before ending this chapter we want to mention a common feature of our plots. We see that the correlator diverges at the turning point $t = t_*$. This divergence occurs due to that fact that when the boundary and the turning point of geodesic come very close and the length (unregulated) of the geodesic becomes of the order of the UV cut off ϵ. This is the standard UV divergence of the theory due to very short distance separation.

3.6 Appendix

Solving of the geodesic equations

Here we discuss a way to solve equations (3.32). It is best to define a new time coordinate

η such that

$$\eta = \int \frac{dt}{a^2(t)}. \tag{3.78}$$

The first equation in (3.32) then reduces to

$$\frac{d^2x}{d\eta^2} - \frac{1}{a^3}\frac{da}{d\eta}\Big(\frac{dx}{d\eta}\Big)^3 = 0. \tag{3.79}$$

Integrating twice, we have

$$x(\eta) = \pm \int \frac{a(\eta)d\eta}{\sqrt{c_1 a(\eta)^2 + 1}} + c_2. \tag{3.80}$$

Here c_1 and c_2 are the integration constants. Now c_1 can be fixed using the boundary condition: at the turning point $\eta = \eta^*$, dx/dt or equivalently $dx/d\eta$ is infinity. This gives

$$c_1 = -\frac{1}{a^2(\eta^*)}. \tag{3.81}$$

Substituting this in (3.80), we can easily integrate the expression. The result is

$$x(\eta) = \frac{(1-2p)^{\frac{1-p}{1-2p}}\eta^{\frac{1-p}{1-2p}}}{1-p}\,{}_2F_1\Big(\frac{1}{2}, \frac{1}{2}(-1+\frac{1}{p}), \frac{1+p}{2p}, \eta^{\frac{2p}{1-2p}}\eta^{*-\frac{2p}{1-2p}}\Big) + c_2, \tag{3.82}$$

where we have used the fact that $a(t) = t^p$. Further, c_2 can be fixed using $x(\eta) = 0$ at $\eta = \eta^*$. This gives

$$c_2 = -\frac{\sqrt{\pi}(1-2p)^{\frac{p}{1-2p}}\eta^{*\frac{1-p}{(1-2p)}}\Gamma(-\frac{1}{2}+\frac{1}{2p})}{\Gamma(-1+\frac{1}{2p})}. \tag{3.83}$$

Going over to the t variable, we can write the above equation as

$$x(t) = \frac{t^{1-p}}{1-p}\,{}_2F_1\Big(\frac{1}{2}, \frac{1}{2}(-1+\frac{1}{p}), \frac{1+p}{2p}, (\frac{t}{t^*})^{2p}\Big) - \frac{\sqrt{\pi}\Gamma(-\frac{1}{2}+\frac{1}{2p})t^{*(1-p)}}{(1-2p)\Gamma(-1+\frac{1}{2p})}. \tag{3.84}$$

Now we turn to the second equation of (3.32). Defining $K = zz'$, we first rewrite it as

$$\frac{dK}{dt} - aa'x'^2K + a^2x'^2 - 1 = 0. \tag{3.85}$$

After going to η variable, above equation can easily be integrated. This gives

$$K = \frac{c_1 a}{\sqrt{c_1 a^2 + 1}} \int \frac{a^3}{\sqrt{c_1 a^2 + 1}} d\eta + c_3, \tag{3.86}$$

where c_3 is an integration constant and c_1 has been defined earlier. More explicitly, we get

$$K = -\frac{(1-2p)^{\frac{2p}{1-2p}} \eta^{\frac{p}{1-2p}}}{\sqrt{\eta^{*\frac{2p}{1-2p}} - \eta^{\frac{2p}{1-2p}}}} \Big[\int d\eta \frac{\eta^{\frac{3p}{1-2p}}}{\sqrt{\eta^{*\frac{2p}{1-2p}} - \eta^{\frac{2p}{1-2p}}}} + c_3 \Big]. \tag{3.87}$$

It is easy to show that, for dz/dx to vanish at $\eta = \eta^*$, the expression inside the brackets has to vanish. This, in turn, fixes c_3. This gives

$$\begin{aligned} K = \quad & -\frac{(1-2p)^{\frac{2p}{1-2p}} \eta^{\frac{p}{1-2p}}}{\sqrt{\eta^{*\frac{2p}{1-2p}} - \eta^{\frac{2p}{1-2p}}}} \Big[(1-2p) \eta^{\frac{1-p}{1-2p}} \{ \eta^{*\frac{p}{1-2p}} {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}(-1+\frac{1}{p}), \frac{1+p}{2p}, (\frac{\eta}{\eta^*})^{\frac{2p}{1-2p}}\Big) \\ & - \sqrt{\eta^{*\frac{2p}{1-2p}} - \eta^{\frac{2p}{1-2p}}} \} - \frac{2p\sqrt{\pi}\eta^{*\frac{1}{1-2p}} \Gamma(\frac{1+p}{2p})}{\Gamma(\frac{1-2p}{2p})} \Big]. \end{aligned} \tag{3.88}$$

Now using the property

$${}_2F_1(a,b,c,z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b, c, z), \tag{3.89}$$

the expression for K can be simplified to

$$K = (1-2p)^{\frac{1}{1-2p}}\eta^{\frac{1}{1-2p}}\Big[1 - {}_2F_1\Big(\frac{1}{2p}, 1, \frac{1+p}{2p}, (\frac{\eta}{\eta^*})^{\frac{2p}{1-2p}}\Big)\Big] + \frac{2p(1-2p)^{\frac{2p}{1-2p}}\eta^{\frac{p}{1-2p}}\sqrt{\pi}\eta^{*\frac{1}{1-2p}}\Gamma(\frac{1+p}{2p})}{\Gamma(\frac{1-2p}{2p})\sqrt{\eta^{*\frac{2p}{1-2p}} - \eta^{\frac{2p}{1-2p}}}}. \tag{3.90}$$

This gives

$$z^2 = \int 2(1-2p)^{\frac{1+2p}{1-2p}}\eta^{\frac{1+2p}{1-2p}}\Big[1 - {}_2F_1\Big(\frac{1}{2p}, 1, \frac{1+p}{2p}, (\frac{\eta}{\eta^*})^{\frac{2p}{1-2p}}\Big)\Big]d\eta + \frac{4\sqrt{\pi}p(1-2p)^{\frac{4p}{1-2p}}\eta^{*\frac{1}{1-2p}}\Gamma(\frac{1+p}{2p})}{\Gamma(\frac{1-2p}{2p})}\int\frac{\eta^{\frac{3p}{1-2p}}}{\sqrt{\eta^{*\frac{2p}{1-2p}} - \eta^{\frac{2p}{1-2p}}}}d\eta + \tilde{c} \tag{3.91}$$

where $\tilde{c}$ is a constant. Carrying out the integrations, we finally get

$$z = \Big[(1-2p)^{\frac{2}{1-2p}}\eta^{\frac{2}{1-2p}}\Big[1 - {}_3F_2\Big(\{1, \frac{1}{2p}, \frac{1}{p}\}, \{\frac{1}{2} + \frac{1}{2p}, 1 + \frac{1}{p}\}, (\frac{\eta}{\eta^*})^{\frac{2p}{1-2p}}\Big)\Big] + \frac{4\sqrt{\pi}p(1-2p)^{\frac{1+2p}{1-2p}}\eta^{*\frac{1}{1-2p}}\Gamma(\frac{1+p}{2p})\eta^{\frac{1-p}{1-2p}}}{\Gamma(\frac{1-2p}{2p})}\Big[-\sqrt{\eta^{*\frac{2p}{1-2p}} - \eta^{\frac{2p}{1-2p}}} + \eta^{*\frac{p}{1-2p}}{}_2F_1\Big(\frac{1}{2}, \frac{1-p}{2p}, \frac{1+p}{2p}, (\frac{\eta}{\eta^*})^{\frac{2p}{1-2p}}\Big)\Big] + \tilde{c}\Big]^{\frac{1}{2}}. \tag{3.92}$$

In terms of variable t, we therefore find

$$z(t) = \Big[t^2\Big[1 - {}_3F_2\Big(\{1, \frac{1}{2p}, \frac{1}{p}\}, \{\frac{1}{2} + \frac{1}{2p}, 1 + \frac{1}{p}\}, (\frac{t}{t^*})^{2p}\Big)\Big] + \frac{4\sqrt{\pi}p\Gamma(\frac{1+p}{2p})t^{*1+p}t^{1-p}}{(1-2p)\Gamma(\frac{1-2p}{2p})}\Big[{}_2F_1\Big(\frac{1}{2}, \frac{1-p}{2p}, \frac{1+p}{2p}, (\frac{t}{t^*})^{2p}\Big) - \sqrt{1 - (\frac{t}{t^*})^{2p}}\Big] + \tilde{c}\Big]^{\frac{1}{2}} \tag{3.93}$$

Equations (3.84) and (3.93) are used in the main text.

