Various Generalizations of BRST Transformations and their Applications

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

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By

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Becchi, Rouet, Stora and Tyutin (BRST) method is one of the most powerful techniques of quantization for the system with constraints. BRST quantization is based on the BRST transformations which are symmetry of the theory. BRST transformations are characterized by an infinitesimal, anti-commuting, global parameter. Such transformations are nilpotent in nature. Because of these properties, the BRST transformation is extremely useful in studying unitarity, renormalizability and other aspects of different effective theories in particle physics. Other nilpotent transformations also play important roles in studying different gauge theories. These transformations are obtained from BRST transformations by interchanging the ghost and anti-ghost fields and known as anti-BRST transformations.

There are several methods to construct BRST transformation. One of these methods is field/anti-field formalism also called Batalin-Vilkovisky (BV) formalism or Lagrangian BRST formalism. This method is more general than usual Faddeev-Popov method and used for wider class of gauge theories (gauge theories with reducible/irreducible gauge algebra). Another method for constructing BRST transformation is Hamiltonian BRST formalism also called Batalin-Fradkin-Vilkovisky (BFV) formalism. This method is used to construct BRST transformation of constrained systems.

Recently, the concept of finite field dependent BRST (FFBRST) transformations have been introduced by generalizing usual BRST transformations. The parameters in such transformations are finite field dependent and anti-commuting. The FFBRST transformations are also symmetry of the effective theory and nilpotent. However, the Jacobians of such transformations are not unity because it involves finite parameter. Therefore the path integral measure is not invariant. This non-trivial Jacobian leads to several new results.

FFBRST transformations have many applications in gauge field theories. A correct prescription for the poles in the gauge field propagators in non-covariant gauges have been derived by connecting effective theories in covariant gauges to the theories in non-covariant gauges by using FFBRST transformation. The divergent energy integrals in Coulomb gauge are regularized by modifying time like propagator using FFBRST transformation.

My thesis work, is primarily focused on BRST and FFBRST of various field theoretic models.

The entire thesis has been divided into seven chapters as given below:

In **Chapter I** we will provide basic information and general introduction of the research work. First we will talk about how BRST transformations came into picture. Then we will discuss their importance in the physical theories. Further we will give introductory idea of our research work done during this period.

The main objective of the **chapter II** is to provide the basic mathematical tools and techniques related to BRST transformation to prepare the necessary background relevant to this thesis. First we will discuss BRST Transformations in Lagrangian formalism where we will discuss BRST quantization of non-Abelian Yang-Mills theory. Later we will discuss field-antifield formalism or Batalin-Vilkovisky (BV) formalism. Then we will discuss Hamiltonian formalism or Batalin-Fradkin-Vilkovisky (BFV) formalism, Dirac Constraint analysis and Batalin-Fradkin-Fradkina-Tyutin (BFFT) technique. Atlast we will discuss FFBRST Transformations.

In **Chapter III**, we apply a generalized Becchi-Rouet-Stora-Tyutin (BRST) formulation to establish a connection between the gauge-fixed SU(2)YangMills (YM) theories formulated in the Lorenz gauge and in the Maximal Abelian (MA) gauge. It is shown that the generating functional corresponding to the FaddeevPopov (FP) effective action in the MA gauge can be obtained from that in the Lorenz gauge by carrying out an appropriate finite and field-dependent BRST (FFBRST) transformation. In this procedure, the FP effective action in the MA gauge is found from that in the Lorenz gauge by incorporating the contribution of non-trivial Jacobian due to the FFBRST transformation of the path integral measure. The present FFBRST formulation might be useful to see how Abelian dominance in the MA gauge is realized in the Lorenz gauge.

In **Chapter IV**, we investigate all possible nilpotent symmetries for a particle on torus. We explicitly construct four independent nilpotent BRST symmetries for such systems and derive the algebra between the generators of such symmetries. We show that such a system has rich mathematical properties and behaves as double Hodge theory. We further construct the finite field dependent BRST transformation for such systems by integrating the infinitesimal BRST transformation systematically. Such a finite transformation is useful in realizing the various theories with toric geometry.

Further we develop BRST symmetry for the first time for a particle on the surface of a torus knot by analyzing the constraints of the system. The theory contains 2nd-class constraints and has been extended by introducing the Wess-Zumino term to convert it into a theory with first-class constraints. BFV analysis of the extended theory is performed to construct BRST/anti-BRST symmetries for the particle on a torus knot. The nilpotent BRST/anti-BRST charges which generate such symmetries are constructed explicitly. The states annihilated by these nilpotent charges consist of the physical Hilbert space. We indicate how various effective theories on the surface of the torus knot are related through the generalized version of the BRST transformation with finite field dependent parameters.

In **Chapter V**, we show how Weyl degree of freedom can be introduced in the Nambu-Goto string in the path-integral formulation using the re-parametrization invariant measure. We first identify Weyl degrees in conformal gauge using BFV formulation. Further we change the Nambu-Goto string action to the Polyakov action. The generating functional in light-cone gauge is then obtained from the generating functional corresponding to the Polyakov action in conformal gauge by using suitably constructed finite field dependent BRST transformation.

In **Chapter VI**, we consider Polyakov theory of Bosonic strings in conformal gauge which exhibits conformal and ghost number anomaly. We show how these anomalies can be removed by connecting this theory to that of in background covariant harmonic gauge by using suitably constructed finite field dependent BRST transformation.

In **Chapter VII**, we will present a brief summary of our entire research work carried out during this research period.

List of Publications:

1. *Maximal Abelian Gauge and Generalized BRST transformation*
   S. Deguchi, **V. K. Pandey** and B. P. mandal, **Phys. Lett. B 756**, 394 (2016).

2. *Double Hodge Theory for a Particle on Torus*
   V. K. Pandey and B. P. Mandal, **Advan. in High Energy Phys., Vol. 2017**, Art. Id 6124189 (2017).

3. *BRST Symmetry for Torus Knot*
   V. K. Pandey and B. P. Mandal, **Eur. Phys. Lett. 119**, 31003 (2017).

4. *Weyl Degree of Freedom in Nambu-Goto String Through Field Transformation*
   V. K. Pandey and B. P. Mandal, **Eur. Phys. Lett. 122**, 21002 (2018).

5. *BRST Quantization on Torus Knot*
   V. K. Pandey and B. P. Mandal, **Sprin. Proc. in Phys. 203**, 513 (2018).

6. *Conformal to Harmonic Gauge for Bosonic Strings*
   V. K. Pandey and B. P. Mandal, accepted for publication in Euro Phys. Letters.

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UNDERTAKING

I hereby declare that I have completed the research work for the full time period prescribed under the clause VIII.1 of the PhD ordinances of the Banaras Hindu University, Varanasi and the research work embodied in this thesis entitled "**Various Generalizations of BRST Transformations and Their Applications**" is my own research work.

I avail myself to responsibility such as an act will be taken on behalf of me, mistakes, errors of fact and misinterpretations are of course entirely my own.

Date: **(Vipul Kumar Pandey)**

ANNEXURE - E

CANDIDATE'S DECLARATION

I, Vipul Kumar Pandey **S/O** Shri Vinod Kumar Pandey, certify that the work embodied in this PhD thesis is my own bonafide work carried out by me under the supervision of Prof. Bhabani Prasad Mandal for a period of September 2013 to January 2019 at the Banaras Hindu University, Varanasi. The matter embodied in this PhD thesis has not been submitted for the award of any other degree/diploma.

I declare that I have faithfully acknowledged, given credit to and referred to the research workers wherever their works have been cited in the text and the body of the thesis. I further certify that I have not willfully lifted up some other's work, para, text, data, results, etc. reported in the journals, books, magazines, reports, dissertations, thesis, etc., or available at websites and included them in this PhD thesis and cited as my own work.

Date:

Place: Varanasi **(Vipul Kumar Pandey)**

Certificate from the Supervisor

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(Prof. Bhabani Prasad Mandal)
Supervisor

(Prof. Rudra Prakash Malik)
Head of the Department

ANNEXURE - F

PRE-SUBMISSION SEMINAR COMPLETION CERTIFICATE

This is to certify that **Mr. Vipul Kumar Pandey**, a bonafide research scholar of this department, has successfully completed the pre-submission seminar requirement which is a part of his PhD program, on his thesis entitled "**Various Generalizations of BRST Transformations and Their Applications**".

Date:
Place: Varanasi

(Prof. Rudra Prakash Malik)
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ANNEXURE G

COPYRIGHT TRANSFER CERTIFICATE

Title of the Thesis: *Various Generalizations of BRST Transformations and Their Applications*

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(Vipul Kumar Pandey)

Contents

Chapter 1

Introduction

In development of modern physics, symmetry principles have been proved to be the most invaluable tools. Gauge field theories which are based on the local gauge invariance of the Lagrangian density of the theories have found enormous importance in describing all the fundamental interactions of nature and play the key role in understanding the particle physics phenomenon. The standard model of particle physics which describes strong, weak and electromagnetic interactions in the unified manner is regarded as the most successful theory because of its ability to explain the varieties of experimental results. The standard model is a non-Abelian gauge theory which serves as a paradigm example of quantum field theory. It illustrates wide range of physics such as spontaneous symmetry breaking, study of anomalies, non-perturbative behavior etc. Recently it has found applications in many other fields such as nuclear physics, astrophysics, cosmology etc.

In 1954, C.N. Yang and R. Mills [1] proposed a theory of the strong interactions between protons and neutrons, which is based on the $SU(3)$ algebra known as non-Abelian gauge theory. Non-Abelian theories are fundamental building blocks for the construction of physical theories. However, one faces various problems to develop the quantum version of such theories with local gauge invariance consistently. In path-integral quantization of these theories the vacuum-vacuum transition amplitude or generating functional is ill defined for such theories. This problem arises due to over counting of physically equivalent gauge configurations grouped together in different gauge orbits. To solve this problem the method of gauge-fixing was used. This helped in removing infinite factor in path integral measure by choosing one gauge field from each orbit. The gauge-fixing was achieved by adding an extra term consisting of arbitrary function of gauge field and arbitrary parameter in the action. The addition of gauge fixing term solved the problem of over counting but introduced other problems like, the physical theory became dependent on arbitrary function of gauge field and/or an arbitrary parameter which is not desirable. To tackle these problems Faddeev-Popov (FP) [2] proposed an effective action by introducing ghost fields. Ghost fields are scalars in nature but behaves like Grassmanians and hence do not follow spin-statistics theorem. These unphysical fields compensate for the effect of arbitrary gauge-fixing function hence preserve the unitarity of the theory. But the total action is no longer gauge invariant which leads to various difficulties in the theory. For example the choice of counter terms in renormalization of the theory is no more restricted to gauge invariant terms as the gauge invariance is broken for the theory itself. This leads to the difficulties in renormalization program.

Four physicists C. Becchi, A. Rout, R. Stora and I. V. Tyutin (independently) [3, 4, 5, 6] found a very interesting symmetry transformation of FP effective action known as BRST Transformation. The analytical form of the BRST transformation and its relevance to renormalization and anomaly cancellation were described by Becchi, Rouet and Stora in a series of papers culminating in the 1976, "Renormalization of gauge theories" [3, 4, 5]. The equivalent transformation and many of its properties were independently discovered by Tyutin [6]. Its significance for rigorous canonical quantization of a Yang-Mills (YM) theory and its correct application to the Fock space of instantaneous field configurations were elucidated by T. Kugo and I. Ojima [7]. These symmetry transformations have following characteristics. They are (i) infinitesimal (ii) global (i.e. independent of space and time) (iii) anti-commuting (iv) nilpotent. Sometimes the nilpotency is proved using equation of motion of one or more fields then it is referred as on-shell nilpotent. However, BRST transformation can be made off-shell nilpotent by introducing Nakanishi-Lautrup (NL) type auxiliary fields to the theory. These transformations are extremely useful in characterizing various field theoretic models and renormalization of gauge theories are known to be greatly facilitated by the use of BRST transformations. These transformations enables one to formulate Slanov-Taylor (ST) [8] identities in a compact and mathematically convenient form. There is another symmetry of gauge fixed action known as anti-BRST symmetry. In this symmetry the role of ghost field changes with anti-ghost field [9, 10]. The anti-BRST symmetry does not add anything substantial to BRST quantization procedure but is important in the geometrical description of the superspace formulation of gauge theories [11, 12]. The foundation of this thesis is based on solid platform of BRST formulation.

It has been found that the usual FP procedure which yields quadratic ghost action is not applicable to some supergravity models where quadratic ghosts are needed to preserve unitarity [13] and nilpotency of the BRST operator is ensured only by using the equation of motion for certain fields in the gauge fixed action. Such theories are said to have open algebra. In some theories the ghost action itself has additional gauge symmetry which needs further gauge fixing. These theories are called reducible gauge theories. For such theories, the field spectrum is enlarged by introducing further ghost of ghosts. FP procedure doesn't work for general reducible theories or when the gauge algebra is not closed. In order to cover a wider class of gauge theories, a powerful technique of BRST quantization was proposed by I. A. Batalin and G. A. Vilkovisky known as field/anti-field (or BV) formalism [14, 15]. In this technique the effective action is extended by introducing anti-fields which satisfy more general and rich mathematical relation known as quantum master equation (QME). The interconnection between BRST formulation and field/anti-field formalism is a very exciting topic of recent research [16]. Field/anti-field formalism is based on the BRST symmetry with an infinitesimal, global and anti-commuting parameter. Field/anti-field formalism is studied in path integral quantization method which uses Lagrangian formalism. This formalism has been reviewed in [17].

Another powerful technique of BRST quantization in the Hamiltonian approach is BFV formalism developed by I. A. Batalin, E. S. Fradkin and G. A. Vilkovisky [18, 19, 20]. This method is used to construct BRST transformation of constrained systems [21]. It is not only applicable to the systems with first class constraints but also applicable to the systems with second class constraints [22]. This technique relies on BRST transformations which are independent of the specific gauge condition. In this technique, the BRST charge is constructed from the set of first class constraints of the theory by introducing a pair of ghost field and corresponding momenta for each set of constraints. For the system of second class constraints, the BRST charge is constructed after converting the second class constraints to first class constraints via various techniques. This method uses the enlarged phase space where Lagrange multipliers and their corresponding momenta are treated as a dynamical variables [23]. The main features of BFV approach are as follows: (i) it does not require closure (off-shell) of the gauge algebra and therefore does not need an auxiliary field, (ii) this formalism relies on BRST transformation which is independent of gauge-fixing condition and (iii) it is also applicable to the first order Lagrangian. Hence it is more general than the strict Lagrangian approach [24]. There are various ways to study BRST formulation such as BV-BRST and BFV-BRST formalism as described above. We mainly consider different generalizations of BRST symmetry in the context of BV-BRST and BFV-BRST formalisms.

To convert second class constraints to first class constraints we use a general method known as BFFT (Batalin-Fradkin-Fradkina-Tyutin) method developed by four physicists I. A. Batalin, E. S. Fradkin, T. E. Fradkina and I. V. Tyutin [25, 26, 27]. This is an iterative technique to change the second class constraint to first class constraint. This method has been used to study many of the mathematical models in recent years [28, 29, 30, 31, 32, 33, 34, 35].

BRST symmetry has been generalized in many ways. In 1993, Lavelle and Macmullan [36] found a generalized BRST symmetry adjoint to usual BRST symmetry in case of QED. This generalized BRST is non-local and non-covariant. The motivation behind the emergence of this symmetry was to refine the characterization of physical states given by the BRST charge. Since locality has been considered to be the main cause of infinities in the usual quantum field theory, people have been turning to non-local quantum field theory [37, 38]. Non-local gauge symmetry plays an important role in non-local quantum field theories. Later, Tang and Finkelstein [39] found another generalized BRST symmetry which is non-local but covariant. Such a BRST is not necessarily nilpotent but can be made nilpotent under certain condition in auxiliary field formulation. This symmetry imposes a constraint on the physical states, which determines the physicality more strongly than previous BRST symmetries. Later two physicists, Yang and Lee [40] also presented a local and non-covariant BRST symmetry in the case of Abelian gauge theories.

S. D. Joglekar and B. P. Mandal [41] further generalized the BRST transformation by allowing the parameter to be finite and field dependent. Such generalized BRST transformations are also symmetry of the effective theory and they are nilpotent. However, the path integral measure is not invariant and give rise to a non-trivial Jacobian. The Jacobian is shown to produce exponential term of local fields which changes effective action to give rise to another new effective action. Such generalized BRST transformations have found many applications [42, 43, 44, 45, 46, 46, 48] namely, to find correct prescriptions for the poles in the axial gauge field propagator [42, 43], to regularize the energy in the Coulomb gauge [44] etc. Recently a new technique of finite BRST transformation has been developed by some some Russian physicists in a series of papers [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59]. Also some important results about BRST for various physical systems have been developed recently [60, 61, 62].

In usual BRST transformation, variation of the kinetic part of the effective action independently vanishes whereas the variation of gauge fixing part cancels with the variation of ghost part of the effective action. One of the important generalizations of the BRST symmetry as local and covariant BRST symmetry is known as dual-BRST symmetry [63]. Under dual-BRST symmetry, the variation of gauge fixing part independently vanishes whereas the variation of the kinetic part cancels with the variation of ghost part of the action. So far in the literature, dual-BRST symmetry has been treated as an independent symmetry because of its analogy to the co-exterior derivative in the language of differential geometry. Therefore sometimes it is referred as co-BRST symmetry. The usual BRST symmetry is analogous to the exterior derivative. The anti-commutators of exterior derivative and co-exterior derivative gives a Laplacian operator analogous to the bosonic symmetry [40, 63, 64, 65, 66].

Another generalization of the BRST transformations can also be made for YM theory in which the anti-commuting parameter is space-time dependent [67]. These are not exact symmetries of the theory, however they do lead to a non-trivial Ward-Takahashi (WT) identity. This non-trivial WT identity could lead to new consequences which are not contained in the usual WT identity. Such generalized BRST transformations are realized as the broken orthosymplectic symmetry found in the superspace formulation of YM theory [68].

BRST and anti-BRST symmetries are treated as an independent symmetry only if they absolutely anti-commute amongst themselves. Similarly, dual BRST and anti-dual BRST symmetries are independent symmetries only if they absolute anti-commute. In order to make them absolutely anti-commutative, a restriction is invoked. Such restrictions are known as Curci-Ferrari (CF) restrictions [9]. Although, it is necessary to invoke these restrictions but reason behind imposing such restrictions are not clear in the Lagrangian framework. It is also not known, what kind of constraints they are in the language of Dirac's prescription of constraint analysis.

The consequences of BRST symmetry, formulated as Slanov-Taylor (ST) identities, are central to the discussion of renormalizability, unitarity, gauge independence of the theory. Any attempt that sheds light on, offer a reformulation of, understanding of BRST symmetry and YM theory is, therefore of significance to particle physics. This motivates us to construct various generalizations of BRST symmetry and their applications in quantum field theory.

This thesis is mainly based on the construction of three important aspects of BRST symmetry. BRST symmetry, dual-BRST symmetry and generalized BRST symmetry with a finite field dependent parameter. The BFV Hamiltonian formalism has been explored in the context of BRST symmetries. This formalism has been applied to mathematical models like particle on a torus, particle on torus knot etc. The FFBRST formalism has also been explored in context of various field and string theory models like Maximal Abelian (MA) gauge in YM theory, Nambu-Goto string in light-cone gauge and bosonic string in harmonic gauge etc. In first chapter we will introduce the important results related to BRST transformation. In II chapter we will discuss about various mathematical techniques required to solve problems related to BRST transformations. In the III chapter we will discuss about maximal Abelian gauge and its use in addressing the confinement problem. Then using the FFBRST transformation we will try to study the confinement problem in more general Lorenz gauges. In IV chapter we will discuss about various nilpotent symmetries related to particle on torus. In the same chapter we will develop BRST and anti-BRST symmetries for particle on torus knot for the first time. In V chapter we will discuss about Weyl degrees of freedom in Nambu-Goto string in light-cone gauge using finite field BRST transformation. In VI chapter we will address the problem of ghost number anomaly in conformal gauge in bosonic string by connecting it to action in harmonic gauge using FFBRST transformation. In The VII chapter we will summarize the total work done. This thesis is divided into the following seven chapters. The detailed content of these chapters are given below.

Chapter I is dedicated to the general introduction of BRST symmetries and related topics like generalized BRST symmetries, basic techniques like field/anti-field formalism in Lagrangian approach and BFV formalism in Hamiltonian approach. Brief discussion about various chapters is also presented.

In **chapter II**, we will discuss mathematical techniques related to BRST formalism in detail, both in Lagrangian as well as Hamiltonian approach. At first we will discuss field/anti-field formalism or BV formalism in detail. There we will discuss about classical/quantum master equations and generation of BRST transformations. Then we will talk about Hamiltonian BRST formalism in which we will discuss about Dirac's constraints analysis where we will discuss about first and second class constraints. Then we will discuss about (BFFT) formalism of conversion of second class constraints to first class constraints. Then we will discuss about BFV formalism. At last we will discuss about FFBRST transformation.

In **Chapter III**, we will apply a generalized (BRST) formulation to establish a connection between the gauge-fixed SU(2) YM theory formulated in the Lorenz gauge and in MA gauge. It is shown that the generating functional corresponding to the FaddeevPopov (FP) effective action in the MA gauge can be obtained from that in the Lorenz gauge by carrying out an appropriate FFBRST transformation. The present FFBRST formulation might be useful to see how quark confinement is realized in the Lorenz gauge.

In **Chapter IV**, we will investigate all possible nilpotent symmetries for a particle on torus. We explicitly construct four independent nilpotent BRST symmetries for such systems and derive the algebra between the generators of such symmetries. We show that such a system have rich mathematical properties and behaves as double Hodge theory. We further construct the FFBRST transformation for such systems by integrating the infinitesimal BRST transformation systematically. Further we develop BRST symmetry for a particle on the surface of a torus knot by analyzing the constraints of the system. The theory contains second class constraint and has been extended by introducing the Wess-Zumino term to convert it into a theory with first class constraints. BFV analysis of the extended theory is performed to construct BRST/anti-BRST symmetries for the particle on a torus knot. We will show how various effective theories on the surface of the torus knot are related through the generalized version of the BRST transformation with finite field dependent parameter. In last section BRST/anti-BRST charge for particle on torus knot will be constructed using the technique used in ref. [148].

In **Chapter V** we will show how Weyl degrees of freedom can be introduced in the Nambu-Goto (NG) string in the path-integral formulation using the re-parametrization invariance of path integral measure. We first identify Weyl degrees of freedom in conformal gauge using BFV formulation. Further we change the NG string action to the Polyakov action. The generating functional in light-cone gauge is then obtained from the generating functional corresponding to the Polyakov action in conformal gauge by using suitably constructed FFBRST transformation.

In **Chapter VI** we consider Polyakov theory of Bosonic strings in conformal gauge which is used to study conformal anomaly. However it exhibits ghost number anomaly. We show how this anomaly can be avoided by connecting this theory to that of in background covariant harmonic gauge which is known to be free from conformal and ghost number current anomaly, by using suitably constructed FFBRST transformation.

Chapter VII has an overall conclusion of the thesis.

Chapter 2

The Mathematical Preliminaries

Renormalization of YM theories revolutionized field of quantum field theory. Even after symmetry breaking it was considered as most complete theory describing particle interactions. But the quantization rules in earlier quantum field theory (QFT) frameworks resembled prescriptions or heuristics more than proofs, especially in non-Abelian QFT, where the use of ghost fields with superficially bizarre properties is almost unavoidable for technical reasons related to renormalization and anomaly cancellation. To avoid these problems a new kind of symmetry transformation also called as BRST transformation was introduced [3, 4, 5, 6].

The main objective of this chapter is to provide the basic mathematical tools and techniques related to BRST transformation to prepare the necessary background relevant to this thesis. First we will discuss BRST transformations in Lagrangian formalism also known as field/anti-field formalism or BV formalism. There we will discuss BRST quantization of non-Abelian YM theory. Then we will discuss Hamiltonian formalism or BFV formalism, Dirac constraint analysis and BFFT technique. At last we will discuss the technique of FFBRST transformations.

2.1 Batalin-Vilkovisky (BV) formalism

The BV formalism (also known as field/anti-field formalism) [14] which is based on Lagrangian framework is a powerful technique to quantize more general gauge theories based on BRST symmetry [15, 16, 17, 23]. This method is applicable to gauge theories with both reducible or open as well as irreducible or closed algebra. This method is also used to analyze the possible symmetry violations in the action due to quantum effects.

