



# Colored knot polynomials for arbitrary pretzel knots and links



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## ARTICLE INFO

### Article history:

Received 9 December 2014

Received in revised form 10 February 2015

Accepted 12 February 2015

Available online 18 February 2015

Editor: M. Cvetič

## ABSTRACT

A very simple expression is conjectured for arbitrary colored Jones and HOMFLY polynomials of a rich  $(g+1)$ -parametric family of pretzel knots and links. The answer for the Jones and HOMFLY is fully and explicitly expressed through the Racah matrix of  $U_q(SU_N)$ , and looks related to a modular transformation of toric conformal block.

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Knot polynomials [1] are among the hottest topics in modern theory. They are supposed to summarize nicely representation theory of quantum algebras and modular properties of conformal blocks [2–6]. The result reported in the present letter, provides a spectacular illustration and support to this general expectation.

The arbitrary pretzel knot/link [7] on genus  $g$  surface is shown in Fig. 1.

It depends on  $g+1$  integers  $n_0, \dots, n_g$ , the algebraic lengths of the constituent 2-strand braids. Orientation of lines does not matter, when one considers the Jones polynomials (not HOMFLY!). Also, these polynomials are defined only for the symmetric representations  $[r]$  and, hence, do not change under arbitrary permutations of parameters  $n_i$  (though the knot/link itself has at best the cyclic symmetry  $n_i \rightarrow n_{i+1}$ , and even this is true only for particular orientations). A particular manifestation of this enhanced symmetry has been recently noted in [8].

The answer for the colored Jones polynomials for this entire family can be written in full generality, and is wonderfully simple

$$J_r^{(n_0, \dots, n_g)}(q) = \sum_{k=0}^r [2k+1] \prod_{i=0}^g \left( \sum_{m=0}^r \mathcal{A}_{km} \lambda_m^{n_i} \right) \quad (1)$$

where

$$\lambda_m = (-)^m q^{m(m+1)} \quad (2)$$

and

$$\mathcal{A}_{km} = \sqrt{\frac{[2m+1]}{[2k+1]}} \cdot S_{km} \begin{pmatrix} r & r \\ r & r \end{pmatrix} \quad (3)$$

is made out of the Racah matrix [9] of  $SU_q(2)$  in representation  $[r]$  (spin  $r/2$ ). Orthogonality of  $S$  implies that

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<http://dx.doi.org/10.1016/j.physletb.2015.02.029>

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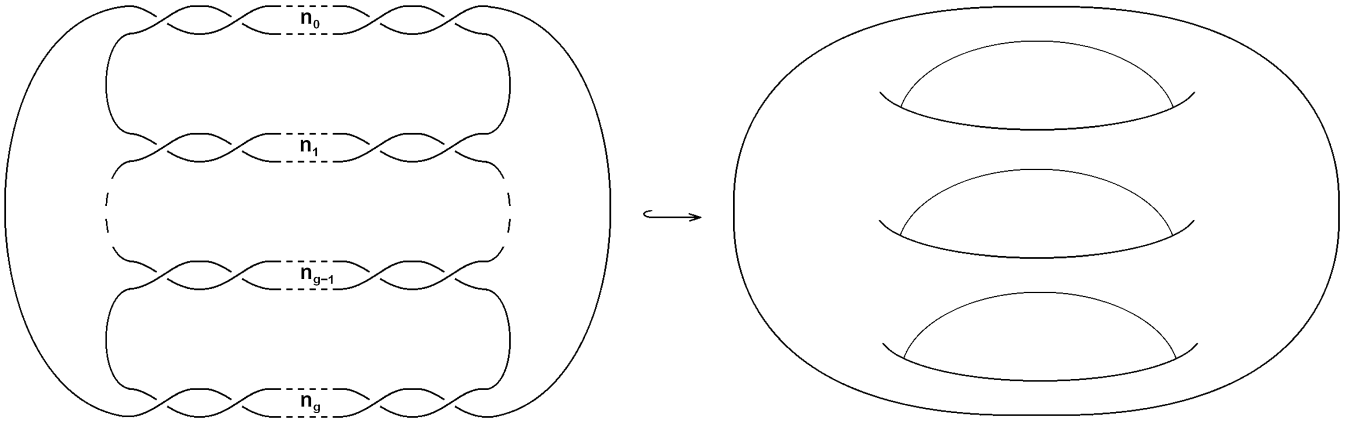


Fig. 1. Arbitrary pretzel knot corresponding to genus  $g$  surface.

$$\sum_{m=0}^r \frac{\mathcal{A}_{km} \mathcal{A}_{k'm}}{[2m+1]} = \frac{\delta_{k,k'}}{[2k+1]},$$

$$\sum_{k=0}^r [2k+1] \mathcal{A}_{km} \mathcal{A}_{k'm'} = [2m+1] \delta_{m,m'} \quad (4)$$

Quantum numbers in these formulas are defined as  $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ .

The first of the  $r+1$  polynomials

$$P_k^{(n)}(q|r) = \sum_{m=0}^r \mathcal{A}_{km} \lambda_m^n \quad (5)$$

is just the Jones polynomial for the 2-strand torus knot/link  $T[2, n]$ :

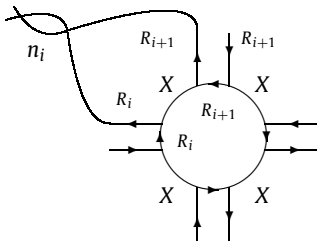
$$P_0^{(n)}(q|r) = J_r^{(n)} = \sum_{m=0}^r [2m+1] \cdot \lambda_m^n \quad (6)$$

The origin of its orthonormal “satellites”  $P_k^{(n)}(q|r)$  with  $k = 1, \dots, r$  and of entire rotation from the monomial basis  $\{\lambda_m^n\}$  is yet unknown.

Naturally, a straightforward generalization exists to the HOMFLY polynomials:

$$H_{\bar{R}}^{\bar{n}}(A, q) = \sum_{X \in \bigcap_i R_i \otimes \bar{R}_i} \dim_X \cdot \prod_{i=0}^g \left( \sum_{Y_i \in R_i \otimes R_{i+1}} \mathcal{A}_{XY_i} \left( \begin{matrix} \bar{R}_i & \bar{R}_{i+1} \\ R_i & R_{i+1} \end{matrix} \right) \cdot \lambda_Y^{n_i} \right) \quad (7)$$

which looks like a modular transformation of *toric* conformal block, summed over intermediate states  $X$  in the loop:



Here  $\dim_X$  and  $\mathcal{A}$  are the universal  $A$ -dependent dimension of representation  $X$  (which is equal at  $A = q^N$  to the quantum dimension of representation  $X$  of  $SU_q(N)$ ) and rescaled Racah matrix respectively, interpolating between those for  $SU_q(N)$  at  $A = q^N$ . This formula is indeed true [10], at least when all the representations are symmetric or their conjugate:  $R_i = [r], [\bar{r}]$ . The answer for antisymmetric representations then follows from the general transposition rule [11,12]  $H_{[1^r]}(A, q) = H_{[r]}(A, q^{-1})$ . It should possess further continuation *à la* [13] to superpolynomials, thus providing a  $\beta$ -deformation [14] of the universal Racah matrix.

Eqs. (1) and (7) result from a tedious calculation in [10,15] with the help of evolution [11,16], modernized Reshetikhin and Turaev [3,5] and modular matrix [2,4,6] methods. Of course, a simple conceptual derivation should exist for such a simple and general formula, but it still remains to be found. The value of (1) is independent of the derivation details, and it is high, because this only formula (together with its lifting to the HOMFLY polynomials in [10]) contains almost all what is currently known about explicit colored knot polynomials beyond

torus links: in particular, all the twist and 2-bridge knots are small subsets in the pretzel family (however, among torus knots with more than two strands, only  $[3, 4]$  and  $[3, 5]$  belong to it).

**Examples.** We give them here mostly for the Jones case, for an exhaustive description of the symmetric HOMFLY polynomials for all pretzel links see [10]. First of all, we list the first few  $S$  and  $\mathcal{A}$  matrices for the lowest representations of  $SU_q(2)$ :