Chapter 4

Space dependent AdS-Kasner

We have seen in the previous chapters that the Kasner geometry, with the space-like singularity, can be extended into the bulk. Consequently, it was possible to study field theory on the Kasner spacetime via its gravity dual. It turns out that much of the computations developed earlier can be used to study field theory on Kasner-like geometry with *timelike* singularity [37]. The metric here is space dependent and inhomogeneous. It is also Ricci flat and can be embedded, in a straightforward manner, into AdS.

In a recent work [70], a space dependent version of the Kasner geometry is discussed wherein the space-like singularity of the Kasner geometry is replaced by a timelike singularity. It turns out that the four dimensional timelike Kasner geometry can be obtained as the IR limit of a deformation of a planar AdS_4 Black hole geometry [71]. Interestingly, as we discuss later, a four dimensional timelike Kasner geometry can be embedded in a five-dimensional AdS spacetime and the resultant configuration can be obtained as a solution of 9 + 1 dimensional type-IIB Supergravity in the presence of a nonzero self-dual five-form field strength and a constant dilaton profile.

In contrast to the space-like singularities, which is relevant to the cosmology, the timelike case may be well suited for the study of black hole interiors. An example of timelike singularity can be found within the structure of Reissner-Nordstrm (RN) black hole. However the RN geometry being a non-vacuum geometry, its extension into the bulk is rather difficult. Also because of the presence of the horizon, holographic study of the black hole interior is conceptually difficult [73]. It however turns out that the near-singularity behaviour of RN black hole geometry can be well approximated by the non-vacuum version of the timelike cousin of the Kasner geometry of the previous chapters. So as a preliminary step, one may wish to investigate if AdS/CFT correspondence helps us to gain some insights into the physics of timelike singularity by studying gauge theory on the timelike Kasner background. This is the aim in this chapter.

This chapter is organized as follows. In the next section we discuss the embedding of the four dimensional space dependent Kasner spacetime with a timelike singularity in five-dimensional locally AdS geometry. Here we show that this AdS-Kasner geometry emerges from a $D3$-brane configuration coupled to a five form gauge field with a purely space dependent profile. Subsequently, we present a study of the timelike singularity of the boundary Kasner spacetime. This singularity is probed by a co-dimension 4 curve. Within certain approximations discussed later, it enables us to extract the correlators of the scalar operators of the dual field theory out of the information encoded in the bulk geometry. Later, we give a description of the co-dimension 2 surface and discuss its relevance in evaluating holographic entanglement entropy near the singularity.

4.1 D3 brane with timelike Kasner world volume

Besides the static D-branes of odd space dimensions, the IIB string theory admits *time dependent* branes as discussed in [13]. In the following we present another solution of type IIB supergravity where brane world volume is Kasner-like but with timelike singularity. Consider, for example, the case of D3 brane. We recall the equations of motion following from the relevant part of standard IIB supergravity action are,

$$
\begin{aligned}
R^{\mu}_{\nu} &= \frac{1}{2}\partial^{\mu}\phi\partial_{\nu}\phi + \frac{1}{2\times 5!}\left(5F^{\mu\xi_2\ldots\xi_5}F_{\nu\xi_2\ldots\xi_5} - \frac{1}{2}\delta^{\mu}_{\nu}F^2_5\right), \\
&\partial_{\mu}(\sqrt{g}F^{\mu\xi_2\ldots\xi_5}) = 0, \\
&\nabla^2\phi = 0. \qquad (4.1)
\end{aligned}
$$

These equations can be solved by the following metric and five form field,

$$
ds^2 = \left(1+\frac{l^4}{\rho^4}\right)^{-\frac{1}{2}}\left[-r^{2p_t}dt^2 + r^{2p_1}dx_1^2 + r^{2p_2}dx_2^2 + dr^2\right] + \left(1+\frac{l^4}{\rho^4}\right)^{\frac{1}{2}}\left[d\rho^2 + \rho^2 d\Omega_5^2\right], \qquad (4.2)
$$

$$
F_{tx_1x_2r\rho} = \frac{4l^4 r^{p_t+p_1+p_2}\rho^3}{(l^4+\rho^4)^2}, \qquad F_{ijklm} = \sqrt{-g}\,\epsilon_{tx_1x_2r\rho ijklm}F^{tx_1x_2r\rho} \qquad (4.3)
$$

$$
\phi = 0, \qquad (4.4)
$$

provided

$$
p_t + p_1 + p_2 = 1 \quad \text{and } p_t^2 + p_1^2 + p_2^2 = 1. \qquad (4.5)
$$

Here, i, j, k, l, m are the indices on S^5. Note here, the world volume has a non-trivial geometry $ds^2 = -r^{p_t}dt^2 + r^{2p_1}dx_1^2 + r^{2p_2}dx_2^2 + dr^2$. This geometry is Ricci flat and singular. The full timelike Kasner-brane geometry (4.2) has a timelike singularity at $r = 0$, one can verify this by calculating Kretschmann scalar $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$.

We now take *near horizon* limit, $\rho \to 0$ of above geometry. We get

$$
ds^2 = \frac{\rho^2}{l^2}\left[-r^{p_t}dt^2 + r^{2p_1}dx_1^2 + r^{2p_2}dx_2^2 + dr^2\right] + \frac{l^2}{\rho^2}d\rho^2 + l^2 d\Omega_5^2\,, \qquad (4.6)
$$

with

$$
F_{tx_1x_2r\rho} = \frac{4r\rho^3}{l^4}, \ \text{ giving potential } C_{tx_1x_2r} = \frac{r\rho^4}{l^4}. \qquad (4.7)
$$

The spacetime geometry expressed in (4.6) is very similar to Kasner-$AdS_5 \times S^5$ found in [13].

We make a coordinate transformation as $z = \frac{1}{\rho}$. We take $l = 1$ for convenience. In this coordinates our new Kasner-AdS_5 geometry becomes

$$ds^2 = \frac{1}{z^2}\left[-r^{2p_t}dt^2 + r^{2p_1}dx_1^2 + r^{2p_2}dx_2^2 + dr^2 + dz^2\right] . \tag{4.8}$$

The boundary of this geometry is at $z = 0$ and it has a singularity at $r = 0$. We consider a quantum field theory resides on the boundary of (4.8). We may think of this QFT to be some deformed CFT whose dual is given by (4.8).

4.2 Investigating Timelike Singularity

Let us consider co-dimension 2 and co-dimension 4 extremal space-like surfaces in the boundary. Both of them extend into the bulk. The co-dimension 4 extremal space-like curve i.e. the geodesic, gives correlator of an operator of the boundary conformal field theory in the limit of large conformal dimensions, whereas extremal co-dimension 2 surface gives entanglement entropy in the same CFT.