The main idea of this approach is to construct an extended action $W_\Psi[\phi, \phi^\star]$ by introducing an anti-field ($\phi^\star$) of opposite statistics corresponding to each field (ϕ) in the theory. The sum of ghost numbers associated to field and its anti-field is equal to -1. The extended action can be expressed as sum of the original gauge invariant action S_0 and the coupling term of anti-field with the BRST transformed field ($\delta_b\phi$)

$$W[\phi, \phi^\star] = S_0 + (\delta_b\phi)\phi^\star \tag{2.1}$$

Here $W[\phi, \phi^\star]$ satisfies following condition

$$0 = \frac{\delta_r W}{\delta\phi^\star}\frac{\delta_l W}{\delta\phi} \tag{2.2}$$

This expression is called master equation where l and r are left and right derivatives.

The generating functional in field/anti-field formulation is written as

$$Z = \int \mathcal{D}\phi e^{iW_\Psi[\phi]} \tag{2.3}$$

where

$$W_\Psi[\phi] = W\big[\phi, \phi^\star = \frac{\partial\psi}{\partial\phi}\big] \tag{2.4}$$

Here ψ is gauge-fixing fermion and has Grassman parity 1 and ghost number -1.

The action function $W_\Psi[\phi]$ satisfies a rich mathematical relation which is known as quantum master equation (QME) [15] and written as:

$$(W_\Psi, W_\Psi) - 2i\Delta W_\Psi = 0, \tag{2.5}$$

where the anti-bracket is defined by

$$(X, Y) \equiv \frac{\delta_r X}{\delta\phi}\frac{\delta_l Y}{\delta\phi^\star} - \frac{\delta_r X}{\delta\phi^\star}\frac{\delta_l Y}{\delta\phi}, \tag{2.6}$$

and the operator Δ is defined as

$$\Delta \equiv \frac{\delta_r}{\delta\phi^\star}\frac{\delta_l}{\delta\phi}. \tag{2.7}$$

The quantum master equation in the zeroth order of ant-fields gives the condition of gauge invariance. On the other hand it reflects the nilpotency of BRST transformation in the first order of anti-fields.

The classical master equation has following form

$$(W_\Psi, W_\Psi) = 0. \tag{2.8}$$

The expectation value of operators which are invariant under the BRST transformations are given by

$$\begin{aligned} \delta_b\phi &= \frac{\delta_r W}{\delta\phi^\star} = -(W, \phi); \\ \delta_b\phi^\star &= \frac{\delta_r W}{\delta\phi} = -(W, \phi^\star) \end{aligned} \tag{2.9}$$

They are independent of change in gauge-fixing fermion ψ. Now we will introduce YM action in this formalism.

The BV action for YM theory is written as [71],

$$\begin{aligned} W_\Psi(\phi,\phi^\star) &= \int d^4x[-\frac{1}{4}F^{\alpha\mu\nu}F^\alpha_{\mu\nu} + A^{\mu\alpha^\star}D^{\alpha\beta}_\mu c^\beta + c^{\alpha^\star}\frac{g}{2}f^{\alpha\beta\gamma}c^\beta c^\gamma \\ &+ B^\alpha\bar{c}^{\alpha^\star}] \end{aligned} \quad (2.10)$$

which is also written as

$$W[\phi,\phi^\star] = S_0(\phi) + \delta_b\Psi \quad (2.11)$$

Here Ψ is the gauge-fixed fermion and can be written in this case as

$$\Psi = \int d^4x\bar{c}^\alpha\big[\frac{\lambda}{2}B^\alpha - \partial.A^\alpha\big] \quad (2.12)$$

In the next section we will discuss canonical BRST quantization of gauge theories based on Hamiltonian formalism.

2.2 Hamiltonian BRST Formalism

In this section, we will discuss Hamiltonian formalism for the quantization of gauge theories. This technique consist of mainly three parts. First, we will perform the constraint analysis of classical systems using Dirac technique. This technique will provide us two different class of constraints, first and second class. First class constraints are generators of gauge transformations. To change the second class constraints to first class, we will use BFFT technique. Therefore in second subsection we will discuss BFFT formalism [25, 26, 27]. At last, we will discuss BFV formalism [18, 19, 20]. These three methods will set the stage for the path integral quantization of gauge theories in Hamiltonian formulation.

2.2.1 Dirac Constraints Analysis

In this subsection, we will discuss briefly a very useful technique of quantization of the systems with first and second class constraints developed by Dirac [22]. In various field theory models, the dynamical phase space variables are not all independent. They satisfy some constraints emerging from the structure of models. In such situations usual procedure for obtaining Hamiltonian from Lagrangian doesn't work. For such models Dirac's technique is used to systematically develop Hamiltonian from Lagrangian of the theory. To illustrate Dirac technique we will consider a classical dynamical system described by a Lagrangian $L[q_n(t), \dot{q}_n(t)]$. Here $\dot{q}_n$ is the time derivative of q_n. We use the notation of a system with discrete set of degrees of freedom, although n may represent a possibly continuous set of indices as done in classical field theory. Momenta canonically conjugate to coordinate for a given Lagrangian is written as

$$p^n(q, \dot{q}) = \frac{\delta L}{\delta \dot{q}_n} \quad n = 1, ..., N \tag{2.13}$$

If the relation in Eq.(2.13) is invertible, then velocities can be expressed in terms of coordinates and momenta.

$$(q_n, \dot{q}_n) \to (q_n, p^n) \tag{2.14}$$

In other words, if for a system

$$\det \frac{\delta^2 L}{\delta \dot{q}_n \delta \dot{q}_{n'}} \neq 0 \tag{2.15}$$

then the Lagrangian is called regular. For such systems, one can obtain Hamiltonian from this Lagrangian using Legendre transformation

$$H(p, q) = \dot{q}_n(p, q)p^n - L[q, \dot{q}(p, q)] \tag{2.16}$$

On the other hand, if the system described by a Lagrangian satisfies

$$\det \frac{\delta^2 L}{\delta \dot{q}_n \delta \dot{q}_{n'}} = 0 \tag{2.17}$$

then the Eq.(2.13) is non-invertible. In such systems, not all the conjugate momenta are independent variables. So, there exist relation between various dynamical variables known as constraints. Such systems are called constrained systems. For these systems, usual Poisson brackets may not represent the true brackets as they need not satisfy the constraints of the theory. To obtain Hamiltonian description systematically for such systems we use Dirac method briefly discussed below.

The primary constraints are defined using the independent relation between the canonical variables as

$$\phi_m(q,p) = 0 \quad m = 1, ..., M \tag{2.18}$$

The Hamiltonian corresponding to these constraints is written as

$$H_p = H + \lambda_m \phi^m \tag{2.19}$$

The time evolution of any dynamical variable $F(q,p)$ in the phase space is written as

$$\dot{F} = \{F, H_p\} \tag{2.20}$$

A basic consistency requirement for primary constraints is that they should be preserved in time. If we consider F in Eq.(2.20) as one of the primary constraints ϕ_m and calculate the time evolution, we obtain a number of secondary constraints expressed as

$$\phi_k = 0, \quad k = M + 1, ..., M + k \tag{2.21}$$

where K is the total number of secondary constraints. We can write all the primary and secondary constraints using a single equation as

$$\phi_j = 0, \quad j = 1, ..., M + k (= J) \tag{2.22}$$

The constraints $F(q, p)$ can further be classified in two categories called as first and second class constraints. F is said to be first class, if its Poisson bracket with every other constraint vanishes weakly.

$$\{F, \phi_j\} \approx 0, \Leftrightarrow \{F, \phi_j\} = \sum_i c^i_j(p, q)\phi_i \tag{2.23}$$

A constraint is said is said to be second class, if there exists at least one other constraint which doesn't have a weakly vanishing Poisson bracket with it. If the second class constraints are absent or eliminated, the set of first class constraints are written as ϕ_α and

$$\{\phi_\alpha, \phi_\beta\} \approx 0, \Leftrightarrow \{\phi_\alpha, \phi_\beta\} = c^\gamma_{\alpha,\beta}\phi_\gamma \tag{2.24}$$

Here $c^\gamma_{\alpha,\beta}$ are called first order structure constants. According to Dirac's theory, first class constraints are generators of gauge symmetries. For theories with second class constraints, Poisson brackets are replaced by Dirac brackets. In this thesis we will deal with first class constraints only. If H_0 is a first class Hamiltonian, its Poisson bracket with first class constraints is written as

$$[H_0, \phi_\alpha] \approx 0, \Leftrightarrow [H_0, \phi_\alpha] = V^\beta_\alpha(p, q)\phi_\beta \tag{2.25}$$

With the set of first class constraints and the Hamiltonian of Eq.(2.25), action can be constructed as

$$S = \int dt \big[\dot{q}_n p^n - H_0 - \sum_\alpha \lambda^\alpha(t)\phi_\alpha\big] \tag{2.26}$$

Here, $\lambda^\alpha(t)$ is a Lagrange multiplier for the system. This action is invariant under the gauge transformation. The transformation for an arbitrary phase space function F is generated by the constraints:

$$\delta_\epsilon F(p,q) = \big[F, \epsilon^\alpha(t)\phi_\alpha\big] \tag{2.27}$$

where ϵ^α are the parameters of gauge transformation.

These were the main features of Dirac's constraint analysis, the first step towards Hamiltonian quantization of gauge systems. The second step in this direction is BFFT formalism. In the next section we will discuss important features of this technique.

2.2.2 Batalin-Fradkin-Fradkina-Tyutin (BFFT) Formalism

BFFT formalism [25, 26, 27] is a technique used to convert second class constraints to first class constraints. The important feature of this technique has been discussed here.

Consider the original phase space variables as (q^i, p^i) where a constrained system has two second class constraints, $\Omega_k(x), k = 1, 2$, obeying the algebra

$$\{\Omega_k(x), \Omega_{k'}(y)\} = \Delta_{kk'}(x,y) \tag{2.28}$$

Where $\Delta_{kk'}$ is a matrix with non-vanishing determinant. To systematically convert these constraints into first class one, two Stuckelberg fields Φ^i will be introduced corresponding to the second class constraints Ω_i with the Poisson brackets

$$\{\Phi^k(x), \Phi^{k'}(y)\} = \omega^{kk'}(x,y), \tag{2.29}$$

where ω^{ij} can be taken as

$$\omega^{kk'}(x,y) = \epsilon^{kk'}\delta(x-y) \tag{2.30}$$

The first class constraints $\tilde{\Omega}_i$ are then constructed as a power series of the Stuckelberg fields

$$\tilde{\Omega}_i = \sum_{n=0}^{\infty} \Omega_i{}^{(n)}, \quad \Omega_i{}^{(0)} = \Omega_i \tag{2.31}$$

where $\Omega_k{}^{(n)}$ are polynomials in the Stuckelberg fields Φ^i of degree n. They can be determined by the requirement that the first class constraints $\tilde{\Omega}_i$ satisfy the closed algebra as follows

$$\{\tilde{\Omega}_k(x), \tilde{\Omega}_{k'}(y)\} = 0 \tag{2.32}$$

We obtain recursive equations which determine the correction terms $\Omega_i{}^{(n)}$. A basic equation in lowest order can be written as

$$\Delta_{kk'} + X_{k\alpha}\omega^{\alpha\beta}X_{\beta k'} = 0 \tag{2.33}$$

and the first order correction term is written as

$$\Omega_k{}^{(1)} = X_{kk'}(q_i, p_i)\Phi^{k'} \tag{2.34}$$

The matrices $\omega_{kk'}$ and $X_{kk'}$ in Eq.(2.29) and Eq.(2.33), which are the inherent arbitrariness of BFFT formalism, can be chosen with the aim of obtaining algebraic simplifications in the determination of the correction terms $\Omega_k{}^{(n)}$.

In the similar way, the gauge invariant Hamiltonian is obtained by the expansion

$$\tilde{H} = \sum_{n=0}^{\infty} H^{(n)}, \quad H^{(0)} = H \tag{2.35}$$

From the Abelian first class algebra

$$\{\tilde{H}, \tilde{\Omega}_k\} = 0 \tag{2.36}$$

we have recursive equations which determine the correction terms $H^{(n)}$ and, consequently, the gauge invariant Hamiltonian.

2.2.3 Batalin-Fradkin-Vilkovisky (BFV) Formalism

We will briefly discuss the BFV formalism [18, 19, 20] which is applicable for the theories with first-class constraints. This method uses an extended phase space where the Lagrange multiplier and the ghosts are treated as dynamical variables. The main features of this approach are as follows: i) it does not require off-shell closure of the gauge algebra and therefore does not require an auxiliary field ii) It heavily depends on BRST transformation which is independent of the gauge condition and iii) it is even applicable to Lagrangian which are not quadratic in velocities and hence is more general than the Lagrangian BRST formalism. First of all, consider a phase space of canonical variables $(q^i, p_i)(i = 1, 2...., n)$ in terms of which the canonical Hamiltonian $\mathcal{H}_0(q^i, p_i)$ and the constraints $(\Omega_a \approx 0)(a = 1, 2, ..., m)$ are given. These constraints satisfy following algebra

$$
\begin{aligned}
\{\Omega_a, \Omega_b\} &= i\Omega_c U^c_{ab} \\
\{\mathcal{H}_0, \Omega_a\} &= i\Omega_b V^b_a
\end{aligned}
\tag{2.37}
$$

where the structure coefficients U^c_{ab} and V^b_a are generally functions of the canonical variables. We also assume that the constraints are irreducible. In order to single out the physical variables, we introduce the additional conditions $\Phi^a(q^i, p_i) \approx 0$. Here the Φ^a play the role of gauge-fixing function. The action, in terms of canonical Hamiltonian density $\mathcal{H}_0$, first class constraints, Ω_a and gauge-fixing function Φ^a can be written in this formalism as

$$
S = \int d^4x(p^i \dot{q}_i - \mathcal{H}_0 - \lambda^a\Omega_a + \pi_a\Phi^a), \tag{2.38}
$$

where (q_i, p^i) are the canonical variables. Lagrange multiplier fields λ^a and π_a are canonically conjugate variables.

In order to make the theory in extended phase space to be consistent with the initial theory, two sets of canonically conjugate anti-commuting ghost coordinate and momenta $(C^a, \bar{P}_a)$ and $(P^a, \bar{C}_a)$ are introduced for each constraint. These canonically conjugate ghost and momenta satisfy the following anti-commutation relation:

$$[C^a, \bar{P}_b] = [P^a, \bar{C}_b] = i\delta^a_b \tag{2.39}$$

where C_α and P_α have ghost number 1 and -1, respectively. The generating functional for this extended theory is then defined as

$$Z_\Psi = \int [\mathcal{D}\phi] e^{iS_{eff}[\phi]}, \tag{2.40}$$

where $[\mathcal{D}\phi]$ is the path integral measure and the effective action S_{eff} is

$$S_{eff} = \int dt(p_i \dot{q}^i + \pi_a \dot{\lambda}^a + \bar{P}_a \dot{C^a} + \bar{C}_a \dot{P^a} - \mathcal{H}_m + i\{Q, \Psi\}). \tag{2.41}$$

Here $\mathcal{H}_m$ is the BRST invariant Hamiltonian which one calls the minimal Hamiltonian,

$$\mathcal{H}_m = \mathcal{H}_0 + \bar{P}_a V^a_b C^b \tag{2.42}$$

where Ψ is the gauge-fixing fermion and Q_b is the nilpotent BRST charge. They have following general form:

$$\begin{aligned} Q_b &= C^a \Omega_a - \frac{1}{2} C^b C^c U^a_{cb} \bar{P}_a + P^a \pi_a, \\ \Psi &= \bar{C}_a \chi^a + \bar{P}_a \lambda^a \end{aligned} \tag{2.43}$$

Here χ^a are gauge-fixing functions that neither depends on the ghosts, anti-ghosts nor on the momenta of both C_a and $\bar{C}_a$. This concludes our brief outline of BFV technique based on Hamiltonian formalism for gauge theories with constraints.

2.3 Finite Field BRST Transformation

The usual BRST transformation for the generic fields ϕ of an effective theory is defined compactly as

$$\delta_b \phi = s_b \phi \Lambda, \tag{2.44}$$

where $s_b\phi$ is the BRST variation of the fields with infinitesimal, anti-commuting and global parameter Λ. Such transformations are on-shell nilpotent, i.e. $s_b^2 = 0$, with the use of some equations of motion for fields and leaves the Fadeev-Popov (FP) effective action invariant. Joglekar and Mandal [41] observed that Λ needs neither to be infinitesimal, nor to be field-independent to maintain the symmetry of the FP effective action of the theory as long as it is anti-commuting and does not depend explicitly on space and time. This observation enabled them to propose finite field-dependent BRST transformation which can be written as

$$\delta_b \phi = s_b \phi \Theta_b[\phi], \tag{2.45}$$

where $\Theta_b[\phi]$ is an x-independent functional of fields ϕ. These transformations are also symmetry of FP effective action. Even though FFBRST transformations are symmetry of the effective action, the path integral measure and hence the generating functional are not invariant under such finite transformations. We briefly mention the important steps to construct FFBRST transformation. We start with the field, $\phi(x, \kappa)$, which is made to depend on some parameter, $\kappa : 0 \leq \kappa \leq 1$, in such a manner that $\phi(x, \kappa = 0) = \phi(x)$ is the initial field and $\phi(x, \kappa = 1) = \phi'(x)$ is the transformed field. The infinitesimal parameter Λ in the BRST transformation is made field dependent and hence the BRST transformation can be written as

$$\frac{d}{d\kappa}\phi(x, \kappa) = s_b \phi(x, \kappa) \Theta'_b[\phi(x, \kappa)], \tag{2.46}$$

where Θ'_b is an infinitesimal field dependent parameter. By integrating these equations from $\kappa = 0$ to $\kappa = 1$, it has been shown [41] that the $\phi'(x)$ are related to $\phi(x)$ by the FFBRST transformation as

$$\phi'(x) = \phi(x) + s_b\phi(x)\Theta_b[\phi(x)], \tag{2.47}$$

where $\Theta_b[\phi(x)]$ is obtained from $\Theta'_b[\phi(x)]$ through the relation [41]

$$\Theta_b[\phi(x,\kappa)] = \Theta'_b[\phi(x,0)]\frac{\exp f[\phi(x,0)] - 1}{f[\phi(x,0)]}, \tag{2.48}$$

where $f[\phi]$ is written as

$$f[\phi] = \sum_i \frac{\delta\Theta'_b(x)}{\delta\phi_i(x)} s_b\phi_i(x) \tag{2.49}$$

This transformation is nilpotent and symmetry of the effective action.

The Jacobian for finite BRST transformations can be evaluated for some specific value of $\Theta'[\phi]$ using the fact that the Jacobian can be written as a succession of infinitesimal transformations of Eq.(2.47).

Now, the path integral measure is defined as

$$\mathcal{D}\phi = J(\kappa)\mathcal{D}\phi(\kappa) = J(\kappa + d\kappa)\mathcal{D}\phi(\kappa + d\kappa) \tag{2.50}$$

Since the transformation $\phi(\kappa)$ to $\phi(\kappa + d\kappa)$ is an infinitesimal one, then the equation reduces to

$$\frac{J(\kappa)}{J(\kappa + d\kappa)} = \int d^4x \sum_\phi \pm \frac{\delta\phi(x,\kappa + d\kappa)}{\delta\phi(x,\kappa)}, \tag{2.51}$$

where $\sum_\phi$ sums over all fields in the measure and $\pm$ refers to whether ϕ is bosonic or fermionic field. Using the Taylor's expansion in the above equation, the expression for infinitesimal change in Jacobian is obtained as follows:

$$\frac{1}{J(\kappa)}\frac{dJ(\kappa)}{d\kappa} = -\int d^4x \sum_{\phi}\left[\pm s_b\phi \frac{\delta\Theta_b'[\phi(x,\kappa)]}{\delta\phi(x,\kappa)}\right] \tag{2.52}$$

The nontrivial Jacobian is the source of new results in the FFBRST formulation.

The generating functional is defined as

$$Z = \int [\mathcal{D}\phi] e^{iS_{eff}} \tag{2.53}$$

It is not invariant under such FFBRST transformation as the Jacobian is not invariant under this transformation as in Eq.(2.50). It has been shown [41] that under certain condition this nontrivial Jacobian can be replaced (within the functional integral) as

$$J[\phi(\kappa)] \rightarrow e^{iS_1[\phi(\kappa)]}, \tag{2.54}$$

where $S_1[\phi(\kappa)]$ is some local functional of $\phi(x)$. The condition for existence of S_1 is

$$\int [\mathcal{D}\phi]\left[\frac{1}{J}\frac{dJ}{d\kappa} - i\frac{dS_1}{d\kappa}\right] \exp i[S_{eff} + S_1] = 0. \tag{2.55}$$

Thus,

$$Z\left(= \int [\mathcal{D}\phi] e^{iS_{eff}}\right) \xrightarrow{FFBRST} Z'\left(= \int [\mathcal{D}\phi] e^{i[S_{eff}(\phi)+S_1(\phi)]}\right). \tag{2.56}$$

$S_1[\phi]$ depends on the finite field dependent parameter. Therefore, the generating functional corresponding to the two different effective theories can be connected through FFBRST transformation with appropriate choices of finite field dependent parameters. The FFBRST transformation has also been used to solve many of the long outstanding problems in quantum field theory. For example, the gauge field propagators in non-covariant gauges contain singularities on the real momentum axis. Proper prescriptions for these singularities in gauge field propagators have been found by using FFBRST transformation [42, 43]. These transformations have been used to establish relation between first class constraint theories to second class constraint theories [69]. These symmetries has been explicitly constructed for pure gauge theories [70]. These symmetries have been studied in both Lagrangian and Hamiltonian formalisms [71, 79]. These symmetry transformations have found applications in many other theories like Chern-Simons theory, BLG theory, ABJM theory, QCD, generalized QED etc. [72, 73, 74, 75, 76, 77, 78, 194, 195].

2.3.1 Conclusion

In this chapter we have provided the basic mathematical techniques which are relevant to the later part of the thesis. First we have discussed Lagrangian formulation also called field/anti-field formalism or BV formalism. After that we have discussed Hamiltonian formulation in which we have discussed about Dirac's constraints analysis and then BFFT formulation for conversion of second class constraints to first class constraints. Further we have discussed about BFV formulation or Hamiltonian formulation for construction of BRST transformation. At last, we have discussed FFBRST transformation.

In the next chapter we will discuss about our first work "Maximal Abelian Gauge and Generalised BRST transformation" [111] in which we will talk about Lorenz and MA gauge. Then we will establish a connection between them. We will also discuss how FFBRST transformation will help us study quark confinement in Lorenz gauge.

Chapter 3

Abelian Projection in Lorenz Gauge and FFBRST transformations

3.1 Introduction

In the previous two chapters we have discussed the introduction and basic mathematical preliminaries. Now we are in position to present the research work carried out in this thesis. The first problem we will discuss about is Maximal Abelian (MA) gauge in Yang-Mills theory and its connection with Lorenz gauge through generalized BRST transformation. It is shown that the generating functional corresponding to the Faddeev-Popov (FP) effective action in the MA gauge can be obtained from that in the Lorenz gauge by carrying out an appropriate finite and field-dependent BRST (FFBRST) transformation. The present FFBRST formulation might be useful to see how Abelian dominance in the MA gauge is realized in the Lorenz gauge.

3.1.1 Maximal Abelian Gauge

In $SU(N)$ YM theory, the MA gauge has been exploited to investigate its non-perturbative features, such as quark confinement [95]. The MA gauge is a nonlinear gauge for a partial gauge fixing, imposed to maintain only the maximal Abelian gauge symmetry specified by $U(1)^{N-1}$. This gauge enables us to extract Abelian degrees of freedom latent in $SU(N)$ YM theory. In fact, in the MA gauge, Abelian dominance [89, 96, 97, 98, 99] and the emergence of magnetic monopoles [82, 83, 84, 90] are realized as remarkable phenomena in the non-perturbative infrared region. The quark confinement is well explicated in $SU(N)$ YM theory formulated in the MA gauge.

