Journal of Physics: Conference Series **411** (2013) 012031 doi:10.1088/1742-6596/411/1/012031

Hopf Algebra Structure of Continual Virasoro Algebra Deformations

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Abstract. The Hopf algebra structure of a few parametric deformations of the continual Virasoro algebra is proven.

1. Introduction

For various purposes of q-deformed structure construction [6] it is important to study deformations of the Virasoro algebra [1], [4], [3]. In [4] four deformations of the continual version of the Witt algebra (the centerless Virasoro algebra) were introduced, and to make connections with Hopf algebras corresponding comultiplications were found, although antipode and counit were missing. In [3], the Hopf algebra structure of the discrete form of one of the q-deformed Virasoro algebra [1] was established. In this paper we unify these two approaches to deformations of the Virasoro algebra and determine the full Hopf algebra structure [2] for all four deformations. We formulate the definition of such deformations with central extensions in terms of a family of deformed algebras parametrized by four parameters. The actual parameter is the difference $\theta = \beta_1 - \beta_2$ of the first two parameters.

Apart from pure algebraic interest for studies of quantum groups and generalizations of vertex algebras, possible applications of the deformations under consideration can be found in the domain of quantum exactly solvable models [5]. Namely, to every such deformation one can associate a pair of q-deformed operators with the zero curvature condition applied giving non-linear equations which could be solvable under some further assumptions on underlying continual Lie algebra.

2. Deformations of Continual Version of Virasoro Algebra

2.1. Continual Virasoro Algebra

In this section we recall two known families of q-deformations [4] of the Virasoro algebra in a continual Lie algebra form [7]. Let us introduce notations. In addition to ordinary commutator we will use the standard (in the sense of [4], [3]) deformed commutator

$$[a, b]_z \equiv zab - z^{-1}ba,$$

as well as a commutator with inserted operation:

$$[a,^{z\cdot} b] \equiv a\ z.b - b\ z.a,$$

where z can be an operator valued in corresponding algebra. Next we define

$$[z] \equiv \frac{q^z - q^{-z}}{q - q^{-1}},$$

$$\langle z \rangle \equiv q^z + q^{-z},$$

for $z \in \mathbb{C}$.

Let E be a commutative associative algebra over a field $\mathcal{K}$. The continual form of the Virasoro algebra, a central extension (denoted by $Vir(E)$) of the Witt algebra [4], [7], is defined by the commutation relations for the generators $\{C(\phi,\psi), X(\phi)|\phi,\psi \in E\}$:

$$\begin{aligned} [X(\phi), X(\psi)] &= X(K(\phi,\psi)) + C(\phi,\psi), \\ [C(\phi,\psi), X(\chi)] &= 0, \end{aligned} \tag{1}$$

$$K(\phi,\psi) = \phi\partial\psi - \psi\partial\phi, \tag{2}$$

where the symbol ∂ stands for a formal differentiation with respect to a parameter. Note that (2) can be written as $X\left(\left[\phi,^{\partial}\psi\right]\right)$. Jocobi identies for generators of a continual Lie algebra dictate cocycle property for the central element $C(\phi,\psi)$,

$$C(\phi, C(\psi,\chi)) + C(\psi, C(\chi,\phi)) + C(\chi, C(\phi,\psi)) = 0,$$

for any ϕ, ψ, $\chi \in E$. The continual Virasoro algebra (1) can be Fourier decomposed to give the commutation relations of the Virasoro algebra Vir with discrete generators $\{c, L_m|m \in \mathbb{Z}\}$, and ordinary relations

$$[L_m, L_n] = (m-n)L_{n+m} + c(m,n), \qquad [c, L_n] = 0 \tag{3}$$

where

$$c(m,n) = \delta_{m,-n}\frac{m^3 - m}{12}\, c. \tag{4}$$

2.2. Deformations

Next we consider the universal enveloping algebra of parametric deformations $Vir_q(E)$ of the continual Lie algebra $Vir(E)$ defined by commutation relations (1) with the mapping (2). Let $U(Vir_q(E))$ be an associative unital algebra over $\mathcal{K}$ with parameters $(\alpha, \beta_1, \beta_2, \gamma)$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta_i \in \{-1,0,1\}$, $\gamma \in \{0,1\}$, and determined by generators $\left\{C_q(\phi,\psi), T^{\pm 1},\ X(\phi)|\phi,\psi,\chi \in E\right\}$ subject to the commutation relations:

