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The Quantum Group Theoretic Approach to Heavy-ion Resonances

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Abstract

We suggest the quantum group as dynamical symmetry group of heavy-ion resonance systems. The rotation-vibrational spectra of the systems are given by the quantum group theoretic approach without going to the detail of the bonding potential of the systems. The corresponding wave functions are obtained. Using the analytic formula of this approach, we fit the experimental data of $^{12}C+{}^{12}C$ *and* $^{12}C+{}^{16}O$ *resonance systems in high accuracies. The pseudo-potential is found in the quantum group symmetric model and discussed in comparison with the conventional nonlinear potential model.*

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I. Introduction

The phenomenon of prominent intermediate-width resonances observed in the energy dependence of cross-sections of heavy-ion reactions at relatively higher energies is at present a major line of development of nuclear physics. While the existence of intermediate nonstatistical processes in the reactions is confirmed beyond reasonable doubt, the problem of their nature is far from being resolved. The first models relating the nature of these phenomena to molecule-like configurations in nuclei date from 1960 [1,2]. The idea of a molecular configuration seemed very natural in describing a long-lived quasi-bound state created in the interaction of two heavy-ions. The angular-momentum barrier and other effects prevent nuclei from coalescing immediately, forcing them to spend all the excess energy on the rotational motion. Also several new approaches have since then been developed, the image of two orbiting clusters forming a quasi-molecular configuration still represents the leading description of the phenomenon of intermediate-width resonances in heavy-ion reactions. The revival of research has established the presence of intermediate-width resonances as a rather general phenomenon in heavy-ion reactions and an impressive array of data of resonances in heavy-ion reactions has been accumulated [3–5]. So, it is no longer considered as isolated phenomena present in a few light nuclear systems but to be a general feature of heavy-ion reactions.

In this paper, we investigate the possibility of quantum group $H_q(4)$ (the quantum Heisenberg group) as dynamical symmetry of the quasi-molecular systems.

The progress of modern physics is accompanied by the studies of symmetries. Recently, much attention has been directed to the investigations upon the quantum groups and Yang-Baxter equations [5–12]. Quantum group $U_q(\alpha)$ is the q-deformation of the universal enveloping algebra $U(\alpha)$ of a finite-dimensional semi-simple Lie algebra α, introduced by Drinfel'd [6] and Jimbo [7] in their studies of the quantum inverse scattering method of the integrable models and Yang-Baxter equations. Quantum group theory is becoming a popular topic in different fields of physics, due to its richer structure than that of the Lie groups.

As a widely accepted opinion, the deformation parameter q is a function of the Planck constant $\hbar$, *e.g.*, $\ln q \propto \hbar$. When $\hbar \to 0$, $q \to 1$ and the quantum group symmetries reduce to Lie group symmetries, while the quantum systems convert to classical ones. As we pointed out [13–18], this viewpoint is not evident. In fact, the q-deformation and canonical quantization can be two independent concepts. It is possible to find the q-deformed symmetries (with nontrivial Hopf algebra structure [13–18] in classical systems, and when the systems are canonically quantized, one obtains quantum group symmetries in quantum systems. The quantum systems described by quantum symmetries reduce to the quantum systems described by the ordinary Lie symmetries, when the deformation parameters q's tend to one.

This new viewpoint brings new possibilities to the studies of quantum groups as potential dynamical symmetries in physical systems. As the quantum groups, with respect to their Lie counterparts, introduce independent parameters q's, the new (and big) symmetries allow violations of the Lie symmetries. When q's are not unity, the new symmetries are exact symmetries in the view of spectrum-generating Hopf algebras and with well established theory of representations, but violations of the Lie symmetries occur. The violations are small if the deviations of q's from unity are small. If the deviations get greater, the violations get greater. The most well known model possesses quantum group symmetry is the Heisenberg spin chain of XXZ type [19,20], where the difference of Z from X induces the violation of $SU(2)$ symmetry. But the violation therein does not break the big symmetry $SU_q(2)$, which is one of the best studied quantum groups.

In the attempts to find other systems in which the quantum groups are possible dynamical symmetries, the authors looked into the century old problem of the rotating and vibrating diatomic molecule. It is shown that the Dunham formulum [21] for the vibration-rotational energy spectrum of the diatomic molecule can be reproduced from the Taylor expansion of the quantum group theoretic formulum [22–25]. The specific character of this approach is that it does not deal with the explicit form of the bonding potential, but demonstrates that the energy levels are appropriate for any diatomic-molecule-like systems that possess the Dunham-like energy spectrum and are dominated by dipole degrees of freedom by using a spectrum generating algebra focusing on the elementary quantity involved in the interaction and not on the interaction potential.

The properties of the quantum group theoretic approach to molecular spectra [22-25] hint us that the approach may be more suitable to the case of heavy-ion resonances. As in nuclear physics, the quasi-molecular configurations are not stable and primary bonding mechanisms have not yet been unambiguously identified. We can not calculate in principle the bonding potential $V(r)$ of the anharmonic oscillator from a knowledge of relevant interactions.

This paper is organized as follows: In section 2, we briefly review the quasi-molecular structures of heavy-ion resonances, concentrating on the Iachello's method and the alternative potential approach. Section 3 is devoted to the Hopf algebra structures of quantum Heisenberg group $H_q(4)$ which we suggest as the possible dynamical symmetry of heavy-ion resonances. In section 4, the quantum group theoretic approach to heavy-ion resonances are given. The fit of the analytic formula of quantum group theoretic approach with experimental data of $^{12}C +^{12} C$ and $^{12}C +^{16} O$ resonances are provided. We conclude with some brief remarks in the last section.