The MA gauge condition is a nonlinear gauge condition and is defined as [82]

$$\nabla^\mu A^i_\mu \equiv \partial^\mu A^i_\mu + g\epsilon^{ij} A^{\mu 3} A^j_\mu = 0 \tag{3.1}$$

This condition partially breaks the SU(2) gauge invariance of the YM action so as to maintain its gauge invariance under the $U(1)$ gauge transformation as given below

$$\begin{aligned} \delta_3 A^i_\mu &= -g\epsilon^{ij} A^j_\mu \lambda^3 \\ \delta_3 A^3_\mu &= \partial_\mu \lambda^3 \end{aligned} \tag{3.2}$$

The MA gauge condition (3.1) can be incorporated into the following gauge-fixing and FP ghost term in a BRST and anti-BRST invariant manner [91, 93, 94]

$$S_{MA} = \int d^4x \big[- s\bar{s}(\frac{1}{2} A^i_\mu A^{\mu i} + \frac{\beta}{2} c^i \bar{c}^i) \big] \tag{3.3}$$

where β is a gauge fixing parameter. The generalized MA gauge condition can be written as

$$\nabla^\mu A^i_\mu - \beta B^i - \beta g \epsilon^{ij} \bar{c}^j c^3 = 0 \tag{3.4}$$

where $c, \bar{c}$ are ghost and anti-ghost fields and B is an auxiliary field. $\beta = 0$ gives the original MA gauge condition. The $U(1)$ gauge transformation rules for the fields B^i, c^i and $\bar{c}^i$ are written as

$$
\begin{aligned}
\delta_3 B^i &= -g\epsilon^{ij} B^j \lambda^3 \\
\delta_3 c^i &= -g\epsilon^{ij} c^j \lambda^3 \\
\delta_3 \bar{c}^i &= -g\epsilon^{ij} \bar{c}^j \lambda^3
\end{aligned}
\tag{3.5}
$$

3.1.2 Lorenz Gauge

The Lorenz gauge condition $\partial^\mu A^a_\mu = 0$ [106] can be used to completely break the $SU(2)$ gauge invariance of the YM action. This gauge condition can be incorporated into the following gauge-fixing and FP ghost term in a BRST and anti-BRST invariant manner [12, 107, 108]

$$
S_L = \int d^4x \big[- s\bar{s}(\frac{1}{2} A^a_\mu A^{\mu a} + \frac{\alpha}{2} c^a \bar{c}^a) \big] \tag{3.6}
$$

where α is a gauge fixing parameter. The generalized Lorenz gauge condition can be written as

$$
\nabla^\mu A^a_\mu - \alpha B^a - \frac{\alpha}{2} g \epsilon^{abc} c^b \bar{c}^c = 0 \tag{3.7}
$$

When $\alpha = 0$, the gauge condition (3.7) reduces to the (original) Lorenz gauge condition.

3.1.3 BRST Symmetry

The total action is defined as

$$
S_T = \int d^2x (\mathcal{L}_0 + \mathcal{L}_{gf} + \mathcal{L}_{gh}) \tag{3.8}
$$

where $\mathcal{L}_0$ is the kinetic part of the total Lagrangian density.

The total action is invariant under following BRST transformation [5, 6]

$$\begin{aligned} s_B A_\mu^a &= -\delta\Lambda D_\mu c^a \\ s_B c^a &= -\frac{1}{2}\delta\Lambda g\epsilon^{abc}c^b c^c \\ s_B \bar{c}^a &= \delta\Lambda B^a \\ s_B B^a &= 0 \end{aligned} \tag{3.9}$$

Corresponding anti-BRST transformation under which the total action is invariant is written as

$$\begin{aligned} \bar{s}_B A_\mu^a &= -\delta\Lambda D_\mu \bar{c}^a \\ \bar{s}_B c^a &= \delta\Lambda(-B^a - g\epsilon^{abc}c^b\bar{c}^c) \\ \bar{s}_B \bar{c}^a &= -\frac{1}{2}\delta\Lambda g\epsilon^{abc}\bar{c}^b\bar{c}^c \\ \bar{s}_B B^a &= \delta\Lambda g\epsilon^{abc}B^b\bar{c}^c \end{aligned} \tag{3.10}$$

3.1.4 Connecting Generating Functionals in MA Gauge to Lorenz Gauge

In this section, we construct the FFBRST transformation with an appropriate finite parameter to obtain the generating functional corresponding to S_{MA} from that corresponding to S_L.

We calculate the Jacobian corresponding to such a FFBRST transformation following the method outlined in chapter 2 and show that it is a local functional of fields and accounts for the difference of the two FP effective actions. The generating functional corresponding to the FP effective action S_L is written as

$$Z_L = \int D\phi \exp(iS_L[\phi]) \tag{3.11}$$

The finite field dependent parameter corresponding to the Lorenz gauge is obtained from the infinitesimal but field dependent parameter, $\Theta'[\phi(k)]$; through $\int_0^\kappa \Theta'[\phi(\kappa)]d\kappa$. We construct $\Theta'[\phi(\kappa)]$ as,

$$\begin{aligned}\Theta'[\phi(k)] &= i\int d^4x[\gamma_1\bar{c}^iB^i+\gamma_2\bar{c}^3B^3+\gamma_3\{\bar{c}^as(\partial^\mu A^a_\mu)-\bar{c}^is(\nabla^\mu A^i_\mu)\} \\ &\quad +\gamma_4 g\epsilon^{abc}\bar{c}^a\bar{c}^bc^c+\gamma_5\epsilon^{ij}\bar{c}^i\bar{c}^jc^3] \qquad (3.12)\end{aligned}$$

Here, $\gamma_p(p=1,2,3,4,5)$ are arbitrary constant parameters and all the fields depend on the parameter k. The infinitesimal change in the Jacobian corresponding to this FFBRST transformation is calculated using Eq.(2.52) as,

$$\begin{aligned}\frac{1}{J}\frac{dJ}{dk} &= -i\int d^4x[-\gamma_1B^iB^i-\gamma_2B^3B^3+\gamma_3\{\bar{c}^as(\partial^\mu A^a_\mu)barc^is(\nabla^\mu A^i_\mu)\} \\ &\quad -\gamma_3B^a\partial^\mu A^a_\mu+\gamma_3B^i\nabla^\mu A^i_\mu+\gamma_4\{-2g\epsilon^{abc}B^a\bar{c}^bc^c+\frac{1}{2}g\epsilon^{abc}\epsilon^{ade}\bar{c}^b\bar{c}^cc^dc^e\} \\ &\quad +\gamma_5\{-2\epsilon^{ij}B^i\bar{c}^jc^3+\frac{1}{2}\epsilon^{ij}\epsilon^{kl}\bar{c}^i\bar{c}^jc^kc^l\}] \qquad (3.13)\end{aligned}$$

To express the Jacobian contribution in terms of a local functional of fields, we make an ansatz for S_1 by considering all possible terms that could arise from such a transformation as

$$\begin{aligned}S_1[\phi(k),k] &= \int d^4x[\xi_1B_a\partial^\mu A^a_\mu+\xi_2B^i\nabla^\mu A^i_\mu+\xi_3B^aB^a+\xi_4B^iB^i \\ &\quad +\xi_5\bar{c}^is(\nabla^\mu A^i_\mu)+\xi_6\bar{c}^as(\partial^\mu A^a_\mu)+\xi_7g\epsilon^{abc}B^ac^b\bar{c}^c \\ &\quad +\xi_8g^2\epsilon^{abc}\epsilon^{ade}\bar{c}^b\bar{c}^cc^dc^e+\xi_9g\epsilon^{ij}B^i\bar{c}^jc^3 \\ &\quad +\xi_{10}g^2\epsilon^{ij}\epsilon^{kl}\bar{c}^i\bar{c}^jc^kc^l] \qquad (3.14)\end{aligned}$$

where all the fields are considered to be k dependent and we have introduced arbitrary k dependent parameters $\xi_n=\xi_n(k)(n=1,2,.....,10)$. It is straight to calculate

$$
\begin{aligned}
\frac{dS_1}{dk} &= \int d^4x\{\xi_1' B_a \partial^\mu A_\mu^a + \xi_2' B^i \nabla^\mu A_\mu^i + \xi_3' B^{a2} + \xi_4' B^{i^2} + \xi_5' \bar{c}^i s(\nabla^\mu A_\mu^i) \\
&+\xi_6' \bar{c}^a s(\partial^\mu A_\mu^a) + \xi_7' g\epsilon^{abc} B^a c^b \bar{c}^c + \xi_8' g^2 \epsilon^{abc}\epsilon^{ade}\bar{c}^b\bar{c}^c c^d c^e + \xi_9' g\epsilon^{ij} B^i \bar{c}^j c^3 \\
&+\xi_{10}' g^2 \epsilon^{ij}\epsilon^{kl}\bar{c}^i c^j c^k c^l + \{\xi_1 B_a s(\partial^\mu A_\mu^a) + \xi_2 B^i s(\nabla^\mu A_\mu^i) + \xi_5 B^i s(\nabla^\mu A_\mu^i) \\
&+\xi_6 B^a s(\partial^\mu A_\mu^a) - \xi_7 g\epsilon^{abc} B^a \epsilon^{bde} c^d c^e \bar{c}^c + 2\xi_8 g^2 \epsilon^{abc}\epsilon^{ade} B^b \bar{c}^c c^d c^e \\
&+\xi_9 \frac{g^2}{2}\epsilon^{ij} B^i \bar{c}^j \epsilon^{kl} c^k c^l + 2\xi_{10} g^2 \epsilon^{ij}\epsilon^{kl} B^i \bar{c}^j c^k c^l\}\Theta'\} \qquad (3.15)
\end{aligned}
$$

where $\xi' = \frac{d\xi}{dk}$. Now using the condition of Eq.(2.55), we can write

$$
\begin{aligned}
&\int D\phi(k) \exp\left[i(S_L([\phi(k)] + S_1[\phi(k), k)]\right] \int d^4x[(-\gamma_1 + \xi_3' + \xi_4') B^i B^i \\
&+(-\gamma_2 + \xi_3') B^3 B^3 + (\gamma_3 + \xi_6')\bar{c}^a s(\partial^\mu A_\mu^a) + (-\gamma_3 + \xi_5')\bar{c}^i s(\nabla^\mu A_\mu^i) \\
&+(-\gamma_3 + \xi_1') B^a \partial^\mu A_\mu^a + (\gamma_3 + \xi_2') B^i \nabla^\mu A_\mu^i + (-2\gamma_4 + \xi_7') g(\epsilon^{ij} B^i c^j \bar{c}^3 \\
&+\epsilon^{ij} B^3 c^i \bar{c}^j) + (-2\gamma_4 - 2\gamma_5 + \xi_7' + \xi_9') g\epsilon^{ij} B^i \bar{c}^j c^3 + \frac{1}{2}(\gamma_4 + \gamma_5 \\
&+2\xi_8' + 2\xi_{10}')\epsilon^{ij}\epsilon^{kl}\bar{c}^i\bar{c}^j c^k c^l + 2(\gamma_4 + 2\xi_8') g^2 \epsilon^{ij}\epsilon^{lk}\bar{c}^j\bar{c}^3 c^k c^3 \\
&+\Theta'\{(\xi_1 + \xi_6) B_a s\partial^\mu A_\mu^a + (\xi_2 + \xi_5) B^i s\nabla^\mu A_\mu^i - \frac{1}{2}(\xi_7 + 4g\xi_8 \\
&+\xi_9 + 4\xi_{10})\epsilon^{abc} B^a \epsilon^{bde} c^d c^e \bar{c}^c + (\xi_7 + 4\xi_8) g^2(\epsilon^{ij}\epsilon^{ik} B^j \bar{c}^3 c^k c^3 \\
&-\epsilon^{ij}\epsilon^{ik} B^3 \bar{c}^j c^k c^3\}] = 0 \qquad (3.16)
\end{aligned}
$$

The terms proportional to Θ' which are regarded as nonlocal, vanishes independently. These equations will give relations between ξ. Making remaining local terms in Eq.(3.16) vanish will give relations between ξ and γ. Solving these Eqs.(A.7,A.8,A.9), we will get following results.

The differential equations for $\xi_n(k)$ can indeed be solved with the initial conditions $\xi_n(0) = 0$, to obtain the solutions

$$
\begin{aligned}
\xi &= \gamma_3 k, \quad \xi_2 = -\gamma_3 k, \quad \xi_3 = \gamma_2 k, \\
\xi_4 &= (\gamma_1 - \gamma_2)k, \quad \xi_5 = \gamma_3 k, \quad \xi_6 = -\gamma_3 k \\
\xi_7 &= 2\gamma_4 k, \quad \xi_8 = -\frac{1}{2}\gamma_4 k, \quad \xi_9 = 2\gamma_5 k \\
\xi_{10} &= -\frac{1}{2}\gamma_5 k
\end{aligned}
\tag{3.17}
$$

Since $\gamma_p (p = 1, 2, 3, 4, 5)$ are arbitrary constant parameters, we can choose them as follows

$$
\begin{aligned}
\gamma_1 &= \frac{1}{2}(\beta - \alpha), \quad \gamma_2 = -\frac{\alpha}{2}, \gamma_3 = 1 \\
\gamma_4 &= -\frac{\alpha}{4}, \quad \gamma_5 = \frac{\beta}{2}
\end{aligned}
\tag{3.18}
$$

Substituting the solutions found in Eq.(3.17) into Eq.(3.14) and considering the specific values of the parameters in Eq.(3.18), we obtain

$$
\begin{aligned}
S_1 = & \int d^4x \big[B_a \partial^\mu A^a_\mu - B^i \nabla^\mu A^i_\mu - \frac{\alpha}{2} B^{a2} + \frac{\beta}{2} B^{i2} + \bar{c}^i \delta(\nabla^\mu A^i_\mu) \\
& - \bar{c}^a \delta(\partial^\mu A^a_\mu) - \frac{\alpha}{2} g \epsilon^{abc} B^a c^b \bar{c}^c + \frac{\alpha}{8} g^2 \epsilon^{abc} \epsilon^{ade} \bar{c}^b \bar{c}^c c^d c^e + \beta g \epsilon^{ij} B^i \bar{c}^j c^3 \\
& - \frac{\beta}{4} g^2 \epsilon^{ij} \epsilon^{kl} \bar{c}^i \bar{c}^j c^k c^l \big]
\end{aligned}
\tag{3.19}
$$

Now the new generating functional is written as in Eq.(2.56)

$$
\begin{aligned}
Z_L &= \int D\phi \exp(iS_L[\phi]) \rightarrow \int D\phi' \exp\{i(S_L[\phi'] + S_1[\phi', 1])\} \\
&= \int D\phi \exp\{i(S_L[\phi] + S_1[\phi, 1])\} \\
&= \int D\phi \exp(iS_{MA}[\phi]) = Z_{MA}
\end{aligned}
\tag{3.20}
$$

In this way, the suitably constructed FFBRST transformation maps $SU(2)$ YM theory in the Lorenz gauge to that in the MA gauge.

3.1.5 Conclusion

We have applied the FFBRST formulation discussed in chapter 2 to clarify the connection between the gauge fixed $SU(2)$ YM theory formulated in the Lorenz and MA gauges. We have explicitly shown that the generating functional corresponding to the FP effective action in the MA gauge can be obtained from that in the Lorenz gauge by carrying out a suitably constructed FFBRST transformation. In this procedure, the FP effective action in the MA gauge is found from that in the Lorenz gauge by taking into account the non-trivial Jacobian arising from the FFBRST transformation in the path integral measure.

In the next chapter we will discuss about various nilpotent symmetries for particle on torus [137]. We will show that these symmetries follow the Hodge algebra and the system follows double Hodge theory. We will also show connection between various gauge conditions for torus system using FFBRST transformation. Then we will discuss BRST and anti-BRST symmetries for particle on a torus knot [147]. We will also show connection between various gauges for torus knot system through FFBRST transformation. At last we will construct BRST and anti-BRST symmetries for particle on torus knot [149] using the technique discussed in ref. [148].

Chapter 4

Torus and Torus Knot: Various Nilpotent Symmetries

In the previous chapter we have observed that YM action in MA gauge can be connected to that in Lorenz gauge through FFBRST transformation. Now in this chapter we are going to extend BRST symmetry and formulate FFBRST symmetry for torus and torus knot system. We investigate all possible nilpotent symmetries for a particle on torus. We explicitly construct four independent nilpotent BRST symmetries for such systems and derive the algebra between the generators of such symmetries. We develop BRST symmetry for the first time for a particle on the surface of a torus knot by analyzing the constraints of the system. The theory contains second class constraints and has been extended by introducing the Wess-Zumino term to convert it into a theory with first-class constraints. BFV analysis of the extended theory is performed to construct BRST/anti-BRST symmetries for the particle on a torus knot. We further construct the FFBRST transformation for such systems by integrating the infinitesimal BRST transformation systematically.

4.1 Double Hodge Theory for a Particle on Torus

Toric geometry which is generalization of the projective identification that defines CP^n corresponding to the most general linear sigma model provides a scheme for constructing Calabi-Yau manifolds and their mirrors [113]. Recently, on the basis of boundary string field theory [114], the brane-antibrane system was exploited [115] in the toroidal background to investigate its thermodynamic properties associated with the Hagedorn temperature [116, 117]. The Nahm transform and moduli spaces of CP^n models were also studied on the toric geometry [118]. In a four-dimensional, toroidally compactified heterotic string, the electrically charged BPS-saturated states were shown to become massless along the hyper surfaces of enhanced gauge symmetry of a two-torus moduli subspace [119].

In the present work we investigate various possible nilpotent symmetries for a particle on torus. Usual BRST symmetry for a particle on torus has already been constructed [120]. In this work we construct four different nilpotent symmetries associated with this system, namely, BRST symmetry, anti-BRST symmetry, dual BRST (also known as co-BRST) symmetry and anti-dual BRST (also known as anti-co-BRST) symmetry [36, 39]. We further construct two different bosonic symmetries using these nilpotent BRST symmetries. Some discrete symmetries associated with ghost number are also written for such systems. Complete algebra satisfied by charges, which generate these symmetries, is derived. Deep mathematical connections of such system with Hodge theory [121, 122, 123, 124] has been established in this work. We found that the system of particle on a torus is realized as Hodge theory with respect to two different sets of operators. The generators for BRST, dual BRST symmetries, and generator for corresponding bosonic symmetries constructed out of BRST and dual BRST symmetries are analogous to exterior derivative, coexterior derivative, and Laplace operator in Hodge theory [64, 125, 126, 127, 128, 129, 130, 131, 132]. On the other hand the charges corresponding to anti-BRST symmetry, anti-dual BRST symmetry and bosonic symmetry constructed out of these two BRST symmetries also form set of de-Rham cohomological operators.

4.1.1 Free Particle on Surface of Torus

A particle moving freely on the surface of a torus is described by Lagrangian [120]:

$$L_0 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m(b + r\sin\theta)^2\dot{\phi}^2 \tag{4.1}$$

where (r, θ, ϕ) are toroidal coordinates related to Cartesian coordinates as

$$x = (b + r\sin\theta)\cos\phi, \quad y = (b + r\sin\theta)\sin\phi, \quad z = r\cos\theta \tag{4.2}$$

Here we have considered a torus with axial circle in the $x - y$ plane centered at the origin, of radius b, having a circular cross section of radius r. The angle θ ranges from $-\pi$ to π and the angle ϕ from 0 to 2π. Since the particle moves on the surface of torus of radius r, it is constrained to satisfy

$$\Omega_1 = r - a \approx 0 \tag{4.3}$$

The canonical Hamiltonian corresponding to the Lagrangian in Eq.(4.1) with the above constraint is then written as

$$H_0 = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2m(b + r\sin\theta)^2} + \lambda(r - a) \tag{4.4}$$

where p_r, p_θ and p_ϕ are the canonical momenta conjugate to the coordinate r, θ and ϕ, respectively, given by

$$p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta}, \quad p_\phi = m(b + r\sin\theta)^2\dot{\phi} \tag{4.5}$$

From Eq.(2.20), the time evolution of the constraint Ω_1 yields the secondary constraint as

$$\Omega_2 = p_r \approx 0 \tag{4.6}$$

4.1.2 Wess-Zumino term and Hamiltonian formation

To construct a gauge invariant theory corresponding to the gauge non-invariant model in Eq.(4.4), we introduce the Wess-Zumino (WZ) term [132] in the Lagrangian density $\mathcal{L}$. For this purpose we enlarge the Hilbert space of the theory by introducing a new quantum field η, called as WZ field, through the redefinition of fields r and λ in the original Lagrangian density $\mathcal{L}$ as follows

$$r \to r - \eta; \quad \lambda \to \lambda + \dot{\eta} \tag{4.7}$$

With this redefinition of the fields, the modified Lagrangian density becomes

$$\begin{aligned} \mathcal{L}^{\mathcal{I}} &= \frac{1}{2}m(\dot{r}-\dot{\eta})^2 + \frac{1}{2}m(r-\eta)^2\dot{\theta}^2 + \frac{1}{2}m(b+(r-\eta)\sin\theta)^2\dot{\phi}^2 \\ &- (\lambda+\dot{\eta})(r-a-\eta) \end{aligned} \tag{4.8}$$

Canonical momenta corresponding to this modified Lagrangian density are then given by

$$\begin{aligned} p_r &= m(\dot{r}-\dot{\eta}), \quad p_\eta = -(m(\dot{r}-\dot{\eta}) + (r-a-\eta)), \quad p_\lambda = 0 \\ p_\theta &= m(r-\eta)^2\dot{\theta}, \quad p_\phi = m(b+(r-\eta)\sin\theta)^2\dot{\phi} \end{aligned} \tag{4.9}$$

The primary constraints for this extended theory is

$$\psi_1 \equiv p_\lambda \approx 0 \tag{4.10}$$

The Hamiltonian density corresponding to $\mathcal{L}^{\mathcal{I}}$ is written as,

$$H^I = p_r\dot{r} + p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + p_\lambda\dot{\lambda} - \mathcal{L}^{\mathcal{I}} \tag{4.11}$$

The total Hamiltonian density after the introduction of a Lagrange multiplier field u corresponding to the primary constraint ψ_1 is then obtained as

$$H_T^I = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m(r-\eta)^2} + \frac{p_\phi^2}{2m(b+(r-\eta)\sin\theta)^2} + \lambda(p_r + p_\eta) + up_\lambda \quad (4.12)$$

Following the Dirac's method of constraint analysis [22], we obtain secondary constraint as in Eq.(2.20),

$$\psi_2 \equiv (p_\eta + p_r) \approx 0 \quad (4.13)$$

4.1.3 BFV Formulation for free Particle on the Surface of Torus

To discuss all possible nilpotent symmetries we further extend the theory using BFV formalism [18, 19, 20, 79, 134, 135]. In the BFV formulation associated with this system, we introduce a pair of canonically conjugate ghost fields (c,p) with ghost number 1 and -1 respectively, for the primary constraint $p_\lambda \approx 0$ and another pair of ghost fields $(\bar{c}, \bar{p})$ with ghost number -1 and 1 respectively, for the secondary constraint, $(p_\eta + p_r) \approx 0$. The effective action for a particle on the surface of torus in extended phase space is then written as in Eq.(2.41)

$$\begin{aligned} S_{eff} &= \int d^4x \big[p_r\dot{r} + p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - p_\lambda\dot{\lambda} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2m(r-\eta)^2} \\ &\quad - \frac{p_\phi^2}{2m(b+(r-\eta)\sin\theta)^2} + \dot{c}p + \dot{\bar{c}}\bar{p} - \{Q_b, \psi\}\big] \quad (4.14) \end{aligned}$$

where Q_b is the BRST charge and ψ is the gauge-fixed fermion. This effective action is invariant under BRST transformation generated by Q_b which is constructed using constraints in the theory as in Eq.(2.43),

$$Q_b = ic(p_r + p_\eta) - i\bar{p}p_\lambda \quad (4.15)$$

The canonical brackets for all dynamical variables are written as

$$[r, p_r] = [\theta, p_\theta] = [\phi, p_\phi] = [\eta, p_\eta] = [\lambda, p_\lambda] = \{\bar{c}, \dot{c}\} = i, \{c, \dot{\bar{c}}\} = -i \quad (4.16)$$

Rest of the brackets are zero. Now, the nilpotent BRST transformation, using the relation $s_b\phi = [\phi, Q_b]_\pm$ ($\pm$ sign represents the fermionic and bosonic nature of the fields ϕ), are explicitly written as

$$\begin{aligned} s_b r &= -c, \quad s_b\lambda = \bar{p}, \quad s_b\bar{p} = 0, \quad s_b\theta = -c \\ s_b p_\phi &= 0, \quad s_b p_\theta = 0, \quad s_b p = (p_r + p_\eta) \\ s_b\bar{c} &= p_\lambda, \quad s_b p_\lambda = 0, \quad s_b c = 0 \end{aligned} \quad (4.17)$$

In BFV formulation the generating functional is independent of gauge-fixed fermion [18, 19, 20, 79, 134, 135], hence we have liberty to choose it in the convenient form as in Eq.(2.43)

$$\psi = p\lambda + \bar{c}(r + \eta + \frac{p_\lambda}{2}) \quad (4.18)$$

Putting the value of ψ in Eq.(4.14) and using Eq.(4.15) and Eq.(4.16), we obtain

$$\begin{aligned} S_{eff} = & \int d^4x \big[p_r\dot{r} + p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - p_\lambda\dot{\lambda} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2m(r-\eta)^2} \\ & - \frac{p_\phi^2}{2m(b + (r-\eta)\sin\theta)^2} + \dot{c}p + \dot{\bar{c}}\bar{p} + \lambda(p_r + p_\eta) + 2c\bar{c} - \bar{p}p \\ & + p_\lambda(r + \eta + \frac{p_\lambda}{2})\big] \end{aligned} \quad (4.19)$$

and the generating functional for this effective theory is represented as

$$Z_\psi = \int D\phi \exp\left[iS_{eff}\right] \quad (4.20)$$

Now integrating this generating functional over p, $\bar{p}$ and p_λ, we get

$$\begin{aligned} Z_\psi &= \int D\phi'' \exp\big[i\int d^4x\big[p_r\dot{r} + p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{p_r^2}{2m} \\ &\quad -\frac{p_\theta^2}{2m(r-\eta)^2} - \frac{p_\phi^2}{2m(b+(r-\eta)\sin\theta)^2} + \dot{\bar{c}}\dot{c} + \lambda(p_r + p_\eta) \\ &\quad -2\bar{c}c - \frac{(\dot{\lambda} - r - \eta)^2}{2}\big]\big] \end{aligned} \tag{4.21}$$

where $D\phi''$ is the path integral measure corresponding to all the dynamical variables involved in the effective action. The BRST transformation for this effective theory is written as

$$\begin{aligned} s_b r &= -c, \quad s_b\lambda = \dot{c}, \quad s_b\eta = -c \\ s_b p_r &= 0, \quad s_b p_\eta = 0 \\ s_b\bar{c} &= -(\dot{\lambda} - \eta - r), \quad s_b c = 0 \end{aligned} \tag{4.22}$$

These transformations are on shell nilpotent.