4.2.1 Extremal Space-like Co-dimension 4 Curve in bulk

We now consider a dual gauge theory in the boundary of the bulk geometry mentioned in the previous section. We would like to compute $\langle\psi|O(x_1', r_0)O(x_1'', r_0)|\psi\rangle$, where $|\psi\rangle$ and $O(x, r_0)$ are a state and an operator of the strongly coupled theory residing on the boundary. We insert two operators in the x_1-direction at the points x_1' and x_1''. This gives us an equal time correlator at a fixed r.

Here we consider only the operators with large conformal dimensions, $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$ (d being spacetime dimensions) in the large N theory. In this limit of the theory and the operators, a well known approximate formula exists for the correlator [15] – [17]. With this approximation, as we have seen in the previous chapter, it is possible to compute the correlators. This formula in the present circumstance reads

$$\langle\psi|O(x', r_0)O(x'', r_0)|\psi\rangle = e^{-m\mathcal{L}_{reg}(x',x'')} \,, \tag{4.9}$$

where $\mathcal{L}_{reg}(x', x'')$ is the regularized length of the geodesic connecting x' and x''. We already mentioned that we would like to fix our two points along x_1 direction. That fixes time also and. consequently, we get equal time correlator. We choose two boundary points x_1' and x_1'' at a fixed radial distance $r = r_0$. Corresponding geodesic must then have two fixed end points x_1', x_1'' at the boundary $z = 0$ at $r = r_0$. For this particular calculation, therefore, the other boundary directions $x_i, i \neq 1$ and t are irrelevant. For the moment, we work with a general scale factor $a_1(r)$ along x_1. Later, we will use the explicit form $a_1 = r^p$. Here we follow the approach used in [33]. Now calling x_1 as x and a_1 as a for notational simplicity, the geodesic equations for (4.8) are given by

$$\begin{aligned} &x'' + 2\frac{a'}{a}x' + aa'x'^3 = 0, \\ &zz'' + z'^2 + x'^2a^2 + aa'zz'x'^2 + 1 = 0. \end{aligned} \tag{4.10}$$

Here, we have taken radial coordinate, r as a parameter and $'$ denotes derivative are with respect to r. General solutions of these equations can be written as

$$x(r) = \pm \int \frac{a(r_*)dr}{a(r)\sqrt{a^2(r) - a^2(r_*)}} \tag{4.11}$$

and

$$z = +\sqrt{-2\int dr\left[\frac{a(r)}{\sqrt{a^2(r) - a^2(r_*)}}\left(\int^r dr' \frac{a(r')}{\sqrt{a^2(r') - a^2(r_*')}}\right)\right]}. \tag{4.12}$$

In the above expressions r_* is the turning point inside the bulk where we impose the boundary conditions that $\frac{dr}{dx}\mid_{r=r_*}= 0$ and $\frac{dz}{dx}\mid_{r=r_*}= 0$.

One can see immediately that, with $a(r) = r^{2p}$, reality of above expression implies $0 < r_* < r_0$ for $p > 0$ and $0 < r_0 < r_*$ for $p < 0$. So for $p > 0$ we can vary r_*, the turning point, near singularity by keeping r_0(i.e the boundary $z = 0$)fixed, and for $p < 0$ we can push r_0, the boundary, near singularity by keeping r_*(the turning point) fixed. In this subsection we take p to be negative. In the next subsection, we will study correlator for positive p.

$p < 0$ case

In case of $p < 0$, our method is a bit different from [33]. The differential equations (4.10) can be converted to hypergeometric differential equations with an argument r^{2p}. But here p being negative and r being close to 0, we need to solve this hypergeometric differential equation around $r = \infty$. That is one of Kummer's 24 solutions. Plugging $p = -q$ in the differential equation (4.10), we proceed to solve it and finally retrieve the solution by replacing $q = -p$. In appendix 4.4, we present a layout of this calculation. The two integration constants appearing in the integration of the x-equation are fixed by the two boundary conditions namely a coordinate shift along the x-direction and by demanding $\frac{dr}{dx}\mid_{r=r_*}= 0$. For the z-equation, the corresponding boundary conditions are (1) $dz/dx\mid_{r=r_*}= 0$ at the turning point of the geodesic in the $z - x$ plane and (2) requiring $z = 0$ for $r = r_0$.

So with $a = r^p$, the integration in (4.11) and (4.12) can be performed. The answers turns out to be

$$x = \pm\frac{r^{1-p}}{1-p}\sqrt{\left(\frac{r}{r_*}\right)^{2p} - 1}\left[1 - {}_2F_1\left\{1, \frac{p-1}{2p}, \frac{2p-1}{2p}, \left(\frac{r}{r_*}\right)^{-2p}\right\}\right], \qquad (4.13)$$

and

$$\begin{aligned} z &= \left\{\frac{2i\sqrt{\pi}{r_*}^{1+p}\Gamma(1-\frac{1}{2p})}{\Gamma(\frac{-1+p}{2p})} r\, {}_2F_1\left[\frac{1}{2}, -\frac{1}{2p}, 1-\frac{1}{2p}, \left(\frac{r}{r_*}\right)^{-2p}\right]\right. \\ &- \left. r^2\, {}_3F_2\left[\left\{1, \frac{1}{2}-\frac{1}{2p}, -\frac{1}{p}\right\}; \left\{1-\frac{1}{p}, 1-\frac{1}{2p}\right\}; \left(\frac{r}{r_*}\right)^{-2p}\right] + c\right\}^{1/2}, \qquad (4.14) \end{aligned}$$

where the constant c is determined by the condition $z = 0$ at $r = r_0$. The integration constants have to be chosen in such a way that the full expressions of the right hand side of equations (4.13) and (4.14) become real.

We use now the scaling symmetry present in the geometry of (4.8). The scaling symmetry of (4.8) is

$$z \longrightarrow \lambda z,\ x_i \longrightarrow \lambda^{1-p_i} x_i,\ t \longrightarrow \lambda^{1-p_t} t,\ r \longrightarrow \lambda r\ . \tag{4.15}$$

We use (4.15) with $\lambda = r_*$, to define new variable $(\bar{z}, \bar{x}, \bar{r})$ as

$$z = r_* \bar{z}, \quad r = r_* \bar{r}, \quad x = {r_*}^{1-p} \bar{x}\ . \tag{4.16}$$

In terms of these re-scaled variables, we can write

$$\bar{x} = \pm \frac{r^{1-p}}{1-p} \sqrt{r^{2p} - 1} \left[1 - {}_2F_1 \left\{ 1, \frac{p-1}{2p}, \frac{2p-1}{2p}, \bar{r}^{-2p} \right\} \right] , \tag{4.17}$$

and

$$\begin{aligned} \bar{z} \;&=\; \left\{ \frac{2i\sqrt{\pi}\,\Gamma(1-\frac{1}{2p})}{\Gamma(\frac{-1+p}{2p})} \bar{r}\, {}_2F_1 \left[\frac{1}{2}, -\frac{1}{2p}, 1 - \frac{1}{2p}, \bar{r}^{-2p} \right] \right. \\ &- \left. \bar{r}^2\, {}_3F_2 \left[\left\{ 1, \frac{1}{2} - \frac{1}{2p}, -\frac{1}{p} \right\} ; \left\{ 1 - \frac{1}{p}, 1 - \frac{1}{2p} \right\} ; \bar{r}^{-2p} \right] + c \right\}^{1/2} . \end{aligned} \tag{4.18}$$

Now $\bar{r}$ varies form $\bar{r}_0$ to 1. These expressions undergo considerable simplification for some specific values of p. We choose such values to do a further calculation.