II. The Quasi-molecular Models of Heavy-ion Resonance

The quasi-molecular models captured the fancy of many physicists by their simplicity and physical transparency. In [26,27] the mechanism of double excitation is introduced as the principal cause of formation of quasi-molecular configurations. Another step forward was the introduction of the resonance-window concept [26,28] which correlated the observability of resonances to the level density of compound system. Recently, group theoretic [29] and anharmonic vibrator [30] approaches were used with considerable success to explain resonance data, giving in both cases energy eigenvalue expressions in the form of a trancated Dunham series. The first effort to organize the resonance data was made by Erb and Bromley [31] under the influence of the group-theoretic results of Iachello.

In the original papers of [29] [31], Iachello and Erb and Bromley proposed that resonances are dominated by dipole rather than the quadrupole degrees of freedom with the fundamental quantity $\vec{R}$ being the distance between two ions. Based on the group-theoretic considerations, they gave the expression

$$E(n,L) = -D + a(n+\frac{1}{2}) - b(n+\frac{1}{2})^2 + cL(L+1) \tag{1}$$

for the center-of-mass energies of the resonances. Where D, a, b and c are parameters to be determined phenomenologically by fitting the experimental data.

To obtain the molecular rotation-vibrational spectra, Iachello introduced a set of four boson operators, divided into a scalar ($J = 0$) operator, denoted by σ, and a vector ($J = 1$) operator denoted by π_μ ($\mu = 0, \pm 1$). They are assumed to have parity $(-)^J$. The introduction of the vector (π) boson reflects the dipole nature of the problem. Expands the Hamiltonian H in terms of boson operators and (for simplicity) stops at quadratic terms, he has

$$\begin{aligned} H &= \epsilon_\sigma(\sigma^+ \cdot \sigma) + \epsilon_\pi(\pi^+ \cdot \tilde{\pi}) \\ &\quad + \sum_{J=0,2} \frac{1}{2}(2J+1)^{\frac{1}{2}} c_J \left[\left[\pi^+ \times \pi^+\right]^{(J)} \times [\tilde{\pi} \times \tilde{\pi}]^{(J)}\right]^{(0)} \\ &\quad + u_0 \left[[\sigma^+ \times \sigma^+]^{(0)} \times [\sigma \times \sigma]^{(0)}\right]^{(0)} + u_1 \left[[\pi^+ \times \sigma^+]^{(1)} \times [\tilde{\pi} \times \sigma]^{(1)}\right]^{(0)} \\ &\quad + u_2 \left[[\pi^+ \times \pi^+]^{(0)} \times [\sigma \times \sigma]^{(0)} + [[\sigma^+ \times \sigma^+]^{(0)} \times [\tilde{\pi} \times \tilde{\pi}]^{(0)}\right]^{(0)} . \end{aligned} \tag{2}$$

Here $\tilde{\pi}_\mu = (-)^{1-\mu}\pi_{-\mu}$ and the square brackets denote tensor product. Energy spectra may be obtained by diagonalizing H in the space $[N]$ of the totally symmetric irreducible representations of the group $U(4)$ generated by the 16 bilinear products $G_{\alpha\alpha'} = b^+_\alpha b_{\alpha'}$ ($\alpha = 1, 2, 3, 4$; $b_\alpha \equiv s, \pi_\mu$). Instead of going to the detail of the bonding potential, Iachello demonstrated that the energy level obtained from the spectrum generating algebra $U(4) \supset O(4) \supset O(3)$ is appropriate for any molecule-type Hamiltonian.

An alternative potential-well approach was also made [32–34]. Satpathy *et al* started with the Morse potential

$$V_M(R) = B\left(\exp\left[-2\beta\left(\frac{R-R_0}{R_0}\right)\right] - 2\exp\left[-\beta\left(\frac{R-R_0}{R_0}\right)\right]\right). \tag{3}$$

The potential has a minimum of $-B$ at $R = R_0$ and approaches zero asymptotically for $R \to \infty$. The effective potential for $J = 0$ (ion-ion potential V_N + Coulomb potential V_c) giving the bound and resonance states can be well represented by the combination of the Morse potential and a constant,

$$V_{eff}(R) = A + B\left(\exp\left[-2\beta\left(\frac{R-R_0}{R_0}\right)\right] - 2\exp\left[-\beta\left(\frac{R-R_0}{R_0}\right)\right]\right). \tag{4}$$

The eigenvalue for the effective potential V_{eff} can be written as

$$\begin{aligned} E \;=\; & A - B + \frac{\hbar^2}{2\mu R_0^2}\left(2\beta\gamma\left(n+\frac{1}{2}\right) - \beta^2\left(n+\frac{1}{2}\right)^2 + J(J+1)\right. \\ & \left. -\frac{3(\beta-1)}{\beta\gamma}(n+1)J(J+1) - \frac{9(\beta-1)^2}{4\beta^4\gamma^2}J^2(J+1)^2\right) \end{aligned} \tag{5}$$

with $\gamma^2 = 2\mu B R_0^2/\hbar^2$.