4.1.4 Nilpotent Symmetries

In this section we will study various other nilpotent symmetries of this model [136]. For this purpose it is convenient to work using Nakanishi-Lautrup type auxiliary field B which linearizes the gauge-fixing part of the effective action in Eq.(4.21). The first order effective action is then given by

$$\begin{aligned} S_{eff} &= \int d^4x\big[p_r\dot{r} + p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2m(r-\eta)^2} \\ &- \frac{p_\phi^2}{2m(b+(r-\eta)\sin\theta)^2} + \dot{\bar{c}}\dot{c} + \lambda(p_r + p_\eta) - 2\bar{c}c \\ &- B(\dot{\lambda} - r - \eta) + \frac{B^2}{2}\big] \end{aligned} \tag{4.23}$$

We can easily show that this action is invariant under the following off-shell nilpotent BRST transformation

$$\begin{aligned} s_b r &= -c, \quad s_b\lambda = \dot{c}, \quad s_b\eta = -c \\ s_b p_r &= 0, \quad s_b p_\eta = 0, \quad s_b\theta = 0 \\ s_b\bar{c} &= B, \quad s_b\bar{c} = 0, s_b p_\phi = 0 \\ s_b\phi &= 0, \quad s_b p_\theta = 0 \end{aligned} \tag{4.24}$$

Corresponding anti-BRST transformation for this theory is then written by interchanging the role of ghost and anti-ghost field as

$$\begin{aligned} s_{ab} r &= -\bar{c}, \quad s_{ab}\lambda = \dot{\bar{c}}, \quad s_{ab}\eta = -\bar{c} \\ s_{ab} p_r &= 0, \quad s_{ab} p_\eta = 0, \quad s_{ab} p_\phi = 0 \\ s_{ab} c &= -B, \quad s_{ab}\bar{c} = 0, \quad s_{ab}\theta = 0 \\ s_{ab}\phi &= 0, \quad s_{ab} p_\theta = 0 \end{aligned} \tag{4.25}$$

The conserved BRST and anti-BRST charges Q_b and Q_{ab} which generate above BRST and anti-BRST transformations are written for this effective theory as

$$Q_b = ic(p_r + p_\eta) - ip_\lambda \dot{c} \tag{4.26}$$

and

$$Q_{ab} = i\bar{c}(p_r + p_\eta) - ip_\lambda \dot{\bar{c}} \tag{4.27}$$

Further by using following equation of motion

$$\begin{aligned} B + \dot{p}_r &= 0, \quad B + \dot{p}_\eta = 0 \quad \dot{r} - p_r + \lambda = 0 \\ \dot{B} &= p_r + p_\eta, \quad \dot{\bar{c}} + 2\bar{c} = 0, \\ \dot{c} + 2c &= 0, \quad B + \dot{\lambda} - r - \eta = 0 \end{aligned} \tag{4.28}$$

it is shown that these charges are constants of motion i.e. $\dot{Q}_b = 0$, $\dot{Q}_{ab} = 0$, and satisfy following relations,

$$Q_b Q_{ab} + Q_{ab} Q_b = 0 \tag{4.29}$$

To arrive on these relations, the canonical brackets [Eq.(4.16)] of the fields and the definition of canonical momenta have been used

$$p_\lambda = B, \quad p_{\bar{c}} = \dot{c}, \quad p_c = -\dot{\bar{c}} \tag{4.30}$$

The physical states of theory are annihilated by the BRST and anti-BRST charges, leading to

$$(p_r + p_\eta)|phys\rangle = 0 \tag{4.31}$$

and

$$p_\lambda|phys\rangle = 0 \tag{4.32}$$

This implies that the operator form of the first class constraint $p_\lambda \approx 0$ and $(p_r + p_\eta) \approx 0$ annihilates the physical state of the theory. Thus the physicality criteria is consistent with Dirac's method of quantization.

4.1.5 Co-BRST and anti co-BRST symmetries

In this section, we investigate two other nilpotent transformations, namely co-BRST and anti co-BRST transformation which are also the symmetry of the effective action in Eq.(4.23). Further these transformations leave the gauge-fixing term of the action invariant independently and the kinetic term (which remains invariant under BRST and anti-BRST transformations) transforms under it to compensate for the transformation of the ghost terms. These transformations are also called as dual and anti dual-BRST transformations [36, 39].

The nilpotent co-BRST ($s_d^2 = 0$) and anti co-BRST transformations

($s_{ad}^2 = 0$) which leave the effective action [in Eq.(4.23)] for a particle on torus, invariant, are given by

$$
\begin{aligned}
s_d r &= -\frac{1}{2}\dot{\bar{c}}, \quad s_d \lambda = -\bar{c}, \quad s_d \eta = -\frac{1}{2}\dot{\bar{c}} \\
s_d p_r &= 0, , \quad s_d p_\eta = 0, \quad s_d \bar{c} = 0 \\
s_d c &= \frac{1}{2}(p_r + p_\eta), \quad s_d B = 0,
\end{aligned} \tag{4.33}
$$

and

$$
\begin{aligned}
s_{ad} r &= -\frac{1}{2}\dot{c}, \quad s_{ad} \lambda = -c, \quad s_{ad} \eta = -\frac{1}{2}\dot{c} \\
s_{ad} p_r &= 0, \quad s_{ad} p_\eta = 0, \quad s_{ad} \bar{c} = 0 \\
s_{ad} c &= -\frac{1}{2}(p_r + p_\eta), \quad s_{ad} B = 0
\end{aligned} \tag{4.34}
$$

These transformations are absolutely anti-commuting as $\{S_d, S_{ad}\} = 0$. The conserved charges for above symmetries are found using Noether's theorem and are written as

$$
Q_d = i\frac{1}{2}(p_r + p_\eta)\dot{\bar{c}} + ip_\lambda \bar{c} \tag{4.35}
$$

and

$$
Q_{ad} = i\frac{1}{2}(p_r + p_\eta)\dot{c} + ip_\lambda c \tag{4.36}
$$

which generate the symmetry transformations in Eq.(4.33) and Eq.(4.34) respectively. It is easy to verify the following relations

$$
\begin{aligned}
s_d Q_d &= -\{Q_d, Q_d\} = 0 \\
s_{ad} Q_{ad} &= -\{Q_{ad}, Q_{ad}\} = 0 \\
s_d Q_{ad} &= -\{Q_{ad}, Q_d\} = 0 \\
s_{ad} Q_d &= -\{Q_d, Q_{ad}\} = 0
\end{aligned} \tag{4.37}
$$

which reflect the nilpotency and anti-commutativity property of s_d and s_{ad} (i.e. $s_d^2 = 0$,$s_{ad}^2 = 0$ and $s_d s_{ad} + s_{ad} s_d = 0$).

4.1.6 Other Symmetries

In this section, we construct other symmetries related to this system. Two different bosonic symmetries are constructed out of four nilpotent symmetries. Discrete symmetry related to ghost number is also constructed.

4.1.7 Bosonic Symmetry

In this part we construct the bosonic symmetry out of these nilpotent BRST symmetries of the theory using ref. [134, 135]. The BRST (s_b), anti-BRST (s_{ab}), co-BRST (s_d), and anti co-BRST(s_{ad}) symmetry operators satisfy the following algebra

$$
\begin{aligned}
\{s_d, s_{ad}\} &= 0, \quad \{s_b, s_{ab}\} = 0 \\
\{s_b, s_{ad}\} &= 0, \quad \{s_d, s_{ab}\} = 0
\end{aligned} \tag{4.38}
$$

and we define bosonic symmetries, s_w and $s_{\bar{w}}$ as,

$$
s_w \equiv \{s_b, s_d\}, \quad s_{\bar{w}} \equiv \{s_{ab}, s_{ad}\} \tag{4.39}
$$

The fields variables transform under bosonic symmetry s_w as,

$$
\begin{aligned}
s_w r &= -\frac{1}{2}(\dot{B} + p_r + p_\eta), \quad s_w \lambda = -\frac{1}{2}(2B - \dot{p}_r - \dot{p}_\eta) \\
s_w \eta &= -\frac{1}{2}(\dot{B} + p_r + p_\eta), \quad s_w p_r = 0, \quad s_w p_\eta = 0 \\
s_w c &= 0, \quad s_w B = 0, \quad s_w \bar{c} = 0
\end{aligned} \tag{4.40}
$$

On the other hand transformation generated by $s_{\bar{w}}$ are,

$$
\begin{aligned}
s_{\bar{w}} r &= -\frac{1}{2}(\dot{B} + p_r + p_\eta), \quad s_{\bar{w}} \lambda = \frac{1}{2}(2B - \dot{p}_r - \dot{p}_\eta) \\
s_{\bar{w}} \eta &= -\frac{1}{2}(\dot{B} + p_r + p_\eta), \quad s_{\bar{w}} p_r = 0, \quad s_{\bar{w}} p_\eta = 0 \\
s_{\bar{w}} c &= 0, \quad s_{\bar{w}} B = 0, \quad s_{\bar{w}} \bar{c} = 0
\end{aligned} \tag{4.41}
$$

However the transformation generated by s_w and $s_{\bar{w}}$ are not independent as it is easy to see from Eq.(4.40) and Eq.(4.41) that the operators s_w and $s_{\bar{w}}$ satisfy the relation $s_w + s_{\bar{w}} = 0$. This implies from Eq.(4.39), that

$$\{s_b, s_d\} = s_w = -\{s_{ab}, s_{ad}\} \tag{4.42}$$

It is clear from above algebra that the operator s_w is analogous of the Laplacian operator in the language of differential geometry and the conserved charge for the above symmetry transformation is calculated as

$$Q_w = -i[B^2 + \frac{1}{2}(p_r + p_\eta)^2] \tag{4.43}$$

which generates the transformation in Eq.(4.40).

Using equation of motion, it can readily be checked that

$$\frac{dQ_w}{dt} = -i \int dx[2B\dot{B} + (p_r + p_\eta)(\dot{p}_r + \dot{p}_\eta)] = 0 \tag{4.44}$$

Hence Q_w is the constant of motion for this theory.

4.1.8 Ghost Symmetry and Discrete Symmetry

Now we consider yet another kind of symmetry of this system called ghost symmetry discussed in ref. [134]. The ghost numbers of the ghost and anti-ghost fields are 1 and -1 respectively. Rest of the variables in the action of this theory have ghost number zero. Keeping this fact in mind we can introduce a scale transformation of the ghost field, under which the effective action is invariant, as

$$\begin{aligned} c &\rightarrow e^{\Lambda} c \\ \bar{c} &\rightarrow e^{-\Lambda} \bar{c} \\ \chi &\rightarrow \chi \end{aligned} \tag{4.45}$$

where $\chi = \{r, \eta, \theta, \phi, u, \lambda, p_r, p_\eta, p_\theta, p_\phi, p_u, B\}$ and Λ is a global scale parameter. The infinitesimal version of the ghost scale transformation can be written as

$$\begin{aligned} s_g \chi &= 0 \\ s_g c &= c \\ s_g \bar{c} &= -\bar{c} \end{aligned} \tag{4.46}$$

The Noether's conserved charge for above symmetry transformation is calculated as

$$Q_g = i[\dot{\bar{c}} c + \dot{c} \bar{c}] \tag{4.47}$$

In addition to above continuous symmetry transformation, the ghost sector respects the following discrete symmetry transformations.

$$c \to \pm i \bar{c}, \quad \bar{c} \to \pm i c \tag{4.48}$$

4.1.9 Geometric Cohomology and Double Hodge Theory

In this section we study the de-Rham cohomological operators [122, 123, 124] and their realization in terms of conserved charges which generate the nilpotent symmetries for the theory of a particle on the surface of torus. In particular we point out the similarities between the algebra obeyed by de-Rham co-homological operators and that by different BRST conserved charges.

Before we proceed to discuss the analogy, we briefly review the essential features of Hodge theory [134, 135]. The de-Rham cohomological operators in differential geometry obey the following algebra

$$\begin{aligned} d^2 &= \delta^2 = 0, \quad \Delta = (d+\delta)^2 = d\delta + \delta d \equiv \{d, \delta\} \\ [\Delta, \delta] &= 0, \quad [\Delta, d] = 0 \end{aligned} \tag{4.49}$$

Where d, δ and Δ are exterior, co-exterior and Laplace-Beltrami operators respectively. The operator d and δ are adjoint or dual to each other and Δ is self-adjoint operator. It is well known that the exterior derivative raises the degree of form by one when it operates on the form ($i.e. df_n \sim f_{n+1}$), whereas the dual-exterior derivative lowers the degree of a form by one when it operates on the form ($i.e. \delta f_n \sim f_{n-1}$). However Δ does not change the degree of form ($i.e. \Delta f_n \sim f_n$). f_n denotes an arbitrary n-form object.

The Hodge-de-Rham decomposition theorem can be stated as follows.

A regular differential form of degree n(α) may be uniquely decomposed into a sum of the harmonic form $(\alpha)_H$, exact form (α_d) and co-exact form (α_δ) i.e.

$$\alpha = \alpha_H + \alpha_d + \alpha_\delta \tag{4.50}$$

where $\alpha \in H^n$, $\alpha_s \in \Lambda^n_\delta$ and $\alpha_d \in \Lambda^n_d$

The generators of all the nilpotent symmetry transformations satisfy the following algebra [134, 135]

$$\begin{aligned} Q_b^2 &= 0, \quad Q_{ab}^2 = 0, \quad Q_d^2 = 0, \quad Q_{ad}^2 = 0 \\ \{Q_b, Q_{ab}\} &= 0, \quad \{Q_d, Q_{ad}\} = 0, \quad \{Q_b, Q_{ad}\} = 0 \\ \{Q_d, Q_{ab}\} &= 0, \quad [Q_g, Q_b] = Q_b, \quad [Q_g, Q_{ad}] = Q_{ad} \\ [Q_g, Q_d] &= -Q_d, \quad [Q_g, Q_{ab}] = -Q_{ab}, \quad [Q_w, Q_r] = 0 \\ \{Q_b, Q_d\} &= -\{Q_{ad}, Q_{ab}\} = Q_w \end{aligned} \tag{4.51}$$

Here the relations between the conserved charges Q_b and Q_{ad} as well as

Q_{ab} and Q_d can be found using equation of motions only. This algebra is similar to the algebra satisfied by de-Rham co-homological operators of differential geometry given in Eq.(4.49). Comparing Eq.(4.49) and Eq.(4.51) we obtain following analogies

$$(Q_b, Q_{ad}) \rightarrow d, \quad (Q_d, Q_{ab}) \rightarrow \delta, \quad Q_w \rightarrow \Delta \tag{4.52}$$

Let n be the ghost number associated with a given state $|\psi\rangle_n$ defined in the total Hilbert space of states, i.e.

$$iQ_g|\psi\rangle_n = n|\psi\rangle_n \tag{4.53}$$

Then it is easy to verify the following relations

$$\begin{aligned}
Q_gQ_b|\psi\rangle_n &= (n+1)Q_b|\psi\rangle_n \\
Q_gQ_{ad}|\psi\rangle_n &= (n+1)Q_{ad}|\psi\rangle_n \\
Q_gQ_d|\psi\rangle_n &= (n-1)Q_b|\psi\rangle_n \\
Q_gQ_{ab}|\psi\rangle_n &= (n-1)Q_{ad}|\psi\rangle_n \\
Q_gQ_w|\psi\rangle_n &= nQ_w|\psi\rangle_n
\end{aligned} \tag{4.54}$$

which imply that the ghost numbers of the states $Q_b|\psi\rangle_n$, $Q_d|\psi\rangle_n$ and $Q_w|\psi\rangle_n$ are (n+1), (n-1) and n respectively. The states $Q_{ab}|\psi\rangle_n$ and $Q_{ad}|\psi\rangle_n$ have ghost numbers (n-1) and (n+1) respectively. The properties of set (Q_b, Q_{ad}) and (Q_d, Q_{ab}) are same as of operators d and δ. It is evident from Eq.(4.54) that the set (Q_b, Q_{ad}) raises the ghost number of a state by one and the set (Q_d, Q_{ab}) lowers the ghost number of the same state by one. Keeping the analogy between charges of different nilpotent symmetries and Hodge-de-Rham differential operators, we express any arbitrary state $|\psi\rangle_n$ in terms of the sets (Q_b, Q_d, Q_w) and $(Q_{ad}, Q_{ab}, Q_{\bar{w}})$ as

$$\begin{aligned}
|\psi\rangle_n &= |w\rangle_n + Q_b|\chi\rangle_{(n-1)} + Q_d|\phi\rangle_{(n+1)} \\
|\psi\rangle_n &= |w\rangle_n + Q_{ad}|\chi\rangle_{(n-1)} + Q_{ab}|\phi\rangle_{(n+1)}
\end{aligned} \tag{4.55}$$

where the most symmetric state is the harmonic state $|w\rangle_n$ that satisfies

$$
\begin{aligned}
Q_w|w\rangle_n &= 0, \quad Q_b|w\rangle_n = 0, \quad Q_d|w\rangle_n = 0 \\
Q_{ab}|w\rangle_n &= 0, \quad Q_{ad}|w\rangle_n = 0
\end{aligned} \tag{4.56}
$$

analogous to the Eq.(4.49). Therefore the BRST charges for a particle on a torus forms two separate set of de-Rham co-homological operators, namely $\{Q_b, Q_{ab}, Q_w\}$ and $\{Q_d, Q_{ad}, Q_{\bar{w}}\}$. Thus we call the theory of a particle on torus as double Hodge theory. Fermionic charges Q_b, Q_{ab}, Q_d and Q_{ad} follow physicality criteria given below,

$$
\begin{aligned}
Q_b|phys\rangle &= 0, \quad Q_{ab}|phys\rangle = 0 \\
Q_d|phys\rangle &= 0, \quad Q_{ad}|phys\rangle = 0
\end{aligned} \tag{4.57}
$$

which lead to

$$
\begin{aligned}
p_\lambda|phys\rangle &= 0 \\
(P_r + P_\eta)|phys\rangle &= 0
\end{aligned} \tag{4.58}
$$

This is the operator form of the first class constraint which annihilates the physical state as a consequence of physicality criteria, which further is consistent with the Dirac's method of quantization of a system with first class constraints.

4.1.10 FFBRST for free particle on surface of torus

The effective action for the free particle on surface of torus using BFV formulation is written in Eq.(4.19) and its BRST transformation is given by Eq.(4.22). In BRST transformation given by Eq.(4.22), $\delta\Lambda$ is global, infinitesimal and anti-commuting parameter. FFBRST transformation corresponding to this BRST transformation is written as

$$\begin{aligned}
s_b r &= c\Theta, \quad s_b\lambda = -\dot{c}\Theta, \quad s_b\eta = c\Theta \\
s_b p_r &= 0, \quad s_b p_\eta = 0, \quad s_b c = 0 \\
s_b\bar{c} &= (\dot{\lambda} - \eta - r)\Theta
\end{aligned} \tag{4.59}$$

where Θ is finite field dependent, global and anti-commuting parameter. Under this transformation too, effective action is invariant.

Generating functional for this effective theory can be written as

$$\begin{aligned}
Z_\psi &= \int D\Phi \exp[i\int d^4x\big[p_r\dot{r} + p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - p_\lambda\dot{\lambda} - \frac{p_r^2}{2m} \\
&\quad -\frac{p_\theta^2}{2m(r-\eta)^2} - \frac{p_\phi^2}{2m(b+(r-\eta)\sin\theta)^2} + \dot{c}p + \dot{\bar{c}}\bar{p} + \lambda(p_r + p_\eta) \\
&\quad +2c\bar{c} - \bar{p}p + p_\lambda(r + \eta + \frac{p_\lambda}{2})\big]\big]
\end{aligned} \tag{4.60}$$

where,

$$D\Phi = dr dp_r d\theta dp_\theta d\phi dp_\phi d\eta dp_\eta d\lambda dp_\lambda dp d\bar{p} dc d\bar{c} \tag{4.61}$$

where $D\Phi$ is the path integral measure integrated over total phase space. The finite BRST transformation given above leaves the effective action invariant but path integral measure in generating functional is not invariant under this transformation. It gives rise to a Jacobian in the extended phase space which can be calculated as,

$$\begin{aligned}
D\Phi &= dr dp_r d\theta dp_\theta d\phi dp_\phi d\eta dp_\eta d\lambda dp_\lambda dp d\bar{p} dc d\bar{c} \\
&= J(k)dr(k)dp_r(k)d\theta(k)dp_\theta(k)d\phi(k)dp_\phi(k)d\eta(k)dp_\eta(k)d\lambda(k)dp_\lambda(k) \\
&\quad dp(k)d\bar{p}(k)dc(k)d\bar{c}(k) \\
&= J(k+dk)dr(k+dk)dp_r(k+dk)d\theta(k+dk)dp_\theta(k+dk) \\
&\quad d\phi(k+dk)dp_\phi(k+dk)d\eta(k+dk)dp_\eta(k+dk)d\lambda(k+dk) \\
&\quad dp_\lambda(k+dk)dp(k+dk)d\bar{p}(k+dk)dc(k+dk)d\bar{c}(k+dk)
\end{aligned} \tag{4.62}$$

Writing it in compact form from chapter 2,

$$= \int d^4x \sum_{\psi} [\frac{\delta\Psi(x,k+dk)}{\delta\Psi(x,k)}] \tag{4.63}$$

Where $\Psi = (r, p_r, \theta, p_\theta, \phi, p_\phi, \eta, p_\eta, \lambda, p_\lambda, p, \bar{p}, c, \bar{c})$. Which can be written as

$$\begin{aligned} &= 1 + dk \int [c\frac{\delta\Theta'(x,k+dk)}{\delta r(x,k)} - \dot{c}\frac{\delta\Theta(x,k+dk)}{\delta\lambda(x,k)} + c\frac{\delta\Theta(x,k+dk)}{\delta\eta(x,k)} \\ &\quad +(\dot{\lambda} - \eta - r)\frac{\delta\Theta(x,k+dk)}{\delta\bar{c}(x,k)}] \\ &= \frac{J(k)}{J(k+dk)} \\ &= 1 - \frac{1}{J(k)}\frac{dJ(k)}{dk}dk \end{aligned} \tag{4.64}$$

Now we consider an example to illustrate the FFBRST formulation. For that purpose we construct finite BRST parameter Θ obtained from,

$$\Theta' = i\gamma \int d^4y \bar{c}(y,k) p_\lambda(y,k) \tag{4.65}$$

through

$$\Theta = \int \Theta'(k) dk \tag{4.66}$$

The Jacobian change is calculated using Eq.(2.52) as,

$$\frac{1}{J(k)}\frac{dJ(k)}{dk} = i\gamma \int d^4y {p_\lambda}^2 \tag{4.67}$$

We make an ansatz for S_1 as,

$$S_1 = i \int d^4x \xi_1(k) {p_\lambda}^2 \tag{4.68}$$

Where $\xi_1(k)$ is a k dependent arbitrary parameter. Now,

$$\frac{dS_1}{dk} = i\int d^4x \xi_1'(k) {p_\lambda}^2 \tag{4.69}$$

Using condition in Eq.(2.55), we will get $\xi_1(k) = \gamma k$. Now the modified generating functional can be written as in Eq.(2.56),

$$\begin{aligned} Z &= \int D\chi'(k) e^{i(S_1+S_{eff})} \\ &= \int D\phi' \exp\Big[i\int d^4x \big[p_r\dot{r} + p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - p_\lambda\dot{\lambda} - \frac{p_r^2}{2m} \\ &\quad -\frac{p_\theta^2}{2m(r-\eta)^2} - \frac{p_\phi^2}{2m(b+(r-\eta)\sin\theta)^2} + \dot{c}p + \dot{\bar{c}}\bar{p} + \lambda(p_r + p_\eta) \\ &\quad +2c\bar{c} - \bar{p}p + p_\lambda(r+\eta) + (\frac{\lambda'}{2} + \gamma k){p_\lambda}^2\big]\Big] \end{aligned} \tag{4.70}$$

Here generating functional at $k = 0$ is the theory for a free particle on the surface of torus with a gauge parameter λ' and at $k = 1$, the generating functional for same theory with a different gauge parameter $\lambda'' = \lambda' + 2\gamma$. These two effective theories with two different gauge parameters on the surface of a torus are related through the FFBRST transformation with finite parameter given in Eq.(4.66). FFBRST transformation is thus helpful in showing the gauge independence of physical quantities.