We intend to keep the discussion general to write down formal expressions. We would like to calculate the regularized geodesic length first. This is given by

$$\mathcal{L} = \int_{\bar{r}=1}^{\bar{r}=\bar{r}_0-\delta} \frac{2d\bar{r}}{\bar{z}(\bar{r})} \sqrt{1 + \left(\frac{d\bar{z}}{d\bar{r}}\right)^2 + \bar{r}^{2p} \left(\frac{d\bar{x}}{d\bar{r}}\right)^2}\ . \tag{4.19}$$

This length turns out to be infinite for $\delta = 0$. To regularize this we subtract from $\mathcal{L}$ the infinite part of geodesic length of pure AdS. This removes $\delta \to 0$ singularity. Consequently

$$\mathcal{L}_{\text{reg}} = \lim_{\delta \to 0} \left[\mathcal{L} - 2 \log \frac{1}{\bar{z}(\bar{r} - \delta)} \right] \tag{4.20}$$

becomes finite and function of r_0 only. The correlator of our probe operator O becomes function of r_0 only because of the scaling symmetry,

$$\langle O(-x, r_0) O(x, r_0) \rangle = \langle O(-\bar{x}, \bar{r}_0) O(\bar{x}, \bar{r}_0) \rangle = e^{-m\mathcal{L}_{\text{reg}}} = f(\bar{r}_0) \, . \tag{4.21}$$

One can investigate now the nature of this correlator in the limit of timelike singularity $r_0 \to 0$.

This general calculation can be done numerically, however we already mentioned before that the expressions of $\bar{x}$ and $\bar{z}$ get simplified for specific values of p. So we have done explicit calculation of the regularized length for $p = -\frac{1}{4}$ and $p = -\frac{1}{6}$. For $p = -\frac{1}{4}$, $\bar{x}$ and $\bar{z}$ get simplified to

$$\bar{x}(\bar{r}) = \pm \frac{4}{15} \sqrt{1 - \sqrt{\bar{r}}} \left(3\bar{r} + 4\sqrt{\bar{r}} + 8 \right) \tag{4.22}$$

$$\bar{z}(\bar{r}) = \frac{4}{3} \sqrt{\bar{r}^{3/2} - \bar{r}_0^{3/2} + 3\,(\bar{r} - \bar{r}_0)}. \tag{4.23}$$

Putting them back in (4.19) one finds

$$\mathcal{L} = 2 \tanh^{-1} \left(\frac{\sqrt{1 - \sqrt{\bar{r}_0 - \delta}} \left(\sqrt{\bar{r}_0 - \delta} + 2 \right)}{\sqrt{1 - \sqrt{\bar{r}_0}} \left(\sqrt{\bar{r}_0} + 2 \right)} \right) , \tag{4.24}$$

and $\mathcal{L}_{\text{reg}}$ turns out to be

$$\mathcal{L}_{\text{reg}} = \log \left[\frac{64}{9} \left(-r_0^{3/2} - 3r_0 + 4 \right) \right] \, . \tag{4.25}$$

So the correlator

$$f(\bar{r}_0) = \left(\frac{64}{9}\right)^{-m} \left(-\bar{r}_0^{3/2} - 3\bar{r}_0 + 4\right)^{-m} . \qquad (4.26)$$

Therefore as we go towards singularity, $\bar{r}_0 \to 0$, $f(\bar{r}_0)$ goes to $\left(\frac{256}{9}\right)^{-m}$, but we consider here only highly massive operator. So we see correlator goes to 0 near singularity. One should observe here that, at $\bar{r}_0 = 1$, $f(\bar{r}_0)$ diverges, but it is not a problem because it is an infrared divergence from gravity point of view. Similar behaviour can be observed for $p > 0$ case for space-like singularity as in [33]. The reason for the occurrence of IR divergence is similar here. This particular singularity appears when r_0 goes close to the turning point. In our parametrization, the boundary ($z = 0$) is situated at $r = r_0$. In that case, the boundary is close to the turning point, and so the length contribution comes from the cutoff δ. In $\delta \to 0$ limit the geodesic length approaches to zero. Normally this produces a divergence in the two-point function. Since this divergence occurs from turning point, which is now very close to the boundary, we can interpret it as usual short distance divergence from dual field theory point of view. We show another exponent $p = -\frac{1}{6}$ to illustrate the result more clearly. Again $\bar{x}$ and $\bar{z}$ get simplified. One gets, after doing the integration (4.19),

$$\mathcal{L} = 2\tanh^{-1}\left(\frac{\sqrt{1-(\bar{r}_0-\delta)^{1/3}}\left\{3\left(\bar{r}_0-\delta\right)^{2/3} + 4\left(\bar{r}_0-\delta\right)^{1/3} + 8\right\}}{\sqrt{1-\bar{r}_0^{1/3}}\left(3\bar{r}_0^{2/3} + 4\bar{r}_0^{1/3} + 8\right)}\right) , \qquad (4.27)$$

And consequently regulated length,

$$\mathcal{L}_{\text{reg}} = \log\left[\frac{16}{25}\left(64 - 9\bar{r}_0^{5/3} - 15\bar{r}_0^{4/3} - 40\bar{r}_0\right)\right] . \qquad (4.28)$$

So the correlator, $f(\bar{r}_0)$ turns out to be

$$f(\bar{r}_0) = \left(\frac{16}{25}\right)^{-m} \left(64 - 9\bar{r}_0^{5/3} - 15\bar{r}_0^{4/3} - 40\bar{r}_0\right)^{-m} . \qquad (4.29)$$

In the limit $\bar{r}_0 \to 0$ this becomes $\left(\frac{1024}{25}\right)^{-m}$, which is almost 0 for highly massive operators. We give here a plot of correlator. We choose $m = 5$ for illustration purpose, for high value of mass it will go faster.

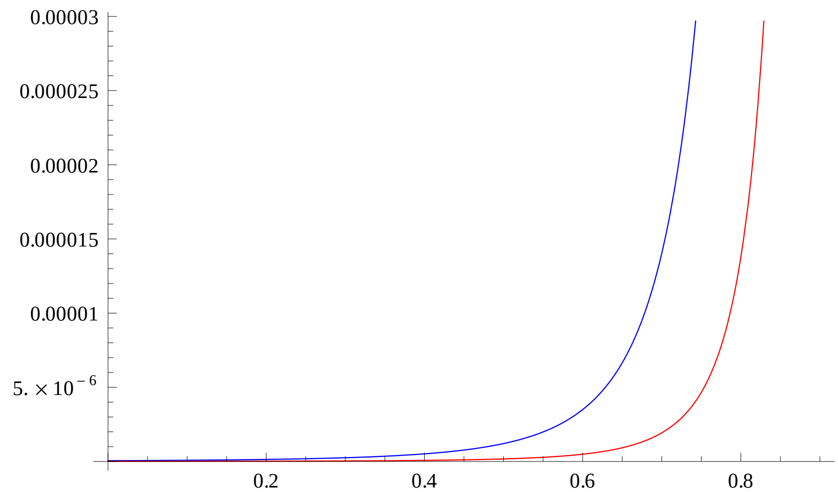

Figure 4.1: They are plots of $f(\bar{r}_0)$ vs. $\bar{r}_0$. Blue and red graphs are for $p = -\frac{1}{4}$ and $p = -\frac{1}{6}$ respectively, $m = 5$.

