# Vertex Operator Algebras, Modular Forms and Moonshine

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**Contents**

# Lecture 1

## §1 The Monster simple group

Group theorists conceived the Monster sporadic simple group $M$ in the early 1970s, although it was not officially born until 1982. Many features of $M$ were understood well before that time, however. In particular the complete character table was already known. Here is a small part of it.

| | 1 | $2A$ | $2B$ |
|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 196883 | 4371 | 275 |
| $\chi_3$ | 21296876 | 91884 | $-2324$ |

Let $V_i$ be the $M$-module that affords the character $\chi_i$. From the character table one can compute branching rules $V_i \otimes V_j = \oplus_k c_{ijk} V_k$. In particular, the tensor square $V_2^{\otimes 2}$ decomposes into the sum of symmetric and exterior squares $S^2(V_2) \oplus \Lambda^2(V_2)$, and the branching rules show that $c_{222} = 1$ with $V_2 \subseteq S^2(V_2)$. So there is a canonical $M$-invariant surjection $V_2 \otimes V_2 \to V_2$, and it gives rise to a *commutative, nonassociatve algebra* structure on $V_1$ whose automorphism group contains $M$. We can formally add an identity element 1 to obtain a unital, commutative algebra

$$B = V_1 \oplus V_2 \tag{1}$$

($V_1 = \mathbb{C}1$) with $M \subseteq \mathrm{Aut}(B)$.

## §2 $J$ and $V^\natural$

Up to an undetermined constant, there is a unique modular function of weight 0 on the full modular group $\Gamma := SL_2(\mathbb{Z})$ with a simple pole of residue 1 at $\infty$. Such functions can be represented as quotients of holomorphic modular forms of equal weight. For example we have

$$J + 744 = \frac{\theta_{E_8}(\tau)^3}{\Delta(\tau)} = q^{-1} + 744 + 196884q + 21493760q^2 + \dots \tag{2}$$

$$J + 24 = \frac{\theta_\Lambda(\tau)}{\Delta(\tau)} = q^{-1} + 24 + 196884q + 21493760q^2 + \dots. \tag{3}$$

Here,

$$J = q^{-1} + 196884q + 21493760q^2 + \dots \tag{4}$$

is the modular function with constant 0, $\Delta(\tau)$ is the discriminant

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \tag{5}$$

$$\theta_L(\tau) = \sum_{\alpha \in L} q^{(\alpha,\alpha)/2} \tag{6}$$

is the theta function of an even lattice $L$, and $E_8, \Lambda$ denote the $E_8$ root lattice and Leech lattice respectively.

John McKay noticed that the first few Fourier coefficients in (4) are simple linear combinations of dimensions of the irreducible $M$-modules $V_i$ with nonnegative coefficients. This suggests that we replace the coefficients by the putative $M$-modules that correspond to them - a process that sometimes goes by the abysmal name of 'categorification'. From (3)-(6), the coefficients of $J$ are all nonnegative, so at least they correspond to linear spaces. Shifting the grading by 1 for later convenience, we obtain a $\mathbb{Z}$-graded linear space

$$V^\natural := V_0^\natural \oplus V_2^\natural \oplus V_3^\natural \oplus \dots \tag{7}$$

with

$$\begin{array}{rcl} V_0^\natural & = & V_1 \\ V_2^\natural & = & V_1 \oplus V_2 \\ V_3^\natural & = & V_1 \oplus V_2 \oplus V_3 \\ & \dots & \end{array} \tag{8}$$

and with $\dim V_n^\natural =$ coefficient of $q^{n-1}$ in (4). McKay's observation was promoted to the conjecture that each $V_n^\natural$ carries a 'natural' action of $M$. Note that $V_2^\natural$ is identified with the algebra $B$.

## §3 Monstrous Moonshine

With the conjectured $\mathbb{Z}$-graded $M$-module $V^\natural$ in hand, for each $g \in M$ we can take the graded trace of $g$ and obtain another $q$-series

$$Z_g = Z_g(q) := q^{-1} \sum_{n=0}^{\infty} \mathrm{Tr}_{V_n^\natural}(g) q^n. \tag{9}$$

It was John Thompson who first asked what one can say about these additional $q$-expansions. (There are 174 of them, one for each conjugacy class of $M$.) We have $Z_{1A}(1, q) = J$ by construction, and from the character table and (8) we see that

$$\begin{aligned} Z_{2A}(q) &= q^{-1} + 4372q + 96256q^2 + \dots \\ Z_{2B}(q) &= q^{-1} + 276q - 2048q^2 + \dots \end{aligned}$$

In a celebrated paper, John Conway and Simon Norton resoundingly answered Thompson's question. They gave overwhelming evidence for the conjecture that each of the trace functions (9) was a *hauptmodul* for a subgroup of $SL_2(\mathbb{Q})$ commensurable with $SL_2(\mathbb{Z})$. This means that for each $g$ we have a subgroup $\Gamma_g \subseteq SL_2(\mathbb{Q})$ with $|\Gamma_g : \Gamma_g \cap \Gamma|, |\Gamma : \Gamma_g \cap \Gamma| < \infty$ such that the following hold:

$(i)$ Each $Z_g$ is the $q$-expansion of a modular function of weight zero on $\Gamma_g$. (10)

$(ii)$ If $\mathbb{H}$ is the complex upper half-plane, the compact Riemann surface $\Gamma_g \setminus \mathbb{H}^*$ is a Riemann sphere whose function field is $\mathbb{C}(Z_g)$.

If $g = 1A$ then of course $Z_1 = J$ and $\Gamma_1 = \Gamma$. Conway-Norton proposed formulae for each $Z_g$. For example,

$$Z_{2B}(q) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24, \tag{11}$$

where $\eta(\tau)$ is the Dedekind eta-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{12}$$

$Z_{2B}(q)$ is a hauptmodul for the index 3 subgroup $\Gamma_0(2) \subseteq \Gamma$.

## §4 **Vertex algebras**

The problem is now to define a natural action of the Monster $M$ on $V^\natural$ (59) so that the graded traces $Z_g$ satisfy the Conway-Norton moonshine conjectures (10). Borcherds' radical proposed solution involved the idea of a *vertex algebra*, which may be defined as follows. It is a pair $(V, \mathbf{1})$ consisting of a nonzero $\mathbb{C}$-linear space $V$ and a distinguished vector $\mathbf{1} \neq 0$. Moreover, $V$ is equipped with bilinear products

$$\begin{aligned} \mu_n : V \otimes V &\to V \qquad (n \in \mathbb{Z}), \\ u \otimes v &\mapsto u(n)v \quad (u, v \in V), \end{aligned}$$

satisfying the following axioms for all $u, v, w \in V$:

1) There is $n_0 = n_0(u, v) \in \mathbb{Z}$ such that $u(n)v = 0$ for $n \geq n_0$, (13)

2) $v(n)\mathbf{1} = 0$ $(n \geq 0)$ and $v(-1)\mathbf{1} = v$, (14)

3) For all $p, q, r \in \mathbb{Z}$ we have

$$\sum_{i=0}^{\infty} \binom{p}{i} \{u(r+i)v\}(p+q-i)w = \tag{15}$$

$$\sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \{u(p+r-i)v(q+i)w - (-1)^r v(q+r-i)u(p+i)w\}.$$

Thanks to 1), both sums in 3) are finite so that the identity in question is sensible.

At this point the reader may well be asking, where did these identities come from, what are they good for, and what do they have to do with Monstrous Moonshine? The point of these lectures is to address these questions.

We begin by specializing (15) in various ways. It is convenient to consider $u(n) \in \mathrm{End}(V)$ to be the linear operator such that $v \mapsto u(n)v$ $(v \in V)$. Taking $r = 0$, the binomial $\binom{r}{i}$ vanishes unless $i = 0$ and (15) reduces to the operator identity

$$[u(p), v(q)] = \sum_{i=0}^{\infty} \binom{p}{i} \{u(i)v\}(p+q-i), \tag{16}$$

called the *commutator formula*. Similarly, taking $p = 0$ yields the *associativity formula*

$$\{u(r)v\}(q) = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \{u(r-i)v(q+i) - (-1)^r v(q+r-i)u(i)\}. \tag{17}$$

With $n_0(u,v)$ as in (13), we obtain

$$\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\{u(p+r-i)v(q+i)-(-1)^r v(q+r-i)u(p+i)\}=0 \quad (18)$$

whenever $r \geq n_0$, which is sometimes referred to as *commutivity*.

Assuming (13), it is not too hard to show that (15) is a consequence of the commutator and associativity formulas, and thus is *equivalent* to them. There are other equivalent ways to reformulate (15) that are useful. We explain one of them (cf. (24)) in the next Section.

## §5 **Locality and Quantum fields**

In the succeeding two Sections we will explain how the idea of a vertex algebra corresponds to the physicist's *conformal field theory*.

The important idea of a *vertex operator*, or *quantum field*, or simply *field*, defined on an arbitrary linear space $V$ is as follows. It is a formal series

$$a(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in \mathrm{End}(V)[[z,z^{-1}]]$$

of operators $a_n$ on $V$ such that if $v \in V$ then $a_n v = 0$ for all large enough $n$. Set

$$\mathfrak{F}(V) = \{a(z) \in \mathrm{End}(V)[[z,z^{-1}]] \mid a(z) \text{ is a field}\}.$$

$\mathfrak{F}(V)$ is a linear subspace of $\mathrm{End}(V)[[z,z^{-1}]]$.