The solution to the Schrödinger equation for the radial motion of the effective potential is of the form

$$\chi_{n,J}(R) = y^{\frac{k_1}{\beta}} \exp\left(-\frac{y}{2}\right) F(\tilde{a},\tilde{c},y) \tag{6}$$

where $F(\tilde{a},\tilde{c},y)$ is a confluent hypergeometric function,

$$\begin{aligned} & y = \rho_2\exp(-\beta x), && \rho_2 = 2\gamma_2/\beta, \\ & \tilde{c} = 2k_1/\beta + 1, && c_0 = 1 - 3/\beta - 6/\beta^2, \\ & \tilde{a} = \tfrac{1}{2}\tilde{c} - \gamma_1^2/\beta\gamma_2, && c_1 = 4/\beta - 6/\beta^2, \\ & && c_2 = -1/\beta + 3/\beta^2, \end{aligned} \tag{7}$$

and

$$\begin{aligned} & k_1^2 = k^2 + J(J+1)c_0, \\ & \gamma_1^2 = \gamma^2 - \tfrac{1}{2}J(J+1)c_1, \\ & \gamma_2^2 = \gamma^2 + J(J+1)c_2. \end{aligned} \tag{8}$$

By assuming that the colliding dinuclear system forms a diatomic-molecule-like rotator and the shock of the collision leads to a surface vibration analogous to that of a deformed system [35]. Neglecting the γ vibration, in [30] Cindro and Greiner postulated a Hamiltonian of the form

$$H = \sum_{k=1}^{3}\frac{M_k'}{J_k(a_\nu)} + \frac{1}{2}B(\dot{a}_0^2 + 2\dot{a}_2^2) + \frac{1}{2}C_2\xi^2 + C_4\xi^4, \tag{9}$$

to describe the resonant colliding system. Where M_k' and $J_k(a_\nu)$ are respectively the components of the angular momentum and the moments of inertia along the three axes of the rotating dinuclear system, the variables a_0, a_2 and ξ are same as defined in [35]. The presence of a quadratic term in ξ introduces the anharmonicity. This leads to an energy spectrum of the form (no γ-vibration)

$$E(n, L, K) = J(J+1)\frac{1}{2}\epsilon + (n+\frac{1}{2})E_\beta + (n+\frac{1}{2})^2 G + \frac{3}{2}G. \tag{10}$$

Expression (1) and (10) are obviously equivalent. These treatments may be a general feature of two-body molecular interactions.

III. Quantum Heisenberg Group $H_q(4)$

A quantum group (or quantum enveloping algebra) is a Hopf algebra which is neither commutative nor co-commutative. Given an associative algebra A with unity, we say that A is Hopf algebra if we can define three operations Δ, S, ϵ on A; the co-multiplication $\Delta : A \to A\otimes A$ is algebra homomorphism, the antipodal map $S : A \to A$ is algebra anti-homomorphism and the counit $\epsilon : A \to C$ is also homomorphism (C is the field of complex numbers). The three operations satisfy the following axioms:

$$\begin{aligned} (id\otimes\Delta)\Delta(a) &= (\Delta\otimes id)\Delta(a), \\ m(id\otimes S)\Delta(a) &= m(S\otimes id)\Delta(a) = \epsilon(a)1 \\ (\epsilon\otimes id)\Delta(a) &= (id\otimes\epsilon)\Delta(a) = a, \end{aligned} \tag{11}$$

where a and b are elements of the algebra, and m is the multiplication in the algebra: $m : A\otimes A \to A$, or $m(a\otimes b) = ab$. The first condition is the associativity of the co-multiplication, the second condition is the definition of the antipode and the third defines the co-unit. If $\sigma = A\otimes A \to A\otimes A$ is the permutation map $\sigma(a\otimes b) = b\otimes a$, it is easy to check that $\Delta' = \sigma o \Delta$ is another co-multiplication in A with antipode $S' = S^{-1}$. A Hopf algebra is a quasi-triangular Yang-Baxter algebra if the co-multiplications Δ, Δ' are related by conjugation:

$$\sigma\Delta(a) = R\Delta(a)R^{-1} \qquad R \in A \otimes A \tag{12}$$

and the following conditions are satisfied:

$$\begin{aligned} (id \otimes \Delta)(R) &= R_{13}R_{12} \\ (\Delta \otimes id)(R) &= R_{13}R_{23} \\ (S \otimes id)(R) &= R^{-1}. \end{aligned} \tag{13}$$

An immediate consequence of these axioms is the Yang-Baxter equation which reads

$$R = \sum_i a_i \otimes b_i, \tag{14}$$

$$R_{13}R_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i b_j. \tag{15}$$

The Yang-Baxter equation is the direct result of the following properties of the universal R-matrix:

$$\begin{aligned} (\sigma \cdot \Delta \otimes id)\, R &= \textstyle\sum_i \Delta'(a_i) \otimes b_i = \sum_i R_{12}\Delta(a_i)R_{12}^{-1} \otimes b_i \\ &= R_{12} \textstyle\sum_i \Delta(a_i) \otimes b_i R_{12}^{-1} = R_{12}(\Delta \otimes id(R))R_{12}^{-1} \\ &= R_{12}R_{13}R_{23}R_{12}^{-1}. \end{aligned} \tag{16}$$

However

$$(\sigma \cdot \Delta \otimes id)R = \sigma_{12}((\Delta \otimes id)R) = \sigma_{12}(R_{13}R_{23}) = R_{23}R_{13} \tag{17}$$

yields the Yang-Baxter equation,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{18}$$