$p > 0$ case

Now we consider the boundary direction whose corresponding Kasner exponent is positive and therefore signify an expansion along that direction. In order to investigate the behaviour of correlators near the singularity we follow the prescriptions of [35, 36]. For the sake of brevity we fix the boundary at the radial slice $r = r_0 = 1$, and the turning point of the probe geodesic in the bulk at $r = r_*$ where we assume $r_* \to 0$. For definiteness, we take $p = \frac{1}{3}$. The geodesic equations read

$$r x'' + \frac{2}{3} x' + \frac{1}{3} r^{2/3} x'^3 = 0 \, , \tag{4.30}$$

$$z z'' + 1 + z'^2 + r^{2/3} x'^2 + \frac{1}{3} r^{-1/3} z z' x'^2 = 0 \, . \tag{4.31}$$

Here the boundary $z = 0$ is at $r = 1$. The boundary conditions $\frac{dz}{dx}|_{r=r_*} = 0$, $\frac{dr}{dx}|_{r=r_*} = 0$ and $x(r_*) = 0$ fix x and z uniquely as

$$x(r) = \pm 3{r_*}^{2/3}\sqrt{\left(\frac{r}{r_*}\right)^{2/3} - 1}\,, \tag{4.32}$$

$$z(r) = \sqrt{(1 - r^2) + 3{r_*}^{2/3}\,(1 - r^{4/3})}\,. \tag{4.33}$$

We can now compute the geodesic length between two points, $-x$ to x on the boundary. This is given by

$$\mathcal{L} = 2\int_{r=r_*}^{1-\delta} \frac{1}{z(r)}\sqrt{1 + r^{\frac{2}{3}}\left(\frac{dx}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2}\,dr\,. \tag{4.34}$$

Note that here we set a cut off near boundary at $r = 1 - \delta$. To regularize the length of the geodesic we subtract from the divergent piece appearing in the unregulated length, the equivalent AdS part and arrive at the following result, namely,

$$\mathcal{L}_{\text{reg}} = \lim_{\delta\to 0}\left[\mathcal{L} - 2\log\left(\frac{1}{z(1-\delta)}\right)\right] = \log\left(4 + 12r_*^{2/3} - 16r_*^2\right)\,. \tag{4.35}$$

The boundary correlator can now be written in terms of the regularized length of the geodesic as,

$$\langle O(-x, r_*)O(x, r_*)\rangle = \left(4 + 12r_*^{2/3} - 16r_*^2\right)^{-m}\,. \tag{4.36}$$

The correlator has a usual short distance singularity for $r_* = 1$ but when we probe the gravitational singularity by taking r_* towards 0, we find no singular behaviour. It goes to 4^{-m}, which is 0 for large mass dimensions of the boundary operator. In figure (4.2) we demonstrate the spatial behaviour of the correlator through a plot.

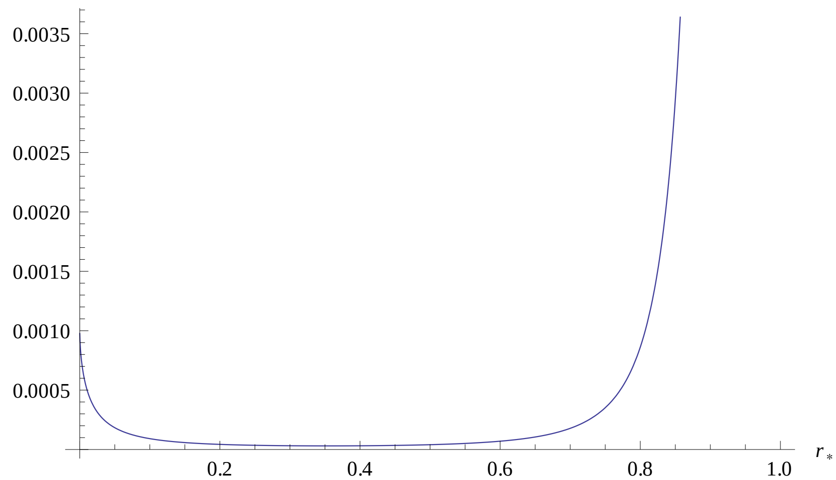

Figure 4.2: Here we plot $\langle \mathcal{O}(-x, r_*)\mathcal{O}(x, r_*)\rangle$ vs. r_* for $p = \frac{1}{3}$ with $m = 5$.

4.2.2 Extremal Space-like Co-dimension 2 Surface in bulk

In this section, we discuss properties of space-like surfaces near the singularity. These surfaces are important in the context of entanglement entropy via a AdS dual. In [76] the authors computed entanglement entropy for a confining gauge theory living on a time dependent Kasner spacetime. Here we consider a space-like surface in the background of (4.8). We consider the following region in the boundary: $x_1 \in [-\infty, \infty]$, $x_2 \in [-\infty, \infty]$ and $r \in [r_0 - \Delta r_0, r_0 + \Delta r_0]$. This surface may extend into the z-direction. To find the area we need to find the induced metric $G_{ab} = g_{\mu\nu}\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}$ on this surface, where ξ^a and ξ^b are coordinates on the surface. The area is given by

$$A = \int d^3\xi\, \sqrt{G}\,. \tag{4.37}$$

We choose a gauge where $\xi^1 = x_1, \xi^2 = x_2$ and $\xi^3 = r$. In this gauge area functional turns out to be

$$A = V\int dr \frac{r^{p_1+p_2}\sqrt{1+z'(r)^2}}{z(r)^3}\,, \tag{4.38}$$

where V is the volume of x_1 and x_2 directions. To find extremal surface one has to find Euler-Lagrange equation for area functional (4.38). It turns out to be

$$rzz'' + \left[1 + z'^2\right]\left[(p_1 + p_2)\, zz' + 3r\right] = 0 \ . \tag{4.39}$$

In the case of pure AdS, $p_1 = p_2 = 0$ and so the Euler-Lagrange equation reduces to a much simpler form which can be solved exactly [77]. In our case, it is difficult to solve the equation of motion analytically. Hence we resort to numerical means. In order to fix the boundary conditions we make use of the fact that the cross-section of the surface in the $z - r$ plane has a turning point at the point $z = z_*$ where $\frac{dz}{dx}|_{z=z^*} = 0$. We initially fix the boundary at a radial slice $r = r_0$ which is finally pushed towards the singularity $r = 0$.

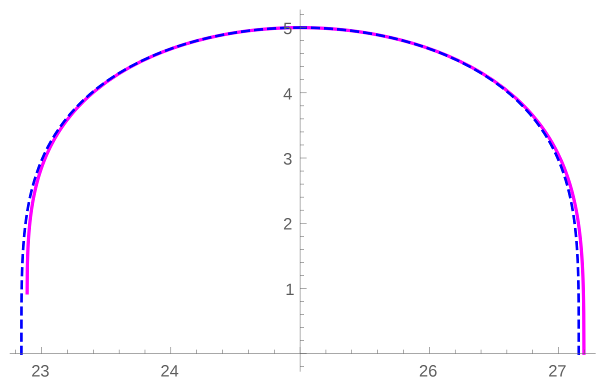

Figure 4.3: Here we plot the cross-section of the extremal surface in $z - r$ plane. The magenta curve is the surface described by 4.39 whereas dashed blue curve shows that of a pure AdS. In this plot, $z_* = 5$ and $r_0 = 25$.

Note that the surface described by (4.39) is slightly shifted from the extremal surface of pure AdS. It is actually expected, because of the presence of $r^{p_1+p_2}$ in the area functional (4.38). In the figure above we take $p_1 + p_2 = 0.33$. Note that as $p_1 + p_2$ becomes smaller, the two curves tend to coincide with each other.

In Figure 4.3, we take $r_0 = 25$. It is clearly seen from the graphs below that, as we decrease r_0, the shift becomes more prominent. But the nature of the curves remain almost same as we go towards the singularity.

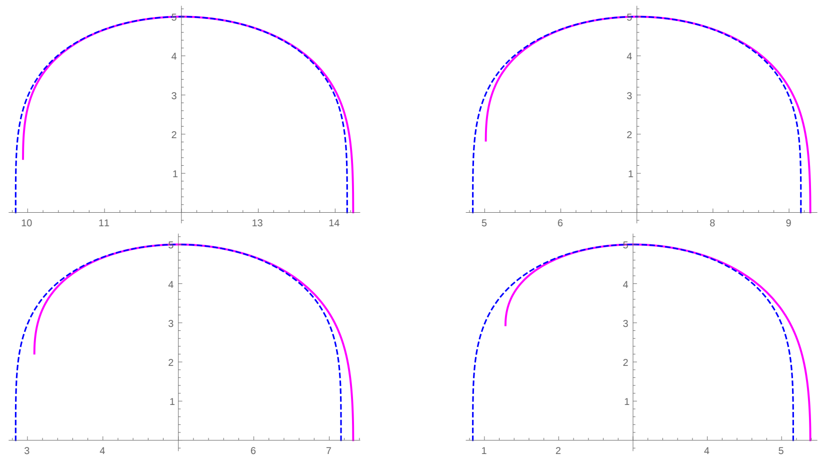

Figure 4.4: In these three plots we take, $z_* = 5$ and (a) $r_0 = 12$, (b) $r_0 = 7$, (c) $r_0 = 5$, (d) $r_0 = 3$.