If $(V,\mathbf{1})$ is a vertex algebra, we set

$$Y(u,z) := \sum_{n\in\mathbb{Z}} u(n)z^{-n-1} \quad (u \in V),$$

where we are using notation introduced in the previous Section. By construction,

$$\{Y(u,z) \mid u \in V\} \subseteq \mathfrak{F}(V), \quad (19)$$

and we can think of $Y$ as a linear map

$$Y : V \to \mathfrak{F}(V),\ u \mapsto Y(u,z). \quad (20)$$

We use obvious notation when manipulating fields, eg.,

$$Y(u,z)v := \sum_n \{u(n)v\} z^{-n-1} \in V[[z]][z,z^{-1}].$$

In this language, (14) reads

$$Y(u,z)\mathbf{1} = u + \sum_{n \le -2} \{u(n)\mathbf{1}\} z^{-n-1}.$$

In particular, it follows that the $Y$ map (20) is *injective.*

A pair of fields $a(z), b(z) \in \mathfrak{F}(V)$ are called *mutually local* if

$$(z_1 - z_2)^k [a(z_1), b(z_2)] = 0 \quad \text{(some integer } k \ge 0). \tag{21}$$

This means that the (operator) coefficients of each monomial $z_1^p z_2^q$ in the following identity coincide:

$$(z_1 - z_2)^k a(z_1) b(z_2) - (z_1 - z_2)^k b(z_2) a(z_1) = 0. \tag{22}$$

Indeed,

$$\begin{aligned} & (z_1 - z_2)^k a(z_1) b(z_2) \\ = \; & \sum_{i=0}^{k} (-1)^i \binom{k}{i} z_1^{k-i} z_2^i \sum_m a_m z_1^{-m-1} \sum_n b_n z_2^{-n-1} \\ = \; & \sum_p \sum_q \left\{ \sum_{k-i-m=-p} \sum_{i-n=-q} (-1)^i \binom{k}{i} a_m b_n \right\} z_1^{-p-1} z_2^{-q-1} \\ = \; & \sum_p \sum_q \left\{ \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_{p+k-i} b_{q+i} \right\} z_1^{-p-1} z_2^{-q-1}. \end{aligned}$$

Therefore also

$$\begin{aligned} & (z_1 - z_2)^k b(z_2) a(z_1) = (-1)^k (z_2 - z_1)^k b(z_2) a(z_1) \\ = \; & (-1)^k \sum_p \sum_q \left\{ \sum_{i=0}^{k} (-1)^i \binom{k}{i} b_{q+k-i} a_{p+i} \right\} z_1^{-p-1} z_2^{-q-1}, \end{aligned}$$

whence locality (21), (22) holds if, and only if, for all integers $p, q$, and some nonnegative integer $k$ we have

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left\{a_{p+k-i}b_{q+i}-(-1)^k b_{q+k-i}a_{p+i}\right\}=0.$$

The last display is identical with the commutivity formula (18) if we take $r=k$. Because (18) holds for all $r\geq n_0$, it certainly holds for some positive integer $k$ in place of $r$. Combining this with (19), we have established

If $(V, \mathbf{1})$ is a vertex algebra then any two vertex operators $Y(u,z_1), Y(v,z_2)$ $(u,v\in V)$ are mutually local fields. (23)

In a similar vein, let $\delta(z) := \sum_{n\in\mathbb{Z}} z^n$ be the formal delta-function, and consider the identity

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)$$
$$=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2). \tag{24}$$

Here, the delta-functions are expanded as power series in the *second variable* in the numerator, e.g.,

$$\delta\left(\frac{z_1-z_2}{z_0}\right)=\sum_{n\in\mathbb{Z}} z_0^{-n}(z_1-z_2)^n$$
$$=\quad\sum_{n=0}^{\infty}(z_1/z_0)^n(1-z_2/z_1)^n+\sum_{n>0}(z_0/z_1)^n\left(\sum_{i\geq 0}(z_2/z_1)^i\right)^n.$$

With this convention, identifying the operator coefficients for each monomial $z_0^p z_1^q z_2^r$ on the lhs and rhs of (24) yields exactly the identity (15).

## §6 **CFT axioms**

(23) is the 'main' axiom for (2-dimensional) *conformal field theory* (CFT). We now discuss the other axioms. Let $(V, \mathbf{1})$ be a vertex algebra, and introduce the endomorphism

$$D: V\to V,\ u\mapsto u(-2)\mathbf{1}. \tag{25}$$

Using (14) and associativity (17) with $q = 2$, we have

$$\begin{aligned}
\{u(n)v\}(-2)\mathbf{1} &= \sum_{i=0}^{\infty}(-1)^i\binom{n}{i}\{u(n-i)v(-2+i)\mathbf{1}\} \\
&= u(n)v(-2)\mathbf{1} - nu(n-1)v(-1)\mathbf{1} \\
&= u(n)v(-2)\mathbf{1} - nu(n-1)v. \qquad (26)
\end{aligned}$$

Therefore,

$$\begin{aligned}
[D, Y(u,z)]v &= \sum_n \{Du(n)v - u(n)Dv\}z^{-n-1} \\
&= \sum_n \{(u(n)v)(-2)\mathbf{1} - u(n)v(-2)\mathbf{1}\}z^{-n-1} \\
&= \sum_n \{-nu(n-1)v\}z^{-n-1} \\
&= \sum_n \{(-n-1)u(n)v\}z^{-n-2} \\
&= \frac{d}{dz}Y(u,z)v,
\end{aligned}$$

where $d/dz$ is the formal derivative. Hence, we obtain

$$[D, Y(u,z)] = \frac{d}{dz}Y(u,z).$$

If we take $u = v = w = \mathbf{1}$ and $p = q = r = -1$ in (15) we find that $\mathbf{1}(-2)\mathbf{1} = \mathbf{1}(-2)\mathbf{1} + \mathbf{1}(-2)\mathbf{1}$. Thus $\mathbf{1}(-2)\mathbf{1} = 0$, that is $D\mathbf{1} = 0$.

We have arrived at the following set-up: a quadruple $(V, Y, \mathbf{1}, D)$ consisting of a linear space $V$, a distinguished nonzero vector $\mathbf{1} \in V$, an endomorphism $D : V \to V$ with $D\mathbf{1} = 0$, and a linear injection $Y : V \mapsto \mathfrak{F}(V)$, satisfying the following for all $u, v \in V$:

$$\begin{aligned}
\underline{\text{Locality:}}\ & Y(u,z_1), Y(v,z_2) \text{ are mutually local fields,} \\
\underline{\text{Creativity:}}\ & Y(u,z)\mathbf{1} = u + O(z), \qquad (27) \\
\underline{\text{Translation covariance:}}\ & [D, Y(u,z)] = d/dzY(u,z).
\end{aligned}$$

The axioms (27) amount to a mathematical formulation of 2-dimensional CFT, and we have shown that a vertex algebra $(V, \mathbf{1})$ naturally defines a

CFT. Conversely if $(V, Y, \mathbf{1}, D)$ is a CFT then it can be shown that $(V, \mathbf{1})$ is a vertex algebra. Basically, this means that the full strength of (15) can be recovered (27).

The nomenclature in (27) is fairly standard in the physical literature, and we use it in what follows. In addition, $\mathbf{1}$ is the *vacuum vector*, $V$ is a *Fock space*, elements in $V$ are *states*, $Y$ is the *state-field correspondence*, $u(n)$ is the $n^{th}$ *mode* of $Y(u, z)$. Creativity is interpreted to mean that the state $u$ is created from the vacuum by the field $Y(u, z)$ corresponding to $u$.

There are several other useful identities that follow without difficulty from our axiomatic set-up. Among them we mention the following.

$$\begin{aligned} Y(\mathbf{1}, z) &= \mathrm{Id}_V, & (28)\\ Y(u, z)\mathbf{1} &= e^{zD}u = \sum_{n=0}^{\infty} \frac{D^n u}{n!} z^n, \\ u(n)v &= (-1)^{n+1} \sum_{i=0}^{\infty} \frac{(-D)^i}{i!} v(n+i)u. & (29) \end{aligned}$$

(29) is called *skew-symmetry*.

# Lecture 2

## §7 Lie algebras and local fields

Certain infinite-dimensional Lie algebras naturally give rise to mutually local fields. In this Section we discuss some important examples that illustrate some of the ideas developed so far.

1. *Affine algebras.*

Let $L$ be a (complex) Lie algebra with bracket $[a, b]$ $(a, b \in L)$, equipped with a *symmetric, invariant, bilinear form* $\langle\ ,\ \rangle : L \otimes L \to \mathbb{C}$. (Invariant means that $\langle [a, b], c\rangle = \langle a, [b, c]\rangle$ for $a, b, c \in L$.) The associated affine Lie algebra is $\widehat{L} := L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ with central element $K$ and bracket

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\delta_{m+n,0}\langle a, b\rangle K.$$

There is a *triangular decomposition*

$$\widehat{L} = \widehat{L}^- \oplus \widehat{L}^0 \oplus \widehat{L}^+$$

with

$\widehat{L}^- := \{a \otimes t^m \mid m < 0\}, \widehat{L}^+ := \{a \otimes t^m \mid m > 0\}, \widehat{L}^0 := \{a \otimes t^0\} \oplus \mathbb{C}K.$

Let $W$ be a (left) $L$-module. Extend $W$ to a $\widehat{L}^+ \oplus \widehat{L}^0$-module by letting $\widehat{L}^+$ *annihilate* $W$; $K$ act as a scalar $l$ called the *level.* The *induced module*

$$V = V(l, W) := \mathrm{Ind}_{\widehat{L}^+ \oplus \widehat{L}^0}^{\widehat{L}}(W) \cong S(\widehat{L}^-) \otimes W \tag{30}$$

is a left $\widehat{L}$-module affording the representation $\pi$, say. (The linear isomorphism in (30) comes from the Poincaré-Birkhoff-Witt theorem.) A typical vector in $V$ is a sum of vectors that look like

$$(b_1 \otimes t^{n_1}) \ldots (b_k \otimes t^{n_k}) \otimes w \ (b_i \in L, w \in W, n_1 \leq \ldots \leq n_k \leq -1),$$

and

$$\begin{aligned} \pi(a \otimes t^n)\{(b_1 \otimes t^{n_1}) \ldots (b_k \otimes t^{n_k}) \otimes w\} = \\ (a \otimes t^n)(b_1 \otimes t^{n_1}) \ldots (b_k \otimes t^{n_k}) \otimes w \end{aligned} \tag{31}$$

where the product on the left is in the universal enveloping algebra of $\widehat{L}$.