For a quantum q-oscillator system, the Hamiltonian is [13,14]

$$H_{vib} = \frac{1}{2}\left(a_q^\dagger a_q + a_q a_q^\dagger\right)\hbar\omega \tag{19}$$

where a_q, $a_q^\dagger$ are annihilation and creation operators for this deformed system. These operators are connected with the operators $a, a^\dagger$ of the simple harmonic oscillator in this way

$$a_q = \sqrt{\frac{[N+1]_q}{N+1}}\,a, \qquad a_q^\dagger = a^\dagger\sqrt{\frac{[N+1]_q}{N+1}}. \tag{20}$$

where $N = a^\dagger a$, $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$. By the basic commutation relations

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0, \tag{21}$$

we have the commutation relations for the q-oscillator system

$$\begin{aligned} \left[a_q, a_q^\dagger\right] &= [N+1]_q - [N]_q, \\ [N, a_q] &= -a_q, \quad \left[N, a_q^\dagger\right] = a_q^\dagger. \end{aligned} \tag{22}$$

This is the q-deformed oscillator algebra $H_q(4)$. The Hopf operations: coproduct, antipode and counit can be defined for this algebra [36].

$$\begin{aligned} \Delta(N) &= N \otimes 1 + 1 \otimes N - \frac{\alpha}{\bar{\gamma}} \cdot 1 \otimes 1; \\ \Delta(a_q^\dagger) &= \left(a_q^\dagger \otimes q^{N/2} + iq^{-N/2} \otimes a_q^\dagger\right) e^{-i\alpha/2}; \\ \Delta(a_q) &= \left(a_q \otimes q^{N/2} + iq^{-N/2} \otimes a_q\right) e^{-i\alpha/2}; \\ S(N) &= -N + i\frac{2\alpha}{\gamma} \cdot 1; \\ S(a_q^\dagger) &= -q^{1/2} a_q^\dagger; \\ S(a_q) &= -q^{-1/2} a_q; \\ \epsilon(N) &= \frac{\alpha}{\bar{\gamma}}; \\ \epsilon(a_q^\dagger) &= \epsilon(a_q) = 0; \\ \epsilon(1) &= 1, \end{aligned} \tag{23}$$

where $\alpha = 2k\pi + \frac{\pi}{2}$, $k \in \mathcal{Z}$ and $\bar{\gamma} = -i\gamma = -i\ln q$. These Hopf operations, together with multi-plication m and identity mapping id satisfy the conditions in (13).

The R-matrix reads explicitly in the following

$$\mathcal{R} = q^{1/2N\otimes N - \frac{\alpha}{\bar{\gamma}}\Delta(N)} \sum_{n\geq 0} i^n \frac{\left(1+q^{-1}\right)^n}{[n]_{q^{1/2}}!} q^{-\frac{n(n+1)}{4}} \left(a_q^\dagger\right)^n \otimes q^{-nN/2} a_q^n, \tag{24}$$

where H_0 is the part of the Hamiltonian from irrelevant motions, and $C_I = J^-J^+ + J^3(J^3+1)$ is the Casimir operator of $SU(2)$ and I is the moment of inertia. The vibrational Hamiltonian for the q-oscillator can be written into the following

$$H_{vib} = \frac{1}{2}\left([N]_q + [N+1]_q\right)\hbar\omega. \tag{32}$$

The representations of quantum group $H_q(4)$ can be constructed in the following way,

$$|n>>= \left([n]_q!\right)^{-1/2}\left(a_q^\dagger\right)^n|0>. \tag{33}$$

The actions of the operators on the Fock states yield

$$\begin{aligned} a_q^\dagger|n>> &= \sqrt{[n+1]_q}|n+1>>, \\ a_q|n>> &= \sqrt{[n]_q}|n-1>>, \\ a_q|0> &= 0. \end{aligned} \tag{34}$$

The energy levels for this system are

$$E_{vib} = \frac{1}{2}\left([n+1]_q + [n]_q\right)\hbar\omega. \tag{35}$$

It is easy to see that the representation for quantum group $H_q(4)$ is isomorphic to that for the ordinary harmonic oscillator algebra $H(4)$. [2]

$$\begin{aligned} |n>> &= \left([n]_q!\right)^{-1/2}\left(a_q^\dagger\right)^n|0>, \\ &= (n!)^{-1/2}\left(a^\dagger\right)^n|0>, \\ &= |n>. \end{aligned} \tag{36}$$

where $|n>$ are the Fock states for the ordinary harmonic oscillator. Therefore the representation in coordinate space can be expressed exactly in Hermite polynomials,

$$\psi_n(x) = N_n H_n(X) e^{-X^2/2}. \tag{37}$$

where $X = \beta x$, $\beta = \left((2\pi)^2 mc\nu_n/h\right)^{1/2}$, $x = r - r_e$, r_e is the equilibrium distance between the two ions and $N_n = (\sqrt{\pi}2^n n!)^{-1/2}$.