$$\begin{aligned} T\,T^{-1} &= T^{-1}T = 1, && (5)\\ T\,C_q(\phi,\psi) &= C_q(\phi,\psi)\,T^{-1}, && (6)\\ C_q(\phi,\psi)\,X(\chi) &= X(\chi)\,C_q(\phi,\psi) && (7)\\ TX(\phi) &= X(q^{-\theta\partial}\phi)T, && (8) \end{aligned}$$

where θ is a parameter. The commutation relations for the generators $X(\phi)$ have the form:

$$\begin{aligned} X(q^{\beta_1\partial}\phi)X(q^{\beta_2\partial}\psi) &- X(q^{\beta_1\partial}\psi)X(q^{\beta_2\partial}\phi) \\ &= T^{\beta_2\delta_{\beta_1,0}}X(K(\phi,\psi))T^{\beta_1\delta_{\beta_2,0}} + C_q(\phi,\psi), \end{aligned} \tag{9}$$

where the mapping is given by

$$\begin{aligned} K(\phi,\psi) &= q^{\gamma(\partial+\alpha)}\phi\cdot[\partial+\alpha+\beta_1\delta_{\beta_2,0}]_q\psi \\ &- q^{\gamma(\partial+\alpha)}\psi\cdot[\partial+\alpha+\beta_1\delta_{\beta_2,0}]_q\phi \\ &= m_E\circ(q^{\gamma(\partial+\alpha)}\otimes[\partial+\alpha+\beta_1\delta_{\beta_2,0}]_q).\left[\phi,^{\otimes}\psi\right], \end{aligned} \tag{10}$$

and m_E is the multiplication $E\otimes E\to E$. We take $\theta=\beta_1-\beta_2$ in (8) as an effective parameter regulating commutation relations (8), (13) and definitions of the the Hopf algebra structure (see (18)–(32)). In [4] four specific deformations *1.–4.* of the continual form of the centerless Virasoro algebra (1) – the Witt algebra, were determined by the following choices of parameters,

$$(\beta_1,\beta_2,\gamma)=\{(1,-1,1),(1,-1,0),(0,1,1),(1,0,1)\},$$

so that $\theta=\{2,2,-1,1\}$ correspondingly. In (9) and (10) it is assumed that the formal series $q^{\beta\partial}$ of powers of differential operators acts on on the second and the first argument of $X(\phi)$ generators in the ordinary bracket. In the limit $q\to1$, the generators $T^{\pm1}$ commute with $X(\phi)$ for every $\phi\in E$, and $C_q(\phi,\psi)$ reduces to the central element $C(\phi,\psi)$ of the undeformed continual Virasoro algebra (1). Thus these deformations $Vir_q(E)$ reduce to $Vir(E)$ in the limit $q\to1$. Note that in the case of the deformations *1.* and *2.* the left hand side bracket can be written as $[X(\phi),X(\psi)]_{q^{\beta\partial}}$, while for the deformations *3.* and *4.* as a bracket $[X(\phi),^{q^{\beta\partial}}X(\psi)]$.

In discrete generator form the deformations $Vir_q(E)$ (9)–(10) of [4] correspond to q-deformed versions Vir_q of the Virasoro algebra Vir (3) [3], [1]. They are determined by the generators $\left\{c_q,T^{\pm1},l_n|n\in\mathbb{Z}\right\}$ subject to the commutation relations (in addition to the relation (5))

$$[c_q,T]_q = 0, \tag{11}$$
$$[l_n,c_q]_{q^n} = 0, \tag{12}$$
$$[T,l_n]_{q^{\frac{\theta}{2}(n+1)}} = 0, \tag{13}$$
$$\left[l_m,^{q^{wt}}l_n\right]_{q^{\beta_1 m+\beta_2 n}} = k_i(m,n)l_{n+m}T^{\delta_{\beta_1,0}+\delta_{\beta_2,0}}+\mathcal{C}[m,n], \tag{14}$$

where the operator wt takes the value of the deformed Virasoro mode index, i.e., $wt(l_n)=n$, and

$$k_1(m,n) = [m-n]_q,\quad k_2(m,n)=[m+\alpha]_q-[n+\alpha]_q, \tag{15}$$
$$k_3(m,n) = [m-n]_q,\quad k_4(m,n)=q^{-1}[m-n]_q, \tag{16}$$

for deformations *1.–4.* correspondingly, and

$$\mathcal{C}[m,n]\equiv c[m,n]\;c=\delta_{m,-n}\,\frac{[m][m-1][m+1]}{[2][3]\langle m\rangle}\,c_q. \tag{17}$$

Note that $m-n$ and (17) are limits of k_i and $\mathcal{C}[m,n]$ when $q\to1$.