Set

$$Y(a, z) := \sum_{n \in \mathbb{Z}} \pi(a \otimes t^n) z^{-n-1} \quad (a \in L). \tag{32}$$

It is easy to see that if $n + \sum_i n_i > 0$ then (31) reduces to 0. In particular, $Y(a, z) \in \mathfrak{F}(V)$. The following calculation, showing that the fields $Y(a, z)$ $(a \in L)$ are mutually local of order 2 (i.e. we may take $k = 2$ in (21)), gives a first insight into how locality comes into play. Thus

$$\begin{aligned} & (z_1 - z_2)^2 [Y(a, z_1), Y(b, z_2)] \\ = \ & (z_1 - z_2)^2 \sum_{m,n \in \mathbb{Z}} [\pi(a \otimes t^m), \pi(b \otimes t^n)] z_1^{-m-1} z_2^{-n-1} \\ = \ & (z_1 - z_2)^2 \sum_{m,n \in \mathbb{Z}} \pi([(a \otimes t^m), (b \otimes t^n)]) z_1^{-m-1} z_2^{-n-1} \\ = \ & (z_1 - z_2)^2 \sum_{m,n \in \mathbb{Z}} \{\pi([a, b] \otimes t^{m+n}) + m\delta_{m+n,0} \langle a, b\rangle \pi(K)\} z_1^{-m-1} z_2^{-n-1} \\ = \ & (z_1 - z_2)^2 \left\{ \sum_{p \in \mathbb{Z}} \pi([a, b] \otimes t^p) \sum_{m \in \mathbb{Z}} z_1^{-m-1} z_2^{m-p-1} + \langle a, b\rangle \pi(K) \sum_{m \in \mathbb{Z}} m z_1^{-m-1} z_2^{m-1} \right\} \end{aligned}$$

$$
\begin{aligned}
&= z_2^{-p-2}(z_1-z_2)^2 \sum_{p\in\mathbb{Z}} \pi([a,b]\otimes t^p) \sum_{m\in\mathbb{Z}} z_1^{-m-1} z_2^{m+1} + \\
&\quad z_2^{-2}(z_1-z_2)^2 \langle a,b\rangle \pi(K) \sum_{m\in\mathbb{Z}} m z_1^{-m-1} z_2^{m+1} \\
&= z_2^{-p}\left(\frac{z_1}{z_2}-1\right)^2 \sum_{p\in\mathbb{Z}} \pi([a,b]\otimes t^p)\delta\left(\frac{z_1}{z_2}\right) - \left(\frac{z_1}{z_2}-1\right)^2 \langle a,b\rangle \pi(K)\delta'\left(\frac{z_1}{z_2}\right).
\end{aligned}
\tag{33}
$$

Here $\delta(z)$ is as in Section 4 (cf. comments preceding (24)), and $\delta'(z) := \sum_{n\in\mathbb{Z}} n z^{n-1}$. Now check that $(z-1)^k\delta(z)=0$ for $k\geq 1, (z-1)^k\delta'(z)=0$ for $k\geq 2$. In particular, (33) vanishes and $(z_1-z_2)^2[Y(a,z_1),Y(b,z_2)]=0$, as asserted.

When $W=\mathbb{C}v_0$ is the trivial 1-dimensional $L$-module we can go further, and see the begginnings of a CFT. Here,

$$
\begin{aligned}
V &= V(l,\mathbb{C}v_0) \cong S(\widehat{L}^{-1})\otimes \mathbb{C}v_0 \\
&= S(\oplus_{m=1}^{\infty} L\otimes t^{-m})\otimes \mathbb{C}v_0 \\
&= \mathbb{C}(1\otimes v_0)\oplus (L\otimes t^{-1})\otimes \mathbb{C}v_0 \\
&\qquad \oplus \big(L\otimes t^{-2}\oplus S^2(L\otimes t^{-1})\big)\otimes \mathbb{C}v_0 \oplus \dots \\
&\cong \mathbb{C}\mathbf{1}\oplus L\oplus (L\oplus S^2(L))\oplus \dots
\end{aligned}
\tag{34}
$$

where we have used the natural identification $L \xrightarrow{\cong} L\otimes t^{-1},\ a\mapsto a\otimes t^{-1}$, set $1\otimes v_0=\mathbf{1}$, and dropped $v_0$ from the notation for convenience.

In this way, the field $Y(a,z)$ (32) is associated with the state $a\in V$. $Y(a,z)$ is creative (cf. (27)) because

$$
\begin{aligned}
Y(a,z)\mathbf{1} &= \sum_{n\in\mathbb{Z}} \{\pi(a\otimes t^n)(1\otimes v_0)\} z^{-n-1} \\
&= \sum_{n=0}^{\infty} \{1\otimes (a\otimes t^n)v_0\} z^{-n-1} + \sum_{n=-1}^{-\infty} \{(a\otimes t^n)\otimes v_0\} z^{-n-1} \\
&= a\otimes t^{-1}\otimes v_0 + \sum_{n=-2}^{-\infty} \{(a\otimes t^n)\otimes v_0\} z^{-n-1} \\
&= a + O(z).
\end{aligned}
$$

(Because $\mathbb{C}v_0$ is the trivial $L$-module then $(a \otimes t^n)v_0 = 0$ for $n \geq 0$.) $Y(a,z)$ is also translation covariant (loc. cit.): if $m \geq 1$ then

$$
\begin{aligned}
& [d/dt, Y(a,z)](b \otimes t^{-m}) \\
= \quad & \frac{d}{dt}\sum_{n\in\mathbb{Z}}\{a\otimes t^n.b\otimes t^{-m}\otimes v_0\}z^{-n-1} + m\sum_{n\in\mathbb{Z}}\{a\otimes t^n.b\otimes t^{-m-1}\otimes v_0\}z^{-n-1} \\
= \quad & \frac{d}{dt}\sum_{n<0}\{a\otimes t^n.b\otimes t^{-m}\otimes v_0\}z^{-n-1} + \\
& \frac{d}{dt}\sum_{n\geq 0}\{[a,b]\otimes t^{n-m}\otimes v_0 + n\delta_{n,m}\langle a,b\rangle K\otimes v_0\}z^{-n-1} + \\
& m\sum_{n\in\mathbb{Z}}\{a\otimes t^n.b\otimes t^{-m-1}\otimes v_0\}z^{-n-1} \\
= \quad & \sum_{n<0}\{na\otimes t^{n-1}.b\otimes t^{-m}\otimes v_0 - ma\otimes t^n.b\otimes t^{-m-1}\otimes v_0\}z^{-n-1} + \\
& \sum_{n\geq 0}\{(n-m)[a,b]\otimes t^{n-m-1}\otimes v_0\}z^{-n-1} + m\sum_{n\in\mathbb{Z}}\{a\otimes t^n.b\otimes t^{-m-1}\otimes v_0\}z^{-n-1} \\
= \quad & \sum_{n<0}\{na\otimes t^{n-1}.b\otimes t^{-m}\otimes v_0\}z^{-n-1} + m\sum_{n\geq 0}\{a\otimes t^n.b\otimes t^{-m-1}\otimes v_0\}z^{-n-1} + \\
& \sum_{n\geq 0}\{(n-m)[a,b]\otimes t^{n-m-1}\otimes v_0\}z^{-n-1} \\
= \quad & \sum_{n<0}\{na\otimes t^{n-1}.b\otimes t^{-m}\otimes v_0\}z^{-n-1} + m\sum_{n\geq 0}\{[a,b]\otimes t^{n-m-1}\otimes v_0\}z^{-n-1} + \\
& m(m+1)\langle a,b\rangle\{K\otimes v_0\}z^{-m-2} + \sum_{n\geq 0}\{(n-m)[a,b]\otimes t^{n-m-1}\otimes v_0\}z^{-n-1} \\
= \quad & \sum_{n<0}\{na\otimes t^{n-1}.b\otimes t^{-m}\otimes v_0\}z^{-n-1} + \sum_{n\geq 0} n\{[a,b]\otimes t^{n-m-1}\otimes v_0\}z^{-n-1} + \\
& m(m+1)\langle a,b\rangle\{K\otimes v_0\}z^{-m-2} \\
= \quad & -\frac{d}{dz}\left\{\sum_{n<0}\{a\otimes t^{n-1}.b\otimes t^{-m}\otimes v_0\}z^{-n} + \sum_{n\geq 0}\{[a,b]\otimes t^{n-m-1}\otimes v_0\}z^{-n} + \right. \\
& \left. \qquad m\langle a,b\rangle\{K\otimes v_0\}z^{-m-1}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{d}{dz}\left\{\sum_{n\in\mathbb{Z}}\{a\otimes t^{n-1}.b\otimes t^{-m}\otimes v_0\}z^{-n}\right\} \\
&= -\frac{d}{dz}Y(a,z)b\otimes t^{-m}.
\end{aligned}
$$

This shows that $[D, Y(a,z)] = d/dzY(a,z)$ where $D = -d/dt$, and because $\mathbf{1}$ is independent of $t$ then $D\mathbf{1} = 0$. It should come as no surprise that in fact $(V(l, \mathbb{C}v_0), Y, 1\otimes v_0, -d/dt)$ is a vertex algebra/CFT. Indeed, based on what we already know, the result follows from the following general result.

$$
\begin{array}{l}
V \text{ is a linear space with } 0\neq \mathbf{1}\in V,\ D\in \mathrm{End}(V), \text{ and mutually local,} \\
\text{translation covariant, creative fields } y(u,z)\in \mathfrak{F}(V)\ (u\in S\subseteq V). \\
\text{If } V \text{ is spanned by states } u_1(n_1)\ldots u_k(n_k)\mathbf{1}\ (u_i\in S, n_i\in\mathbb{Z}) \text{ then} \\
\text{there is a vertex algebra } (V, Y, \mathbf{1}, D) \text{ with } Y(u,z) := y(u,z)\ (u\in S).
\end{array}
\tag{35}
$$

In this situation, we say that $S$ *generates* $V$. Thus $(V(l, \mathbb{C}v_0), Y, 1\otimes v_0, -d/dt)$ is a vertex algebra generated by $L = L\otimes t^{-1}$. We will denote this vertex algebra by $V(L, l)$.