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[2] For q a root of unity, the properties of the representation are more complicated, see [15],[16],[36].

where m_1 and m_2 are masses for the two nuclei, V is the effective potential between them. The Schrödinger equation in the center of mass frame is

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\Psi(\vec{r}) = E\Psi(\vec{r}), \tag{27}$$

where $m = \dfrac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. As the variables can be separated, we have

$$\Psi(\vec{r}) = R(r)Y_{JM}(\theta,\phi), \quad R(r) = \frac{1}{r}\psi(r), \tag{28}$$

where $Y_{JM}(\theta,\phi)$ are spheric harmonics. The tangent motion is rigid rotation described by $SU(2)$ group. While the radial motion satisfies the following equation,

$$\left\{-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{J(J+1)\hbar^2}{2mr^2} + V(r)\right\} R = ER, \tag{29}$$

or

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{J(J+1)\hbar^2}{2mr^2} + V(r)\right\}\psi = E\psi. \tag{30}$$

Under the lowest order approximation, the second term in the above equation, *i.e.*, $\dfrac{J(J+1)\hbar^2}{2mr^2}$, can be neglected and the above equation is treated as the motion equation for simple harmonic oscillator. And therefore the relative motion of the nuclei can be decomposed into rigid rotation and linear vibration. However this model can not provide a correct picture of the fine structure of vibrations and rotations.

As remarked in the above phrases, generally, one can not ignore the influence of the interaction among vibrations and rotations. In a better approximation, the term proportional to $J(J+1)$ in the equation of radial motion must be handled carefully. In this paper, we propose that the q-deformed oscillation model that possesses symmetry of quantum group $H_q(4)$ to describe the radial vibration of heavy-ion resonances.

The Hamiltonian of the quasi-molecular system may be of the following form

$$\begin{aligned} H &= H_0 + H_{vib} + H_{rot} \\ &= H_0 + \frac{1}{2}(a_q^+ a_q + a_q a_q^+)\hbar\omega + \frac{C_I}{2I}, \end{aligned} \tag{31}$$

where the convention

$$[n]_{q^{1/2}}! = [1]_{q^{1/2}}\,[2]_{q^{1/2}} \cdots [n]_{q^{1/2}} \tag{25}$$

has been applied.

IV. Quantum Group Symmetry and Quasi-molecular System

A vibration-rotational model based on a quasi-molecular picture with rotational resonances fragmented by harmonic surface vibrations was proposed about ten years ago to account for the observed $^{12}C +^{12} C$ resonances. The two colliding nuclei is proved to be a molecule-type rotator, on the other hand the shock between the nuclei leads to surface vibration of the system. Because of the interacting between vibration and rotation, one can't separate the Schrödinger equation of the two body system into vibrational one and rotational one. In fact, in the radial part of the Schrödinger equation there is a term which is proportional to $J(J+1)$, the eigenvalues of rotational moves. In [22–25], we suggested that the two body system is described by the so called q-oscillator model and rotator model, and possesses $H_q(4) \otimes SU(2)$ quantum group symmetry. It is successful to apply the model to describe the spectra of diatomic molecules [22–25]. Now, we use the model to investigate properties of heavy-ion resonance systems.

Under the adiabatic condition, the Schrödinger equation for a dinuclear quasi-molecule is

$$\left[-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(r)\right]\Phi = E_t\Phi, \tag{26}$$

It is well known that the coordinate representation of $SU(2)$ are Y_{JM}, the spherical harmonics, *i.e.*,

$$\psi_{JM} = Y_{JM}(\theta,\phi). \tag{38}$$

The generators of $SU(2)$ algebra are

$$\begin{aligned} J^{\pm} &= i\hbar e^{\mp i\phi}(\pm i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}), \\ J^{3} &= -i\hbar\frac{\partial}{\partial\phi}. \end{aligned} \tag{39}$$

When the generators act on Ψ_{JM}, we have

$$\begin{aligned} J^{\pm}\psi_{JM} &= \hbar\sqrt{(J\mp M)(J\pm M+1)}\psi_{J,M\pm 1}, \\ J^{3}\psi_{JM} &= \hbar M\psi_{JM}. \end{aligned} \tag{40}$$

So the Casimir has eigenvalues $J(J+1)\hbar^2$, *i.e.*, the eigenvalues of rotational Hamiltonian are

$$E_{rot} = \frac{\hbar^2 J(J+1)}{2I}. \tag{41}$$

Then we obtain the rotation-vibrational spectra of the heavy-ion resonances systems as

$$E = E_0 + \frac{1}{2}\left([n+1]_q + [n]_q\right)\hbar\omega + \frac{\hbar^2}{2I}J(J+1). \tag{42}$$

The corresponding wavefunctions of the systems have the following form

$$\psi = \psi_n\psi_{JM} = N_n H_n(X)e^{-\frac{X^2}{2}}Y_{JM}(\theta,\phi). \tag{43}$$

To fit experimental data conveniently, we rewrite the spectrum (42) into the following form

$$E = E_0 + \frac{\hbar\omega}{2\sin(\bar{\gamma}/2)}\sin\left(\bar{\gamma}\left(n+\frac{1}{2}\right)\right) + \frac{\hbar^2}{2I}J\left(J+1\right). \tag{44}$$

The above equation is the general form of the rotation-vibrational spectrum for heavy-ion resonances obtained in the quantum group theoretic approach. The fitting of the data is made by firstly forming a dense of curves in the E_{CM} against n spaces using the above equation. In selecting the parameters the following procedure was applied: First, the data were fitted at comparatively higher J values where there are more known resonances; then using the so obtained parameters to form the mess, a reasonable fit to all J values was assured. This fitting method is almost the same used by other authors [31-34].

