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Generalized second law of thermodynamics for a phantom energy accreting BTZ black hole

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Abstract In this paper, we have studied the accretion of phantom energy on a $(2 + 1)$ -dimensional stationary Banados–Teitelboim–Zanelli (BTZ) black hole. It has already been shown by Babichev et al. that for the accretion of phantom energy onto a Schwarzschild black hole, the mass of black hole would decrease and the rate of change of mass would be dependent on the mass of the black hole. However, in the case of $(2 + 1)$ -dimensional BTZ black hole, the mass evolution due to phantom accretion is independent of the mass of the black hole and is dependent only on the pressure and density of the phantom energy. We also study the generalized second law of thermodynamics at the event horizon and construct a condition that puts an lower bound on the pressure of the phantom energy.

Keywords BTZ black hole, Accretion, Dark energy, Generalized second law of thermodynamics

1 Introduction

It has been found by various astronomical and cosmological observations [1; 2; 3; 4; 5] that our universe is currently in the phase of accelerated expansion. In the framework of Einstein's gravity, this accelerated expansion has been explained by the presence of a 'cosmological constant' bearing negative pressure which results in the stretching of the spacetime [6; 7; 8; 9]. Many other theoretical models have been presented to explain the present accelerated expansion of the universe including based on homogeneous and time dependent scalar field like the quintessence [10; 11; 12; 13; 14], Chaplygin gas [15; 16; 17; 18; 19; 20; 21; 22; 23; 24; 25; 26] and phantom energy [27; 28; 29; 30; 31; 32; 33], to name a few. The phantom

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energy is characterized by the equation of state $p = \omega\rho$, with $\omega < -1$. It possesses some weird properties: the cosmological parameters like energy density and scale factor become infinite in a finite time; all gravitationally bound objects lose mass with the accretion of phantom energy; the fabric of spacetime is torn apart at the big rip; and that it violates the standard relativistic energy conditions. The astrophysical data coming from the microwave background radiation categorically favors the phantom energy [34]. Motivated from the dark energy models, we model phantom energy by an ideal fluid with negative pressure.

The accretion of dark energy onto a black hole has been studied by many authors [35; 36; 37; 38; 39; 40] after the seminal work of Babichev et al. [41; 42] who have shown that the mass of the black hole will decrease with time when we consider the accretion of phantom energy. In the Einstein theory of gravity, the accretion of the phantom energy onto Schwarzschild black hole and evaporation of primordial black hole has been discussed [41; 42; 43]. It will be interesting to investigate the accretion dynamics in low and higher dimensional gravities. It is also important to investigate accretion dynamics in the extended theories of gravity.

In this paper we investigate the accretion of exotic phantom energy onto a static uncharged 3-dimensional BTZ black hole. We will show that the expression of the evolution of BTZ black hole mass is independent of its mass and depends only on the energy density and pressure of the phantom energy. It is well-known that the horizon area of the black hole decreases with the accretion of phantom energy, hence it is essential to study the generalized second law of thermodynamics (GSL) in this case [44; 45; 46]. We show that the validity of GSL in the present model yields an lower bound on the phantom energy pressure. We also demonstrate that the first law of thermodynamics holds in the present construction.

The plan of the paper is as follows: In second section we model the accretion of phantom energy onto three dimensional BTZ black hole. In third section, we study the GSL for BTZ black hole. Finally we conclude our results.

2 Model of accretion

Consider the field equations for a $(2 + 1)$ -dimensional spacetime with a negative cosmological constant Λ

$$G_{ab} + \Lambda g_{ab} = \pi T_{ab}, \quad (a, b = 0, 1, 2) \quad (1)$$

where G_{ab} is the Einstein tensor in $(2 + 1)$ -dimension while T_{ab} is the stress energy tensor of the matter field. The units are chosen such that $c = 1$ and $G_3 = 1/8$. Considering the stress-energy tensor to be vacuum, one can obtain the following spherically symmetric metric, a $(2+1)$ -dimensional BTZ black hole [47; 48]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \quad (2)$$

where $f(r) = -M + r^2/l^2$, M is the dimensionless mass of the black hole and $l^2 = -1/\Lambda$, is a positive constant. The coefficient g_{00} is termed as the lapse function.

The event horizon of the BTZ black hole is obtained by setting $f(r) = 0$, which turns out, $r_e = l\sqrt{M}$. Also we have $\sqrt{|g|} = r$, where g is the determinant of the metric. To analyze the accretion of phantom energy onto the BTZ black hole, we here employ the formalism from the work by Babichev et al. [41; 42]. The stress energy momentum tensor representing the phantom energy is the perfect fluid