2. *Virasoro algebra*. (Several aspects of this case are similar to the previous one, so we give less detail.)

The Virasoro algebra is the Lie algebra with underlying linear space $Vir := \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n \oplus \mathbb{C}K$ with central element $K$ and bracket

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}K. \tag{36}
$$

(The denominator 12 is conventional here; it can be removed by rescaling.) There is a triangular decomposition

$$
Vir = Vir^+ \oplus Vir^0 \oplus Vir^-
$$

with

$$
Vir^+ := \oplus_{n>0}\mathbb{C}L_n,\ Vir^0 := \mathbb{C}L_0\oplus\mathbb{C}K,\ Vir^- = \oplus_{n<0}\mathbb{C}L_n.
$$

Let $W = \mathbb{C}v_0$ be the 1-dimensional $Vir^0$-module such that $L_0v_0 = hv_0, Kv_0 = cv_0$, extend to a $Vir^+\oplus Vir^0$-module by letting $Vir^+$ *annihilate* $v_0$, and form the induced module

$$\begin{aligned}
V = V(c,h) &= \mathrm{Ind}_{Vir^+\oplus Vir^0}^{Vir} W \\
&\cong S(Vir^-)\otimes \mathbb{C}v_0 \\
&= S(\oplus_{n<0}\mathbb{C}L_n)\otimes \mathbb{C}v_0 \\
&\cong \mathbb{C}\mathbf{1}\oplus \mathbb{C}L_{-1}\oplus \dots
\end{aligned}$$

where $\mathbf{1} := 1\otimes v_0$. $h$ and $c$ are called the *conformal weight* and *central charge* respectively. Introduce

$$Y(\omega, z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{37}$$

One sees easily that $Y(\omega, z)\in \mathfrak{F}(V)$. Note the slight change in convention regarding powers of $z$ in (37), which is standard. The reader may enjoy proving that $Y(\omega, z)$ is a (self-) local field. Indeed, we have

$$(z_1 - z_2)^4[Y(\omega, z_1), Y(\omega, z_2)] = 0. \tag{38}$$

Note that

$$Y(\omega, z)\mathbf{1} = \sum_{n\in\mathbb{Z}}\{L_n\mathbf{1}\}z^{-n-2} = h\mathbf{1}z^{-2} + L_{-1}\mathbf{1}z^{-1} + L_{-2}\mathbf{1} + \dots. \tag{39}$$

So there is no chance that $Y(\omega, z)$ is creative, because $L_{-1}\mathbf{1}$ is nonzero by construction. Furthermore, as it stands $\omega$ is just an abstract symbol, not a state in $V$. We do not deal systematically with these issues here, but move on to the definition of *vertex operator algebra*, where in some sense they get resolved.

## §8 **Vertex operator algebras**

A *vertex operator algebra* (VOA) is a vertex algebra with additional structure that arises from a special Virasoro field of the type discussed in Section 7. Specifically, a VOA is a vertex algebra/CFT $(V, Y, \mathbf{1}, D)$ together with a distinguished state $\omega\in V$ (called the *conformal* or *Virasoro* vector) such that the following hold:

1) $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ and the modes $L(n)$ generate an action of the Virasoro algebra $Vir$ (36) in which $K$ acts on $V$ as a scalar $c$, called the *central charge* of $V$.
2) $L(0)$ is a *semisimple* operator on $V$. Its eigenvalues lie in $\mathbb{Z}$, are *bounded below*, and have *finite-dimensional* eigenspaces.
3) $D = L(-1)$.

This definition requires some discussion. Because $(V, Y, \mathbf{1}, D)$ is a vertex algebra, the fields $Y(u, z)$ $(u \in V)$ are required to be mutually local and creative. In particular, $Y(\omega, z)$ is necessarily self-local - a condition that can be independently verified (38). Furthermore, comparison with (39) shows that in the present situation we must have $L(0)\mathbf{1} = L(-1)\mathbf{1} = 0$ (otherwise $Y(\omega, z)$ is not creative) and $\omega = L(-2)\mathbf{1}$ (because $\omega$ is created from the vacuum by the field which corresponds to it). Note that $L(n) = \omega(n + 1)$.

The associativity formula (17) yields

$$(L(-1)u)(n) = (\omega(0)u)(n) = \omega(0)u(n) - u(n)\omega(0) = [L(-1), u(n)].$$

Thanks to 3) and the last display, translation covariance may then be written

$$d/dzY(u, z) = [L(-1), Y(u, z)] = Y(L(-1)u, z).$$

In particular, $Du = L(-1)u = u(-2)\mathbf{1}$, and 3) is consistent with (25).

For $n \in \mathbb{Z}$ we let $V_n$ be the $L(0)$-eigenspace with eigenvalue $n$. According to 2), we have the fundamental *spectral decomposition* (into finite-dimensional graded pieces)

$$V = \bigoplus_{n=n_0}^{\infty} V_n \tag{40}$$

where $n_0$ is the smallest eigenvalue of $L(0)$. Because $L(0)\mathbf{1} = 0$ then $\mathbf{1} \in V_0$.

We usually denote a VOA by the quadruple $(V, Y, \mathbf{1}, \omega)$. It is a model for the creation and annihilation of *bosons* (particles of integer spin).

The vertex algebra $V(L, l)$ can sometimes be given the structure of a VOA - we just have to find the right conformal vector. We describe two important cases where this can be achieved.

1. *Heisenberg algebra, or free bosonic theories.*

Here, the Lie algebra $L$ is *abelian* (i.e. $[a, b] = 0$ $(a, b \in L)$) of dimension $l$, equipped with the (unique) nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ (which is automatically invariant). The level (the scalar by which $K$ acts) is also $l$. The conformal vector is $\omega := 1/2 \sum_{i=1}^{l} v_i(-1)v_i$ where $\{v_i\}$ is an orthonormal basis of $L$, and it transpires that the central charge is $c = l$. The grading by $L(0)$-eigenvalues (40) coincides with the natural tensor product grading in which $L \otimes t^{-m}$ has degree $m$ (cf. (34)). This is the *rank $l$ Heisenberg* VOA. It models $l$ free (noninteracting) bosons. The special case when $l = 24$ underlies the *bosonic string.*

2. *Kac-Moody theories, or WZW models.*

In this case, $L$ is a *finite-dimensional simple* Lie algebra, and $\langle\ ,\ \rangle$ is the *Killing form* (which is unique up to an overall scalar). The conformal vector is similar to the last case, namely $\omega = 1/2 \sum_{i=1}^{\dim L} v_i(-1)v_i$ for an orthonormal basis $\{v_i\}$ of $L$. The central charge is $c = l \dim L/(l + h^\vee)$, and we obtain a VOA as long as $l + h^\vee \neq 0$ ($h^\vee$ is the *dual Coxeter number* of $L$).

## §9 Super vertex algebras

Physically realistic theories incorporate both bosons and fermions. Axiomatically, this corresponds to *super vertex (operator) algebras* (SV(O)A). We limit ourselves here to the basic definitions.

The Fock space for a SVA is a linear super space, i.e. a linear space $V$ equipped with a $\mathbb{Z}_2$-grading $V = V^0 \oplus V^1$, and a nonzero vacuum vector $\mathbf{1} \in V^0$. Here and below, superscripts will always lie in $\{0, 1\}$ regarded as the two elements of $\mathbb{Z}/2\mathbb{Z}$. We write $|u| = p$ if $u \in V^p$. $V^0$ and $V^1$ are called the *even* and *odd* parts of $V$ respectively.

There is a correspondence $u \mapsto Y(u, z)$ between states $u \in V$ and mutually local, creative fields $Y(u, z) := \sum_{n\in\mathbb{Z}} u(n)z^{-n-1}$, and we have

$$u(n) : V^p \to V^{p+|u|}.$$

Finally, we require the super version of the basic identity (15), namely

$$\sum_{i=0}^{\infty}\binom{p}{i}\{u(r+i)v\}(p+q-i)w=$$

$$\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\{u(p+r-i)v(q+i)w-(-1)^{r+|u||v|}v(q+r-i)u(p+i)w\}.$$

The delta-function version of this (cf. (24)) reads

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)-(-1)^{|u||v|}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1) \\ =z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2). \qquad (41) \end{aligned}$$

Note that the substructure $(V^0,Y,\mathbf{1})$ is a vertex algebra. As in the case of vertex algebras, these axioms are equivalent to a SCFT for which super locality, the super analog of (22), is as follows:

$$(z_1-z_2)^k[Y(u,z_1),Y(v,z_2)]=(-1)^{|u||v|}(z_1-z_2)^k[Y(v,z_2),Y(u,z_1)]$$

A SVOA is a quadruple $(V,Y,\mathbf{1},\omega)$ such that analogs of 1)-3) of Section 8 hold. The only change is that eigenvalues of $L(0)$ are allowed to lie in $1/2\mathbb{Z}$. $L(0)$ leaves $V^0$ invariant, and on this subspace the eigenvalues lie in $\mathbb{Z}$. Thus $(V^0,Y,\mathbf{1},\omega)$ is a VOA.

There are further variations on this theme, where it is assumed that additional special states and fields exist. These lead to so-called $N=1$ SCFT, $N=2$ SCFT, etc. They play a rôle in certain geometric and physical applications, although we will not discuss them here.

