ESSENTIALLY IRREDUCIBLE REPRESENTATIONS OF THE LIE SUPERALGEBRAS *sl*(n/1) and *sl*(n/2)

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ABSTRACT

We consider a class of finite-dimensional irreducible representations of the Lie superalgebras *sl*(n/1) and *sl*(n/2) for any n, which we call essentially irreducible. Within each such representation space we introduce a basis and write down explicit expressions for the transformation of the basis under the action of the generators

The Lie superalgebras(LS's) *sl*(n/m) belong to the class of the basic LS's in the classification of Kac[1].These LS's and their representations have been intensively studied in the last years (see, for instance, Refs.2 - 7 and the references therein). Nevertheless, there is still much to be done. In particular, the important from a physical point of view question to compute the matrix elements of the generators has been solved so far only for some LS's[8-12]. In the present paper we show how one can introduce a basis in a class of modules of the general linear LS *gl*(n/m) and hence of *sl*(n/m) for m=1,2 , which we call essentially typical.We introduce appropriate basis whithin each module and write down expressions for the transformation of the basis under the action of the generators.

The LS *gl*(n/m) can be defined as the set of all squared (n+m)-dimensional matrices. We label the rows and the columns of these matrices with indices $i, j, k=1,2,\dots,n+m$. Assign to each index i a degree (i), which is zero for $i=1,\dots,n$ and 1 for $i=n+1,\dots,n+m$. Let $e_{ij} \in gl(n/m)$ be a matrix with 1 on the i-th row and the j-th column and zero elsewhere. The even (resp. the odd) part of $gl(n/m)$ is defined to be the linear span of all e_{ij}, for which $(i)+(j)$ is an even (resp. an odd) number. The multiplication (= the supercommutator) $[\![\ ,\ ]\!]$ on $gl(n/m)$ is given with the linear extension of the relations

$$[\![e_{ij}, e_{kl}]\!] = \delta_{jk} e_{il} - (-1)^{[(i)+(j)][(k)+(l)]} \delta_{il} e_{kj} \tag{1}$$

The LS $sl(n/m)$ is a subalgebra of $gl(n/m)$ consisting of all those matrices $a \in gl(n/m)$, whose supertrace (= str) vanishes, i.e.

$$sl(n/m) = \{a \mid a \in gl(n/m),\ str(a) = \textstyle\sum_{i=1}^{r} (-1)^{(i)} a_{ii} = 0\}. \tag{2}$$

The even subalgebra $gl(n/m)_0$ of $gl(n/m)$ is

$$gl(n/m)_0 = gl(n) \oplus gl(m) = lin.env.\{e_{ij} \mid (i)+(j) = \text{even number}\}. \tag{3}$$

Let $e_{11}, e_{22}, \dots, e_{rr}$, $r=n+m$, be a basis in the Cartan subalgebra H of $gl(n/m)$. Denote by e^1, $e^2, \dots, e^r$ the conjugate to it basis from the dual space H^*.

Lemma. The finite-dimensional irreducible modules (=fidirmodes) of $gl(n/m)$ are labelled by the set of all complex r-tuples

$$[m]_r \equiv [m_{1r},\ldots,m_{rr}]\ ,\ m_{ir}-m_{i+1,r}\in \mathbb{Z}_+ \quad \forall\ i\neq n=1,\ldots,r. \tag{4}$$

Then $\Lambda = \sum_{i=1}^{r} m_{ir} e^{i}$ is the highest weight of the corresponding fidirmod, which we denote by $W([m]_r)$.

Here and throughout the paper we omit the proofs.

Let

$$l_{ij}= (-1)^{(i)}m_{ij} + |n-i|+1-(i) \quad \forall\ i\leq j=1,\ldots,r. \tag{5}$$

Proposition 1. The fidirmod $W([m]_r)$ is typical[2] if and only if all numbers $l_{1r}, l_{2r},\ldots,l_{rr}$ are different.