3. Hopf Algebra Structure

In this section we define the comultiplication Δ, antipode S and counity ε for the deformations *1.–4.* (9)–(10) of the continual version of the Virasoro algebra $Vir(E)$ and in the discrete form Vir_q (11)–(17), and prove their Hopf algebra structure. For Vir_q these operations are given by

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(for the deformation *1.*, $\theta = 2$, they were introduced in [3])

$$\Delta(T) = T \otimes T, \tag{18}$$
$$\Delta(c_q) = c_q \otimes 1 + 1 \otimes c_q, \tag{19}$$
$$\Delta(l_n) = l_n \otimes T^n + T^n \otimes l_n, \tag{20}$$
$$\varepsilon(T) = q^{\pm(2-|\theta|)\partial}, \tag{21}$$
$$\varepsilon(c_q) = 0, \tag{22}$$
$$\varepsilon(l_n) = 0, \tag{23}$$
$$S(T) = T^{-1}, \tag{24}$$
$$S(c_q) = -c_q, \tag{25}$$
$$S(l_n) = -T^{(1-\theta)n} l_n T^{(1-\theta)n}. \tag{26}$$

The $+/-$ sign in (21) corresponds to the left/right action of $\varepsilon(T)$ in the tensor product. For the continual form of (5)–(9), in addition to (18), (21), (24) we introduce

$$\Delta(C_q(\phi,\psi)) = C_q(Q_1\phi, Q_1\psi) \otimes T^{2-|\theta|} + T^{2-|\theta|} \otimes C_q(Q_2\phi, Q_2\psi), \tag{27}$$
$$\Delta(X(\phi)) = X(Q_1\phi) \otimes T^{2-|\theta|} + T^{2-|\theta|} \otimes X(Q_2\phi), \tag{28}$$
$$\varepsilon(C_q(\phi,\psi)) = 0, \tag{29}$$
$$\varepsilon(X(\phi)) = 0, \tag{30}$$
$$S(C_q(\phi,\psi)) = -T^{-(2-|\theta|)} C_q(Q_0\phi, Q_0\psi) T^{-(2-|\theta|)}, \tag{31}$$
$$S(X(\phi)) = -T^{-(2-|\theta|)} X(Q_0\, \phi) T^{-(2-|\theta|)}, \tag{32}$$

for any ϕ, $\psi \in E$. Here the operators $Q_i = Q_i(q)$, $i = 0, 1, 2$ are distributions on E, commuting with the formal differentiation ∂ with respect to q. As in [4] we demand (see the proof of Proposition 3.1) the commutativity and projectivity of the operators Q_i, $i = 1, 2$, i.e.,

$$X(Q_i^2\phi) = X(Q_i\phi), \qquad X([Q_1, Q_2]\phi) = 0, \tag{33}$$

and for $\beta \in \mathbb{Z}$,

$$Y(Q_i q^{\beta\partial}) = \pm Y(Q_i q^{-\beta\partial}). \tag{34}$$

where $Y \in \{C_q(\phi,\psi), X(\phi) | \phi, \psi \in E\}$, and $\lim_{q\to 1} X(Q_i\phi) = X(\phi)$, for any $\phi \in E$. We also require that $Y(Q_0^2) = Y(1)$, for $i = 1, 2$, and

$$Y(Q_a) = Y(Q_0 Q_{\bar{a}}), \tag{35}$$

where $\bar{a}$ is the opposite to a in the set $\{1, 2\}$, and $Y(Q_1 q^{-(2-|\theta|)^2\partial}) = Y(1) = Y(q^{(2-|\theta|)^2\partial} Q_2)$, for $Y = C_q$. The actual form of the distributions $Q_i(q)$, $i = 0, 1, 2$ can be determined [4] for each deformation *1.–4.* by the coinvariance condition of the comultiplication Δ and antipode S with respect to the defining relations (11)–(9) on generators of $Vir_q(E)$. Then taking into account conditions (33)–(35), we formulate the following