$$\begin{aligned}
 r=1: \quad S &= \frac{1}{[2]} \begin{pmatrix} 1 & \sqrt{[3]} \\ \sqrt{[3]} & -1 \end{pmatrix}, \quad \mathcal{A} = \frac{1}{[2]} \begin{pmatrix} 1 & [3] \\ 1 & -1 \end{pmatrix}, \\
 r=2: \quad S &= \frac{1}{[3]} \begin{pmatrix} 1 & \sqrt{[3]} & \sqrt{[5]} \\ \sqrt{[3]} & \frac{[6]}{[4]} & -\frac{[2]\sqrt{[3][5]}}{[4]} \\ \sqrt{[5]} & -\frac{[2]\sqrt{[3][5]}}{[4]} & \frac{[2]}{[4]} \end{pmatrix}, \quad \mathcal{A} = \frac{1}{[3]} \begin{pmatrix} 1 & [3] & [5] \\ 1 & \frac{[6]}{[4]} & -\frac{[2][5]}{[4]} \\ 1 & -\frac{[2][3]}{[4]} & \frac{[2]}{[4]} \end{pmatrix}, \\
 r=3: \quad S &= \frac{1}{[4]} \begin{pmatrix} 1 & \sqrt{[3]} & \sqrt{[5]} & \sqrt{[7]} \\ \sqrt{[3]} & \frac{[2][6]-1}{[5]} & ([5]-[2]^2)\sqrt{\frac{[3]}{[5]}} & -\frac{[3]^2[7]}{\sqrt{[3][5]}} \\ \sqrt{[5]} & \frac{[7]-[2]^2}{\sqrt{[3][5]}} & -\frac{[2]([2][6]-[3])}{[6]} & \frac{[2][3]}{[6]}\sqrt{\frac{[7]}{[5]}} \\ \sqrt{[7]} & -\frac{[3]}{[5]}\sqrt{[3][7]} & \frac{[2][3]}{[6]}\sqrt{\frac{[7]}{[5]}} & -\frac{[2][3]}{[5][6]} \end{pmatrix}, \quad \mathcal{A} = \frac{1}{[4]} \begin{pmatrix} 1 & [3] & [5] & [7] \\ 1 & \frac{[2][6]-1}{[5]} & [5]-[2]^2 & -\frac{[3][7]}{[5]} \\ 1 & \frac{[7]-[2]^2}{[5]} & -\frac{[2]([2][6]-[3])}{[6]} & \frac{[2][3][7]}{[5][6]} \\ 1 & -\frac{[3]^2}{[5]} & \frac{[2][3]}{[6]} & -\frac{[2][3]}{[5][6]} \end{pmatrix}, \\
 &\dots
 \end{aligned}$$

In general, from (3) and [9] one has:

$$\mathcal{A}_{km} = (-1)^{r+k+m} [2m+1] \cdot \frac{([k]![m]!)^2 [r-k]! [r-m]!}{[r+k+1]! [r+m+1]!} \sum_j \frac{(-1)^j [j+1]!}{([j-r-k]! [j-r-m]! [r+k+m-j]!)^2 [2r-j]!} \quad (8)$$

Given these matrices, Eq. (1) provides absolutely explicit expressions for all pretzel knots/links in the corresponding representations [15]:

$$J_1^{(n_0, \dots, n_g)} = \frac{1}{[2]^{g+1}} \left\{ \prod_{i=0}^g (1 + [3] \cdot (-q^2)^{n_i}) + [3] \cdot \prod_{i=0}^g (1 - (-q^2)^{n_i}) \right\}, \quad (9)$$

$$\begin{aligned}
 J_2^{(n_0, \dots, n_g)} &= \frac{1}{[3]^{g+1}} \left\{ \prod_{i=0}^g (1 + [3] \cdot (-q^2)^{n_i} + [5] \cdot q^{6n_i}) \right. \\
 &\quad \left. + [3] \cdot \prod_{i=0}^g \left( 1 + \frac{[6]}{[4]} \cdot (-q^2)^{n_i} - \frac{[2][5]}{[4]} \cdot q^{6n_i} \right) + [5] \prod_{i=0}^g \left( 1 - \frac{[2][3]}{[4]} \cdot (-q^2)^{n_i} + \frac{[2]}{[4]} \cdot q^{6n_i} \right) \right\}, \quad (10) \\
 &\dots
 \end{aligned}$$