# Lecture 3

## §10 Modules over a VOA

Suppose that $(V,Y,\mathbf{1})$ is a VA. A module over this structure, i.e. a $V$-module, is a linear space $W$ and a linear map $Y_W : V \to \mathfrak{F}(W)$, $v \mapsto$

$Y_W(v,z) = \sum_{n\in\mathbb{Z}} v_W(n)z^{-n-1}$ such that $Y_W(\mathbf{1},z) = Id_W$ and the analog of (15) holds, i.e. for all $u,v \in V, w \in W$ we have

$$\sum_{i=0}^{\infty} \binom{p}{i} \{u(r+i)v\}_W (p+q-i)w = \tag{42}$$
$$\sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \{u(p+r-i)_W v(q+i)_W w - (-1)^r v(q+r-i)_W u(p+i)_W w\}.$$

As before, there are a number of auxiliary consequences of this identity. We mention only that the fields $Y_W(u,z)$ $(u \in V)$ are *mutually local*. Generally $W$ has no analog of the vacuum vector, so creativity has no meaning for the fields $Y_W(u,z)$.

Suppose that $(V, Y, \mathbf{1}, \omega)$ is a VOA. A module over this structure is a $V$-module (in the previous sense) such that $L_W(0)$ is a semisimple operator (on $W$) with finite-dimensional eigenspaces. The eigenvalues of $L_W(0)$ are truncated below in the following sense: given an eigenvalue $\lambda$, there are only *finitely many* eigenvalues of the form $\lambda - n$ $(n \in \mathbb{N})$. In particular, we have a spectral decomposition of $W$ analogous to (40).

$W$ is called *irreducible*, or *simple*, if the only subspaces invariant under all modes $u_W(q)$ $(u \in V, q \in \mathbb{Z})$ are 0 and $W$. It is easy to see that $\sum_{n\in\mathbb{Z}} W_{\lambda+n}$ is always an invariant subspace. Hence, if $W$ is a simple $V$-module then the spectral decomposition takes the form

$$W = \bigoplus_{n=0}^{\infty} W_{h+n} \tag{43}$$

for some uniquely determined $h = h_W \in \mathbb{C}$ called the *conformal weight* of $W$.

We give a few examples of $V$-modules.

1. If $(V, Y, \mathbf{1}, \omega)$ is a VOA then $V$ is itself a $V$-module, called the *adjoint* module.

2. Suppose that $V$ is a VOA and $W \subseteq V$ satisfies $u(n)w \in W$ $(u \in V, w \in W)$. Then $W$ is a $V$-module. $W$ is called an *ideal* of $V$, because we also have $w(n)u \in W$ (use skew-symmetry (29)). It follows that $V/W$ is the Fock space of a VOA (the *quotient VOA of* $V$) in which the mode $(u+W)(q)$ of $Y(u+W,z)$ is the operator induced on $V/W$ by $u(q)$. We call $V$ *simple* if

the only ideals are the trivial ones $V$ and 0. For example, any Heisenberg VOA $V(l, \mathbb{C}v_0)$ is simple.

3. If $(V, Y, \mathbf{1}, \omega)$ is a SVOA (cf. Section 9) the odd part $V^1$ is a module over the even part $V^0$. In this case the conformal weight of $V^1$ is necessarily a half-integer. (A VOA may have *no* modules with half-integral conformal weight except for (direct sums of) the adjoint module. Thus the VOAs that can occur as the even part of a SVOA are severely restricted.)

4. Recall the rank $l$ Heisenberg VOA $V(l, \mathbb{C}v_0)$ (cf. Section 8) generated by a rank $l$ abelian Lie algebra $L$. For an $L$-module $W$ we constructed (Section 7) a space $V(l, W)$ and mutually local fields $Y(a, z) \in \mathfrak{F}(V(l, W))$ $(a \in L)$. It is not hard to see that $V(l, W)$ is a $V(l, \mathbb{C}v_0)$-module, and it is simple whenever $\dim W = 1$.

## §11 **Lattice theories**

An *integral lattice* $L$ is a finitely generated free abelian group equipped with a nondegenerate, symmetric, $\mathbb{Z}$-valued bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$. Let $l := \mathrm{rk}L$. Set $H := \mathbb{C} \otimes_{\mathbb{Z}} L$ and let $\langle\ ,\ \rangle$ also denote the linear extension to $H$. We regard $H$ as an abelian Lie algebra equipped with a symmetric invariant bilinear form. As such, there is the associated Heisenberg VOA $V(l, \mathbb{C}v_0)$ (cf. Sections 7 and 8).

Fix $\beta \in L$. Let $\mathbb{C}e^\beta$ be the 1-dimensional linear space spanned by $e^\beta$, regarded as an $H$-module through the action

$$\alpha.e^\beta := \langle \alpha, \beta \rangle e^\beta \quad (\alpha \in H). \tag{44}$$

Associated to this $L$-module is the simple $V(l, \mathbb{C}v_0)$-module $V(l, \mathbb{C}e^\beta)$. Note that $\mathbb{C}e^0$ is the trivial $L$-module, so that it can be identified with $\mathbb{C}v_0$. Also, we have a linear isomorphism $V(l, \mathbb{C}e^\beta) \cong S(\widehat{H}^-) \otimes \mathbb{C}e^\beta$ (cf. (30)). We form the Fock space

$$\begin{aligned} V_L &:= \bigoplus_{\beta \in L} V(l, \mathbb{C}e^\beta) \\ &\cong S(\widehat{H}^-) \otimes \bigoplus_{\beta \in L} \mathbb{C}e^\beta \\ &= S(\widehat{H}^-) \otimes \mathbb{C}[L]. \end{aligned} \tag{45}$$

(It is convenient to identify the *group algebra* $\mathbb{C}[L]$ of $L$ with $\oplus_\beta \mathbb{C}e^\beta$.) We discuss the following result:

> $V_L$ carries the structure of a SVOA; if $L$ is an *even* lattice (i.e. $\langle\beta,\beta\rangle \in 2\mathbb{Z}$ for $\beta \in L$), then $V_L$ is a VOA.

$S(\widehat{H}^-)$ is naturally identified with the Heisenberg VOA itself, and in particular it is generated (cf. (35)) by the fields $Y(\alpha, z)$ ($\alpha \in H$). Because each $V(l, \mathbb{C}e^\beta)$ is a Heisenberg module, the $Y(\alpha, z)$ naturally extend to (mutually local) fields on $V_L$. To get a generating set of fields for $V_L$ (loc. cit.) we would need to extend the set of $Y(\alpha, z)$ to a larger set of mutually (super) local fields by defining fields $Y(1 \otimes e^\beta, z)$ ($\beta \in L$) directly. We will skip the details here. Recall (cf. Section 8) that the conformal vector for the Heisenberg VOA is $\omega := 1/2 \sum_{i=1}^{l} v_i(-1)v_i$ for an orthonormal basis $\{v_i\}$ of $H$. This state is also taken as the conformal vector of $V_L$. In particular, the central charge of $V_L$ is the rank $l$ of $L$. The field $Y(\omega, z) = \sum_n L(n)z^{-n-2}$ determined by $\omega$ is defined in the natural way, i.e. on $V(l, \mathbb{C}e^\beta)$ it acts as $Y_{V(l,\mathbb{C}e^\beta)}(\omega, z)$. Since each summand in (45) is a Heisenberg module, $L(0)$ acts semisimply on each of them, and therefore on $V_L$. We consider the eigenvalues and eigenspaces of $L(0)$ in the next Section. Finally, we note that $V_L$ is a simple VOA if $L$ is even.

## §12 **Partition functions**

Suppose that $(V, Y, \mathbf{1}, \omega)$ is a VOA of central charge $c$ (cf. Section 8, axiom 1)), and spectral decomposition (40) into $L(0)$-eigenspaces. The *partition function* of $V$ is the formal $q$-series

$$Z(q) = Z_V(q) := q^{-c/24} \sum_{n=n_0}^{\infty} \dim V_n q^n. \tag{46}$$

(This is the first place that $c$ has played a rôle in the proceedings.) More generally, for a simple $V$-module $W$ with spectral decomposition (43), the corresponding partition function is

$$Z(q) = Z_W(q) := q^{h-c/24} \sum_{n=0}^{\infty} \dim V_n q^n. \tag{47}$$

These expressions make sense because $L(0)$-eigenspaces in both cases are finite-dimensional. Indeed, it will be convenient to define the partition function for any graded space in the same way, as long as it too makes sense. One can often check the VOA axioms regarding the conformal vector (Section 8, axiom 2)) by directly computing the corresponding partition function. We will carry this out in the case of the Fock spaces for the Heisenberg VOA and the lattice theory $V_L$.

For the rank $l$ Heisenberg theory $V = V(l, \mathbb{C}v_0)$ we saw (34) that $V$ has a tensor decomposition $S(\oplus_{m=1}^{\infty} L \otimes t^{-m}) \otimes \mathbb{C}v_0$ ($L$ is the abelian Lie algebra of rank $l$). It is not hard to see that the $L(0)$-grading respects this decomposition, and that $L \otimes t^{-m}$ is an eigenspace with eigenvalue $m$. Since symmetric powers are multiplicative over direct sums, we obtain

$$\begin{aligned} Z_{V(l,\mathbb{C}v_0)}(q) &= q^{-l/24} \prod_{m=1}^{\infty} \big(\text{partition function of } S(L \otimes t^{-m})\big) \\ &= q^{-l/24} \prod_{m=1}^{\infty} (1 + q^m + q^{2m} + \ldots)^l \\ &= q^{-l/24} \prod_{m=1}^{\infty} (1 - q^m)^{-l} = \eta(q)^{-l}, \end{aligned}$$

$\eta(q)$ being the eta function (12).