Proposition 3.1 *The universal enveloping algebra*

$$\mathcal{U}(Vir_q(E)) = [T; T^{-1}]_{\mathcal{K}} \otimes_{\mathcal{K}} U(Vir_q(E)),$$

of $(\alpha, \beta_1, \beta_2, \gamma)$-parameter q-deformation $Vir_q(E)$ defined by the generators $\{C_q(\phi,\psi), T^{\pm 1}, X(\phi) \mid \phi, \psi \in E\}$, subject to the commutation relations (5)–(9) with the mapping (10) (and for deformations Vir_q with discrete generators $\{c_q, T^{\pm 1}, l_n | n \in \mathbb{Z}\}$, relations (11)–(14), and mappings (15)–(16)) possesses a non-commutative co-commutatvie Hopf algebra structure given by (18), (21), (24), (27)–(32) ((18)–(26) correspondingly).

Proof. We first prove the Hopf algebra structure in the discrete form (11)–(17) $U(Vir_q)$. Note that the case *1.* was proven in [3]. It is clear that ε is an algebraic homomorphism, $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$, and $\mu(1\otimes\varepsilon)\Delta = \mu(\varepsilon\otimes 1)\Delta = \mathrm{Id}$, where μ is the multiplication. One can easily verify the coassoativity of Δ: $(\Delta\otimes 1)\Delta = (1\otimes\Delta)\Delta$. We next check coinvariance of the relations (11)–(14) with respect to Δ. This is easy to see that for the relations (5), (11), (12) and (13). Then we apply the comultiplication Δ to the commutation relations (14) of generators l_n:

$$\begin{aligned}\Delta(l_n)\Delta(l_m) &= l_n l_m \otimes T^{n+m} + T^{n+m} \otimes l_n l_m\\ &+ l_n T^m \otimes T^n l_m + T^n l_m \otimes l_n T^m\\ &= l_n l_m \otimes T^{n+m} + T^{n+m} \otimes l_n l_m\\ &+ q^{-\theta n(m+1)} l_n T^m \otimes l_m T^n\\ &+ q^{-\theta n(m+1)} l_m T^n \otimes l_n T^m,\end{aligned}$$

where $\theta = 2$ for the deformations *1.–2.*, and $\theta = -1$, $\theta = 1$ for the deformations *3.–4.*. Therefore for all deformations we obtain

$$\begin{aligned}q^{\beta_1\, wt}\Delta(l_m)q^{\beta_2\, wt}\Delta(l_n) &- q^{\beta_1\, wt}\Delta(l_n)q^{\beta_2\, wt}\Delta(l_m)\\ &= \left(q^{\beta_1\, wt}l_m q^{\beta_2\, wt}l_n - q^{\beta_1\, wt}l_n q^{\beta_2\, wt}l_m\right)\otimes T^{n+m}\\ &\quad +T^{n+m}\otimes\left(q^{\beta_1\, wt}l_m q^{\beta_2\, wt}l_n - q^{\beta_1\, wt}l_n q^{\beta_2\, wt}l_m\right)\\ &= \Delta\left(q^{\beta_1\, wt}l_m q^{\beta_2\, wt}l_n - q^{\beta_1\, wt}l_n q^{\beta_2\, wt}l_m\right)\\ &= \Delta\left(\left[l_m, {}^{q^{wt}} l_n\right]_{q^{\beta_1 m+\beta_2 n}}\right)\\ &= k_i(m,n)\Delta(l_{n+m})\Delta(T^{\delta_{\beta_1,0}+\delta_{\beta_1,0}}) + c[m,n]\Delta(c_q).\end{aligned}$$

We conclude that the commutation relations (14) are preserves by the comultiplication. Thus Δ is an algebraic homomorphism.

It is also clear that $\mu(1\otimes S)\Delta = \mu(S\otimes 1)\Delta = \eta.\varepsilon$, for some η, and $S^2 = 1$, where μ is the multiplication in $\mathcal{U}(Vir_q)$. Next we check that S is an algebraic anti-homomorphism. The antipode S preserves (5) and (11)–(13),

$$[S(c_q), S(T^n)]_{q^n} = 0, \qquad [S(c_q), S(l_n)]_{q^n} = 0,$$

for any $n \in \mathbb{Z}$. We also have

$$\begin{aligned}q^{\beta_1 m+\beta_2 n}S(l_m l_n) &= q^{\beta_1 m+\beta_2 n}S(l_n)S(l_m)\\ &= q^{\beta_1 m+\beta_2 n}T^{(1-\theta)n}l_n T^{(1-\theta)(n+m)}l_m T^{(1-\theta)m}\\ &= q^{-\beta_2 n-\beta_1 m}T^{-(1-\theta)(n+m)}l_n l_m T^{-(1-\theta)(n+m)}.\end{aligned}$$