Similarly, the fundamental HOMFLY from [10] is

$$\begin{aligned}
 H_{[1]}^{(n_0, \dots, n_g)} &= \frac{1}{\chi_{[1]}^{g+1}} \left\{ \prod_{a=0}^{g-2g_{||}} (1 + \Delta_1 \cdot (-A)^{n_a}) \prod_{i=1}^{2g_{||}} (\chi_{[11]} + \chi_{[2]} \cdot (-q^2)^{n_i}) \right. \\
 &\quad \left. + \Delta_1 \prod_{a=0}^{g-2g_{||}} (1 - (-A)^{n_a}) \prod_{i=1}^{2g_{||}} \left( \frac{\chi_{[11]}}{[2]} (1 - (-q^2)^{n_i}) \right) \right\} \quad (11)
 \end{aligned}$$

where  $\chi_m$  and  $\Delta_k$  are quantum dimensions of the representations appearing in the products  $[r] \otimes [r]$  and  $[r] \otimes \overline{[r]}$  respectively (i.e. restrictions of the Schur functions to the topological locus, described by the hook formulas). One more new parameter is the number  $2g_{||}$  of parallel braids ( $g_{||} = g_{\uparrow\uparrow} = g_{\downarrow\downarrow}$ ), the remaining  $g_{\uparrow\downarrow} = g + 1 - 2g_{||}$  are antiparallel.

As to practical applications, formula (10) for, say,

$$\begin{aligned}
 J_{[2]}^{\overbrace{(1, \dots, 1)}^a, \overbrace{(n, \dots, n)}^b} &= \frac{1}{[3]^b} \left\{ q^{8a} \cdot (1 \mp q^{2n}[3] + q^{6n}[5])^b \right. \\
 &\quad \left. + [3] \cdot q^{6a} \left( 1 - q^{6n} - (q^4 + 1 + q^{-4}) \cdot q^{2n+2} \cdot \frac{q^{4n} \pm 1}{q^4 + 1} \right)^b + [5] \cdot q^{2a} \left( 1 \pm q^{2n} + q^{2n+2} \cdot \frac{q^{4n} \pm 1}{q^4 + 1} \right)^b \right\} \quad (12)
 \end{aligned}$$

(upper/lower signs are for odd/even  $n$ ) can look a little too long as compared to the usual

$$1 \mp [3] \cdot q^{2n+1} + [5] \cdot q^{6n+6}$$

for the two-strand knots/links  $[2, n+1] = (n+1)_1$  (when  $a=1, b=1$ ) or

$$[3] - [3][4][n+1]\{q\}^2 \cdot q^{n+3} + [2][5][n+1]\{q\}^4 \cdot q^{2n+6} \left( [n+2] + q^{n+3}[2n+1] \right), \quad \{q\} = q - q^{-1}$$

for the twist knots  $(n+2)_2$  or  $(|n|+1)_1$  (when  $a=2, b=1, n$  odd): a lot of cancelations happen in these special cases. However, in generic situation such cancelations do not take place, and just the same expression describes the colored Jones polynomials for, say, the pretzel  $(1, 1, 1, 1, 1, 1, 1, 1, 19, 19, 19, 19, 19, 19, 19)$  with  $a=8, b=7, n=19$ , which is about one page long.

To avoid possible confusion, we note that the Jones polynomials are unreduced and appear in this formalism in the vertical framing, the one consistent with the orientation independence. Conversion to the topological framing inserts the factor  $q^{-8}$  for each vertex in the parallel braids, this means a total  $q^{-8(n+1)}$  for  $(n+1)_1$ , while for the twist knots the total factor is either  $q^{16}$  or just 1, if  $tw_k$  is represented as  $(-1, -1, 2k)$  or as  $(1, 1, 2k-1)$ .

## Acknowledgements

We are indebted for hospitality to the International Institute of Physics (Natal, Brazil), where a significant part of this work was done.

Our work is partly supported by grants NSH-1500.2014.2, by RFBR grants 13-02-00457 (D.G., A.Mir. & A.S.), 13-02-00478 (A.Mor.), 14-02-00627 (D.M.), by the joint grants 13-02-91371-ST-a and 15-51-52031-NSC-a (A.Mir., A.Mor. & A.S.), by 14-01-92691-Ind-a (D.M., A.Mir., A.Mor. & A.S.) and by the young-scientist grants 14-01-31395-young-a (D.G.) and 14-02-31446-young-a (A.S.). Also we are partly supported by the Brazilian National Counsel of Scientific and Technological Development (D.M. & A.Mor.), by foundation FUNPEC-UFRN (A.Mir. & A.S.), by the Quantum Topology Lab of Chelyabinsk State University (Russian Federation Government grant 14.Z50.31.0020) (A.S.) and by DOE grants SC0010008, ARRA-SC0003883, DE-SC0007897 (D.G.).

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