To evaluate the entanglement entropy we need to compute the area of the extremal surface (4.38). The integral in (4.38) however yields a divergent result which needs to be regularized. One may contemplate a suitable scheme of regularization in the following way. Let us consider the case of pure AdS where one can set a cut off *viz.* $z = \delta$ near the boundary. The area of the extremal surface is consequently found to be

$$A = \frac{\sqrt{1 - \frac{\epsilon^6}{z_*^6}}}{\epsilon^2} - \frac{\sqrt{\pi}\Gamma\left(\frac{5}{3}\right)}{4z_*^2\Gamma\left(\frac{7}{6}\right)} + \frac{\epsilon^4\, {}_2F_1\left(\frac{1}{2}, \frac{2}{3}; \frac{5}{3}; \frac{\delta^6}{z_*^6}\right)}{4z_*^6} \,. \tag{4.40}$$

In the limit $\epsilon \to 0$, the last term in (4.40)goes to zero, whereas the first term diverges as $\frac{1}{\epsilon^2}$. In the same way one should be able to compute the regularized area of the extremal surface for space dependent AdS-Kasner geometry by subtracting the contribution $\sim \frac{1}{\epsilon^2}$ of the pure AdS part from the numerically calculated area. However the computations turn out to be very hard near the singularity, even on the numerical front, although it is possible to carry out the same calculations in a region which reasonably far from the singularity.

4.3 Discussion

In this chapter, we have studied the space dependent version of Kasner geometries within the premises of gauge/gravity duality. Our main result is the correlator calculated in section 4.2.1. A striking difference of the present computation from the case of the space-like singularity is the absence of any pole of the correlator near the timelike singularity. From our discussion in the previous chapter, we know that for time dependent Kasner geometries with negative Kasner exponents, the correlation functions are plagued by divergences from a pole at the singularity. On the other hand, for positive Kasner exponents the correlator smoothly goes to zero as it approaches the singularity, at least for large conformal dimensions. On the contrary in the timelike case, correlators are free from poles for both $p > 0$ and $p < 0$ cases. This is clear from equations (4.26) and (4.36) and from the figures (4.1) and (4.2). Hence we argue that in the present case, there is no "particle creation" near the singularity. This is somewhat expected though for a static spacetime. On the other hand, our result shows that this is true even in the case of strongly coupled theory. One possible reason for the smoothness of the correlator may be the following. The metric (4.8) near the boundary, say at a radial point $z = \epsilon$, is given by

$$ds_b^2 = \frac{1}{\epsilon^2}\left[-r^{2p_t}dt^2 + r^{2p_1}dx_1^2 + r^{2p_2}dx_2^2 + dr^2\right] . \tag{4.41}$$

Now a coordinate transformation $r = e^{r'}$ forces the boundary metric to become

$$ds_b^2 = \frac{e^{2r'}}{\epsilon^2}\left[-e^{2r'(p_t-1)}dt^2 + e^{2r'(p_1-1)}dx_1^2 + e^{2r'(p_2-1)}dx_2^2 + dr'^2\right] . \tag{4.42}$$

which implies that the boundary field theory lives on the conformal frame shown above. Along a spatial direction, say x_1, the proper boundary separation between the two points with coordinates $\pm\bar{x}_1$ is $L_{bdy} = 2e^{r'(p_1-1)}\bar{x}_1 = 2r_0^{(p_1-1)}\bar{x}_1$. However, since $p_1 < 1$ always, therefore as $r_0 \to 0$ we find $L_{bdy} \to \infty$. As this proper boundary separation, in the conformal frame, diverges, we may expect that the two-point correlation function should vanish.

In [36] the authors argued that the appearance of pole in boundary-gauge theory implies that the boundary field theory chooses non-normalizable states as a basis for the correlation functions. In the same spirit, the absence of any pole in the correlators in the present case leads us to expect the boundary basis states to be normalizable. Again from the bulk geometry, we observe that the full isometry group is broken. So, for the boundary gauge theory, the conformal group is also broken and it possibly implies that the boundary gauge theory is not in ground state but in some excited state. The precise nature of the dual field theory for the space dependent AdS-Kasner background is not very clear to us. It needs further study to understand the nature and properties of the dual gauge theory.

4.4 Appendix

Solution of Geodesic Equations

Here we give a method to solve equations (4.10). As already mentioned, these are equations convertible to a hypergeometric equation. we need to solve this equations around $r = 0$ that means the argument of hypergeometric function $(\frac{r}{r_*})^{2p}$ should be around ∞. So to do that we substitute $a(r) = r^{-q}$ right here. After solving we substitute back $q = -p$. Our first equation now becomes

$$r x''(r) - q r^{-2q} x'(r)^3 - 2q x'(r) = 0 \; . \tag{4.43}$$

Here $'$ denotes derivative with respect to argument. This equation can be solved.

$$x(r) = \pm \frac{r^{1-p}}{1-p} \sqrt{\left(\frac{r}{r_*}\right)^{2p} - 1} \left[1 - {}_2F_1\left\{1, \frac{p-1}{2p}, \frac{2p-1}{2p}, \left(\frac{r}{r_*}\right)^{-2p}\right\}\right] + c_1 \; . \tag{4.44}$$

The integration constant c_1 can be found by setting $x(r_*) = 0$.

$$c_1 = 0. \tag{4.45}$$

To solve the second equation of (4.10), Define $K = zz'$. We can rewrite the equation as

$$\frac{dK}{dr} + aa'x'^2K + a^2x'^2 + 1 = 0\ . \tag{4.46}$$

Again substituting $a = r^{-q}$ one finds

$$K'(r) - qr^{-2q-1}x'(r)^2K(r) + r^{-2q}x'(r)^2 + 1 = 0\ . \tag{4.47}$$

We know solution for x. So substituting $x'(r)$ we find

$$r\left[\left(r^{-2q} - r_*{}^{-2q}\right)K'(r) + r^{-2q}\right] - q\,r_*{}^{-2q}K(r) = 0\ . \tag{4.48}$$

Solving above equation one finds

$$K(r) = \frac{c_2}{\sqrt{r^{2q} - r_*{}^{2q}}} - r\,{}_2F_1\left(1, \frac{q+1}{2q}; 1 + \frac{1}{2q}; r^{2q}r_*{}^{-2q}\right)\ . \tag{4.49}$$

c_2 is fixed by the boundary condition $\frac{dz}{dx}\Big|_{r=1} = 0$,

$$c_2 = i\frac{\sqrt{\pi}r_*\Gamma(1 - \frac{1}{2p})}{\Gamma(\frac{-1+p}{2p})}\ . \tag{4.50}$$

Here we use $q = -p$. z is now given in terms of K as $z^2 = 2\int drK(r)$. Doing this integration and then substituting $q = -p$ we have

$$\begin{aligned} z^2(r) &= 2c_2 r {r_*}^p \, {}_2F_1\left[\frac{1}{2}, -\frac{1}{2p}, 1-\frac{1}{2p}, \left(\frac{r}{r_*}\right)^{-2p}\right] \\ &- r^2 \, {}_3F_2\left[\left\{1, \frac{1}{2}-\frac{1}{2p}, -\frac{1}{p}\right\}; \left\{1-\frac{1}{p}, 1-\frac{1}{2p}\right\}; \left(\frac{r}{r_*}\right)^{-2p}\right] + c_3 \, . \end{aligned} \tag{4.51}$$

Here the constant c_3 has to be fixed from the condition that at $r = r_0$ we have to reach boundary $z = 0$.