4.1.11 Conclusion

We have constructed nilpotent BRST, dual-BRST, anti-BRST and anti-dual BRST transformations for this system. Dual-BRST transformations are also the symmetry of effective action and leaves gauge-fixing part of the effective action invariant. Interchanging the role of ghost and anti-ghost fields the anti-BRST and anti-dual BRST symmetry transformations are constructed. We have shown that the nilpotent BRST and anti dual-BRST charges are analogous to the exterior derivative operators as the ghost number of the state $|\psi\rangle_n$ on the total Hilbert space is increased by one when these charges operate on this state and algebra followed by these operators is same as the algebra obeyed by the de-Rham cohomological operators. Similarly the dual-BRST and anti-BRST charges are analogous to co-exterior derivative. The anti-commutators of BRST and dual-BRST and anti-BRST and anti dual-BRST charges lead to bosonic symmetry. The corresponding charges are analogous to Laplacian operator. Further, this theory has another nilpotent symmetry called ghost symmetry under which the ghost term of the effective action is invariant. We further have shown that this theory behaves as double Hodge theory as the charges for BRST (Q_b), dual BRST (Q_d) and the charges for the bosonic symmetry generated out of these two symmetries (Q_w) form the algebra for Hodge theory. On the other hand charges for anti-BRST (Q_{ab}), anti-dual BRST (Q_{ad}) and $Q_{\bar{w}}$, charge for bosonic symmetry generalized out of these nilpotent symmetries also satisfy the Hodge algebra. Thus particle on the surface of torus has very rich mathematical structure.

We further constructed the FFBRST transformation for this system. By constructing appropriate field dependent parameter we have explicitly shown that such generalized BRST transformations are capable of connecting different theories on torus. It will be interesting to construct finite version of dual BRST transformation and study its consequences on the system with constraints.

4.2 BRST Symmetry for a Torus Knot

Knot [138, 139] theory, based on mathematical concepts has found immense applications in various branches of frontier physics. Knot invariants in physical systems were introduced long ago and have got considerable impact during last one and half decades [138, 139, 140, 141, 142, 143, 144, 145, 146], especially when interpreted as Wilson loop observable in Chern-Simons(CS) theory [144]. The discussion on topological string approach to the torus knot invariants are presented in ref. [144]. In the context of gauge theory, knot invariant theories relate 3d symmetry on the CS sub-manifold and 3d SUSY gauge theory. It also plays important role in various other problems like, inequivalent quantization problem [141], in the role of topology in defining vacuum state in gauge theories [142], in understanding band theory of solids [143]. We will discuss here the BRST symmetry for particle on a torus knot. We will also discuss FFBRST symmetry for this system.

4.2.1 Particle on a Torus Knot

In knot theory, a torus knot is a special kind of knot that lies on the surface of un-knotted torus in R^3. It is specified by a set of co-prime integers p and q. A torus knot of type (p, q) winds p times around the rotational symmetry axis of the torus and q times around a circle in the interior of the torus. The toroidal co-ordinate system is a suitable choice to study this system. Toroidal co-ordinates are related to Cartesian co-ordinates (x_1, x_2, x_3) in following ways,

$$x_1 = \frac{a\sinh\eta\cos\phi}{\cosh\eta - \cos\theta}, \quad x_2 = \frac{a\sinh\eta\sin\phi}{\cosh\eta - \cos\theta}, \quad x_3 = \frac{a\sin\theta}{\cosh\eta - \cos\theta} \tag{4.71}$$

where, $0 \leq \eta \leq \infty$, $-\pi \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. A toroidal surface is represented by some specific value of η (say η_0). Parameters a and η_0 are written as $a^2 = R^2 - d^2$ and $\cosh\eta_0 = \frac{R}{D}$, where R and D are major and minor radius of torus respectively.

Similarly, toroidal coordinates can be represented in the form of Cartesian co-ordinates as,

$$\eta = \ln\frac{d_1}{d_2}, \quad \cos\theta = \frac{r^2 - a^2}{((r^2 - a^2)^2 + 4a^2{x_3}^2)^{\frac{1}{2}}}, \quad \phi = \tan^{-1}\frac{x_2}{x_1} \tag{4.72}$$

where

$$\begin{aligned} d_1^2 &= (\sqrt{x_1^2 + x_2^2} + a)^2 + x_3^2, \quad d_2^2 = (\sqrt{x_1^2 + x_2^2} - a)^2 + x_3^2 \\ r^2 &= x_1^2 + x_2^2 \end{aligned} \tag{4.73}$$

r is cylindrical radius.

Lagrangian for a particle residing on the surface of torus knot is given by [140, 146]

$$L = \frac{1}{2}ma^2\frac{\dot{\eta}^2 + \dot{\theta}^2 + \sinh\eta^2\dot{\phi}^2}{(\cosh\eta - \cos\theta)^2} - \lambda(p\theta + q\phi) \tag{4.74}$$

where (η, θ, ϕ) are the toroidal coordinates for toric geometry.

Constraint that forces the particle to move in knot is imposed as

$$\Omega_1 = (p\theta + q\phi) \approx 0 \tag{4.75}$$

The Hamiltonian corresponding to this Lagrangian is then written as,‘

$$H = \frac{(\cosh\eta - \cos\theta)^2}{ma^2}[p_\eta^2 + p_\theta^2 + \frac{p_\phi^2}{\sinh^2\eta}] + \lambda(p\theta + q\phi) \tag{4.76}$$

Here p_η, p_θ and p_ϕ are canonical momenta corresponding to the coordinates η, θ and ϕ.

Using Eq.(2.20), the time evolution of the constraint Ω_1 gives additional secondary constraint

$$\Omega_2 = \frac{(\cosh\eta - \cos\theta)^2}{ma^2}[pp_\theta + \frac{qp_\phi}{\sinh^2\eta}] \approx 0 \tag{4.77}$$

Constraints Ω_1 and Ω_2 form second-class constraint algebra [22]

$$\begin{aligned}\Delta_{kk'}(x,y) &= \{\Omega_k(x), \Omega_{k'}(y)\} \\ &= \epsilon^{kk'}\frac{(\cosh\eta - \cos\theta)^2}{ma^2}[p^2 + \frac{q^2}{\sinh^2\eta}]\delta(x-y)\end{aligned} \tag{4.78}$$

with $\epsilon^{12} = -\epsilon^{21} = 1$. In the next section we will convert this gauge variant theory to the gauge invariant theory in an extended Hilbert space.

4.2.2 Wess-Zumino term and Hamiltonian Formation

To construct a gauge invariant theory corresponding to a gauge non-invariant model of particle on a torus knot, we introduce Wess-Zumino (WZ) term in Lagrangian in Eq.(4.74). For this purpose we enlarge the Hilbert space of the theory by introducing a new co-ordinate α, called as WZ term, through the redefinition of co-ordinates θ, ϕ and λ in the Lagrangian L as follows

$$\theta \rightarrow \theta - \frac{\alpha}{2p}, \quad \phi \rightarrow \phi - \frac{\alpha}{2q}, \quad \lambda \rightarrow \lambda + \dot{\alpha} \tag{4.79}$$

With this redefinition of co-ordinates, modified Lagrangian is written as

$$\begin{aligned} L^I \;=\; & \frac{1}{2}\frac{ma^2}{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}[\dot{\eta}^2 + \dot{\theta}^2 + \sinh^2\eta\dot{\phi}^2 + \frac{\dot{\alpha}^2}{4}(\frac{1}{p^2} + \frac{\sinh^2\eta}{q^2}) \\ & -\dot{\alpha}(\frac{\dot{\theta}}{p} + \sinh^2\eta\frac{\dot{\phi}}{q})] - (\lambda + \dot{\alpha})(p\theta + q\phi - \alpha) \end{aligned} \tag{4.80}$$

which is invariant under following time-dependent gauge transformations

$$\begin{aligned} \delta\lambda = \dot{f}(t), \quad \delta\theta = -\frac{f(t)}{2p}, \quad \delta\phi = -\frac{f(t)}{2q}, \quad \delta\alpha = -f(t) \\ \delta p_\eta = \delta p_\theta = \delta p_\phi = 0, \quad \delta b = \delta\Pi_\alpha = \delta\eta = \delta\Pi_\lambda = 0 \end{aligned} \tag{4.81}$$

where $f(t)$ is an arbitrary function of time. To construct the Hamiltonian for this gauge invariant theory we construct the canonical momenta corresponding to this modified Lagrangian as

$$
\begin{aligned}
p_\eta &= \frac{ma^2}{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}\dot{\eta}, \quad \Pi_\lambda = 0, \\
p_\theta &= \frac{ma^2}{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}(\dot{\theta} - \frac{\dot{\alpha}}{2p}), \\
p_\phi &= \frac{ma^2\sinh^2\eta}{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}(\dot{\phi} - \frac{\dot{\alpha}}{2q}), \\
\Pi_\alpha &= \frac{ma^2}{2(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}\{\frac{\dot{\alpha}}{2}(\frac{1}{p^2} + \frac{\sinh^2\eta}{q^2}) + (\frac{\dot{\theta}}{p} + \sinh^2\eta\frac{\dot{\phi}}{q})\} \\
&\quad -(p\theta + q\phi - \alpha)
\end{aligned}
\tag{4.82}
$$

The only primary constraint for this extended theory is

$$\Psi_1 \equiv \Pi_\lambda \approx 0 \tag{4.83}$$

The Hamiltonian corresponding to Lagrangian L^I is written as

$$H^I = p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + \Pi_\alpha\dot{\alpha} - L^I \tag{4.84}$$

The total Hamiltonian after using Lagrange multiplier β corresponding to the primary constraint Π_λ is obtained as

$$
\begin{aligned}
H_T^I &= \frac{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2}[p_\eta^2 + p_\theta^2 + \frac{p_\phi^2}{\sinh^2\eta}] \\
&- \lambda(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) + \beta\Pi_\lambda
\end{aligned}
\tag{4.85}
$$

Using Dirac's method of constraint analysis [22], we obtain secondary constraint

$$\Psi_2 \equiv (\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) \approx 0 \tag{4.86}$$

There is no tertiary constraint corresponding to this total Hamiltonian as

$$\Psi_3 = \dot{\Psi}_2 = [H_T^I, (\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q})] = 0 \tag{4.87}$$

This extended theory thus has only first class constraints.

4.2.3 BFV Formalism for Torus Knot

To discuss all possible nilpotent symmetries we further extend the theory using BFV formalism [18, 19, 20, 79]. In the BFV formulation associated with this system, we introduce a pair of canonically conjugate ghost fields (c,p) with ghost number 1 and -1 respectively, for the primary constraint $\Pi_\lambda \approx 0$ and another pair of ghost fields $(\bar{c}, \bar{p})$ with ghost number -1 and 1 respectively, for the secondary constraint, $(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) \approx 0$. Then using Eq.(2.41), effective action for particle on the surface of torus knot in extended phase space is written as

$$\begin{aligned} S_{eff} &= \int dt \big[p_\eta \dot{\eta} + p_\theta \dot{\theta} + p_\phi \dot{\phi} + \Pi_\alpha \dot{\alpha} - \Pi_\lambda \dot{\lambda} - \frac{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2} \{ p_\eta^2 \\ &\quad + p_\theta^2 + \frac{p_\phi^2}{\sinh^2\eta} \} + \dot{c}P + \dot{\bar{c}}\bar{P} - \{Q_b, \psi\} \big] \end{aligned} \tag{4.88}$$

where Q_b is BRST charge and has been constructed using the constraints of the system as in Eq.(2.43)

$$Q_b = ic(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) - i\bar{P}\Pi_\lambda \tag{4.89}$$

The canonical brackets for all dynamical variables are written as

$$\begin{aligned} [\eta, p_\eta] &= [\theta, p_\theta] = [\phi, p_\phi] = [\alpha, \Pi_\alpha] = [\lambda, \Pi_\lambda] = \{\bar{c}, \dot{c}\} = i, \\ \{c, \dot{\bar{c}}\} &= -i \end{aligned} \tag{4.90}$$

Nilpotent BRST transformation corresponding to this action is constructed using the relation $s_b\Phi = -[Q_b, \Phi]_\pm$, which is related to infinitesimal BRST transformation as $\delta_b\Phi = s_b\Phi\delta\Lambda$. Here $\delta\Lambda$ is infinitesimal BRST parameter. Here -ve sign is for bosonic and +ve is for fermionic variable. The BRST transformation for the particle on a torus knot is then written as

$$
\begin{aligned}
s_b\lambda = \bar{P}, \quad s_b\theta = -\frac{c}{2p}, \quad s_b\phi = -\frac{c}{2q}, \quad s_b\alpha = -c \\
s_bp_\eta = s_bp_\theta = s_bp_\phi = 0, \quad s_bP = (\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) \\
s_b\bar{c} = \Pi_\lambda = b, \quad s_bc = s_bb = s_b\Pi_\alpha = s_b\eta = s_b\Pi_\lambda = 0
\end{aligned} \tag{4.91}
$$

One can check that these transformations are nilpotent. In BFV formulation the generating functional is independent of gauge-fixing fermion [18, 19, 20], hence we have liberty to choose it in the convenient form as in Eq.(2.43)

$$
\Psi = p\lambda + \bar{c}(p\theta + q\phi + \alpha + \frac{\Pi_\lambda}{2}) \tag{4.92}
$$

Using the expressions for Q_b and Ψ, effective action (4.88) is written as

$$
\begin{aligned}
S_{eff} &= \int dt\big[p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + \Pi_\alpha\dot{\alpha} - \Pi_\lambda\dot{\lambda} - \frac{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2}\{p_\eta^2 \\
&\quad + p_\theta^2 + \frac{p_\phi^2}{\sinh^2\eta}\} + \dot{c}P + \dot{\bar{c}}\bar{P} - \bar{P}P + \lambda(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) + 2c\bar{c} \\
&\quad + \Pi_\lambda(p\theta + q\phi + \alpha + \frac{\Pi_\lambda}{2})\big]
\end{aligned} \tag{4.93}
$$

and the generating functional for this effective theory is represented as

$$
Z_\psi = \int D\phi \exp\left[iS_{eff}\right] \tag{4.94}
$$

The measure $D\phi = \prod_i d\xi_i$, where ξ_i are all dynamical variables of the theory. Now integrating this generating functional over P and $\bar{P}$, we get

$$\begin{aligned} Z_\psi &= \int D\phi' \exp\Big[i\int dt\big[p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + \Pi_\alpha\dot{\alpha} - \Pi_\lambda\dot{\lambda} \\ &\quad -\frac{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2}\{p_\eta^2 + p_\theta^2 + \frac{p_\phi^2}{\sinh^2\eta}\} + \dot{\bar{c}}\dot{c} + \lambda(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) \\ &\quad -2\bar{c}c + \Pi_\lambda(p\theta + q\phi + \alpha + \frac{\Pi_\lambda}{2})\big]\Big] \end{aligned} \tag{4.95}$$

where $D\phi'$ is the path integral measure for effective theory when integrations over fields P and $\bar{P}$ are carried out. Further integrating over Π_λ we obtain an effective generating functional as

$$\begin{aligned} Z_\psi &= \int D\phi'' \exp\Big[i\int dt\big[p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + \Pi_\alpha\dot{\alpha} \\ &\quad -\frac{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2}\{p_\eta^2 + p_\theta^2 + \frac{p_\phi^2}{\sinh^2\eta}\} + \dot{\bar{c}}\dot{c} + \lambda(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) \\ &\quad -2\bar{c}c - \frac{\{\dot{\lambda} - (p\theta + q\phi + \alpha)\}^2}{2}\big]\Big] \end{aligned} \tag{4.96}$$

where $D\Phi''$ is the path integral measure corresponding to all the dynamical variables involved in the effective action. The BRST symmetry transformation for this effective theory is written as

$$\begin{gathered} s_b\theta = -\frac{c}{2p}, \quad s_b\phi = -\frac{c}{2q}, \quad s_b\alpha = -c \\ s_b p_\eta = s_b p_\theta = s_b p_\phi = 0, \quad s_b\bar{c} = -\{\dot{\lambda} - (p\theta + q\phi + \alpha)\} \\ s_b\lambda = \dot{c}, \quad s_b c = s_b b = s_b\Pi_\alpha = s_b\eta = s_b\Pi_\lambda = 0 \end{gathered} \tag{4.97}$$

4.2.4 BRST and Anti-BRST charge

In this section we show that physical subspace of the system is consistent with the constraints of the system. Nilpotent charge for BRST symmetry in Eq.(4.97) is constructed as

$$Q_b = ic(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) - i\dot{c}\Pi_\lambda \tag{4.98}$$

This BRST charge generates BRST transformations in Eq.(4.97) through the following commutation and anti-commutation relations

$$\begin{aligned} s_b\theta = -[Q_b,\theta] &= -\frac{c}{2p} \\ s_b\phi = -[Q_b,\phi] &= -\frac{c}{2q} \\ s_b\alpha = -[Q_b,\alpha] &= -c \\ s_b\lambda = -[Q_b,\lambda] &= \dot{c} \\ s_b\bar{c} = -\{Q_b,\bar{c}\} &= -\Pi_\lambda \end{aligned} \tag{4.99}$$

The physical states are annihilated by the BRST charge in Eq.(4.98) as

$$\begin{aligned} Q_b|\psi\rangle &= 0 = \{ic(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) - i\dot{c}\Pi_\lambda\}|\psi\rangle \\ &= ic(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q})|\psi\rangle - i\dot{c}\Pi_\lambda\}|\psi\rangle \end{aligned} \tag{4.100}$$

This implies that

$$(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q})|\psi\rangle = 0, \quad \Pi_\lambda|\psi\rangle = 0 \tag{4.101}$$

The Hamiltonian (4.85) is also invariant under anti-BRST transformation in which role of c and $-\bar{c}$ are interchanged. Anti-BRST transformations for this theory are written as

$$\begin{aligned} \bar{s}_{ab}\theta = \frac{\bar{c}}{2p}, \quad \bar{s}_{ab}\phi = \frac{\bar{c}}{2q}, \quad \bar{s}_{ab}\alpha = \bar{c} \\ \bar{s}_{ab}p_\eta = \bar{s}_{ab}p_\theta = \bar{s}_{ab}p_\phi = 0, \quad \bar{s}_{ab}c = \{\dot{\lambda} - (p\theta + q\phi + \alpha)\} \\ \bar{s}_{ab}\lambda = -\dot{\bar{c}}, \quad \bar{s}_{ab}\bar{c} = \bar{s}_{ab}b = \bar{s}_{ab}\Pi_\alpha = \bar{s}_{ab}\eta = \bar{s}_{ab}\Pi_\lambda = 0 \end{aligned} \tag{4.102}$$

The nilpotent charge for the anti-BRST symmetry in Eq.(4.102) is constructed as

$$Q_{ab} = -i\bar{c}(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) + i\dot{\bar{c}}\Pi_\lambda \tag{4.103}$$

Like BRST charge, anti-BRST charge Q_{ab} also generates the anti-BRST transformations in (4.102) through the following commutation and anti-commutation relations

$$
\begin{aligned}
s_{ab}\theta = -[Q_{ab}, \theta] &= \frac{\bar{c}}{2p} \\
s_{ab}\phi = -[Q_{ab}, \phi] &= \frac{\bar{c}}{2q} \\
s_{ab}\alpha = -[Q_{ab}, \alpha] &= \bar{c} \\
s_{ab}\lambda = -[Q_{ab}, \lambda] &= -\dot{\bar{c}} \\
s_{ab}c = -\{Q_{ab,}c\} &= -\Pi_\lambda
\end{aligned} \tag{4.104}
$$

Anti-BRST charge too annihilates the states of physical Hilbert space.

$$
\begin{aligned}
Q_{ab}|\psi\rangle &= 0 \\
-i\bar{c}(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) + i\dot{\bar{c}}\Pi_\lambda|\psi\rangle &= 0
\end{aligned} \tag{4.105}
$$

or

$$
(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q})|\psi\rangle = 0, \quad \Pi_\lambda|\psi\rangle = 0 \tag{4.106}
$$

Anti-BRST charge too project on the physical subspace of total Hilbert space. Thus anti-BRST charge plays exactly same role as BRST charge. It is straight forward to check that these charges are nilpotent i.e. $Q_b^2 = 0 = Q_{ab}^2$ and satisfy

$$
\{Q_b, \quad Q_{ab}\} = 0 \tag{4.107}
$$

4.2.5 FFBRST for Torus Knot

The finite version of the BRST for torus knot is written as

$$
\begin{aligned}
\delta_b\lambda = \bar{P}\Theta, \quad \delta_b\theta = -\frac{c}{2p}\Theta, \quad \delta_b\phi = -\frac{c}{2q}\Theta, \quad \delta_b\alpha = -c\Theta \\
\delta_b p_\eta = \delta_b p_\theta = \delta_b p_\phi = 0, \quad \delta_b P = (\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q})\Theta \\
\delta_b\bar{c} = \Pi_\lambda\Theta = b\Theta, \quad \delta_b c = \delta_b b = \delta_b\Pi_\alpha = \delta_b\eta = \delta\Pi_\lambda = 0
\end{aligned} \tag{4.108}
$$

where Θ is finite field dependent, global and anti-commuting parameter. It is straight forward to check that under this transformation too, effective action in Eq.(4.93) is invariant. Generating functional for this effective theory is then written as

$$
\begin{aligned}
Z_\psi \;=\; & \int D\Phi \exp[i \int dt\big[p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + \Pi_\alpha\dot{\alpha} - \Pi_\lambda\dot{\lambda} \\
& -\frac{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2}\{p_\eta^2 + p_\theta^2 + \frac{p_\phi^2}{\sinh^2\eta}\} + \dot{c}P + \dot{\bar{c}}\bar{P} - \bar{P}P \\
& +\lambda(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) - 2\bar{c}c + \Pi_\lambda(p\theta + q\phi + \alpha + \frac{\Pi_\lambda}{2})\big]\big]
\end{aligned} \tag{4.109}
$$

where,

$$
D\Phi = \prod d\eta dp_\eta d\theta dp_\theta d\phi dp_\phi d\lambda d\Pi_\lambda dP d\bar{P} dc d\bar{c} \tag{4.110}
$$

is the path integral measure in the total phase space. This path integral measure is not invariant under such FFBRST transformation as already mentioned. It gives rise to a Jacobian in the extended phase space which is calculated using Eqs.(2.51, 2.52). Using the condition in Eq.(2.55), one can calculate the extra part in the action S_1 for some specific choices of the finite parameter Θ.