The values of th fitting parameters are $E_0 = 0.34 \ \ (MeV)$, $\bar{\gamma} = 0.18$, $\omega = 1.20 \ \ (MeV/\hbar)$, and $I = 6.61 \ \ (\hbar^2/MeV)$ for $C^{12} + C^{12}$, and $E_0 = -0.015 \ \ (MeV)$, $\bar{\gamma} = 0.19$, $\omega = 1.98 \ \ (MeV/\hbar)$, and $I = 7.14 \ \ (\hbar^2/MeV)$ for $C^{12} + O^{16}$.

In other words, the rotation-vibrational spectra of $C^{12} + C^{12}$ and $C^{12} + O^{16}$ are of the form

$$\begin{aligned} E &= 0.34 + 6.7 \sin\left(0.18\left(n + \tfrac{1}{2}\right)\right) + 0.0757 J(J+1), \quad \text{for} \ \ C^{12} + C^{12}, \\ E &= -0.015 + 10.45 \sin\left(0.19\left(n + \tfrac{1}{2}\right)\right) + 0.07 J(J+1), \quad \text{for} \ \ C^{12} + O^{16}. \end{aligned} \tag{45}$$

Figures 1, 2 and 3 show the obtained fits.

V. Discussions and Remarks

The quantum group as a beautiful mathematical structure has been investigated in some particular quantum systems, such as exactly solved statistical model, integrable quantum systems and conformal field theory [11,12,37]. More and more Physicists pay attentions to possible applications of quantum groups in physics. In the aobve, we found that the quantum group $H_q(4)$ may be possible dynamical symmetry of heavy-ion resonance systems.

In this quantum group theoretic approach, if the deformation parameter $q \to 1$, harmonic oscillator plus rigid rotator model recovered. This differs from the usual quantum group models discussed by many other authors. In those well known systems which possessing quantum group symmetries, when $q \to 1$, $\hbar \to 0$, *i.e.*, quantum systems get to classical ones, but not quantum systems possessing Lie group symmetries. Although these group theoretic approaches do not care the detail of the interactions, but go to the nature of the Hamiltonians and energy spectra, it is natural to raise the question as what kind of the effective potential the quantum group theoretic approach imply. In the following, for completeness, we will give a discussion the corresponding effective potential implied in the quantum group theoretic approach in some detail. We choos to begin with the Taylor expansion of the analytic formula of the quantum group theoretic approach for resonance spectrum

$$E_{vib} = \hbar\omega\left(n + \frac{1}{2}\right) + c_1\hbar\omega\left(n + \frac{1}{2}\right)^2 + c_2\hbar\omega\left(n + \frac{1}{2}\right)^3 + \cdots, \tag{46}$$

where c_1, c_2, $\cdots$; are the coefficients of Taylor expansion. This is (a special case of) Dunham formulum [21], which stimulates one to propose the following Hamiltonian for the quasi-molecule in the obviously most direct way,

$$H_{vib} = \hbar\omega\left(N + \frac{1}{2}\right) + c_1\hbar\omega\left(N + \frac{1}{2}\right)^2 + c_2\hbar\omega\left(N + \frac{1}{2}\right)^3 + \cdots. \tag{47}$$

This Hamiltonian describes a system with nonlinear potential depending on the energy level. If one rewrites H_{vib} into coordinates, one obtains

$$H_{vib} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \hat{V}, \tag{48}$$

and

$$\hat{V} = \frac{1}{2}\omega^2 m x^2 + c_1\hbar\omega\left(N + \frac{1}{2}\right)^2 + c_2\hbar\omega\left(N + \frac{1}{2}\right)^3 + \cdots, \tag{49}$$

which is a the pseudo-potential that depends on the energy eigenvalues. This point is clear if we rewrites the Schrödinger equation into

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V_{eff}(x)\psi(x) = E\psi(x), \tag{50}$$

where $V_{eff}(x) = \left(\hat{V}\psi\right)\psi^{-1}$ is the effective potential. The pseudo-potential is currently a common concept frequently applied in theories of nuclear physics, though its dependence on the states that the system sits is against intuition.

The conventional model of anharmonic vibrator applied to explain the Dunham formula (46) introduces local potential of the following type

$$U = \frac{1}{2}m\omega^2 x^2 + ax^3 + bx^4 + \cdots, \tag{51}$$

where

$$a = \frac{1}{6}\left(\frac{d^3U}{dx^3}\right)_{r=r_e}, \quad b = \frac{1}{24}\left(\frac{d^4U}{dx^4}\right)_{r=r_e}. \tag{52}$$

The energy eigenvalues are obtained by 2-nd order perturbation (to reproduce c_1) and 3-rd order perturbation (to reproduce c_2), while the wavefunctions of the linear oscillator are modified by 1-st and 2-nd order perturbations. As we require that the energy spectra of the pseudo-potential model and the local potential model coincide, these two models are essentially identically effective, if one ignores the differences between the wavefunctions of the pseudo-potential model (which are just those of the linear oscillator) and those of the local potential model (which are the wavefunctions of the linear oscillator modified by 1-st and/or 2-nd order perturbations). Because the small differences between the wavefunctions yielded from the two models are difficult to detect experimentally (if not in principle), one has the freedom to favour either one of these models. After all, the new Hamiltonian in (47) is the most direct conjecture from the Dunham formula (46), and is therefore unreasonable to refuse, if one is not prejudiced against theories involving pseudo-potentials.