We turn to the lattice theory $V_L$. The partition function for $V_L$ is the product of those for the two factors $S(\widehat{H}^-)$ and $\mathbb{C}[L]$ in (45). Moreover, the first of these is just the partition function for the Heisenberg theory that we just computed. As for the second factor, using the module version of associativity (17) we have

$$\begin{aligned} L(0).1 \otimes e^{\beta} &= 1/2 \sum_{i=1}^{l} (v_i(-1)v_i)(1).1 \otimes e^{\beta} \\ &= 1/2 \sum_{i=1}^{l} \sum_{j=0}^{\infty} \{(v_i(-1-j)v_i)(1+j) + v_i(-j)v_i(j)\} 1 \otimes e^{\beta} \\ &= 1/2 \sum_{i=1}^{l} \{v_i(0)v_i(0)\} 1 \otimes e^{\beta} \\ &= 1/2 \sum_{i=1}^{l} \langle v_i, \beta\rangle^2 1 \otimes e^{\beta} = 1/2 \langle \beta, \beta\rangle 1 \otimes e^{\beta}. \end{aligned}$$

(Here, we used that $v_i(j)$ moves across the tensor sign if $j \geq 0$ and annihilates $e^{\beta}$ if $j \geq 1$, as well as (44). The last equality holds because $\{v_i\}$ is an orthonormal basis of $H$.) The upshot is that $1 \otimes e^\beta$ is an eigenvector for $L(0)$ with eigenvalue $1/2\langle\beta,\beta\rangle$. We therefore see that

$$\text{partition function of } \mathbb{C}[L] = \sum_{\beta \in L} q^{1/2\langle\beta,\beta\rangle} = \theta_L(q)$$

is the theta function of $L$ (6). Altogether then, we obtain

$$Z_{V_L}(q) = \frac{\theta_L(q)}{\eta(q)^l}, \tag{48}$$

and in particular the $L(0)$-eigenspaces are indeed finite-dimensional.

Let $L_0 \subseteq L$ consist of those $\beta \in L$ such that $\langle\beta,\beta\rangle \in 2\mathbb{Z}$. Because $L$ is an integral lattice, $L_0$ is a sublattice of $L$ with $|L : L_0| \leq 2$. If $L = L_0$ then $L_0$ is an even lattice and $V_L$ is a VOA. If $|L : L_0| = 2$, choose $\gamma \in L \setminus L_0$. Then, with an obvious notation, there is a decomposition

$$V_L = S(\widehat{H}^-) \otimes \mathbb{C}[L_0] \oplus S(\widehat{H}^-) \otimes \mathbb{C}[L_0 + \gamma],$$

where $S(\widehat{H}^-) \otimes \mathbb{C}[L_0], S(\widehat{H}^-) \otimes \mathbb{C}[L_0 + \gamma]$ are the parts of $V_L$ graded by $\mathbb{Z}$ and $1/2 + \mathbb{Z}$ respectively. In this case, $V_L$ is a SVOA and $S(\widehat{H}^-) \otimes \mathbb{C}[L_0]$ and $S(\widehat{H}^-) \otimes \mathbb{C}[L_0 + \gamma]$ are the even and odd parts.

With this discussion, we have at last made contact with the ideas of Section 2. For if we take $L$ to be the *Leech lattice* $\Lambda$ (a (self-dual) even lattice of rank 24), then according to (48) we have

$$Z_{V_\Lambda}(q) = \frac{\theta_\Lambda(q)}{\Delta(q)}, \tag{49}$$

and (using (5)) this is the partition function (3). Similar comments apply to (2), which is now seen to be the partition function for $V_{3E_8}$.

Thanks to (48) and known transformation properties of $\theta$- and $\eta$-functions, it follows that the partition function $Z_{V_L}(q)$ of a lattice theory is a modular function of weight zero on a congruence subgroup of the modular group. We derived this result only after explicitly computing the partition function, but in fact there is a large class of VOAs for which *a priori* results about the partition function and its transformation properties can be proved without explicitly knowing what the partition function is. This is the class of *regular* VOAs.

One point that we will not pursue but that deserves mention is this: the partition function of a VOA is a *formal* $q$-expansion, with no *a priori* convergence properties. On the other hand, at least for a regular VOA, the partition function turns out to be *holomorphic* in the complex upper half-plane $\mathfrak{H}$ when we think of it as a function $Z_V(\tau)$ with $q = e^{2\pi i\tau}$, $\tau \in \mathfrak{H}$. For this reason, we now write partition functions as functions of $\tau$ rather than $q$.

Although there will be no time to develop the general theory of regular VOAs in these lectures, we can illustrate some of the ideas using the lattice theory $V_L$. If $V$ is an arbitrary VOA, the set of modules over $V$ are the objects of a category $V$-Mod. A *morphism* $f : W_1 \to W_2$ between two $V$-modules $W_1, W_2$ is a linear map such that

$$f(u(n)w) = u(n)(f(w)) \quad (u \in V, n \in \mathbb{Z}, w \in W_1).$$

In terms of fields, this reads $fY_{W_1}(u,z) = Y_{W_2}(u,z)f$. Roughly speaking, $V$ is called *rational* if $V$-Mod is *semisimple*, i.e. every $V$-module is a direct sum of simple $V$-modules. (In fact, one has to include additional types of modules that we did not discuss in Section 10.) It can be shown that a rational VOA has only *finitely many* (isomorphism classes of) simple $V$-modules. A VOA is regular if it is both rational in the above sense and satisfies an additional condition that we will not discuss here.

If $L$ is an even lattice as before then $V_L$ is indeed a regular VOA. It therefore has only finitely many inequivalent simple modules, and in fact they are enumerated by the *quotient group* $L^0/L$ where $L^0$ is the *dual lattice* of $L$. If we set $E := \mathbb{R} \otimes_{\mathbb{Z}} L$ then the *dual lattice* is

$$L^0 := \{\alpha \in E \mid \langle \alpha, \beta \rangle \in \mathbb{Z} \ (\beta \in L)\}.$$

Because $L$ is integral and positive-definite then $L \subseteq L^0$ is a subgroup of finite index. The simple $V_L$-modules have a structure that is parallel to $V_L$ itself. The Fock spaces are

$$\begin{aligned} V_{L+\gamma} &:= \bigoplus_{\beta \in L+\gamma} V(l, \mathbb{C}e^{\beta}) \\ &\cong S(\widehat{H}^-) \otimes \bigoplus_{\beta \in L+\gamma} \mathbb{C}e^{\beta} \\ &= S(\widehat{H}^-) \otimes \mathbb{C}[L+\gamma], \end{aligned} \tag{50}$$

(compare with (45)), where $L+\gamma \in L^0/L$. The partition function is

$$Z_{V_{L+\gamma}}(\tau) = \frac{\theta_{L+\gamma}(\tau)}{\eta(\tau)^l},$$

which is once again a modular function of weight zero on a congruence subgroup of the modular group. Indeed, one knows that the linear space

$$P := \langle \theta_{L+\gamma}(\tau)/\eta(\tau)^l \mid L+\gamma \in L^0/L \rangle$$

spanned by these partition functions furnishes a representation of the modular group (through the usual action $\tau \mapsto \frac{a\tau+b}{c\tau+d}$). This set-up is conjectured to hold for all rational VOAs $V$; that is, if $P$ is the span of the partition functions of the (finitely many) simple $V$-modules then $P$ affords a representation of the modular group that factors through a principal congruence subgroup. This phenomenon is often called *modular-invariance* of rational VOAs.

An important special case of these ideas arises when the VOA $V$ is not only rational, but has (up to isomorphism) a *unique* simple module, namely the adjoint module $V$. We call such a $V$ *holomorphic.* Then our discussion of modular-invariance shows that the partition function $Z_V(\tau)$ of a holomorphc VOA is a modular function on the full modular group (perhaps with a character). For example, since the simple $V_L$-modules are indexed by the cosets of $L$ in $L^0$, it follows that $V_L$ is holomorphic if, and only if, $L = L^0$ is *self-dual.* The Leech lattice $\Lambda$ and orthogonal sums of the $E_8$ root lattice are examples of self-dual lattices, and indeed their partition functions (2), (3) are modular functions on the full modular group.

# Lecture 4

## §13 The Lie algebra on $V_1$

We have seen that a regular VOA that is holomorphic (i.e. has a unique simple module) has a partition function that is a modular function of weight 0 on the full modular group. A case in point is the Leech lattice theory $V_\Lambda$, which has central charge 24 ($=\mathrm{rk}\Lambda$) and partition function $Z_{V_\Lambda}(\tau) = J+24 = q^{-1} + 24 + 196884q + \ldots$. Our goal now is to construct a holomorphic VOA

$V^\natural$, also of central charge 24, whose partition function is $J$ (4), which has constant term 0. This is the[1] *Moonshine module*.

Although the VOAs $V_\Lambda$ and $V^\natural$ have partition functions differing only in their constant term, many of their algebraic properties are quite different. Indeed, these properties are to a large extent governed by the constant term. For this reason, we begin with a general discussion of this point. We restrict attention to VOAs of *CFT-type*, which means that in the spectral decomposition (40) the pieces $V_n$ vanish for $n < 0$ and $V_0 = \mathbb{C}\mathbf{1}$. (Recall that we always have $\mathbf{1} \in V_0$.) There are many interesting VOAs that are *not* of CFT-type, nevertheless CFT-type theories are natural from a physical standpoint because they arise from 'unitarity' assumptions. Be that as it may, our basic assumption here is that the spectral decomposition of $V$ has the shape

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus V_2 \oplus \ldots$$

For states $u, v \in V_1$ we define a bracket $[\ ]$ by setting $[uv] := u(0)v$. Now

$$L(0)u(0)v \quad = \quad [L(0), u(0)]v + u(0)L(0)v = u(0)v.$$

($[L(0), u(0)] = 0$ by translation covariance and $L(0)v = v$ because $v \in V_1$). This shows that $u(0)v \in V_1$, so that we have a bilinear product $[\ ] : V_1 \times V_1 \to V_1$. One can check that this makes $V_1$ into a Lie algebra. (Use (29) for skew-symmetry $[uv] = -[vu]$ and the associativity formula (17) for the Jacobi identity $[[uv]w] + [[wu]v] + [[vw]u] = 0$.)