Now we consider a simple example of FFBRST transformation to show the connection between two effective theories explicitly. For that we choose finite BRST parameter $\Theta = \int dk \Theta'(k)$, where Θ' is given as

$$\Theta' = i\gamma \int dt' \bar{c}(y,k)\Pi_\lambda(y,k) \tag{4.111}$$

Using Eq.(2.52), the change in Jacobian is calculated for this particular parameter as,

$$\frac{1}{J(k)}\frac{dJ(k)}{dk} = -i\gamma \int dt' \Pi_\lambda^2(y,k) \tag{4.112}$$

We make an ansatz for S_1 as,

$$S_1 = i \int dt \xi_1(k)\Pi_\lambda^2 \tag{4.113}$$

Where $\xi_1(k)$ is a k dependent arbitrary parameter. Now,

$$\frac{dS_1}{dk} = i \int dt \xi_1'(k)\Pi_\lambda^2 \tag{4.114}$$

By satisfying the condition in Eq.(2.55), we find $\xi_1 = -\gamma k$. The FFBRST transformation with finite parameter Θ as given in Eq.(4.111) changes this generating functional as in Eq.(2.56),

$$\begin{aligned} Z &= \int D\phi(k) e^{i(S_1+S_{eff})} \\ &= \int D\Phi \exp[i \int dt \big[p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + \Pi_\alpha\dot{\alpha} - \Pi_\lambda\dot{\lambda} \\ &\quad - \frac{(\cosh\eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2}\{p_\eta^2 + p_\theta^2 + \frac{p_\phi^2}{\sinh^2\eta}\} + \dot{c}P + \dot{\bar{c}}\bar{P} \\ &\quad - \bar{P}P + \lambda(\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) - 2\bar{c}c + \Pi_\lambda(p\theta + q\phi + \alpha) \\ &\quad + (\frac{\lambda'}{2} - \gamma k)\frac{\Pi_\lambda^2}{2})\big]\big] \end{aligned} \tag{4.115}$$

Here generating functional at $k = 0$ will give pure theory for a free particle on the surface of torus knot with a gauge parameter λ' and at $k = 1$, the generating functional for same theory with a different gauge parameter $\lambda'' = \lambda' - 2\gamma$. Even though we have considered a very simple example, our formulation is valid to connect any two generating functionals corresponding to different effective actions on the surface of torus knot using FFBRST transformation with suitable parameter.

4.2.6 Conclusion

Mathematical concept of knot theory is very useful in describing various physical systems and it has been extensively used to study many different phenomena in physics. However there was no BRST formulation for particle on the surface of torus knot. In this work we systematically developed the BRST/anti-BRST formulation for the first time for a particle moving on a torus knot. Using Dirac's constraint analysis we found all the constraints of this system. Further we have extended this theory to include WZ term to recast this theory as gauge theory. Using BFV formulation BRST/anti-BRST invariant effective action for a particle moving on a torus knot has been developed. Nilpotent charges which generate these symmetries have been calculated explicitly. The physical states which are annihilated by these nilpotent charges are consistent with the constraints of the system. Our formulation is independent of particular choice of a torus knot. We further have extended the BRST formulation by considering the transformation parameter finite and field dependent. We indicate how different effective theories on the surface of torus knot are related through such a finite transformation through the non-trivial Jacobian factor. In support of our result we explicitly relate the generating functionals of two effective theories with different gauge-fixing parameters. Using FFBRST with suitable finite parameter the connection between any two effective theories can be made in a straight forward manner following the prescriptions outlined in this work.

4.3 BRST Qantization on Torus Knot

4.3.1 Particle on a Torus Knot

Particle on surface of torus knot is given by [140, 146]. This model is analogous to the rigid rotor model discussed in ref. [148].

To quantize the system and realize the physics, we have to diagonalize the system in presence of the constraints $\Phi = (p\theta + q\phi) \approx 0$. To execute this, we will discard the [148] terms proportional to p_θ^2 and p_ϕ^2 as they don't commute with the constraint equation. The effective Hamiltonian for this system keeping the essential physical content of the theory intact is then written as

$$H_c = \frac{(\cosh\eta - \cos\theta)^2}{ma^2} p_\eta^2 + \lambda(p\theta + q\phi) \tag{4.116}$$

4.3.2 Particle on torus knot as gauge theory

The constraint which doesn't commute with unphysical operators is to be identified with the generator of gauge transformations. In this case the gauge transformation is an unitary operator of the form [148]

$$U_f = e^{-if(t)\Phi} \tag{4.117}$$

where $f(t)$ is an arbitrary c number and function of t. Under this transformation only p_θ and p_ϕ transform non-trivially. First order Lagrangian for torus knot is written as

$$L_c = p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{(\cosh\eta - \cos\theta)^2}{ma^2} p_\eta^2 - \lambda(p\theta + q\phi) \tag{4.118}$$

Here we have enlarged the dynamical degrees of freedom to include the Lagrange multiplier λ and its canonically conjugate momentum p_λ. This Lagrangian is invariant under following time dependent gauge transformation [148]

$$
\begin{aligned}
\delta\lambda &= \dot{f}(t), \quad \delta p_\theta = pf(t), \quad \delta p_\phi = qf(t) \\
\delta p_\eta &= \delta\eta = \delta\theta = \delta\phi = \delta p_\lambda = 0
\end{aligned} \tag{4.119}
$$

as under this gauge transformation the Lagrangian corresponding to canonical Hamiltonian H_c changes only by a total time derivative, $\frac{d}{dt}[f(t)(p\theta + q\phi)]$. Hence the action is invariant under gauge transformation in Eq.(4.119).

4.3.3 Gauge fixing and BRST transformation

To construct the BRST and anti-BRST symmetry for the particle on the surface of a torus knot we consider the gauge invariant Lagrangian of the torus knot obtained in last section and replace the gauge transformation in Eq.(4.119) by introducing new anti-commuting variables c and $\bar{c}$ and a commuting variable b such that the BRST transformations

$$
\begin{aligned}
\delta\lambda &= -\dot{c}, \quad \delta p_\theta = pc, \quad \delta p_\phi = qc, \quad \delta\bar{c} = -p_\lambda = b \\
\delta p_\eta &= \delta\eta = \delta\theta = \delta\phi = \delta p_\lambda = \delta c = \delta b = 0
\end{aligned} \tag{4.120}
$$

are nilpotent in nature. The gauge fixed effective Lagrangian which is invariant under this kind of BRST transformation is then constructed as

$$
\begin{aligned}
L_{eff} &= p_\eta\dot{\eta} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{(\cosh\eta - \cos\theta)^2}{ma^2}p_\eta^2 - \lambda(p\theta + q\phi) \\
&- b(\dot{\lambda} - \frac{p_\theta}{2p} - \frac{p_\phi}{2q}) - \frac{b^2}{2} + \dot{\bar{c}}\dot{c} - \bar{c}c
\end{aligned} \tag{4.121}
$$

The BRST charge which generates the transformation in (4.120) is written as

$$
Q_b = -ic(p\theta + q\phi) - i\dot{c}p_\lambda. \tag{4.122}
$$

The Lagrangian (4.121) is also invariant under anti-BRST transformation in which role of c and $-\bar{c}$ are interchanged.

The nilpotent charge for the anti-BRST symmetry is written as

$$Q_{ab} = i\bar{c}(p\theta + q\phi) + i\dot{\bar{c}}p_\lambda \tag{4.123}$$

It is straight forward to check that these charges are nilpotent i.e. $Q_b^2 = 0 = Q_{ab}^2$ and satisfy

$$\{Q_b, \quad Q_{ab}\} = 0 \tag{4.124}$$

4.3.4 Conclusion

We for the first time constructed BRST/anti-BRST transformations for particle moving on torus knot. The nilpotent BRST/anti-BRST charges are constructed which produce such nilpotent transformations. Our formulation will be helpful in characterizing various field theoretic models defined on torus knot manifold.

In the next chapter we will discuss about "Weyl Degree of freedom in Nambu-Goto string through field Transformation" [163]. We will show how Weyl degrees of freedom can be introduced in the Nambu-Goto string in the path-integral formulation using the re-parametrization invariant measure. First, we will identify Weyl degrees of freedom in conformal gauge using BFV formulation. Further we will change the NG string action to the Polyakov action. The generating functional in light-cone gauge will then be obtained from the generating functional corresponding to the Polyakov action in conformal gauge by using suitably constructed FFBRST transformation.

Chapter 5

Nambu-Goto String and Weyl Symmetry

5.1 Weyl Degree of Freedom in Nambu-Goto String through Field Transformaion

In this chapter we will study, how Weyl degree of freedom can be incorporated in Nambu-Goto (NG) string in light-cone gauge. Bosonic strings are formulated using two alternative actions namely the NG action [150] and the Polyakov action [151]. In the Polyakov formulation, one uses the metric of the string world-sheet, $g_{\alpha\beta}$, to manifest the Weyl and reparametrization symmetries at the classical level. These symmetries are useful to eliminate the degrees of freedom of $g_{\alpha\beta}$. On the other hand, in the NG string only string coordinates $X_\mu(\mu = 0, ..., D-1)$ are used as dynamical variables and hence no Weyl freedom is present from the beginning. We consider the NG string theory in BFV formulation at the sub-critical dimensions, where the BF fields appear as the conformal degrees of freedom [155, 156]. Using FFBRST transformations we will connect the action in conformal gauge to light cone gauge. In this way we will be able to incorporate Weyl degrees of freedom in NG string.

5.1.1 BRST for Nambu-Goto action

NG action for the string coordinates X^μ, $\mu = 0, 1, ...D-1$ on a two-dimensional world-sheet, parametrized by $x^\alpha = (\tau, \sigma)$, $\alpha = 0, 1$ is written as [150]

$$S_0 = \int d^2x(-\det G_{\alpha\beta})^{\frac{1}{2}} \tag{5.1}$$

where

$$G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu \tag{5.2}$$

The momentum conjugate to X^μ for this theory is then written as

$$P_\mu = \sqrt{-G}\partial_\alpha X_\mu G^{\alpha 0} \tag{5.3}$$

where $G = \det G_{\alpha\beta}$. The Hamiltonian corresponding to this system vanishes. This system has two primary constraints which generate two reparameterizations of string world-sheet and are written as,

$$\phi_\pm = \frac{1}{4}(P_\mu^2 + (\partial_\sigma X^\mu)^2) \pm \frac{1}{2}\partial_\sigma X^\mu P_\mu \tag{5.4}$$

These constraints are first class at the classical level but appears as second class at quantum level due to conformal anomaly. To convert them into first class constraints, we will introduce the new field θ and its momentum conjugate Π_θ in the action. The new effective constraints then take the form

$$\tilde{\phi}_\pm = \phi_\pm + \frac{k}{\sqrt{2}}(\partial_\sigma \Pi_\theta \pm (\partial_\sigma^2 \theta)) + \frac{1}{4}(\Pi_\theta \pm \partial_\sigma \theta)^2 \tag{5.5}$$

where k is a constant which is fixed as [159]

$$k = \frac{(25-D)}{24\pi} \tag{5.6}$$

We further extend the phase space by introducing following pair of fields

$$(C^\pm, \bar{P}_\pm), \quad (P^\pm, \bar{C}_\pm), \quad (N^\pm, B_\pm) \tag{5.7}$$

Now, the action is written in the extended phase space as

$$S = \int d^2\sigma[\dot{X}^\mu P_\mu + \dot{\theta}\Pi_\theta + \dot{C}^a \bar{P}_a + \{\psi, Q\}] \tag{5.8}$$

where BRST charge Q is given as

$$Q = \int d\sigma[C^\pm(\tilde{\phi}_\pm + \bar{P}_\pm \partial_\sigma C^\pm) + B^\pm P_\pm] \tag{5.9}$$

One can easily find out that Q^2 is nilpotent in nature and gauge-fixing functional takes the form

$$\Psi = \int d\sigma(i\bar{C}_a \chi^a + \bar{P}_a N^a) \tag{5.10}$$

where X^a does not depend on ghost, anti-ghost, B and N fields.

After eliminating all the non-dynamical variables, BRST transformation for the dynamical variables is written as [155, 159]

$$\begin{aligned}
\delta X^\mu &= -\frac{1}{2}(C^a\partial_a X^\mu) \\
\delta\theta &= -\frac{1}{2}(C^a\partial_a\theta) + \frac{k}{2\sqrt{2}}(\partial_+ C^+ - \partial_- C^-) \\
\delta C^\pm &= -\frac{1}{4}C^\pm\partial_\pm C^\pm \\
\delta\bar{C}_\pm &= -\frac{1}{4}\partial_\pm X^\mu\partial_\pm X_\mu \pm \bar{C}_\pm\partial_\pm C^\pm \pm \frac{1}{2}\partial_\pm\bar{C}_\pm C^\pm - \frac{1}{4}\partial_\pm\theta\partial_\pm\theta \\
&\mp\frac{k}{2\sqrt{2}}\partial_\pm\partial_\pm\theta
\end{aligned} \tag{5.11}$$

which leaves action in Eq.(5.8) invariant.

Now, total Lagrangian density has the form

$$\mathcal{L} = \mathcal{L}_x + \mathcal{L}_{gf} + \mathcal{L}_{gh} \tag{5.12}$$

where $\mathcal{L}_x$ denote the string part of the Lagrangian density and gauge-fixing and ghost terms are defined as

$$\lambda(\mathcal{L}_{gf} + \mathcal{L}_{gh}) = -i\delta(\bar{C}_a \chi^a) \tag{5.13}$$

where λ is infinitesimal Grassmann parameter. In the next section we are going to discuss BRST symmetric Polyakov action in conformal as well as in light-cone gauges.

5.1.2 Polyakov Action

Following the technique in ref. [158] we convert NG action to a Polyakov action as

$$\mathcal{L}_x = -\frac{1}{2}\tilde{g}^{ab}\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\tilde{g}^{ab}\partial_a\theta\partial_b\theta \tag{5.14}$$

Here θ dependent term in the above Lagrangian density brings extra degrees of freedom in the system. We need a re-parametrization invariant measure in the path integral formulation to construct the BRST symmetry of this theory. Using the methods described in [160, 161] we construct the BRST transformation as

$$\begin{aligned}
\delta X^\mu &= -(C^a\partial_a X^\mu) \\
\delta\theta &= -(C^a\partial_a\theta) + \frac{k}{\sqrt{2}}(\partial_a C^a) \\
\delta C^a &= -C^b\partial_b C^a \\
\delta\bar{C}_a &= iB_a \\
\delta\tilde{g}^{ab} &= \partial_c C^a\tilde{g}^{cb} + \partial_c C^b\tilde{g}^{ac} - \partial_c(C^c\tilde{g}^{ab}) + 2\partial_c C^c\theta\tilde{g}^{ab}
\end{aligned} \tag{5.15}$$

We define the generating functional in path-integral formulation as

$$Z = \int D\phi \exp\left(i\int d^2x(\mathcal{L}_x + \mathcal{L}_{gf} + \mathcal{L}_{gh})\right) \tag{5.16}$$

where Lagrangian density is given by Eqs.(5.14, 5.13) and $D\phi$ is the generic notation for path integral measure. Transformation in Eq.(5.15) leaves the effective action invariant. Now we fix the gauge more specifically and discuss BRST invariant effective theories in conformal as well as light cone gauges.

5.1.3 Conformal Gauge

Conformal gauge has been used extensively in the discussion of various problems. It has been used to study strings, gravity etc in path-integral and covariant operator formalism. This gauge is very useful to remove conformal anomaly, to introduce Weyl symmetry and in renormalizing the theory [157, 159, 160].

The conformal gauge condition is expressed as $\tilde{g}^{ab} = \eta^{ab}$ [156, 160] and is incorporated into the following gauge-fixing and FP ghost term in a BRST invariant manner,

$$\mathcal{L}_{cf} = \lambda(\mathcal{L}_{gf} + \mathcal{L}_{gh}) = -i\delta^B(\bar{C}^0\tilde{g}^{++} + \bar{C}^1\tilde{g}^{--}) \tag{5.17}$$

Here $\bar{C}^0$ and $\bar{C}^1$ are anti-ghost fields.

5.1.4 Light-cone Gauge

On the other hand, light cone gauge is used to eliminate unphysical degrees of freedom and also in decoupling of ghost fields. Light-cone gauge has also been used in Kaku-Kikkawa string field theory, in showing the ultraviolet finiteness of $N = 4$ supersymmetric YM theory, in dimensional regularization, in gravity, supergravity, string and superstrings theories [162].

The light-cone gauge condition, ($X^+ = f(\sigma)$, $\tilde{g}^{++} = 0$) [156, 161] is incorporated into the following gauge-fixing and ghost term in a BRST invariant manner,

$$\mathcal{L}_{lc} = \lambda(\mathcal{L}_{gf} + \mathcal{L}_{gh}) = -i\delta^B(\bar{C}^0\tilde{g}^{++} + \bar{C}^1(X^+ - f(\sigma))). \tag{5.18}$$

Here $f(\sigma)$ is an arbitrary function of σ^0 and σ^1.

Now we proceed to use FFBRST to address the Weyl degree of freedom in NG string formulation.

5.1.5 Connection between generating functionals in conformal and light-cone gauges

In this subsection, we construct the FFBRST transformation with an appropriate finite parameter to obtain the generating functional corresponding to $\mathcal{L}_{cf}$ from that of corresponding to $\mathcal{L}_{lc}$. We calculate the Jacobian corresponding to such a FFBRST transformation following the method outlined in chapter 2 and show that it is a local functional of fields and accounts for the difference of the two FP effective actions.

The generating functional corresponding to the FP effective action S_{cf} is written as

$$Z_{cf} = \int D\phi \exp(iS_{cf}[\phi]) \tag{5.19}$$

where S_{cf} is given by

$$S_{cf} = \int d^2x(\mathcal{L}_x + \mathcal{L}_{cf}) \tag{5.20}$$

Now, to obtain the generating functional corresponding to S_{lc}, we apply the FFBRST transformation with a finite parameter $\Theta[\phi]$, which is obtained from the infinitesimal but field dependent parameter, $\Theta'[\phi(k)]$; through $\int_0^\kappa \Theta'[\phi(\kappa)]d\kappa$. We construct $\Theta'[\phi(\kappa)]$ as,

$$\Theta'[\phi] = i\int d^2x[\gamma\bar{C}^1\{(X^+ - f(\sigma)) - \tilde{g}^{--}\}] \tag{5.21}$$

Here γ is arbitrary constant parameter and all the fields depend on the parameter k. The infinitesimal change in the Jacobian corresponding to this FFBRST transformation is calculated using Eq.(2.52)

$$
\begin{aligned}
\frac{1}{J(k)}\frac{dJ(k)}{dk} &= -i\int d^2x\gamma[-iB^1\{(X^+ - f(\sigma)) - \tilde{g}^{--}\} - (C^a\partial_a X^+)\bar{C}^1 \\
&\quad -\delta\tilde{g}^{--}\bar{C}^1]
\end{aligned} \tag{5.22}
$$

To express the Jacobian contribution in terms of a local functional of fields, we make an ansatz for S_1 by considering all possible terms that could arise from such a transformation as

$$
\begin{aligned}
S_1[\phi(k), k] &= \int d^2x[\xi_1 iB^1(X^+ - f(\sigma)) + \xi_2 iB^1\tilde{g}^{--} + \xi_3(C^a\partial_a X^+)\bar{C}^1 \\
&\quad +\xi_4\delta\tilde{g}^{--}\bar{C}^1 + \xi_5 iB^0\tilde{g}^{++} + \xi_6\bar{C}^0\delta\tilde{g}^{++}]
\end{aligned} \tag{5.23}
$$

where all the fields are considered to be k dependent and we have introduced arbitrary k dependent parameters $\xi_n = \xi_n(k)(n = 1, 2, .., 6)$ with initial condition $\xi_n(k = 0) = 0$. It is straight forward to calculate

$$
\begin{aligned}
\frac{dS_1}{dk} &= \int d^2x[\xi_1' iB^1(X^+ - f(\sigma)) + \xi_2' iB^1\tilde{g}^{--} + \xi_3'(C^a\partial_a X^+)\bar{C}^1 \\
&\quad +\xi_4'\delta\tilde{g}^{--}\bar{C}^1 + \xi_5' iB^0\tilde{g}^{++} + \xi_6'\bar{C}^0\delta\tilde{g}^{++} + \Theta'\{-\xi_1 C^a\partial_a X^+ iB^1 \\
&\quad +\xi_2\delta\tilde{g}^{--}iB^1 + \xi_3(iB^1)C^a\partial_a X^+ + \xi_3(C^b\partial_b C^a)\partial_a X^+\bar{C}^1 \\
&\quad +\xi_3 C^a\partial_a(-C^b\partial_b X^+)\bar{C}^1 - \xi_4 iB^1\delta\tilde{g}^{--} + \xi_5\delta\tilde{g}^{++}iB^0 \\
&\quad -\xi_6 iB^0\delta\tilde{g}^{++}\}]
\end{aligned} \tag{5.24}
$$

where $\xi_n' = \frac{d\xi_n}{dk}$. Now we will use the condition of Eq.(2.55).

$$
\begin{aligned}
&\int D\phi\exp[i(S_{cf}[\phi(k)] + S_1[\phi(k), k])]\int d^2x[(-\gamma + \xi_1')iB^1(X^+ - f(\sigma)) \\
&+(\gamma + \xi_2')iB^1\tilde{g}^{--} + (-\gamma + \xi_3')(C^a\partial_a X^+)\bar{C}^1 + (-\gamma + \xi_4')\delta\tilde{g}^{--}\bar{C}^1 \\
&+\xi_5' iB^0\tilde{g}^{++} + \xi_6'\bar{C}^0\delta\tilde{g}^{++} + \Theta'\{(-\xi_1 + \xi_3)C^a\partial_a X^+ iB^1 \\
&+(\xi_2 + \xi_4)\delta\tilde{g}^{--}iB^1 + (\xi_5 - \xi_6)\delta\tilde{g}^{++}iB^0\}] = 0
\end{aligned} \tag{5.25}
$$

The terms proportional to Θ' which are regarded as nonlocal, vanishes independently. These will give relations between ξ. Making remaining local terms in Eq.(5.25) vanish will give relations between ξ and γ. Solving these Eqs.(A.18, A.19), we will get following results.

The differential equations for $\xi_n(k)$ can be solved with the initial conditions $\xi_n(0) = 0$, to obtain the solutions

$$\xi_1 = \gamma k, \quad \xi_2 = -\gamma k, \quad \xi_3 = \gamma k, \quad \xi_4 = \gamma k, \quad \xi_5 = \xi_6 = 0 \tag{5.26}$$

Putting values of these parameters in expression of S_1, and choosing arbitrary parameter $\gamma = -1$, we obtain,

$$\begin{aligned} S_1[\phi(1), 1] &= \int d^2x[-iB^1(X^+ - f(\sigma)) + iB^1\tilde{g}^{--} - (C^a\partial_a X^+)\bar{C}^1 \\ &\quad -\delta\tilde{g}^{--}\bar{C}^1] \end{aligned} \tag{5.27}$$

Thus the FFBRST transformation with the finite parameter Θ that is defined by Eq.(5.21) changes the generating functional Z_{cf} as in Eq.(2.56)

$$\begin{aligned} Z_{cf} &= \int D\phi \exp(iS_{cf}[\phi]) \\ &= \int D\phi' \exp[i(S_{cf}[\phi'] + S_1[\phi', 1])] \\ &= \int D\phi \exp[i(S_{cf}[\phi] + S_1[\phi, 1])] \\ &= \int D\phi \exp(iS_{lc}[\phi]) \equiv Z_{lc} \end{aligned} \tag{5.28}$$

Here S_{lc} is defined as

$$S_{lc} = \int d^2x(\mathcal{L}_x + \mathcal{L}_{lc}) \tag{5.29}$$

In this way FFBRST transformation with the finite field dependent parameter in Eq.(5.21) connects generating functional for the Polyakov action in conformal gauge to that of in the light-cone gauge.

5.1.6 Conclusion

In this present work we have demonstrated how Weyl degrees of freedom are incorporated in the formulation of NG string through certain field transformation. Weyl degrees of freedom are first identified in conformal gauge using BFV formulation. Then we have established the connection between conformal gauge to light-cone in Polyakov type action for NG string using the technique of FFBRST transformation, which connects various theories through the non-trivial Jacobian of path integral measure. The non-local BRST transformation by Igarashi etal in [158] is nothing but a particular type of FFBRST transformation. The parameter λ in the non-local transformation in [158] is identified with FFBRST parameter Θ'.

In the next chapter we will discuss about issue of ghost number current anomaly in bosonic string in conformal gauge and its removal using FFBRST transformation [196]. We consider Polyakov theory of Bosonic strings in conformal gauge which are used to study conformal anomaly. However it exhibits ghost number anomaly. We show how this anomaly can be avoided by connecting this theory to that of in background covariant harmonic gauge which is known to be free from conformal and ghost number current anomaly, by using suitably constructed finite field dependent BRST transformation.

Chapter 6

Harmonic Gauge in Bosonic String and FFBRST Transformation

6.1 Background Covariant Harmonic Gauge and Finite Field Dependent BRST Transformation

Bosonic string in path integral formulation [151, 164, 165] has been studied in many gauges. The simplest choice of gauge is conformal gauge [151, 157, 160, 165]. Renormalizing effective action in this gauge gives conformal anomaly [166]. The effective action is defined as whole by one loop Feynman diagram. Besides the conformal anomaly, there is another important anomaly associated with conformal gauge is the ghost number current anomaly on curved world-sheet [157]. Bosonic string has been investigated in harmonic gauge [167] (a choice similar to Lorentz gauge in QED). In harmonic gauge standard $(D-26)$ answer [151] has been produced. It has been also found that ghost number current anomaly in curved world-sheet is absent in background covariant harmonic gauge [168] but the absence of a ghost number anomaly is achieved at the expense of a new anomaly in the sector involving the Nakanishi-Lautrup field [169]. The BRST analysis of the bosonic string in this perspective have been discussed in [170].