Consider the chain of subalgebras

$$gl(n/m)\supset gl(n/m-1)\supset\ldots\supset gl(n/k)\supset\ldots\supset gl(n/1)\supset gl(n). \tag{6}$$

Denote by $W(n/k)$ any $gl(n/k)$ module and let

$$W(n/m)\supset W(n/m-1)\supset\ldots\supset W(n/k)\supset\ldots\supset W(n/1)\supset W(n) \tag{7}$$

be a flag of submodules, corresponding to the chain (6).

Definition. We say that the typical $gl(n/m)$ module $W([m]_r)\equiv W(n/m)$ is essentially typical, if there exists no flag (7) containing a nontypical $gl(n/k)$ module $W(n/k)$.

Proposition 2. The fidirmod $W([m]_r)$ is essentially typical if and only if

$$l_{kr}\notin(l_{n+1,r}, l_{n+1,r}+1, l_{n+1,r}+2,\ldots,l_{rr}) \quad \forall \quad k=1,\ldots,n. \tag{8}$$

Proposition 3. As a basis in the essentially typical fidirmod $W([m]_r)$ with a highest weight $\Lambda=\sum_{i=1}^{r} m_{ir} e^{i}$ one can choose the set of all schemes

$$(m) \equiv \begin{bmatrix} m_{1r}, & m_{2r} & ,\dots, & m_{r-1,r} & ,m_{rr} \\ m_{1,r-1}, & m_{2,r-1} & ,\dots, & m_{r-1,r-1} & \\ \dots\dots\dots & & & & \\ m_{1i} & m_{2i} & ,\dots, & m_{ii} & \\ \dots\dots\dots & & & & \\ m_{12} & m_{22} & & & \\ m_{11} & & & & \end{bmatrix}, \tag{9}$$

where the numbers $m_{1r},\dots,m_{rr}$ are fixed ; the other numbers m_{ij} take all possible values, consistent with the conditions

(1) $\forall$ $j=1,\dots,n-1$ and $\forall$ $p=n+1,\dots,r$

$$m_{ip} - m_{i,p-1} \equiv \theta_{i,p-1} = 0,1 \ ; \tag{10}$$

(2) $\forall$ $j=1,\dots,n-1$ and $\forall$ $q=n+1,\dots,r$

$$m_{jq} - m_{j+1,q} \in \mathbb{Z}_+ \ ; \tag{11}$$

(3) $\forall$ $i\le j=1,2,\dots,n-1$ and $i\le j=n+1,\dots,r-1$

$$m_{i,j+1} - m_{ij} \in \mathbb{Z}_+ \ , \quad m_{ij} - m_{i+1,j+1} \in \mathbb{Z}_+ \ . \tag{12}$$

We call this basis a Gel'fand-Zetlin basis(GZ basis) in $W([m]_r)$.Each GZ basis vector (m) is a weight vector. More precisely,

$$e_{ii}(m) = (\textstyle\sum_{k=1} m_{ki} - \sum_{k=1} m_{k,i-1})(m). \tag{13}$$

The transformation of the GZ basis under the action of the $gl(n)$ generators e_{ij} ,$i,j = 1,\dots,n$ is completely determined from the relations $(i = 2,\dots,n)$[13]

$e_{i,i-1}(m)=$

$$\sum_{j=1}^{i-1}\left|\frac{\prod_{k=1}^{i}(l_{ki}-l_{j,k-1}+1)\prod_{k=1}^{i-2}(l_{k,i-2}-l_{j,i-1})}{\prod_{k\neq j=1}^{i-1}(l_{k,i-1}-l_{j,i-1}+1)(l_{k,i-1}-l_{j,i-1})}\right|^{1/2}(m)_{-j,i-1},$$

(14)

$e_{i-1,i}(m)=$

$$\sum_{j=1}^{i-1}\left|\frac{\prod_{k=1}^{i}(l_{ki}-l_{j,i-1})\prod_{k=1}^{i-2}(l_{k,i-2}-l_{j,i-1}-1)}{\prod_{k\neq j=1}^{i-1}(l_{k,i-1}-l_{j,i-1})(l_{k,i-1}-l_{j,i-1}-1)}\right|^{1/2}(m)_{j,i-1}.$$

The Lie superalgebra *gl*(n/1).In this case [see (9)] r=n+1 .