Chapter 5

In Lieu of a Conclusion

In this thesis, we made an attempt to study the time dependent (and static) Kasner-class of geometries within the framework of the AdS/CFT correspondence. Our investigation is evidently preliminary. For the cosmological backgrounds, due to the lack of time translational symmetry, the traditional way of computing the boundary correlators does not seem to work. In order to compute the spacelike correlators, we had to resort to the geodesic approximation. Within this approximation, it is encouraging to find that in the strong coupling gauge theory two point scalar correlators show non-singular behaviour even if the backgrounds suffer from the initial curvature singularities. However, we should remember that the geodesic approximation hides many details. For example, as mentioned in the first chapter, consider the formula

$$\langle\psi|\mathcal{O}(x)\mathcal{O}(x')|\psi\rangle \sim e^{-\Delta\mathcal{L}_{reg}(x,x')}, \tag{5.1}$$

where Δ is the dimension of the scalar operator $\mathcal{O}$ and $\mathcal{L}_{reg}$ is the regularized length of the geodesic. The computations provide us with a result but, as it is noticeable from our previous discussions, we do not yet know the details of the state $|\psi\rangle$ that the geodesic picks. The background breaks the conformal symmetry and consequently $|\psi\rangle$ is not the usual gauge theory state. It will surely be instructive to understand this better.

Can we compute some higher point correlators even within the geodesic approximation? It seems to us that this can be possible. Consider, for example, the three point scalar correlator. We take three points on the boundary and construct the geodesics taking the points pairwise. These geodesics intersect at some point in the bulk. We simply vary this point to minimize certain weighted sum of the individual geodesic lengths. In this regard, see the section (3.2) of [84] for computations in the AdS. Our preliminary analysis suggests that it is likely to work for AdS-Milne too. We leave that for our future analysis.

Finally, let us note that all our gauge theory computations are carried out at infinite 't Hoot coupling. Consequently, we have not kept the higher derivative terms in the gravitational bulk action. But since we are working with the singular gravitational backgrounds, these terms are important and we need to incorporate it in our analysis. In the presence of these terms, re-computation of even the two point correlator may shed some important light on the subject.

Bibliography

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2**(1998) 231, [hep-th/9711200]

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B **428**, 105 (1998) doi:10.1016/S0370-2693(98)00377-3 [hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**(1998) 253, [hep-th/9802150]

[4] G t Hooft, “Dimensional Reduction in Quantum Gravity”, Utrecht Preprint THU-93/26, gr-qc/9310006

[5] L. Susskind, The World As A Hologram, J. Math. Phys. **36** (1995) 6377, [hep-th/9409089]

[6] E. D’Hocker and D.Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT correspondence”, [hep-th/0201253]

[7] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, “Large N Field Theories, String Theory And Gravity,” Phys. Rept. **323** (2000) 183-386, [hep-th/9905111]

[8] J. L. Petersen, “Introduction to the Maldacena conjecture on AdS / CFT,” Int. J. Mod. Phys. A **14**, 3597 (1999) [hep-th/9902131]

[9] N.D. Birrell and P.C.W. Davies, “Quantum fields in curved space”, Cambridge University Press, 1999.

[10] L. Parker and D. Toms, “Quantum Field Theory in Curved Spacetime”, Cambridge University Press, 2009.

[11] L.C.B Crispino , A. Higuchi and G.E.A Matsas, “The Unruh effect and its applications” Rev. Mod. Phys.**80** 787 838(2008) [arxiv:0710.5373]

[12] E. Kasner, Am. J. Math. 43, 217 (1921)

[13] S. Banerjee, S. Bhowmick and S. Mukherji, “Anisotropic branes,” Phys. Lett. B **726**, 461 (2013) [arXiv:1301.7194 [hep-th]]

[14] S.R. Das, J. Michelson, K. Narayan and S.P. Trivedi, “Time dependent cosmologies and their duals”, Phys. Rev.D **74** (2006) 026002 [hep-th/0602107]

[15] V. Balasubramanian, S. F. Ross, “Holographic Particle Detection,” Phys. Rev. D **61**(2000) 044007 [hep-th/9906226]

[16] T. Banks, M.R. Douglous, G. T. Horowitz, E. J. Martinec, [hep-th/9808016]

[17] J. Louko, D. Marolf and S. F. Ross, Phys. Rev. D **62**, 044041 (2000) doi:10.1103/PhysRevD.62.044041 [hep-th/0002111]

[18] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, JHEP **0402**, 014 (2004) doi:10.1088/1126-6708/2004/02/014 [hep-th/0306170].

[19] G. Festuccia and H. Liu, JHEP **0604**, 044 (2006) doi:10.1088/1126-6708/2006/04/044 [hep-th/0506202].

[20] S. Fischetti, D. Kastor and J. Traschen, Non-Vacuum AdS Cosmologies and the Approach to Equilibrium of Entanglement Entropy, Class. Quant. Grav. **31**, no. 23, 235007 (2014) [arXiv:1407.4299 [hep-th]].

[21] I.M. Khalatnikov and S.L. Parnovsky, Phys. Lett. 66A (1978) 466

[22] B. Zwiebach, A first course in string theory, Cambridge, UK: Univ. Pr. (2004) 558 p.

[23] K. Becker, M. Becker, J. H. Schwarz, String theory and M-theory: A modern introduction, Cambridge, UK: Cambridge Univ. Pr. (2007) 739 p.

[24] M. B. Green, J. H. Schwarz, E. Witten, Superstring Theory. Vol. 1: Introduction, Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).

[25] M. B. Green, J. H. Schwarz, E. Witten, Superstring Theory. Vol. 2: Loop Ampli- tudes, Anomalies And Phenomenology, Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics)

[26] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges", Phys. Rev. Lett.**75** (1995) [hep-th/9510017]

[27] K. Skenderis,"Lecture Notes on Holographic Renormalization", Class.Quant.Grav. **19** (2002) 5849-5876 [hep-th/0209067]

[28] T. Hertog and G. T. Horowitz, "Towards a big crunch dual," JHEP **0407**, 073 (2004) doi:10.1088/1126-6708/2004/07/073 [hep-th/0406134].

[29] T. Hertog and G. T. Horowitz, "Holographic description of AdS cosmologies," JHEP **0504**, 005 (2005) doi:10.1088/1126-6708/2005/04/005 [hep-th/0503071].

[30] Ya. B. Zel'dovich, JETP Lett. 12, 307 (1970).

[31] E. Kasner, Am. J. Math. 43, 217 (1921).

[32] S.A. Fulling, L. Parker and B.L. Hu, " Conformal energy-momentum tensor in curved spacetime: Adiabatic regularization and renormalization", Phys Rev D10, 3905 (1974).

[33] S. Banerjee, S. Bhowmick, S. Chatterjee and S. Mukherji, "A note on AdS cosmology and gauge theory correlator," JHEP **1506** (2015) 043 doi:10.1007/JHEP06(2015)043 [arXiv:1501.06317 [hep-th]].

[34] S. Chatterjee, S. P. Chowdhury, S. Mukherji and Y. K. Srivastava, "Non-vacuum AdS cosmology and comments on gauge theory correlator," Phys. Rev. D **95**, 046011 (2017) arXiv:1608.08401 [hep-th].

[35] N. Engelhardt, T. Hertog and G. T. Horowitz, "Holographic Signatures of Cosmological Singularities," Phys. Rev. Lett. **113** (2014) 121602 doi:10.1103/PhysRevLett.113.121602 [arXiv:1404.2309 [hep-th]].

[36] N. Engelhardt, T. Hertog and G. T. Horowitz, "Further Holographic Investigations of Big Bang Singularities," JHEP **1507** (2015) 044 doi:10.1007/JHEP07(2015)044 [arXiv:1503.08838 [hep-th]].

[37] S. Bhowmick, S. Chatterjee "Towards time-like singularity via AdS dual," Int. J. of Mod. Phys. A 32, 1750122 (2017) arxiv:1610.0584 [hep-th]

[38] I. S. Gradshteyn and I. M. Byzhik, Table of Integvals, Series, and Products (Academic, New York, 1965).