6.1.1 Bosonic String Action

In the path integral formalism bosonic string action is written as [151]

$$S_0 = \int d^2x \frac{1}{2}\sqrt{-g} g^{ab}\partial_a X^\mu \partial_b X_\mu \tag{6.1}$$

where $X^\mu, \mu = 0, 1, 2....D-1$ is string co-ordinate and $g^{ab}, (a, b = 0, 1)$ is world-sheet metric. This action is invariant under both diffeomorphisms and Weyl transformations in following ways

$$g_{ab} \to g'_{ab} = g_{ab} + \nabla_a \xi_b + \nabla_b \xi_a, \quad g_{ab} \to g'_{ab} = (1+2\sigma)g_{ab} \tag{6.2}$$

To factor out the infinite factors in the functional integral associated with these transformations, usually we choose the conformal gauge condition and introduce the Jacobian for the change of the gauge condition under the infinitesimal diffeomorphisms (ξ^a) and Weyl transformations (σ).

6.1.2 Conformal Gauge

Conformal gauge has been used extensively in the discussion of various problems. It has been used to study strings, gravity etc in path-integral and covariant operator formalism. This gauge is very useful to remove conformal anomaly, to introduce Weyl symmetry and in re-normalizing the theory [151, 157, 160, 165].

The conformal gauge condition is expressed as $h_{ab}(\equiv g_{ab} - \hat{g}_{ab} = 0)$ and is incorporated into the following gauge-fixing and FP ghost term in a BRST invariant manner [160],

$$\mathcal{L}_{cf} = \lambda(\mathcal{L}_{gf} + \mathcal{L}_{gh}) = -i\delta_B[\bar{C}^a(\frac{1}{2}h_{ab})] = -i\delta_B[\bar{C}^a(\frac{1}{2}A_a)] \tag{6.3}$$

Total Lagrangian density in conformal gauge in extended form is written as

$$\mathcal{L}^t_{cf} = \frac{1}{2}\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\sqrt{-\hat{g}}b^a h_{ab} - \frac{1}{2}\sqrt{-\hat{g}}\delta_B(A^a)\bar{C}^a \quad (6.4)$$

6.1.3 Harmonic Gauge

The harmonic gauge condition for bosonic string action is written as [167, 168, 169]

$$\hat{\nabla}_a(\sqrt{-g}g^{ab}) = 0 \quad (6.5)$$

where the Christoffel connection in $\hat{\nabla}$ is calculated for an arbitrary background metric g^{ab}.

In order to calculate gauge-fixing and ghost part of the action we split the metric g_{ab} into a classical background field $\hat{g}_{ab}$ and a (quantum) metric perturbation h_{ab} :

$$g_{ab} = \hat{g}_{ab} + h_{ab} \quad (6.6)$$

We need to fix the gauge invariance only for the quantum field h_{ab}

$$\begin{aligned} \delta h_{ab} &= \hat{\nabla}_a\xi_b + \hat{\nabla}_b\xi_a + h_{bc}\hat{\nabla}_a\xi^c + h_{ac}\hat{\nabla}_b\xi^c + \xi^c\hat{\nabla}_c h_{ab} \\ \delta h_{ab} &= 2\sigma(\hat{g}_{ab} + h_{ab}) \end{aligned} \quad (6.7)$$

Here the classical field $\hat{g}_{ab}$ is invariant under general coordinate invariance together with standard tensor transformation rules for the other fields.

The linearized form of gauge-fixing condition in harmonic gauge is

$$\frac{1}{2}\hat{\nabla}_a h - \hat{\nabla}^b h_{ab} = 0 \quad (6.8)$$

where $h = \hat{g}^{ab}h_{ab}$. The gauge condition in Eq.(6.8) as well as in Eq.(6.5) are not quantum Weyl covariant. To make them Weyl covariant we need a special gauge fixing condition. The gauge-fixing condition is given by

$$\hat{g}^{ab}h_{ab} = 0 \tag{6.9}$$

Now the gauge-fixing and ghost part of the action can be written in BRST invariant manner as,

$$\mathcal{L}_{gf} + \mathcal{L}_{fp} = \delta_B[\bar{C}^a(\frac{1}{2}\hat{\nabla}_a h - \hat{\nabla}^b h_{ab}) + \bar{\tau}\hat{g}^{ab}h_{ab}] \tag{6.10}$$

This action can be further simplified using the technique in ref. [168]. The simplified gauge-fixing and ghost term can be written in BRST invariant manner as

$$\mathcal{L}_{hm} = \lambda(\mathcal{L}_{gf} + \mathcal{L}_{fp}) = -i\delta_B[\bar{C}^a(\frac{1}{2}\hat{\nabla}_a h - \hat{\nabla}^b h_{ab})] \tag{6.11}$$

Total Lagrangian density in background covariant harmonic gauge is written in extended form as

$$\begin{aligned}\mathcal{L}^t_{hm} \;=\; & \frac{1}{2}\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu - \sqrt{-\hat{g}}b^a\hat{\nabla}^b h_{ab} - i\sqrt{-\hat{g}}[\hat{\nabla}^b\bar{C}^a\hat{\nabla}_b\bar{C}_a \\ & -\bar{C}^a\hat{R}_{ab}C^b + (\hat{\nabla}^b\bar{C}^a + \hat{\nabla}^a\bar{C}^b - \hat{g}^{ab}\nabla.\bar{C})h_{bc}\hat{\nabla}_a C^c + \hat{\nabla}^b\bar{C}^a C^c\hat{\nabla}_c h_{ab} \\ & -h_{ab}\hat{\nabla}^a\bar{C}^b(\hat{\nabla}.\bar{C} + h_{ij}\hat{\nabla}^i C^j)]\end{aligned} \tag{6.12}$$

6.1.4 BRST Symmetry

The total action is defined as

$$S^t_0 = \int d^2x(\mathcal{L}_0 + \mathcal{L}_{gf} + \mathcal{L}_{gh}) \tag{6.13}$$

where $\mathcal{L}_0$ is the kinetic part of the total Lagrangian density.

This total action is invariant under following BRST transformation

[168],

$$
\begin{aligned}
\delta_B h_{ab} &= i\lambda[\hat{\nabla}_a C_b + \hat{\nabla}_b C_a + h_{bc}\hat{\nabla}_a C^c + h_{ac}\hat{\nabla}_b C^c + C^c\hat{\nabla}_c h_{ab} \\
&\quad -(\hat{\nabla}.C + h_{ij}\hat{\nabla}C^j)(h_{ab} + \hat{g}_{ab})], \quad \delta_B X^\mu = i\lambda C^a \partial_a X^\mu, \\
\delta_B C^a &= i\lambda C^b \hat{\nabla}_b C^a, \quad \delta_B \bar{C}^a = \lambda b^a, \quad \delta_B b^a = 0 \qquad (6.14)
\end{aligned}
$$

where λ is infinitesimal, anti-commuting BRST parameter.

6.1.5 Connection between generating functionals in Background Covariant Harmonic and Conformal gauges

In this section, we construct the FFBRST transformation with an appropriate finite parameter to obtain the generating functional corresponding to $\mathcal{L}^t_{hm}$ from that of corresponding to $\mathcal{L}^t_{cf}$. We calculate the Jacobian corresponding to such a FFBRST transformation following the method outlined in chapter 2 and show that it is a local functional of fields and accounts for the difference of the two FP effective actions.

The generating functional corresponding to the FP effective action S^t_{cf} is written as

$$
Z^t_{cf} = \int D\phi \exp(iS^t_{cf}[\phi]) \qquad (6.15)
$$

where S^t_{cf} is given by

$$
S^t_{cf} = \int d^2x[\mathcal{L}_0 + \mathcal{L}_{cf}] \qquad (6.16)
$$

Now, to obtain the generating functional corresponding S^t_{hm}, we apply the FFBRST transformation with a finite parameter $\Theta[\phi]$ which is obtained from the infinitesimal but field dependent parameter, $\Theta'[\phi(k)]$ through

$\int_0^\kappa \Theta'[\phi(\kappa)]d\kappa$. We construct $\Theta'[\phi(\kappa)]$ as,

$$\Theta'[C,h] = i\int d^2x[\gamma\bar{C}^a\{\frac{1}{2}A_a - (\frac{1}{2}\hat{\nabla}_a h - \hat{\nabla}^b h_{ab})\}] \tag{6.17}$$

Here γ is arbitrary constant parameter and all the fields depend on the parameter k. The infinitesimal change in the Jacobian corresponding to this FFBRST transformation is calculated using Eq.(2.52) as

$$\begin{aligned}\frac{1}{J(k)}\frac{dJ(k)}{dk} &= -i\int d^2x\gamma[\delta(\bar{C}^a)\{\frac{1}{2}A_a - (\frac{1}{2}\hat{\nabla}_a h - \hat{\nabla}^b h_{ab})\} + \frac{1}{2}\delta(A_a)\bar{C}^a \\ &\quad -\frac{1}{2}\hat{\nabla}^a\delta h\bar{C}^a + \hat{\nabla}^b\delta(h_{ab})\bar{C}^a]\end{aligned} \tag{6.18}$$

To express the Jacobian contribution in terms of a local functional of fields, we make an ansatz for S_1 by considering all possible terms that could arise from such a transformation as

$$\begin{aligned}S_1[\phi(k),k] &= \int d^2x[\frac{\xi_1}{2}\delta(\bar{C}^a)A_a + \frac{\xi_2}{2}\delta(\bar{C}^a)\hat{\nabla}_a h + \xi_3\delta(\bar{C}^a)\hat{\nabla}^b h_{ab} \\ &\quad +\frac{\xi_4}{2}\delta(A_a)\bar{C}^a + \frac{\xi_5}{2}\hat{\nabla}_a\delta h(\bar{C}^a) + \xi_6\hat{\nabla}^b\delta(h_{ab})\bar{C}^a]\end{aligned} \tag{6.19}$$

where all the fields are considered to be k dependent and we have introduced arbitrary k dependent parameters $\xi_n = \xi_n(k)(n = 1,2,..,6)$ with initial condition $\xi_n(k=0) = 0$. It is straight forward to calculate

$$\begin{aligned}\frac{dS_1}{dk} &= \int d^2x[\frac{\xi_1'}{2}\delta(\bar{C}^a)A_a + \frac{\xi_2'}{2}\delta(\bar{C}^a)\hat{\nabla}_a h + \xi_3'\delta(\bar{C}^a)\hat{\nabla}^b h_{ab} + \xi_4'\frac{1}{2}\delta(A_a)\bar{C}^a \\ &\quad +\frac{\xi_5'}{2}\hat{\nabla}_a\delta h(\bar{C}^a) - \xi_6'\hat{\nabla}^b\delta(h_{ab})\bar{C}^a + \Theta'\{\frac{\xi_1}{2}\delta(A_a)\delta(\bar{C}^a) + \frac{\xi_2}{2}\hat{\nabla}_a\delta h\delta(\bar{C}^a) \\ &\quad +\xi_3\hat{\nabla}^b\delta h_{ab}\delta(\bar{C}^a) + \frac{\xi_4}{2}\delta(\bar{C}^a)\delta(A_a) + \frac{\xi_5}{2}\delta(\bar{C}^a)\hat{\nabla}_a\delta h \\ &\quad +\xi_6\delta(\bar{C}^a)\hat{\nabla}^b\delta(h_{ab})\}]\end{aligned} \tag{6.20}$$

where $\xi_n' = \frac{d\xi_n}{dk}$. Now we will use the condition of Eq.(2.55).

$$
\begin{aligned}
&\int D\phi \exp[i(S_{CF}[\phi(k)] + S_1[\phi(k),k])] \int d^2x[\frac{(\gamma+\xi_1')}{2}\delta(\bar{C}^a)A_a \\
&+\frac{(-\gamma+\xi_2')}{2}\delta(\bar{C}^a)\hat{\nabla}_a h + (\gamma+\xi_3')\delta(\bar{C}^a)\hat{\nabla}^b h_{ab} + \frac{(\gamma+\xi_4')}{2}\delta(A_a)\bar{c}^a \\
&+\frac{(-\gamma+\xi_5')}{2}\hat{\nabla}_a\delta h(\bar{C}^a) + (\gamma+\xi_6')\hat{\nabla}^b\delta(h_{ab})\bar{c}^a \\
&+\Theta'\{\frac{(\xi_1-\xi_4)}{2}\delta(A_a)\delta(\bar{C}^a) + \frac{(\xi_2-\xi_5)}{2}\hat{\nabla}_a\delta h\delta(\bar{C}^a) \\
&+(\xi_3-\xi_6)\hat{\nabla}^b\delta h_{ab}\delta(\bar{C}^a)\}] = 0
\end{aligned}
\tag{6.21}
$$

The terms proportional to Θ' which are regarded as nonlocal, vanishes independently. These will give relations between ξ. Making remaining local terms in Eq.(6.21) vanish will give relations between ξ and γ. Solving these Eqs.(A.27, A.28), we will get following results.

The differential equations for $\xi_n(k)$ can be solved with the initial conditions $\xi_n(0) = 0$, to obtain the solutions

$$
\begin{aligned}
\xi_1 &= -\gamma k, \quad \xi_2 = \gamma k, \quad \xi_3 = -\gamma k, \quad \xi_4 = -\gamma k, \quad \xi_5 = \gamma k, \\
\xi_6 &= -\gamma k
\end{aligned}
\tag{6.22}
$$

Putting values of these parameters in expression of S_1, and choosing arbitrary parameter $\gamma = -1$, we obtain,

$$
\begin{aligned}
S_1[\phi(1),1] &= \int d^2x[-\frac{1}{2}\delta(\bar{C}^a)A_a + \frac{1}{2}\delta(\bar{C}^a)\hat{\nabla}_a h - \delta(\bar{C}^a)\hat{\nabla}^b h_{ab} \\
&\quad -\frac{1}{2}\delta(A_a)\bar{C}^a + \frac{1}{2}\hat{\nabla}_a\delta h(\bar{C}^a) - \hat{\nabla}^b\delta(h_{ab})\bar{C}^a]
\end{aligned}
\tag{6.23}
$$

Thus the FFBRST transformation with the finite parameter Θ that is defined by Eq.(6.17) changes the generating functional Z_{CF} as in Eq.(2.56).

$$
\begin{aligned}
Z^t_{cf} &= \int D\phi \exp(iS^t_{cf}[\phi]) \\
&= \int D\phi' \exp[i(S^t_{cf}[\phi'] + S_1[\phi', 1])] \\
&= \int D\phi \exp[i(S^t_{cf}[\phi] + S_1[\phi, 1])] \\
&= \int D\phi \exp(iS^t_{hm}[\phi]) \equiv Z^t_{hm}
\end{aligned} \tag{6.24}
$$

Here S^t_{hm} is defined as

$$
S^t_{hm} = \int d^2x(\mathcal{L}_x + \mathcal{L}_{hm}) \tag{6.25}
$$

In this way FFBRST transformation with the finite field dependent parameter in Eq.(6.17) connects generating functional for the Polyakov action in the conformal gauge to that of in background covariant harmonic gauge.

6.1.6 Conclusion

In this chapter we have shown how ghost number current anomaly can be removed in conformal gauge in curved world-sheet. Ghost number current anomaly is absent in bosonic string in background covariant harmonic gauge. Using the finite field dependent BRST transformation we have obtained action in background covariant harmonic gauge from action in conformal gauge. In this way we have avoided ghost number current anomaly in conformal gauge.

In the next chapter we will present overall summary of this thesis.

Chapter 7

Concluding Remarks

This thesis is based on the construction of various generalizations of BRST transformations and their applications in different type of gauge field theories and string theories. We have started with the techniques of BRST quantization in both Lagrangian and Hamiltonian formalism. In Lagrangian formulation we have discussed field/anti-field formalism applicable to gauge theories with irreducible/reducible or closed/open gauge algebra. In Hamiltonian formalism first we have discussed Dirac's constraints analysis. Then we have discussed the (BFFT) formalism, which is used to convert second class constraints to first class constraints. Finally we have discussed (BFV) formalism, to construct BRST transformation. In this thesis, BRST symmetry has been generalized in various ways and some applications of such generalizations have been discussed. The usual BRST transformation under which the FP effective action remains invariant, is characterized by an infinitesimal, global and anti-commuting parameter. One of the important generalizations of BRST transformations is FFBRST transformation in which infinitesimal BRST parameter is generalized to be finite field dependent. Various implications of FFBRST transformations have been discussed. Under the generalized BRST transformation with a finite field dependent parameter, the effective action remains invariant but the path integral measure gives rise to a nontrivial Jacobian which under certain condition can be written as exponential of some local functional of fields. In the usual BRST transformation, the variation of kinetic part independently vanishes whereas the variation of gauge fixing part cancels with the variation of ghost part of the effective action. Another important generalization of BRST transformation which is local and covariant is dual-BRST transformation. It has been discussed in the context of particle on a torus. Under dual-BRST transformations, the variation of gauge fixing part independently vanishes whereas the variation of kinetic part cancels with the variation of ghost part of the effective action. In the Lagrangian framework, dual BRST symmetry is generally considered to be an independent symmetry because of its analogy with co-exterior derivative in the language of differential geometry. The absolute anti-commutativity of BRST and anti-BRST symmetry as well as dual-BRST and anti-dual BRST symmetry is essential for these symmetries to be independent. Independency of these symmetries can be insured through application of Curci-Ferrari condition. In the Lagrangian framework of BRST formalism, it is not clear why it is important to impose CF condition. It also doesn't answer the question "What kind of constraint CF condition is" in the Dirac's classification of constraints.

In the chapter I, we have discussed about introductory idea of BRST quantization in the gauge theory. BRST transformation is a very powerful technique since its origin in non-Abelian gauge theories. We have also given various methods of BRST quantization. We have also discussed various types of BRST transformations. Later we have given introductory idea of our work.

In the II chapter we have discussed various techniques of BRST quantization. First we have discussed Lagrangian formalism also called field/anti-field formalism or BV formalism. Then we have discussed Hamiltonian formalism also called BFV formalism. This formalism is mainly based on Dirac's method of constraints analysis. Then we have discussed about BFFT formalism which is a technique of converting second class constraints to first class constraints. At last we have discussed generalized BRST transformation also called FFBRST transformation.

In the III chapter we have discussed about MA gauge and generalized BRST transformation [111]. The main idea of this chapter is to show how quark confinement can be realized in Lorenz gauge. In $SU(N)$ YM theory, the MA gauge has been used to study its non-perturbative features, such as quark confinement. The MA gauge is a nonlinear partial gauge. It partially fixes the the system to maintain only the MA gauge symmetry specified by $U(1)^{N-1}$. This gauge helps us to extract Abelian degrees of freedom latent in $SU(N)$ YM theory. In MA gauge, Abelian dominance and the emergence of magnetic monopoles are realized as remarkable phenomena in the non-perturbative infrared region. We have applied the FFBRST formulation discussed in chapter 2, to connect the action in the gauge-fixed $SU(2)$ YM theories formulated in the Lorenz and MA gauges. We have clearly shown, how the generating functional corresponding to the FP effective action in the MA gauge can be obtained from that in the Lorenz gauge by carrying out a suitably constructed FF-BRST transformation.

In the IV chapter we have discussed about various nilpotent symmetries for a particle on torus [137]. We have also discussed about BRST and anti-BRST symmetries for a particle on torus knot. For particle on torus, we have used BFV technique to study all the symmetries. We have constructed nilpotent BRST, dual BRST, anti-BRST, and anti-dual BRST transformations for this system. Dual BRST transformations are also the symmetry of effective action and leave gauge fixing part of the effective action invariant. Interchanging the role of ghost and anti-ghost fields the anti-BRST and anti-dual BRST symmetry transformations are constructed. We have shown that the nilpotent BRST and anti-dual BRST charges are analogous to the exterior derivative operators as the ghost number of the state $|\psi\rangle$ on the total Hilbert space is increased by one when these charges operate on these states and algebra followed by these operators is the same as the algebra obeyed by the de-Rham cohomological operators. Similarly the dual BRST and anti-BRST charges are analogous to coexterior derivative. The anti-commutators of BRST and dual BRST and anti-BRST and anti-dual BRST charges lead to bosonic symmetry. The corresponding charges are analogous to Laplacian operator. Further, this theory has another nilpotent symmetry called ghost symmetry under which the ghost term of the effective action is invariant. We further have shown that this theory behaves as double Hodge theory as the charges for BRST Q_b and dual BRST Q_d and the charges for the bosonic symmetry Q_w generated out of these two symmetries, form the algebra for Hodge theory. On the other hand charges for anti-BRST, Q_{ab} anti-dual BRST Q_{ad}, and $Q_{\bar{w}}$, charge for bosonic symmetry generated out of these nilpotent symmetries, also satisfy the Hodge algebra. Thus a particle on the surface of the torus has very rich mathematical structure. We further constructed the FFBRST transformation for this system. By constructing appropriate field dependent parameter we have explicitly shown that such generalized BRST transformations are capable of connecting different theories on torus.

For a particle on the surface of torus knot we have systematically developed the BRST/anti-BRST formulation for the first time [147]. Using Diracs constraint analysis we found all the constraints of this system. Further, we have extended this theory to include the Wess-Zumino term to recast this theory as gauge theory. Using BFV formulation, the BRST/anti-BRST invariant effective action for a particle moving on a torus knot has been developed. Nilpotent charges which generate these symmetries have been calculated explicitly. The physical states which are annihilated by these nilpotent charges are consistent with the constraints of the system. Our formulation is independent of a particular choice of a torus knot. We have further extended the BRST formulation by considering the transformation parameter finite and field dependent. We indicate how different effective theories on the surface of the torus knot are related through such a finite transformation through the nontrivial Jacobian factor. In support of our result we explicitly relate the generating functionals of two effective theories with different gauge fixing parameters. Using FFBRST with a suitable finite parameter the connection between any two effective theories can be made in a straightforward manner following the prescriptions outlined in this work.

We have also developed BRST and anti-BRST symmetries for a particle on torus knot [149] using the technique used in ref. [148].

In chapter V we have discussed Weyl degree of freedom in NG string in light-cone gauge through field transformation [163]. In this work we have demonstrated how Weyl degrees of freedom are incorporated in the formulation of NG string through a certain field transformation. Weyl degrees of freedom are first identified in conformal gauge using BFV formulation. Then we have established the connection between conformal gauge to light-cone gauge in a Polyakov-type action for NG string using the technique of FFBRST transformation, which connects various theories through the nontrivial Jacobian of the path-integral measure. The nonlocal BRST transformation by Igarashi et al. in [158] is nothing but a particular type of FFBRST transformation. The parameter λ in the nonlocal transformation in [158] is identified with the FFBRST parameter Θ'.

In the chapter VI we have discussed connection between conformal and harmonic gauges for Bosonic Strings [196]. In this present work we have shown how ghost number current anomaly present in conformal gauge in curved world-sheet is avoided using field transformation. By constructing appropriate FFBRST transformation we obtain the generating functional in harmonic gauge from that of conformal gauge. This provides a convenient way to go from a theory with ghost number anomaly to the theory where there is no ghost number current anomaly. Further harmonic gauge which is a complicated gauge is directly connected through the suitably constructed field transformation to conformal gauge theory which is simpler to use.

In the chapter VII we have presented conclusion of the whole thesis. We have specifically discussed about work done in each chapter.

In future we will try to work on more mathematical models similar to torus and knot, which have some implications on physical phenomena. It will be interesting to generalize the FFBRST formulation for other theories like superstring, supergravity etc. It will also be interesting to construct finite version of dual BRST transformation and study its impact on the system with constraints.