Thus

$$\begin{aligned}\left[S(l_n), {}^{q^{wt}} S(l_m)\right]_{q^{\beta_1 m+\beta_2 n}} &= S\left([l_n, {}^{q^{wt}} l_m]_{q^{\beta_1 m+\beta_2 n}}\right)\\ &= k_i(m,n)\, S\left(T^{\delta_{\beta_1,0}+\delta_{\beta_2,0}}\right)S(l_{n+m})\\ &\quad +c[n,m]S(c_q).\end{aligned}$$

Therefore the commutation relations (14) are preserved by the antipode S. Thus we conclude that $\mathcal{U}(Vir_q)$ is a Hopf algebra.

Now we show that the universal enveloping algebra $\mathcal{U}(Vir_q(E))$ in the continual form of generators has a Hopf algebra structure. The algebraic homomorphisity of ϵ is obvious. The coassociativity of the comultiplication, $(1\otimes\Delta)\Delta = (\Delta\otimes 1)\Delta$, and the condition $S^2 = 1$ (since $Y(Q_0^2) = Y(1)$), lead to the commutativity and projectivity of operators Q_i, $i = 1$, 2. It is easy to check $\mu(1\otimes\varepsilon)\Delta = \mu(\varepsilon\otimes 1)\Delta = \mathrm{Id}$, using the projectivity of Q_i, $i = 1$, 2, i.e., $C_q(Q_i\phi, Q_i\psi) = Q_i C_q(\phi,\psi)$, (35), and the left or right action of (21). Next for $\mu(1\otimes S)\Delta = \mu(S\otimes 1)\Delta = \eta.\varepsilon$, we use (6) and (35). Indeed,

$$\begin{aligned}\mu(1\otimes S)\Delta(C_q(\phi,\psi)) &= C_q(Q_1\phi, Q_1\psi)\,T^{-(2-|\theta|)}\\ &\quad -C_q(Q_0Q_2\phi, Q_0Q_2\psi)\,T^{-(2-|\theta|)}\\ &= 0 = \varepsilon(C_q(\phi,\psi)),\end{aligned}$$

$$\begin{aligned}\mu(S\otimes 1)\Delta(C_q(\phi,\psi)) &= -T^{-(2-|\theta|)}\,C_q(Q_0Q_1\phi, Q_0Q_1\psi)\\ &\quad +T^{-(2-|\theta|)}\,C_q(Q_2\phi, Q_2\psi)\\ &= 0 = \varepsilon(C_q(\phi,\psi)),\end{aligned}$$

$$\begin{aligned}\mu(1\otimes S)\Delta(X(\phi)) &= X(Q_1\phi)\,T^{-(2-|\theta|)} - X(Q_0Q_2\phi)\,T^{-(2-|\theta|)}\\ &= 0 = \varepsilon(X(\phi)),\end{aligned}$$

$$\begin{aligned}\mu(S\otimes 1)\Delta(X(\phi)) &= -T^{-(2-|\theta|)}\,X(Q_0Q_1\phi) + T^{-(2-|\theta|)}\,X(Q_2\phi)\\ &= 0 = \varepsilon(X(\phi)).\end{aligned}$$

Using (6) we also have $S(TC_q(\phi,\psi)) = S(C_q(\phi,\psi)T^{-1})$, and similarly for (7) and (8). Applying the comultiplication to the commutation relations (9) we obtain