Proposition 4. Every typical *gl*(n/1) module is essentially typical.

The representation of *gl*(n/1) is uniquely determined from Eqs. (14) and the transformation of $W([m]_{n+1})$ under the action of the odd generators $e_{n,n+1}$ and $e_{n+1,n}$, which read

$$e_{n,n+1}(m)=\sum_{i}^{n}\theta_{in}(-1)^{i-1}(-1)^{\theta_{1n}+\dots+\theta_{i-1,n}}$$

$$*\left|\frac{\prod_{k=1}^{n-1}(l_{k,n-1}-l_{in}-1)}{\prod_{k\neq i=1}^{n}(l_{k,n+1}-l_{i,n+1})}\right|^{1/2}(m)_{in}\ , \qquad (15)$$

$$e_{n+1,n}(m)=\sum_{i=1}^{n}(1-\theta_{in})(-1)^{i-1}(-1)^{\theta_{1n}+\ldots+\theta_{i-1,n}}(l_{i,n+1}-l_{n+1,n+1})$$

$$*\left|\frac{\prod_{k=1}^{n-1}(l_{k,n-1}-l_{in})}{\prod_{k\neq i=1}^{n}(l_{k,n+1}-l_{i,n+1})}\right|^{1/2}(m)_{-in}\,. \qquad (16)$$

The Lie superalgebra *gl*(n/2). To obtain the GZ basis we set in (9) r=n+2. The transformation of a given essentially typical fidirmod $W([m]_{n+2})$ is determined from the relations (13)-(16) and the expressions

$$e_{n+1,n+2}(m) =$$

$$\sum_{i=1}^{n}\theta_{i,n+1}(1-\theta_{in})(-1)^{\theta_{1,n+1}+\ldots+\theta_{i-1,n+1}}(-1)^{\theta_{i+1,n}+\ldots+\theta_{nn}}$$

$$*\prod_{k\neq i=1}^{n}\left|\frac{(l_{i,n+2}-l_{k,n+1})(l_{i,n+2}-l_{k,n+1}-1)}{(l_{i,n+2}-l_{k,n+2})(l_{i,n+2}-l_{kn}-1)}\right|^{1/2}(m)_{i,n+1}\ +$$

$$+\ |(l_{n+1,n+2}-l_{n+1,n+1})(l_{n+1,n+1}-l_{n+2,n+2})|^{1/2}$$

$$* \prod_{k=1}^{n} \frac{(l_{k,n+1}-l_{n+1,n+1}+1)(l_{k,n+1}-l_{n+1,n+1})}{(l_{kn}-l_{n+1,n+1}+1)(l_{k,n+2}-l_{n+1,n+1}+1)} (\mathbf{m})_{n+1,n+1}, \qquad (17)$$

$$e_{n+2,n+1}(\mathbf{m}) =$$

$$\sum_{i=1}^{n} (1-\theta_{i,n+1})\theta_{in}(-1)^{\theta_{1,n+1}+\dots+\theta_{i-1,n+1}}(-1)^{\theta_{i+1,n}+\dots+\theta_{nn}}$$

$$* \frac{(l_{i,n+2}-l_{n+1,n+2})(l_{i,n+2}-l_{n+2,n+2})}{(l_{i,n+2}-l_{n+1,n+1}-1)(l_{i,n+2}-l_{n+1,n+1})}$$

$$* \prod_{k\neq i=1}^{n} \left| \frac{(l_{i,n+2}-l_{k,n+1})(l_{i,n+2}-l_{k,n+1}-1)}{(l_{i,n+2}-l_{k,n+2})(l_{i,n+2}-l_{kn}-1)} \right|^{1/2} (\mathbf{m})_{-i,n+1} +$$

$$+ |(l_{n+1,n+1}-l_{n+1,n+1})(l_{n+1,n+1}-l_{n+2,n+2})|^{1/2} (\mathbf{m})_{-(n+1),n+1} \qquad (18)$$

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