[39] C. Fefferman, C. R. Graham, Conformal invariants, in *Élie Cartan et les Mathématiques d'Aujourd'hui*, (Astérisque, 1985), 95.

[40] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys.**2**, 505 (1998) [hep-th/9803131].

[41] G. T. Horowitz and R. C. Myers, "The AdS / CFT correspondence and a new positive energy conjecture for general relativity," Phys. Rev. D **59**, 026005 (1998) [hep-th/9808079]

[42] G. T. Horowitz and A. R. Steif, "Space-time Singularities in String Theory", Phys. Rev. Lett. **64** 260 (1990).

[43] V. Balasubramanian, S. F. Hassan, E. Keski-Vakkuri and A. Naqvi, "A space-time orbifold: A toy model for a cosmological singularity", [hep-th/0202187].

[44] J. Karczmarek, A. Strominger, "Matrix cosmology", [hep-th/0309138]

[45] M. Berkooz, B. Pioline, "Strings in an electric field, and the Milne universe", [hep-th/0307280].

[46] B. Craps, S. Sethi, E. Verlinde, "A Matrix Big Bang", [hep-th/0506180].

[47] M. Li, "A class of cosmological matrix models", [hep-th/0506260].

[48] B. Chen, "The time-dependent supersymmetric configurations in M-theory and matrix models", [hep-th 0508191].

[49] T. Ishino, H. Kodama and N. Ohta, "Time-dependent Solutions with Null Killing Spinor in M-theory and Superstrings", Phys. Lett. **B631** (2005) 68, [hep-th/0509173].

[50] S. Kalyana Rama, "Branes in a time dependent universe", [hep-th/0510008]

[51] S.R. Das and J. Michelson, "Matrix Membrane Big Bangs and D-brane Production", [hep-th/0602099]

[52] R. H. Brandenberger, Y. F. Cai, S. R. Das, E. G. M. Ferreira, I. A. Morrison and Y. Wang, "Fluctuations in a Cosmology with a Space-Like Singularity and their Gauge Theory Dual Description," arXiv:1601.00231 [hep-th].

[53] S. F. Ross and G. Titchener, "Time-dependent spacetimes in AdS/CFT: Bubble and black hole," JHEP **0502**, 021 (2005) [hep-th/0411128].

[54] V. Balasubramanian, K. Larjo and J. Simon, "Much ado about nothing," Class. Quant. Grav. **22**, 4149 (2005) [hep-th/0502111].

[55] J. A. Hutasoit, S. P. Kumar and J. Rafferty, "Real time response on dS(3): The Topological AdS Black Hole and the Bubble," JHEP **0904**, 063 (2009) [arXiv:0902.1658 [hep-th]].

[56] S. B. Giddings and A. Nudelman, "Gravitational collapse and its boundary description in AdS," JHEP **0202**, 003 (2002) [hep-th/0112099].

[57] J. Aparicio and E. Lopez, "Evolution of Two-Point Functions from Holography," JHEP **1112**, 082 (2011) [arXiv:1109.3571 [hep-th]].

[58] W. Fischler, S. Kundu and J. F. Pedraza, "Entanglement and out-of-equilibrium dynamics in holographic models of de Sitter QFTs", JHEP**07**(2014)021 arXiv:1311.5519[hep-th]

[59] K. Koyama and J. Soda, "Strongly coupled CFT in FRW universe from AdS / CFT correspondence", JHEP **0105**, 027 (2001) [hep-th/0101164].

[60] K. Ghoroku, M. Ishihara and A. Nakamura, "Gauge theory in de Sitter space-time from a holographic model," Phys. Rev. D **74**, 124020 (2006) [hep-th/0609152].

[61] D. Marolf, M. Rangamani and M. Van Raamsdonk, "Holographic models of de Sitter QFTs," Class. Quant. Grav. **28**, 105015 (2011) [arXiv:1007.3996 [hep-th]].

[62] M. Li and Y. Pang, "Holographic de Sitter Universe," JHEP **1107**, 053 (2011) [arXiv:1105.0038 [hep-th]].

[63] J. Maldacena and G. L. Pimentel, "Entanglement entropy in de Sitter space," JHEP **1302**, 038 (2013) [arXiv:1210.7244 [hep-th]].

[64] J. K. Erickson, D. H. Wesley, P. J. Steinhardt and N. Turok, "Kasner and mixmaster behavior in universes with equation of state $w \geq 1$," Phys. Rev. D **69**, 063514 (2004) [hep-th/0312009].

[65] S. Deser and O. Levin, "Accelerated detectors and temperature in (anti)-de Sitter spaces," Class. Quant. Grav. **14**, L163 (1997) doi:10.1088/0264-9381/14/9/003 [gr-qc/9706018].

[66] S. Deser and O. Levin, "Equivalence of Hawking and Unruh temperatures through flat space embeddings," Class. Quant. Grav. **15**, L85 (1998) doi:10.1088/0264-9381/15/12/002 [hep-th/9806223].

[67] S. Deser and O. Levin, "Mapping Hawking into Unruh thermal properties," Phys. Rev. D **59**, 064004 (1999) doi:10.1103/PhysRevD.59.064004 [hep-th/9809159].

[68] A. Kehagias and A. Riotto, "Conformal Symmetries of FRW Accelerating Cosmologies," Nucl. Phys. B **884**, 547 (2014) doi:10.1016/j.nuclphysb.2014.05.006 [arXiv:1309.3671 [hep-th]].

[69] S. A. Paston and A. A. Sheykin, "Embeddings for solutions of Einstein equations," Theor. Math. Phys. **175**, 806 (2013) doi:10.1007/s11232-013-0067-4 [arXiv:1306.4826 [gr-qc]].

[70] E. Shaghoulian and H. Wang, Class. Quant. Grav. **33** (2016) no.12, 125020 doi:10.1088/0264-9381/33/12/125020 [arXiv:1601.02599 [hep-th]].

[71] J. Ren, JHEP **1607**, 112 (2016) doi:10.1007/JHEP07(2016)112 [arXiv:1603.08004 [hep-th]].

[72] E. Mefford, arXiv:1605.09369 [hep-th].

[73] A. Ishibashi, K. Maeda and E. Mefford, arXiv:1703.09743 [hep-th].

[74] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96** (2006) 181602 doi:10.1103/PhysRevLett.96.181602 [hep-th/0603001].

[75] S. Ryu and T. Takayanagi, JHEP **0608** (2006) 045 doi:10.1088/1126-6708/2006/08/045 [hep-th/0605073].

[76] N. Engelhardt, G. T. Horowitz, "Entanglement Entropy Near Cosmological Singularities" JHEP**06**(2013) 041 [arxiv: 1303.4442[hep-th]]

[77] V. E. Hubeny, JHEP **1207**, 093 (2012) doi:10.1007/JHEP07(2012)093 [arXiv:1203.1044 [hep-th]].

[78] L.D Landau and E.M Lifshitz, Classical Theory of Fields, Pergamon Press(1987).

[79] E. Witten, "Quantum Gravity In De Sitter Space", hep-th/0106109.

[80] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, "Correlation functions in the CFT(d) / AdS(d+1) correspondence", Nucl. Phys. B**546** (1999) 96118, [hep-th/9804058]

[81] E. Hijano, P. Kraus, E. Perlmutter, and R. Snively, "Witten Diagrams Revisited: The AdS Geometry of Conformal Blocks", JHEP **01** (2016) 146, [arxiv:1508.00501].

[82] E. Dyer, D. Z. Freedman and J. Sully, "Spinning Geodesic Witten Diagrams", arxiv:1702.06139

[83] H.-Y. Chen, E.-J. Kuo and H. Kyono, "Anatomy of Geodesic Witten Diagrams", JHEP **05** (2017)070, arxiv:1702.08818

[84] V. Keranen, W. Sybesma, P. Szepietowski and L. Thorlacius, JHEP **1705**, 033 (2017) doi:10.1007/JHEP05(2017)033 [arXiv:1611.09371 [hep-th]].