$$\begin{aligned}&\Delta(X(q^{\beta_1\partial}\phi))\,\Delta(X(Q_1q^{\beta_2\partial}\psi)) - \Delta(X(q^{\beta_1\partial}\psi))\,\Delta(X(Q_1q^{\beta_2\partial}\phi))\\ =\;& X(Q_1q^{\beta_1\partial}\phi)\,X(Q_1q^{\beta_2\partial}\psi)\otimes T^{2(2-|\theta|)} - X(Q_1q^{\beta_1\partial}\psi)\,X(Q_1q^{\beta_2\partial}\phi)\otimes T^{2(2-|\theta|)}\\ +\;& T^{2(2-|\theta|)}\otimes X(Q_2q^{\beta_1\partial}\phi)\,X(Q_2q^{\beta_2\partial}\psi) - T^{2(2-|\theta|)}\otimes X(Q_2q^{\beta_1\partial}\psi)\,X(Q_2q^{\beta_2\partial}\phi)\\ &+T^{2-|\theta|}\,X(Q_1q^{\beta_1+(2-|\theta|)\theta)\partial}\phi)\otimes X(Q_2q^{\beta_2-(2-|\theta|)\theta)\partial}\psi)\,T^{2-|\theta|}\\ &\quad+T^{2-|\theta|}\,X(Q_1q^{\beta_2\partial}\psi)\otimes X(Q_2q^{\beta_1\partial}\phi)\,T^{2-|\theta|}\\ &\quad-T^{2-|\theta|}\,X(Q_1q^{\beta_1+(2-|\theta|)\theta)\partial}\psi)\otimes X(Q_2q^{\beta_2-(2-|\theta|)\theta)\partial}\phi)\,T^{2-|\theta|}\\ &\qquad-T^{2-|\theta|}\,X(Q_1q^{\beta_2\partial}\phi)\otimes X(Q_2q^{\beta_1\partial}\psi)\,T^{2-|\theta|}\\ =\;& T^{\beta_2\delta_{\beta_1,0}}\,X(Q_1K(\phi,\psi))\,T^{\beta_1\delta_{\beta_2,0}}\otimes T^{2-|\theta|}\\ &+T^{2-|\theta|}\otimes T^{\beta_2\delta_{\beta_1,0}}\,X(Q_2K(\phi,\psi))\,T^{\beta_1\delta_{\beta_2,0}}\\ &\quad+T^{2-|\theta|}\otimes C_q(Q_2\phi,Q_2\psi) + C_q(Q_1\phi,Q_1\psi)\otimes T^{2-|\theta|}\\ =\;& \Delta(T^{\beta_2\delta_{\beta_1,0}}\,X(K(\phi,\psi))\,T^{\beta_1\delta_{\beta_2,0}} + C_q(\phi,\psi)),\end{aligned} \tag{36}$$

where it was assumed that Q_i, $i = 1$, 2 commutes with ∂. Cross terms in (36) cancel when the condition (34) is satisfied.

By similar computations one checks that S given by (31)–(32) is an algebraic anti-homomorphism. Indeed, when the distribution Q_0 satisfies $X(K(Q_0\phi, Q_0\psi)) = X(Q_0K(\phi,\psi))$, for any ϕ, $\psi \in E$, then

$$\begin{aligned}
&S(X(q^{\beta_1\partial}\phi)\ X(q^{\beta_2\partial}\psi)) - S(X(q^{\beta_1\partial}\psi)\ X(q^{\beta_2\partial}\phi)) \\
&= T^{-2(2-|\theta|)}\ X(q^{(\beta_2-(2-|\theta|)\theta)\partial}Q_0\psi)\ X(q^{(\beta_1+(2-|\theta|)\theta)\partial}Q_0\phi)\ T^{-2(2-|\theta|)} \\
&\qquad -T^{-2(2-|\theta|)}\ X(q^{(\beta_2-(2-|\theta|)\theta)\partial}Q_0\phi)\ X(q^{(\beta_1+(2-|\theta|)\theta)\partial}Q_0\psi)\ T^{-2(2-|\theta|)} \\
&= T^{-(2-|\theta|)}\ X(q^{\beta_1\partial}Q_0\psi)\ X(q^{\beta_2\partial}Q_0\phi)\ T^{-2(2-|\theta|)} \\
&\qquad -T^{-2(2-|\theta|)}\ X(q^{\beta_1\partial}Q_0\phi)\ X(q^{\beta_2\partial}Q_0\psi)\ T^{-2(2-|\theta|)} \\
&= S\left(T^{\beta_1\delta_{\beta_2,0}}\ X(K(\phi,\psi)\ T^{\beta_2\delta_{\beta_1,0}} + C_q(\phi,\psi)\right).
\end{aligned}$$

We proved that the commutation relations (5)–(9) are preserved by the comultiplication and antipode actions when operators $Q_i(q)$ satisfy the property (34). Thus $\mathcal{U}(Vir_q(E))$ possesses a Hopf algebra structure.

□

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