For a *VOA* $V$ of CFT-type and central charge $c = 24$, the partition function has the general shape $Z_V(\tau) = q^{-1} + \dim V_1 + \ldots$. So for such theories, the constant term is the *dimension* of the Lie algebra on $V_1$.

If $L$ is an even lattice of rank $l$, the nature of the partition function (48) of the lattice theory shows that

$$Z_{V_L}(\tau) = q^{-l/24}(1 + (l + |L_2|)q + \ldots), \tag{51}$$

where $L_2 = \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$ are the *roots* of $L$. In particular, $V_L$ is of CFT-type. The Lie algebra on $(V_L)_1$ is *reductive*, being a direct sum $\mathfrak{a} \oplus \mathfrak{g}$ where $\mathfrak{a}$ is abelian and $\mathfrak{g}$ is semisimple with root system $L_2$. (The set of roots in an even lattice always carries the structure of a semisimple

[1] It is expected that there is a *unique* VOA with partition function $J$, but this remains open.

root system embedded in the ambient Euclidean space $E = \mathbb{R} \otimes L$.) For example, if $L = 3E_8$ then the Lie algebra on $(V_L)_1$ is semisimple, being the sum of three copies of the $E_8$ Lie algebra. (Note that $\dim E_8 = 248$, so that $\dim(V_L)_1 = 744$, in agreement with (2).) Similarly, the Leech lattice $\Lambda$ has *no* roots, whence $(V_\Lambda)_1$ is abelian of rank $l = 24$.

Because $J$ has no constant term, a VOA $V^\natural$ with partition function $J$ and central charge $c = 24$ necessarily has *no* corresponding Lie algebra. In particular, $V^\natural$ cannot be a lattice theory, because the weight one piece never vanishes for a lattice theory (cf. (51)).

§14 **Automorphisms**

Let $V$ be a (S)VOA. An *automorphism* of $V$ is an invertible linear map $g : V \to V$ such that $g(\omega) = \omega$ and $gv(q)g^{-1} = g(v)(q)$ for all $v, q$, i.e.

$$gY(v,z)g^{-1} = Y(g(v), z) \quad (v \in V). \tag{52}$$

We give some basic examples of automorphisms.

1. One checks (use induction and (16) or (17)) that for $n \geq 0$,

$$(u(0)^n v)(q) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} u(0)^{n-i} v(q) u(0)^i \quad (u, v \in V,\ q \in \mathbb{Z}).$$

Therefore,

$$\begin{aligned} \left(e^{u(0)}.v\right)(q) &= \sum_{n=0}^{\infty} \frac{1}{n!} (u(0)^n v)(q) \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!} u(0)^{n-i} v(q) u(0)^i \\ &= e^{u(0)} v(q) e^{-u(0)}, \end{aligned}$$

showing that (52) holds with $g = e^{u(0)}$. Furthermore, if $V$ is of CFT-type (cf. Section 12) and $u \in V_1$ then

$$\begin{aligned} u(0)\omega &= u(0)L(-2)\mathbf{1} \\ &= [u(0), L(-2)]\mathbf{1} + L(-2)u(0)\mathbf{1} \\ &= [u(0), \omega(-1)]\mathbf{1} \\ &= -[\omega(-1), u(0)]\mathbf{1} \\ &= -\sum_{i=0}^{\infty} (-1)^i (\omega(i)u)(-1-i)\mathbf{1} \end{aligned}$$

$$\begin{aligned} &= -\{(L(-1)u)(-1) - (L(0)u)(-2) + (L(1)u)(-3)\}\,\mathbf{1} \\ &= 0. \end{aligned}$$

(For the last two equalities use translation covariance, $L(0)u = u$ (because $u \in V_1$), $L(1)u \in V_0 = \mathbb{C}\mathbf{1}$, $L(n)u \in V_{1-n} = 0$ for $n \geq 2$, and $\mathbf{1}(q) = \delta_{q+1,0}\mathrm{Id}_V$ (cf. (28).)

It follows from this calculation that if $V$ is a VOA of CFT-type then $\{e^{u(0)} \mid u \in V_1\}$ is a set of automorphisms of $V$. In the previous Section we learned that $V_1$ carries the structure of a Lie algebra with bracket $[uv] = u(0)v$. Now we see that the usual action of the associated Lie group $\mathfrak{G}$ generated by exponentials $e^{ad_u}$ extends to an action of $\mathfrak{G}$ as automorphisms of $V$.

2. Suppose that $V$ is a SVOA. Then there is a canonical involutorial automorphism which acts as $+1$ on the even part of $V$ and $-1$ on the odd part.

3. A related example (and the one we will need later) is an involutorial automorphism $t$ of a lattice VOA $V_L$, defined to be a lifting of the $-1$ automorphism of the lattice $L$. $t$ also acts as $-1$ on the abelian Lie algebra $\mathbb{C} \otimes L$ and then acts as naturally on the associated Heisenberg VOA (cf. (34) - where $L$ is the Lie algebra, not the lattice!) and on $V_L$, where

$$t(u \otimes e^{\beta}) = t(u) \otimes e^{-\beta} \quad (u \in S(\widehat{H}^{-}) \tag{53}$$

(cf. (45)).

If $g$ is an automorphism of $V$ then $gY(\omega, z)g^{-1} = Y(g(\omega), z) = Y(\omega, z)$, in particular $gL(0)g^{-1} = L(0)$. Therefore $g$ acts on the eigenspaces of $V$, i.e. the homogeneous pieces $V_n$. We may therefore define additional partition functions

$$Z_V(g, \tau) := q^{-c/24} \sum_{n=n_0}^{\infty} (\mathrm{Tr}_{V_n} g) q^n.$$

Let's compute this trace function for the automorphism $t$ of $V_L$. It is clear from (53) that the only contributions to the trace arise from states $u \otimes e^0$, i.e.

from states in the Heisenberg VOA Fock space $S(\widehat{H}^-)$. Therefore by (34),

$$\begin{aligned}
Z_{V_L}(t,\tau) &= \text{Trace } t \text{ on } q^{-l/24}S(\oplus_{m>0}H\otimes t^{-m}) \\
&= \text{Trace } t \text{ on } q^{-l/24}\bigotimes_{m>0} S(\mathbb{C}u\otimes t^{-m})^l \\
&= q^{-l/24}\prod_{m>0}(1-q^m+q^{2m}-\ldots)^l \\
&= q^{-l/24}\prod_{m>0}(1+q^m)^{-l} \\
&= \left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^l. \qquad (54)
\end{aligned}$$

This is a modular function of weight 0. If $l=24$ it is almost equal to (11)!

## §15 Twisted sectors

Let $(V,Y)$ be a VOA and $g$ an automorphism of $V$ of finite order $R$. A *$g$-twisted* $V$-module, or *$g$-twisted sector*, is a generalization of $V$-module (to which it reduces if $g=1$). Precisely, it is a pair $(W_g,Y_g)$ consisting of a Fock space $W_g$ and a $Y$-map $Y_g: V\to\mathfrak{F}(W_g),\ u\mapsto Y_g(u,z)$ where

$$Y_g(u,z) := \sum_{n\in r/R+\mathbb{Z}} u(n)z^{-n-1}\in \mathrm{End}(W)[[z^{1/R},z^{-1/R}]]$$

whenever $g(u)=e^{-2\pi i r/R}$ $(r\in\mathbb{Z})$, and $Y_g(\mathbf{1},z)=\mathrm{Id}_{W_g}$. The *twisted vertex operators* $Y_g(u,z)$ are required to satisfy twisted analogs of the basic identity (15). In the delta-function formulation (cf. (24)) this reads

$$\begin{aligned}
z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_g(u,z_1)Y_g(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_g(v,z_2)Y_g(u,z_1)\\
= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r/R}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_g(Y(u,z_0)v,z_2). \qquad (55)
\end{aligned}$$

Finally, the operator $L_g(0)$ (the zero mode of $Y_g(\omega,z)$) is required to be semisimple with finite-dimensional eigenspaces. The eigenvalues satisfy a truncation condition analogous to that for $V$-modules (cf. the discussion in Section 10 preceding display (43)). There is an obvious notion of irreducible (or simple) $g$-twisted module, and as in the untwisted case (cf. (43)) the spectral decomposition of a simple $g$-twisted module takes the form

$$W_g = \bigoplus_{n=0}^{\infty} (W_g)_{h_g+n/R} \tag{56}$$

for a scalar $h_g$ (the conformal weight). Needless to say, the twisted sector has an associated partition function

$$Z_{W_g}(\tau) := q^{-c/24+h_g} \sum_{n=0}^{\infty} \dim(W_g)_n q^{n/R}.$$

Let us specialize to the case of the (even, self-dual) Leech lattice $\Lambda$ with its associated VOA $V_\Lambda$ and canonical involution $t$ (cf. Section 14). In this case there is (up to isomorphism) a *unique* simple $t$-twisted module, denoted by $V_\Lambda(t,\tau)$. The following transformation law can be proved:

$$Z_{V_\Lambda}(t, -1/\tau) = Z_{V_\Lambda(t)}(\tau).$$

Using (54), the partition function of the $t$-twisted sector must be

$$\begin{aligned} Z_{V_\Lambda(t)}(\tau) &= \left(\frac{\eta(-1/\tau)}{\eta(-2/\tau)}\right)^{24} \\ &= 2^{12}\left(\frac{\eta(\tau)}{\eta(\tau/2)}\right)^{24} \\ &= 2^{12} q^{1/2} \prod_{n=1}^{\infty} (1+q^{n/2})^{24}, \end{aligned} \tag{57}$$

(using the transformation law $\eta(-1/\tau) = (\sqrt{\tau}/i)\eta(\tau)$). Because the central charge is $c = 24$, it follows that the conformal weight of $V_\Lambda(t,\tau)$ is $3/2$.