It should also be noted that the Yang-Baxter equation constructed from the $H_q(4)$ symmetry is apparently hidden symmetry, but the explicit interpretation remains to be clarified. This is an interesting topic still in progress.

It is also worth noting that the nuclear shell model of isotropic 3-dimensional simple harmonic oscillator can also be extended by replacing some or all of the three components by the q-deformed harmonic oscillator(s), and the extended model will certainly give exact descriptions to the violations of the spherical symmetry of the spherically symmetric system and the equidistancy of the energy levels, and therefore may be an *exactly solvable* model suitable in dealing with the structures of some nuclei.

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Figure captions

Figure 1. Plot of energy in MeV in center of mass frame against the assumed vibrational quantum number (n) for resonance data of $^{12}C+ ^{12}C$ system.

Figure 2. Plot of energy in MeV in center of mass frame against the assumed vibrational quantum number (n) for resonance data of $^{12}C+ ^{16}O$ system with even spins.

Figure 3. Plot of energy in MeV in center of mass frame against the assumed vibrational quantum number (n) for resonance data of $^{12}C+ ^{16}O$ system with odd spins.

Figure 1.

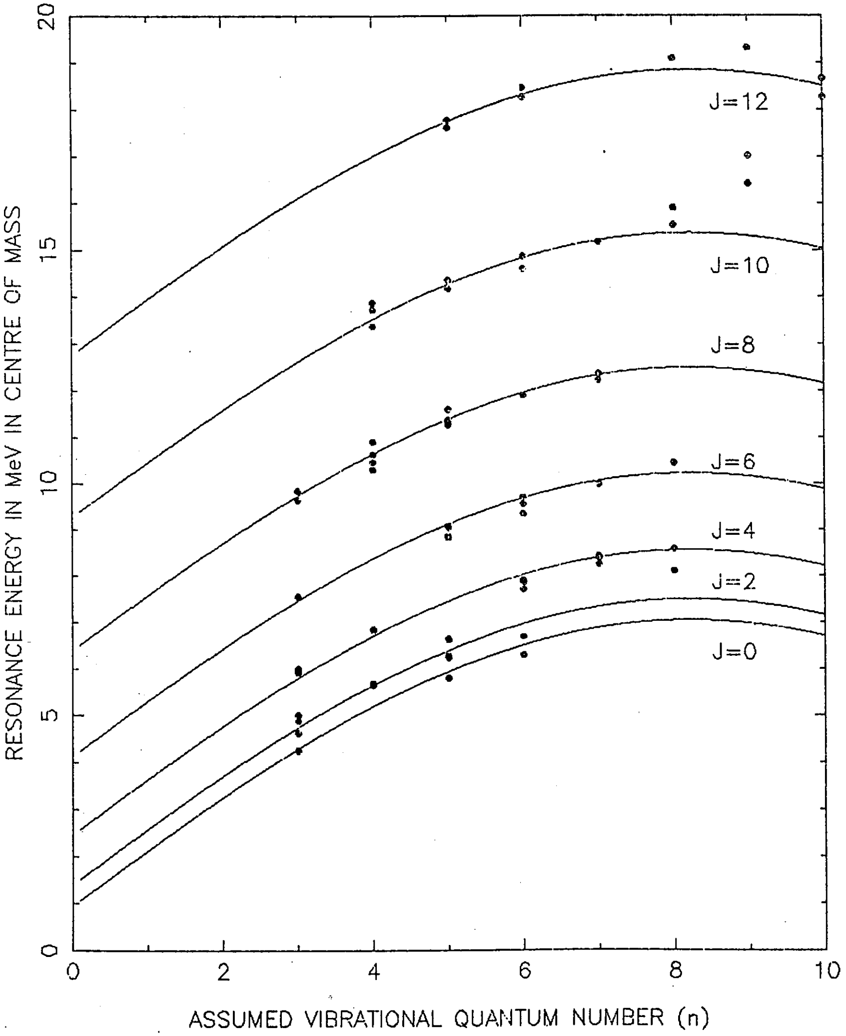

RESONANCE ENERGY IN MeV IN CENTRE OF MASS
0
5
10
15
20
ASSUMED VIBRATIONAL QUANTUM NUMBER (n)
0
2
4
6
8
10
J=12
J=10
J=8
J=6
J=4
J=2
J=0

Figure 2.