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}, \quad (3)$$

Here ρ and p are the energy density and pressure of the phantom energy while $u^a = (u^0, u^1, 0)$ is the velocity three vector of the fluid flow. Also $u^1 = u$ is the radial velocity of the flow while the third component u^2 is zero due to spherical symmetry of the BTZ black hole. There are two important equations of motion in our model: one which controls the conservation of mass flux is $J^a_{;a} = 0$, where J^a is the current density and the other that controls the energy flux $T^a_{0;a} = 0$, across the horizon. Since the black hole is stationary, the only component of stress energy tensor of interest is T^{01} . Thus the equation of energy conservation $T^0_{a;a} = 0$ is

$$ur(\rho + p)\sqrt{f(r) + u^2} = C_1, \quad (4)$$

where C_1 is an integration constant. Since the flow is inwards the black hole therefore $u < 0$. Also the projection of the energy momentum conservation along the velocity three vector $u_a T^a_{;b} = 0$ (the energy flux equation) is

$$ur \exp \left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p} \right] = -A_1. \quad (5)$$

Here A_1 is a constant and the associated minus sign is taken for convenience. Also ρ_h and ρ_∞ are the energy densities of phantom energy at the BTZ horizon and at infinity respectively. From Eqs. (4) and (5), we obtain

$$(\rho + p)\sqrt{f(r) + u^2} \exp \left[- \int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p} \right] = C_2, \quad (6)$$

where $C_2 = -C_1/A_1 = \rho_\infty + p(\rho_\infty)$. The rate of change in the mass of black hole $\dot{M} = -2\pi r T^1_0$, is given by

$$dM = 2\pi A_1 (\rho_\infty + p_\infty) dt. \quad (7)$$

Note that $\rho_\infty + p_\infty < 0$ (violation of null energy condition) leads to decrease in the mass of the black hole. Moreover, the above expression is also independent of mass contrary to the Schwarzschild black hole and the Reissner–Nordström black hole [35; 36; 37; 38; 39; 40]. Further, the last equation is valid for any general ρ and p violating the null energy condition, thus we can write

$$dM = 2\pi A_1 (\rho + p) dt. \quad (8)$$

3 Critical accretion

We are interested only in those solutions that pass through the critical point as these correspond to the material falling into the black hole with monotonically increasing speed. The falling fluid can exhibit variety of behaviors near the critical point of accretion, close to the compact object. The equation of mass flux or the continuity equation $J^a_{;a} = 0$ is

$$\rho u r = k_1. \quad (9)$$

Here k_1 is integration constant. From Eqs. (4) and (9), we have

$$\left(\frac{\rho+p}{\rho}\right)^2 (f(r)+u^2) = \left(\frac{C_1}{k_1}\right)^2 = C_3. \quad (10)$$

Taking differentials of (9) and (10) and after simplification, we obtain

$$\frac{du}{u} \left[-V^2 + \frac{u^2}{f(r)+u^2} \right] + \frac{dr}{r} \left[-V^2 + \frac{r^2}{l^2(f(r)+u^2)} \right] = 0. \quad (11)$$

Here

$$V^2 \equiv \frac{d \ln(\rho+p)}{d \ln \rho} - 1, \quad (12)$$

From (11) if one or the other bracket factor is zero, one gets a turnaround point corresponding double-valued solution in either r or u . The only solution that passes through a critical point is feasible. The feasible solution will correspond to material falling into the object with monotonically increasing velocity. The critical point is obtained by taking the both bracketed factors in Eq. (11) to be zero. This will give us the critical points of accretion. We obtain

$$V_c^2 = \frac{r_c^2}{(f(r_c)+u_c^2)l^2}, \quad (13)$$

$$V_c^2 = \frac{u_c^2}{f(r_c)+u_c^2}. \quad (14)$$

Above the subscript c refers to the critical quantity. On comparing Eqs. (13) and (14), we get

$$u_c^2 = \frac{r_c^2}{l^2}, \quad V_c^2 = \frac{u_c^2}{-M+2u_c^2}. \quad (15)$$

Here u_c is the critical speed of flow at the critical points which we determine below. For physically acceptable solution, we require $V_c^2 > 0$, hence we get the following restrictions on speeds and the location of the critical points

$$u_c^2 > \frac{M}{2}, \quad r_c^2 > \frac{r_+^2}{2}. \quad (16)$$

4 Generalized second law of thermodynamics and BTZ black hole

In this section we will discuss the thermodynamic of phantom energy accretion that crosses the event horizon of BTZ black hole. Let us first write the BTZ metric in the form

$$ds^2 = h_{mn} dx^m dx^n + r^2 d\phi^2, \quad m, n = 0, 1 \quad (17)$$

where $h_{mn} = \text{diag}(-f(r), 1/f(r))$, is a 2-dimensional metric. From the condition of normalized velocities $u^a u_a = -1$, one can obtain the relations

$$u^0 = f(r)^{-1} \sqrt{f(r) + u^2}, \quad u_0 = -\sqrt{f(r) + u^2}. \quad (18)$$