Similarly to the rank 24 Heisenberg VOA, the product term in (57) is the partition function of a symmetric algebra $S(\oplus_{n>0} H \otimes t^{-n/2})$ (cf. (34)). This suggests how one might try to construct the twisted sector, though we must skip the details here. (The curious factor $2^{12}$ turns out to correspond to a Clifford algebra. Cf. Section 17 for further comment.)

§16 **The Moonshine Module**

Retaining the notation of the previous Section, consider

$$V_\Lambda \oplus V_\Lambda(t). \tag{58}$$

The involution $t$ acts naturally on the twisted sector: in the 'usual way' on $S(\oplus_{n>0} H \otimes t^{-n/2})$ and as $-1$ on the $2^{12}$ constant part. The Moonshine Module is then defined to be the space of $t$-invariants

$$V^\natural := V_\Lambda^+ \oplus V_\Lambda(t)^+. \tag{59}$$

Now every state $u \otimes e^\beta \in V_\Lambda$ ($\beta \neq 0$) produces a $t$-invariant $u \otimes e^\beta + t(u) \otimes e^{-\beta}$. On the other hand, the partition function of the Heisenberg VOA (consisting of states $u \otimes e^0$) is $1/\Delta(\tau)$ and the graded trace of $t$ is $\Delta(\tau)/\Delta(2\tau)$ (the case $l = 24$ of (54)). It follows that

$$\begin{aligned}
& Z_{V_\Lambda^+}(\tau) \\
= \ & (Z_{V_\Lambda}(\tau) - 1/\Delta(\tau))/2 + (1/\Delta(\tau) + \Delta(\tau)/\Delta(2\tau))/2 \\
= \ & (Z_{V_\Lambda}(\tau) + \Delta(\tau)/\Delta(2\tau))/2 \\
= \ & ((q^{-1} + 24 + 196884q + \ldots) + q^{-1}(1 - 24q + 276q^2 + \ldots))/2 \\
= \ & q^{-1} + 98580q + \ldots
\end{aligned}$$

On the other hand, a similar calculation using (54) and the nature of the twisted sector as a symmetric algebra shows that

$$\begin{aligned}
Z_{V_\Lambda(t)^+}(\tau) &= 2^{12}(\Delta(\tau)/\Delta(\tau/2) - q\Delta(\tau/2)/\Delta(\tau))/2 \\
&= 2^{11} q^{1/2} \left( \prod_{n=1}^{\infty} (1 + q^{n/2})^{24} - \prod_{n=1}^{\infty} (1 - q^{n-1/2})^{24} \right) \\
&= 98304q + \ldots
\end{aligned}$$

Altogether, we have

$$\begin{aligned}
Z_{V^\natural}(\tau) &= Z_{V_\Lambda^+}(\tau) + Z_{V_\Lambda(t)^+}(\tau) \\
&= (q^{-1} + 98580q + \ldots) + (98304q + \ldots) \\
&= q^{-1} + 196884q + \ldots
\end{aligned} \tag{60}$$

It is clear from the above that $Z_{V^\natural}(\tau)$ is a modular function of weight 0 and level at most 2, and it is easy to check that in fact it is invariant under the full modular group. Thus from the $q$-expansion we arrive at the identity

$$Z_{V^\natural}(\tau) = J.$$

The space (58) has the structure of an *abelian intertwining algebra*, a generalization of VOA and SVOA. The main missing ingredient, which we cannot go into here, is the definition of fields $Y_{V_\Lambda \oplus V_\Lambda(t)}(u, z)$ for states $u$ in the twisted sector satisfying an appropriate variation of the basic identity (15), (24). Once this is done, $t$ is seen to be an automorphism of this larger structure. Then it is easy to see that the $t$-invariant subspace $V^\natural$, together with the restriction of the fields to this subspace, defines the structure of a VOA on $V^\natural$ with central charge 24. Furthermore, $V_\Lambda^+ \oplus V_\Lambda(t)^-$ is a SVOA with even part $V_\Lambda^+$. (Indeed, it is an $N = 1$ superconformal field theory, a term we alluded to but did not define in Section 9.)

Consider a VOA $V$ of CFT-type (cf. Section 13) with *trivial* Lie algebra $V_1$:

$$V = V_0 \oplus V_2 \oplus \dots$$

($V^\natural$ satisfies these conditions, as follows from (60).) If $u, v \in V_2$, define $u.v := u(1)v$. It is easy to check that $u(1)v \in V_2$, so that we have a (nonassociative) bilinear product on $V_2$. By skew-symmetry (29), $v(1)u = u(1)v - L(-1)u(2)v + L(-1)^2 u(3)v/2 - \dots$ But $u(2)v \in V_1 = 0, u(3)v \in \mathbb{C}\mathbf{1}$ and $L(-1)\mathbf{1} = 0$, and all other $u(q)v$ ($q \geq 4$) lie in $V_n$ with $n < 0$ and hence also vanish. The upshot is that $u(1)v = v(1)u$, so that $V_2$ has the structure of a commutative, nonassociative algebra. In the case of $V^\natural$, this is precisely the algebra $B$ that we discussed in Section 1.

## §17 **Aut**$V^\natural$

Consider the CFT

$$V_\Lambda = \mathbb{C}\mathbf{1} \oplus (V_\Lambda)_1 \oplus \dots$$

where $\Lambda$ is, as before, the Leech lattice. Because $\Lambda$ has no roots, it follows from (51) that $\dim(V_\Lambda)_1 = 24$, and the Lie algebra on $(V_\Lambda)_1$ is *abelian*. So the automorphisms $e^{u(0)}$ ($u \in (V_\Lambda)_1$) generate a 24-dimensional complex torus $T$. Additional automorphisms of $V_\Lambda$ arise from the automorphism group $Co_0 := \mathrm{Aut}(\Lambda)$ of the Leech lattice, and there is a (nonsplit) short exact sequence

$$1 \to T \to \mathrm{Aut}V_\Lambda \to Co_0 \to 1.$$

The automorphism $t$ of $\Lambda$ (or of $V_\Lambda$) is a central involution of $Co_0$, and the quotient $Co_1 := Co_0/\langle t\rangle$ is the largest sporadic (simple) Conway group of order $2^{21}\ldots$.

Because $t$ acts as $-1$ on $T$, its only fixed elements are those of order at most 2. So the *centralizer* $C(t)$ of $t$ in $\mathrm{Aut}V_\Lambda$ (the elements that commute with $t$) is described by another short exact sequence (also nonsplit)

$$1 \to 2^{24} \to C(t) \to Co_0 \to 1.$$

($2^{24}$ is a direct product of 24 copies of $\mathbb{Z}_2$, the 2-torsion of $T$.) Note that $|C(t)| = 2^{46}\ldots$

As regards the Monster, the relevance of $C(t)$ is that it *preserves* the decomposition (59). This is a bit subtle: $t$ acts trivially by definition, but the action of $C(t)/\langle t\rangle$ is *projective* on $V_\Lambda(t)^+$. When it is linearized, we obtain a third group $\widehat{C}$ occuring as the middle term of a short exact sequence

$$1 \to 2^{1+24} \to \widehat{C} \to Co_1 \to 1,$$

where now $2^{1+24}$ is the (nonabelian) linearization of the projective action of the 2-torsion of $T$. $2^{1+24}$ is a so-called *extra-special* group. It is familiar in physics ($24 \times 24$ Pauli matrices) and the theory of theta-functions. It has a unique faithful irreducible representation, realizable on the $2^{12}$-dimensional Clifford algebra that we identified at the end of Section 15. $|\widehat{C}| = 2^{46}\ldots$ and

$$\widehat{C} \subseteq \mathrm{Aut}V^\natural$$

It turns out that the decomposition (59) breaks the symmetry of $V^\natural$ in the sense that there are further automorphisms that do *not* preserve (59) and hence do not lie in $\widehat{C}$. The Monster $M$ is the full automorphism group of $V^\natural$ and also of the algebra $B$, and

$$|M| = 2^{46}\ldots 47.59.71$$

These results are not easily obtained, and we say no more about them here.

The graded traces $Z_{V^\natural}(g,\tau)$ for $g \in M$ turn out to be hauptmoduln as described in Section 1. This result is also difficult. We end these Notes with the computation for a single automorphism $g$ of order 2 that acts trivially on $V_\Lambda^+$ and as $-1$ on $V_\Lambda(t)^+$. A previous calculation shows that its graded trace is a modular function of weight 0 and level 2. Specifically,

$$\begin{aligned} Z_{V^\natural}(g,\tau) &= Z_{V_\Lambda^+}(\tau) - Z_{V_\Lambda(t)^+}(\tau) \\ &= (q^{-1} + 98556q + \ldots) - (98304q + \ldots) \\ &= q^{-1} + 276q + \ldots \end{aligned}$$

is the hauptmodul for the Monster element $2B$ (11).

**References**

The following textbooks and monographs cover the material in these Notes and much more.


J. Conway et al, *ATLAS of Finite Groups*, Clarendon Press, OUP, 1985.

C. Dong and J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Birkhäuser, 1993.

I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, 1988.

T. Gannon, *Moonshine Beyond the Monster: The Bridge Connecting Algebra, Modular Forms, and Physics*, CUP, 2006.

V. Kac, *Vertex Algebras for Beginners*, $2^{nd}$ ed., Univ. Lect. Ser. Vol. 10, AMS, 1998.

J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Birkäuser, 2004.

A. Matsuo and K. Nagatomo, *Axioms for a Vertex Algebra and the Locality of Quantum Fields*, Math. Soc. of Japan Memoirs, Vol. 4, 1999.

G. Mason and M. Tuite, *Vertex operators and modular forms*, in *A Window into Zeta and Modular Physics*, MSRI Publ. No. 57, CUP, 2010.

*Moonshine, The First Quarter Century and Beyond*, J. Lepowsky, J. McKay and M. Tuite eds., LMS Lect. Note Series No. 372, CUP, 2010.

*Moonshine, the Monster, and Related Topics*, C. Dong and G. Mason eds., Contemp. Math. No. 193, AMS, 1996.