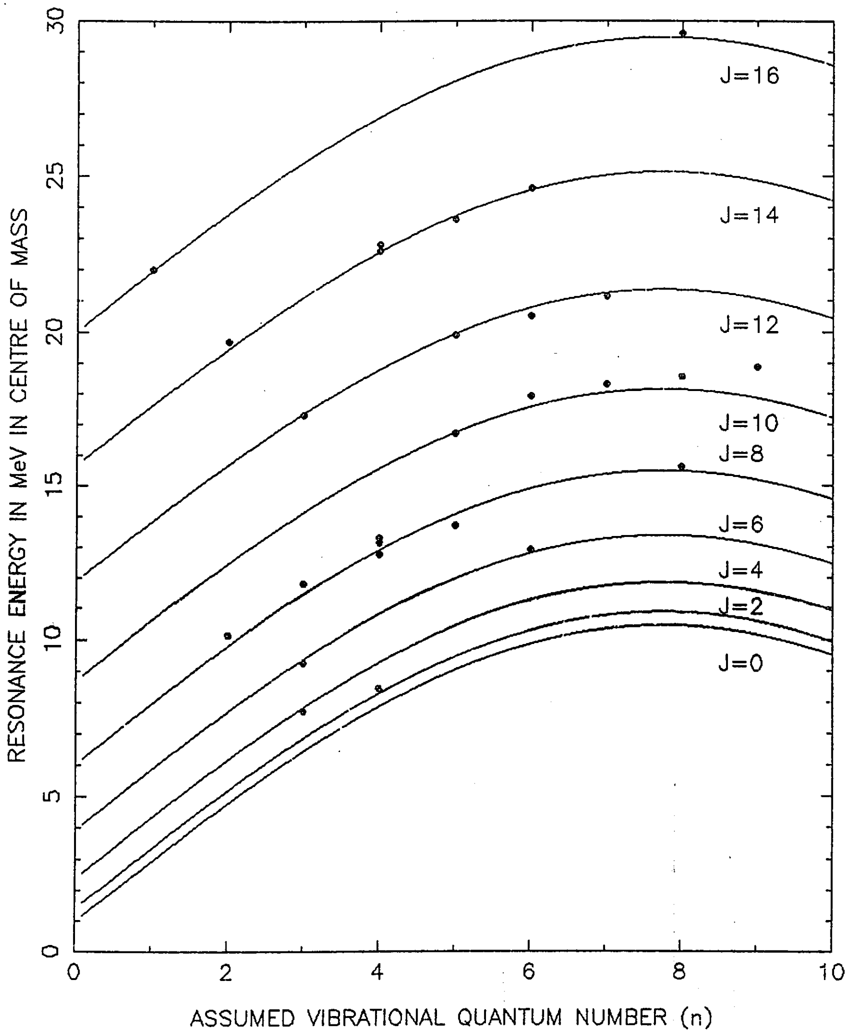
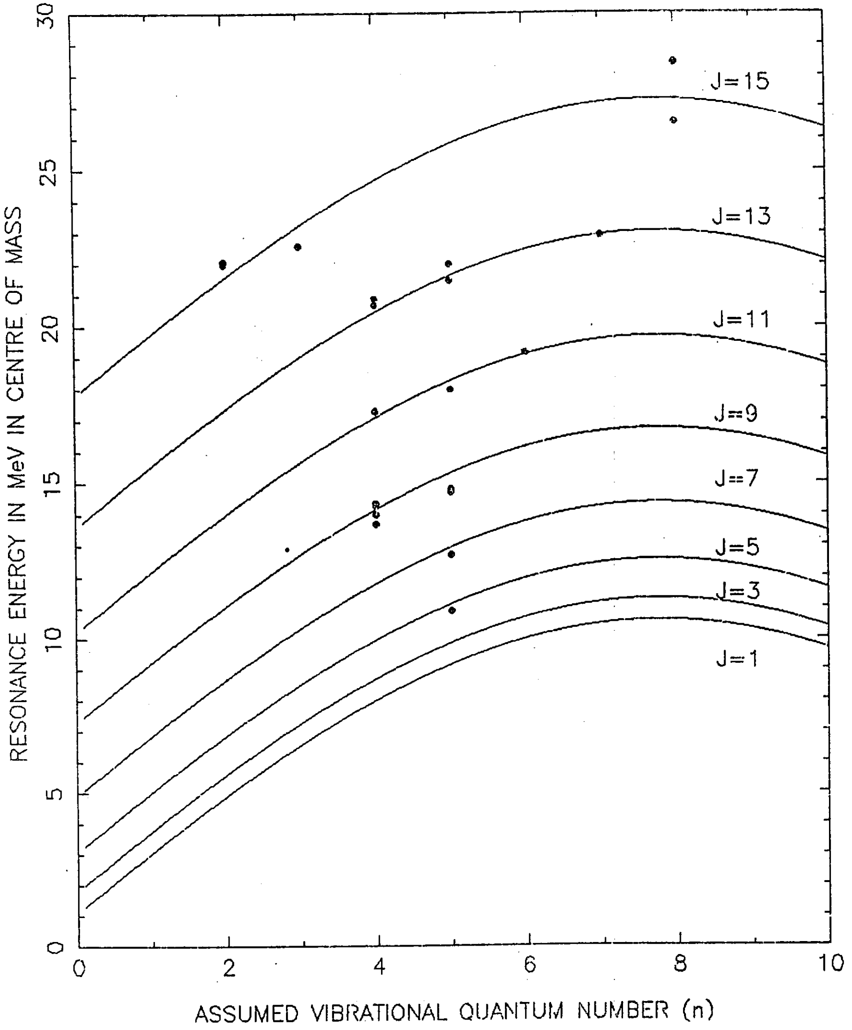

RESONANCE ENERGY IN MeV IN CENTRE OF MASS
0
5
10
15
20
25
30
ASSUMED VIBRATIONAL QUANTUM NUMBER (n)
0
2
4
6
8
10
J=16
J=14
J=12
J=10
J=8
J=6
J=4
J=2
J=0

Figure 3.


RESONANCE ENERGY IN MeV IN CENTRE OF MASS
0
5
10
15
20
25
30
ASSUMED VIBRATIONAL QUANTUM NUMBER (n)
0
2
4
6
8
10
J=15
J=13
J=11
J=9
J=7
J=5
J=3
J=1