The components of stress energy tensor are $T^{00} = f(r)^{-1}[(\rho + p)(\frac{f(r)+u^2}{f(r)}) - p]$, and $T^{11} = (\rho + p)u^2 + f(r)p$. These two components help us in calculating the work density which is defined by $W = -\frac{1}{2}T^{mn}h_{mn}$ [49]. In our case it comes out

$$W = \frac{1}{2}(\rho - p). \quad (19)$$

The energy supply vector is defined by

$$\Psi_n = T_n^m \partial_m r + W \partial_n r. \quad (20)$$

The components of the energy supply vector are $\Psi_0 = T_0^1 = -u(\rho + p)\sqrt{f(r) + u^2}$, and $\Psi_1 = T_1^1 + W = (\rho + p)(\frac{1}{2} + \frac{u^2}{f(r)})$. The change of energy across the apparent horizon is determined through $-dE \equiv -A\Psi$, where $\Psi = \Psi_0 dt + \Psi_1 dr$. The energy crossing the event horizon of the BTZ black hole is given by

$$dE = 4\pi r_e u^2 (\rho + p) dt. \quad (21)$$

Assuming $E = M$ and comparing Eqs. (8) and (21), we can determine the value of constant $A_1 = 2u^2 l \sqrt{M}$.

The entropy of BTZ black hole is

$$S_h = 4\pi r_e. \quad (22)$$

It can be shown easily that the thermal quantities, change of phantom energy dE , horizon entropy S_h and horizon temperature T_h satisfy the first law $dE = T_h dS_h$, of thermodynamics. After differentiation of last equation w.r.t. t , and using Eq. (8), we have

$$\dot{S}_h = 8\pi^2 l^2 u^2 (\rho + p). \quad (23)$$

Since all the parameters are positive in the above Eq. (23) except that $\rho + p < 0$, it shows that the second law of thermodynamics is violated i.e. $\dot{S}_h < 0$, as a result of accretion of phantom energy on a BTZ black hole.

Now we proceed to the generalized second law of thermodynamics (GSL). It is defined by

$$\dot{S}_{tot} = \dot{S}_h + \dot{S}_{ph} \geq 0. \quad (24)$$

In other words, the sum of the rate of change of entropies of black hole horizon and phantom energy must be positive. We consider event horizon of the BTZ black hole as a boundary of thermal system and the total matter energy within the event horizon is the mass of the BTZ black hole. We also assume that the horizon temperature is in equilibrium with the temperature of the matter-energy enclosed by the event horizon, i.e. $T_h = T_{ph} = T$, where T_{ph} is the temperature of the phantom energy. Similar assumptions for the temperatures T_h and T_{ph} has been studied in [50; 51; 52; 53; 54]. We know that the Einstein field equations satisfy first law of thermodynamics $T_h dS_h = p dA + dE$, at the event horizon [55; 56]. We also assume that the matter-energy enclosed by the event horizon of BTZ black hole also satisfy the first law of thermodynamics given by

$$T_{ph} dS_{ph} = p dA + dE. \quad (25)$$

Here the horizon temperature is given by

$$T_h = \left. \frac{f'(r)}{4\pi} \right|_{r=r_e} = \frac{\sqrt{M}}{2\pi l}. \quad (26)$$

In this paper, we are assuming that $T_h = T_{ph} = T$. Therefore Eq. (24) gives

$$T \dot{S}_{tot} = T(\dot{S}_h + \dot{S}_{ph}) = 4\pi l^2 u(\rho + p)(2\sqrt{M} + \pi l p). \quad (27)$$

From the above equation, note that $u < 0$ and $\rho + p < 0$ the GSL holds provided $2\sqrt{M} + \pi l p > 0$ which implies

$$p \geq -\frac{2\sqrt{M}}{\pi l}. \quad (28)$$

Since the pressure of the phantom energy is negative ($p < 0$), therefore the GSL gives us the lower bound on the pressure of the phantom energy.

$$-\frac{2\sqrt{M}}{\pi l} \leq p < 0. \quad (29)$$

The GSL in the phantom energy accretion holds within the inequality (29). Otherwise GSL does not hold which forbid evaporation of BTZ black hole by the phantom accretion [57; 58]. In addition, it is not clear whether the GSL should be valid in presence of the phantom fluid not respecting the dominant energy condition [57; 58].

5 Conclusion

In this paper, we have investigated the accretion of exotic phantom energy onto a BTZ black hole. The motivation behind this work is to study the accretion dynamics in low dimensional gravity. Our analysis has shown that evolution of mass of a BTZ black hole would be independent of its mass and will be dependent only on the energy density and pressure of the phantom energy in its vicinity. Due to spherical symmetry, the accretion process is simple since the phantom energy falls radially on the black hole. The accretion would be much more interesting when

additional parameters like charge and angular momentum are also incorporated in the BTZ spacetime. Similarly, it would be of much interest to perform the above analysis in higher $(n + 1)$ dimensional black hole spacetimes.

We also discussed GSL in the BTZ black hole spacetime. We assumed that the event horizon of BTZ black hole acts as a boundary of the thermal system and the phantom energy crossing the event horizon will change the mass of the black hole. We assumed that the horizon temperature is in local equilibrium with the temperature of the matter energy at the event horizon. Under these constraints it is shown that the GSL holds provided the pressure of the phantom energy p has an lower bound $p \geq -\frac{2\sqrt{M}}{\pi l}$, on the black hole parameters (M and l).

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