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Appendix A

Appendix

A.1 FFBRST in MA Gauge

The finite field dependent paramter corresponding to the Lorenz gauge is obtained from the infinitesimal but field dependent parameter, $\Theta'[\phi(k)]$; through $\int_0^\kappa \Theta'[\phi(\kappa)]d\kappa$. We construct $\Theta'[\phi(\kappa)]$ as,

$$\begin{aligned}\Theta'[\phi(k)] &= i\int d^4x[\gamma_1\bar{c}^iB^i + \gamma_2\bar{c}^3B^3 + \gamma_3\{\bar{c}^a s(\partial^\mu A^a_\mu) - \bar{c}^i s(\nabla^\mu A^i_\mu)\} \\ &\quad +\gamma_4 g\epsilon^{abc}\bar{c}^a\bar{c}^b c^c + \gamma_5\epsilon^{ij}\bar{c}^i\bar{c}^j c^3]\end{aligned} \tag{A.1}$$

Here, $\gamma_p(p = 1, 2, 3, 4, 5)$ are arbitrary constant parameters and all the fields depend on the parameter k. The infinitesimal change in the Jacobian corresponding to this FFBRST transformation is calculated using Eq.(2.52) and obtained as,

$$\begin{aligned}\frac{1}{J}\frac{dJ}{dk} &= -i\int d^4x[-\gamma_1B^iB^i - \gamma_2B^3B^3 + \gamma_3\{\bar{c}^a s(\partial^\mu A^a_\mu)\bar{c}^i s(\nabla^\mu A^i_\mu)\} \\ &\quad -\gamma_3B^a\partial^\mu A^a_\mu + \gamma_3B^i\nabla^\mu A^i_\mu + \gamma_4\{-2g\epsilon^{abc}B^a\bar{c}^bc^c \\ &\quad +\frac{1}{2}g\epsilon^{abc}\epsilon^{ade}\bar{c}^b\bar{c}^c c^d c^e\} + \gamma_5\{-2\epsilon^{ij}B^i\bar{c}^jc^3 \\ &\quad +\frac{1}{2}\epsilon^{ij}\epsilon^{kl}\bar{c}^i\bar{c}^jc^kc^l\}]\end{aligned} \tag{A.2}$$

To express the Jacobian contribution in terms of a local functional of fields, we make an ansatz for S_1 by considering all possible terms that could arise from such a transformation as

$$\begin{aligned}
S_1[\phi(k),k] &= \int d^4x[\xi_1 B_a\partial^\mu A^a_\mu + \xi_2 B^i\nabla^\mu A^i_\mu + \xi_3 B^a B^a + \xi_4 B^i B^i \\
&+\xi_5\bar{c}^i s(\nabla^\mu A^i_\mu) + \xi_6\bar{c}^a s(\partial^\mu A^a_\mu) + \xi_7 g\epsilon^{abc}B^a c^b\bar{c}^c \\
&+\xi_8 g^2\epsilon^{abc}\epsilon^{ade}\bar{c}^b\bar{c}^c c^d c^e + \xi_9 g\epsilon^{ij}B^i\bar{c}^j c^3 \\
&+\xi_{10}g^2\epsilon^{ij}\epsilon^{kl}\bar{c}^i\bar{c}^j c^k c^l]
\end{aligned} \tag{A.3}$$

where all the fields are considered to be k dependent and we have introduced arbitrary k dependent parameters $\xi_n = \xi_n(k)(n = 1, 2, ..., 10)$. It is straight to calculate

$$\begin{aligned}
\frac{dS_1}{dk} &= \int d^4x\{\xi_1' B_a\partial^\mu A^a_\mu + \xi_2' B^i\nabla^\mu A^i_\mu + \xi_3' B^{a2} + \xi_4' B^{i^2} + \xi_5'\bar{c}^i s(\nabla^\mu A^i_\mu) \\
&+\xi_6'\bar{c}^a s(\partial^\mu A^a_\mu) + \xi_7' g\epsilon^{abc}B^a c^b\bar{c}^c + \xi_8' g^2\epsilon^{abc}\epsilon^{ade}\bar{c}^b\bar{c}^c c^d c^e + \xi_9' g\epsilon^{ij}B^i\bar{c}^j c^3 \\
&+\xi_{10}' g^2\epsilon^{ij}\epsilon^{kl}\bar{c}^i c^j c^k c^l + \{\xi_1 B_a s(\partial^\mu A^a_\mu) + \xi_2 B^i s(\nabla^\mu A^i_\mu) + \xi_5 B^i s(\nabla^\mu A^i_\mu) \\
&+\xi_6 B^a s(\partial^\mu A^a_\mu) - \xi_7 g\epsilon^{abc}B^a\epsilon^{bde}c^d c^e\bar{c}^c + 2\xi_8 g^2\epsilon^{abc}\epsilon^{ade}B^b\bar{c}^c c^d c^e \\
&+\xi_9\frac{g^2}{2}\epsilon^{ij}B^i\bar{c}^j\epsilon^{kl}c^k c^l + 2\xi_{10}g^2\epsilon^{ij}\epsilon^{kl}B^i\bar{c}^j c^k c^l\}\Theta'\}
\end{aligned} \tag{A.4}$$

where $\xi' = \frac{d\xi}{dk}$. Now using the condition of Eq.(2.55), we can write

$$\begin{aligned}
&\int D\phi(k)\exp\left[i(S_L([\phi(k)] + S_1[\phi(k),k)]\right]\int d^4x[-\gamma_1 B^{i^2} - \gamma_2 B^{3^2} \\
&+\gamma_3\{\bar{c}^a s(\partial^\mu A^a_\mu) - \bar{c}^i s(\nabla^\mu A^i_\mu)\} - \gamma_3 B^a\partial^\mu A^a_\mu + \gamma_3 B^i\nabla^\mu A^i_\mu \\
&+\gamma_4 g\{-2\epsilon^{abc}B^a\bar{c}^b c^c + \frac{1}{2}\epsilon^{abc}\bar{c}^a\bar{c}^b\epsilon^{cde}c^d c^e\} + \gamma_5 g\{-2\epsilon^{ij}B^i\bar{c}^j c^3 \\
&+\frac{1}{2}\epsilon^{ij}\bar{c}^i\bar{c}^j\epsilon^{lk}c^l c^k\} - \{\xi_1' B_a\partial^\mu A^a_\mu + \xi_2' B^i\nabla^\mu A^i_\mu + \xi_3' B^{a2} + \xi_4' B^{i^2} \\
&+\xi_5'\bar{c}^i s\nabla^\mu A^i_\mu + \xi_6'\bar{c}^a s(\partial^\mu A^a_\mu) + \xi_7' g\epsilon^{abc}B^a c^b\bar{c}^c + \xi_8' g^2\epsilon^{abc}\epsilon^{ade}\bar{c}^b\bar{c}^c c^d c^e \\
&+\xi_9' g\epsilon^{ij}B^i\bar{c}^j c^3 + \xi_{10}' g^2\epsilon^{ij}\epsilon^{kl}\bar{c}^i\bar{c}^j c^k c^l + \{\xi_1 B_a s\partial^\mu A^a_\mu + \xi_2 B^i s\nabla^\mu A^i_\mu \\
&+\xi_5\bar{c}^i s\nabla^\mu A^i_\mu + \xi_6 B^a s\partial^\mu A^a_\mu - \xi_7 g\epsilon^{abc}B^a\epsilon^{bde}c^d c^e\bar{c}^c \\
&+2\xi_8 g^2\epsilon^{abc}\epsilon^{ade}B^b\bar{c}^c c^d c^e + \xi_9\frac{g^2}{2}\epsilon^{ij}B^i\bar{c}^j\epsilon^{kl}c^k c^l \\
&+2\xi_{10}g^2\epsilon^{ij}\epsilon^{kl}B^i\bar{c}^j c^k c^l\}\Theta'\}] = 0
\end{aligned} \tag{A.5}$$

which can be written as

$$
\begin{aligned}
&\int D\phi(k)\exp\left[i(S_L([\phi(k)]+S_1[\phi(k),k)]\int d^4x[(-\gamma_1+\xi_3'+\xi_4')B^iB^i\right.\\
&+(-\gamma_2+\xi_3')B^3B^3+(\gamma_3+\xi_6')\bar{c}^a s(\partial^\mu A_\mu^a)+(-\gamma_3+\xi_5')\bar{c}^i s(\nabla^\mu A_\mu^i)\\
&+(-\gamma_3+\xi_1')B^a\partial^\mu A_\mu^a+(\gamma_3+\xi_2')B^i\nabla^\mu A_\mu^i+(-2\gamma_4+\xi_7')g(\epsilon^{ij}B^ic^j\bar{c}^3\\
&+\epsilon^{ij}B^3c^i\bar{c}^j)+(-2\gamma_4-2\gamma_5+\xi_7'+\xi_9')g\epsilon^{ij}B^i\bar{c}^jc^3+\frac{1}{2}(\gamma_4+\gamma_5\\
&+2\xi_8'+2\xi_{10}')\epsilon^{ij}\epsilon^{kl}\bar{c}^i\bar{c}^jc^kc^l+2(\gamma_4+2\xi_8')g^2\epsilon^{ij}\epsilon^{lk}\bar{c}^j\bar{c}^3c^kc^3+\Theta'\{(\xi_1\\
&+\xi_6)B_a s\partial^\mu A_\mu^a+(\xi_2+\xi_5)B^is\nabla^\mu A_\mu^i-\frac{1}{2}(\xi_7+4g\xi_8\\
&+\xi_9+4\xi_{10})\epsilon^{abc}B^a\epsilon^{bde}c^dc^e\bar{c}^c+(\xi_7+4\xi_8)g^2(\epsilon^{ij}\epsilon^{ik}B^j\bar{c}^3c^kc^3\\
&-\epsilon^{ij}\epsilon^{ik}B^3\bar{c}^jc^kc^3\}]=0
\end{aligned}\tag{A.6}
$$

The terms proportional to Θ' which are regarded as nonlocal, vanishes independently if,

$$
\begin{aligned}
\xi_1+\xi_6=0,\quad \xi_2+\xi_5 &= 0,\quad \xi_7+4\xi_8=0\\
\xi_7+4\xi_8+\xi_9+4\xi_{10} &= 0
\end{aligned}\tag{A.7}
$$

To make the remaining local terms in Eq.(A.6) vanish, we need the following conditions:

$$
\begin{aligned}
\xi_1'-\gamma_3=0,\quad \xi_2'+\gamma_3=0,\quad \xi_3'-\gamma_2 &= 0\\
\xi_3'+\xi_4'-\gamma_1=0,\quad \xi_5'-\gamma_3=0,\quad \xi_6'+\gamma_3 &= 0\\
\xi_7'-2\gamma_4=0,\quad \xi_7'+\xi_9'-2(\gamma_4+\gamma_5) &= 0\\
\xi_8'-\frac{1}{2}\gamma_4=0,\quad \xi_8'+\xi_{10}'+\frac{1}{2}(\gamma_4+\gamma_5) &= 0
\end{aligned}\tag{A.8}
$$

from which we also have

$$
\xi_4'-\gamma_1+\gamma_2=0,\quad \xi_9'-2\gamma_5=0,\quad \xi_{10}'+\frac{1}{2}\gamma_5=0 \tag{A.9}
$$

The differential equations for $\xi_n(k)$ can indeed be solved with the initial conditions $\xi_n(0) = 0$, to obtain the solutions

$$
\begin{aligned}
\xi &= \gamma_3 k, \quad \xi_2 = -\gamma_3 k, \quad \xi_3 = \gamma_2 k, \\
\xi_4 &= (\gamma_1 - \gamma_2) k, \quad \xi_5 = \gamma_3 k, \quad \xi_6 = -\gamma_3 k \\
\xi_7 &= 2\gamma_4 k, \quad \xi_8 = -\frac{1}{2}\gamma_4 k, \quad \xi_9 = 2\gamma_5 k \\
\xi_{10} &= -\frac{1}{2}\gamma_5 k
\end{aligned} \tag{A.10}
$$

Since $\gamma_p (p = 1, 2, 3, 4, 5)$ are arbitrary constant parameters, we can choose them as follows

$$
\begin{aligned}
\gamma_1 &= \frac{1}{2}(\beta - \alpha), \quad \gamma_2 = -\frac{\alpha}{2}, \quad \gamma_3 = 1 \\
\gamma_4 &= -\frac{\alpha}{4}, \quad \gamma_5 = \frac{\beta}{2}
\end{aligned} \tag{A.11}
$$

Substituting the solutions obtained in Eq.(A.10) into Eq.(A.3) and considering the specific values of the parameters in Eq.(A.11), we obtain

$$
\begin{aligned}
S_1 = & \int d^4x \big[B_a \partial^\mu A^a_\mu - B^i \nabla^\mu A^i_\mu - \frac{\alpha}{2} B^{a2} + \frac{\beta}{2} B^{i2} + \bar{c}^i \delta(\nabla^\mu A^i_\mu) \\
& -\bar{c}^a \delta(\partial^\mu A^a_\mu) - \frac{\alpha}{2} g \epsilon^{abc} B^a c^b \bar{c}^c + \frac{\alpha}{8} g^2 \epsilon^{abc} \epsilon^{ade} \bar{c}^b \bar{c}^c c^d c^e + \beta g \epsilon^{ij} B^i \bar{c}^j c^3 \\
& -\frac{\beta}{4} g^2 \epsilon^{ij} \epsilon^{kl} \bar{c}^i \bar{c}^j c^k c^l \big]
\end{aligned} \tag{A.12}
$$

A.2 FFBRST in Light-cone Gauge

The finite field dependent paramter corresponding to the Light-cone gauge is obtained from the infinitesimal but field dependent parameter, $\Theta'[\phi(k)]$; through $\int_0^\kappa \Theta'[\phi(\kappa)] d\kappa$. We construct $\Theta'[\phi(\kappa)]$ as,

$$
\Theta'[\phi] = i \int d^2x [\gamma \bar{C}^1 \{ (X^+ - f(\sigma)) - \tilde{g}^{--} \}] \tag{A.13}
$$

Here γ is arbitrary constant parameter and all the fields depend on the parameter k. The infinitesimal change in the Jacobian corresponding to this FFBRST transformation is calculated using Eq.(2.52)

$$\begin{aligned} \frac{1}{J(k)}\frac{dJ(k)}{dk} &= -i\int d^2x\gamma[-iB^1\{(X^+ - f(\sigma)) - \tilde{g}^{--}\} - (C^a\partial_a X^+)\bar{C}^1 \\ &\quad -\delta\tilde{g}^{--}\bar{C}^1] \end{aligned} \tag{A.14}$$

To express the Jacobian contribution in terms of a local functional of fields, we make an ansatz for S_1 by considering all possible terms that could arise from such a transformation as

$$\begin{aligned} S_1[\phi(k), k] &= \int d^2x[\xi_1 iB^1(X^+ - f(\sigma)) + \xi_2 iB^1\tilde{g}^{--} + \xi_3(C^a\partial_a X^+)\bar{C}^1 \\ &\quad +\xi_4\delta\tilde{g}^{--}\bar{C}^1 + \xi_5 iB^0\tilde{g}^{++} + \xi_6\bar{C}^0\delta\tilde{g}^{++}] \end{aligned} \tag{A.15}$$

here all the fields are considered to be k dependent and we have introduced arbitrary k dependent parameters $\xi_n = \xi_n(k)(n = 1, 2, .., 6)$ with initial condition $\xi_n(k = 0) = 0$. It is straight forward to calculate

$$\begin{aligned} \frac{dS_1}{dk} &= \int d^2x[\xi_1' iB^1(X^+ - f(\sigma)) + \xi_2' iB^1\tilde{g}^{--} + \xi_3'(C^a\partial_a X^+)\bar{C}^1 \\ &\quad +\xi_4'\delta\tilde{g}^{--}\bar{C}^1 + \xi_5' iB^0\tilde{g}^{++} + \xi_6'\bar{C}^0\delta\tilde{g}^{++} + \Theta'\{-\xi_1 C^a\partial_a X^+ iB^1 \\ &\quad +\xi_2\delta\tilde{g}^{--}iB^1 + \xi_3(iB^1)C^a\partial_a X^+ + \xi_3(C^b\partial_b C^a)\partial_a X^+\bar{C}^1 \\ &\quad +\xi_3 C^a\partial_a(-C^b\partial_b X^+)\bar{C}^1 - \xi_4 iB^1\delta\tilde{g}^{--} + \xi_5\delta\tilde{g}^{++}iB^0 \\ &\quad -\xi_6 iB^0\delta\tilde{g}^{++}\}] \end{aligned} \tag{A.16}$$

where $\xi_n' = \frac{d\xi_n}{dk}$. Now we will use the condition of Eq.(2.55).

$$\begin{aligned} &\int D\phi\exp[i(S_{cf}[\phi(k)] + S_1[\phi(k), k])]\int d^2x[(-\gamma + \xi_1')iB^1(X^+ - f(\sigma)) \\ &+(\gamma + \xi_2')iB^1\tilde{g}^{--} + (-\gamma + \xi_3')(C^a\partial_a X^+)\bar{C}^1 + (-\gamma + \xi_4')\delta\tilde{g}^{--}\bar{C}^1 \\ &+\xi_5' iB^0\tilde{g}^{++} + \xi_6'\bar{C}^0\delta\tilde{g}^{++} + \Theta'\{(-\xi_1 + \xi_3)C^a\partial_a X^+ iB^1 \\ &+(\xi_2 + \xi_4)\delta\tilde{g}^{--}iB^1 + (\xi_5 - \xi_6)\delta\tilde{g}^{++}iB^0\}] = 0 \end{aligned} \tag{A.17}$$

The terms proportional to Θ', which are nonlocal due to Θ', vanish independently if

$$-\xi_1 + \xi_3 = 0, \quad \xi_2 + \xi_4 = 0, \quad \xi_5 - \xi_6 = 0 \tag{A.18}$$

To make the remaining local terms in Eq.(A.17) vanish, we need the following conditions:

$$\begin{aligned} -\gamma + \xi_1' = 0 \quad \gamma + \xi_2' = 0 \quad \xi_5' &= 0 \\ -\gamma + \xi_3' = 0 \quad -\gamma + \xi_4' = 0, \quad \xi_6' &= 0 \end{aligned} \tag{A.19}$$

The differential equations for $\xi_n(k)$ can be solved with the initial conditions $\xi_n(0) = 0$, to obtain the solutions

$$\xi_1 = \gamma k, \quad \xi_2 = -\gamma k, \quad \xi_3 = \gamma k, \quad \xi_4 = \gamma k, \quad \xi_5 = \xi_6 = 0 \tag{A.20}$$

Putting values of these parameters in expression of S_1, and choosing arbitrary parameter $\gamma = -1$, we obtain,

$$\begin{aligned} S_1[\phi(1), 1] &= \int d^2x[-iB^1(X^+ - f(\sigma)) + iB^1\tilde{g}^{--} - (C^a\partial_a X^+)\bar{C}^1 \\ &\quad -\delta\tilde{g}^{--}\bar{C}^1] \end{aligned} \tag{A.21}$$

A.3 FFBRST in Harmonic Gauge

The finite field dependent paramter corresponding to the conformal gauge is obtained from the infinitesimal but field dependent parameter, $\Theta'[\phi(k)]$; through $\int_0^\kappa \Theta'[\phi(\kappa)]d\kappa$. We construct $\Theta'[\phi(\kappa)]$ as,

$$\Theta'[C, h] = i\int d^2x[\gamma\bar{C}^a\{\frac{1}{2}A_a - (\frac{1}{2}\hat{\nabla}_a h - \hat{\nabla}^b h_{ab})\}] \tag{A.22}$$

Here γ is arbitrary constant parameter and all the fields depend on the parameter k. The infinitesimal change in the Jacobian corresponding to this FFBRST transformation is calculated using Eq.(2.52)

$$\begin{aligned}\frac{1}{J(k)}\frac{dJ(k)}{dk} &= -i\int d^2x\gamma[\delta(\bar{C}^a)\{\frac{1}{2}A_a-(\frac{1}{2}\hat{\nabla}_a h-\hat{\nabla}^b h_{ab})\}\\ &\quad +\frac{1}{2}\delta(A_a)\bar{C}^a-\frac{1}{2}\hat{\nabla}^a\delta h\bar{C}^a+\hat{\nabla}^b\delta(h_{ab})\bar{C}^a] \qquad \text{(A.23)}\end{aligned}$$

To express the Jacobian contribution in terms of a local functional of fields, we make an ansatz for S_1 by considering all possible terms that could arise from such a transformation as

$$\begin{aligned}S_1[\phi(k),k] &= \int d^2x[\frac{\xi_1}{2}\delta(\bar{C}^a)A_a+\frac{\xi_2}{2}\delta(\bar{C}^a)\hat{\nabla}_a h+\xi_3\delta(\bar{C}^a)\hat{\nabla}^b h_{ab}\\ &\quad +\frac{\xi_4}{2}\delta(A_a)\bar{C}^a+\frac{\xi_5}{2}\hat{\nabla}_a\delta h(\bar{C}^a)+\xi_6\hat{\nabla}^b\delta(h_{ab})\bar{C}^a] \qquad \text{(A.24)}\end{aligned}$$

where all the fields are considered to be k dependent and we have introduced arbitrary k dependent parameters $\xi_n=\xi_n(k)(n=1,2,..,6)$ with initial condition $\xi_n(k=0)=0$. It is straight forward to calculate

$$\begin{aligned}\frac{dS_1}{dk} &= \int d^2x[\frac{\xi_1'}{2}\delta(\bar{C}^a)A_a+\frac{\xi_2'}{2}\delta(\bar{C}^a)\hat{\nabla}_a h+\xi_3'\delta(\bar{C}^a)\hat{\nabla}^b h_{ab}\\ &\quad +\xi_4'\frac{1}{2}\delta(A_a)\bar{C}^a+\frac{\xi_5'}{2}\hat{\nabla}_a\delta h(\bar{C}^a)-\xi_6'\hat{\nabla}^b\delta(h_{ab})\bar{C}^a+\Theta'\{\frac{\xi_1}{2}\delta(A_a)\delta(\bar{C}^a)\\ &\quad +\frac{\xi_2}{2}\hat{\nabla}_a\delta h\delta(\bar{C}^a)+\xi_3\hat{\nabla}^b\delta h_{ab}\delta(\bar{C}^a)+\frac{\xi_4}{2}\delta(\bar{C}^a)\delta(A_a)+\frac{\xi_5}{2}\delta(\bar{C}^a)\hat{\nabla}_a\delta h\\ &\quad +\xi_6\delta(\bar{C}^a)\hat{\nabla}^b\delta(h_{ab})\}] \qquad \text{(A.25)}\end{aligned}$$

where $\xi_n'=\frac{d\xi_n}{dk}$. Now we will use the condition of Eq.(2.55).

$$\begin{aligned}&\int D\phi\exp[i(S_{CF}[\phi(k)]+S_1[\phi(k),k])]\int d^2x[\frac{(\gamma+\xi_1')}{2}\delta(\bar{C}^a)A_a\\ &+\frac{(-\gamma+\xi_2')}{2}\delta(\bar{C}^a)\hat{\nabla}_a h+(\gamma+\xi_3')\delta(\bar{C}^a)\hat{\nabla}^b h_{ab}+\frac{(\gamma+\xi_4')}{2}\delta(A_a)\bar{c}^a\\ &+\frac{(-\gamma+\xi_5')}{2}\hat{\nabla}_a\delta h(\bar{C}^a)+(\gamma+\xi_6')\hat{\nabla}^b\delta(h_{ab})\bar{c}^a\\ &+\Theta'\{\frac{(\xi_1-\xi_4)}{2}\delta(A_a)\delta(\bar{C}^a)+\frac{(\xi_2-\xi_5)}{2}\hat{\nabla}_a\delta h\delta(\bar{C}^a)\\ &+(\xi_3-\xi_6)\hat{\nabla}^b\delta h_{ab}\delta(\bar{C}^a)\}]=0 \qquad \text{(A.26)}\end{aligned}$$

The terms proportional to Θ', which are nonlocal due to Θ', vanish independently if

$$\xi_1 - \xi_4 = 0, \quad \xi_2 - \xi_5 = 0, \quad \xi_3 - \xi_6 = 0 \tag{A.27}$$

To make the remaining local terms in Eq.(A.26) vanish, we need the following conditions:

$$\begin{aligned} \gamma + \xi_1' = 0, \quad -\gamma + \xi_2' = 0, \quad \gamma + \xi_3' &= 0 \\ \gamma + \xi_4' = 0, \quad -\gamma + \xi_5' = 0, \quad \gamma + \xi_6' &= 0 \end{aligned} \tag{A.28}$$

The differential equations for $\xi_n(k)$ can be solved with the initial conditions $\xi_n(0) = 0$, to obtain the solutions

$$\begin{aligned} \xi_1 &= -\gamma k, \quad \xi_2 = \gamma k, \quad \xi_3 = -\gamma k, \quad \xi_4 = -\gamma k, \quad \xi_5 = \gamma k \\ \xi_6 &= -\gamma k \end{aligned} \tag{A.29}$$

Putting values of these parameters in expression of S_1, and choosing arbitrary parameter $\gamma = -1$, we obtain,

$$\begin{aligned} S_1[\phi(1), 1] &= \int d^2x[-\frac{1}{2}\delta(\bar{C}^a)A_a + \frac{1}{2}\delta(\bar{C}^a)\hat{\nabla}_a h - \delta(\bar{C}^a)\hat{\nabla}^b h_{ab} \\ &\quad -\frac{1}{2}\delta(A_a)\bar{C}^a + \frac{1}{2}\hat{\nabla}_a \delta h(\bar{C}^a) - \hat{\nabla}^b\delta(h_{ab})\bar{C}^a] \end{aligned} \tag{A.30}$$