Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 952 (2020) 114911

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Derivatives of any Horn-type hypergeometric functions with respect to their parameters

Vladimir V. Bytev [a,b], Bernd A. Kniehl [a,*]

[a] *II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany*
[b] *Joint Institute for Nuclear Research, 141980 Dubna (Moscow Region), Russia*

Received 14 November 2018; received in revised form 12 December 2019; accepted 20 December 2019
Available online 2 January 2020
Editor: Stephan Stieberger

Abstract

We consider the derivatives of Horn hypergeometric functions of any number of variables with respect to their parameters. The derivative of such a function of n variables is expressed as a Horn hypergeometric series of $n+1$ infinite summations depending on the same variables and with the same region of convergence as for the original Horn hypergeometric function. The derivatives of Appell functions, generalized hypergeometric functions, confluent and non-confluent Lauricella series, and generalized Lauricella series are explicitly presented. Applications to the calculations of Feynman diagrams are discussed, especially the series expansions in ϵ within dimensional regularization. Connections with other classes of special functions are discussed as well.

1. Introduction

In recent years, a lot of attention [1–10] has been devoted to hypergeometric series containing the digamma or psi function,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}\,. \tag{1}$$

* Corresponding author.
E-mail addresses: bvv@jinr.ru (V.V. Bytev), kniehl@desy.de (B.A. Kniehl).

https://doi.org/10.1016/j.nuclphysb.2019.114911

On the other hand, series with gamma functions have been known for a long time, see for example the definitions in the book by Hansen [11]. Recently, however, some new summations for hypergeometric-type series which contain digamma functions have been established. In a series of papers by Miller and by Cvijović [1–3], summation formulae have been derived for hypergeometric-type series which contain a digamma function as a factor by using certain transformation and reduction formulae in the theory of Kampé de Fériet double hypergeometric functions.

Renewed interest in hypergeometric-type series containing digamma functions has emerged in connection with derivatives of hypergeometric functions with respect to their parameters. The first derivatives for some special values of parameters were already known a long time ago [12–14]. Later on, Ancarani et al. have found in a series of papers [4–6] the derivatives of Gaussian hypergeometric functions and some derivatives of two-variable series, namely the Appell series and four degenerate confluent series. Moreover, it has been shown that the first derivatives of generalized hypergeometric functions are expressible in terms of Kampé de Fériet functions [8], and, with the same technique, derivatives of the Appell hypergeometric function have been obtained in Ref. [7].

In another approach based on the expression of the Pochhammer symbol,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\,, \tag{2}$$

and its reciprocal value, the derivatives in terms of Stirling numbers have been provided in the papers of Greynat et al. [9,10]. With such an approach, one has the possibility to express some special parameter cases of Appell or generalized hypergeometric functions in terms of finite sums of well-known special functions as nested harmonic sums [15]. Also, one may formulate the derivatives as series suitable for numerical evaluation.

In all the above-cited papers except for Refs. [9,10], the technique of infinite-series resummation [16] is used,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k)\,, \tag{3}$$

which might not necessarily be correct if applied to series which are not absolutely convergent. When the derivative of a hypergeometric function is written as a power series in its arguments, of course, some discussion of the convergence of the function obtained is needed. Only in some special cases, explicit formulae for derivatives have been presented (Appell function, some confluent hypergeometric functions of two variables), but there are no explicit results for mixed derivatives of generalized Kampé de Fériet functions.

In various mathematical and physical applications, one finds hypergeometric series which belong to classes of functions different from generalized Kampé de Fériet functions, for example, Horn hypergeometric series of two variables, $H_3(a,b,c;x,y)$ (see Eq. (A.7) below for the definition), where a Pochhammer symbol $(a)_{2m+n}$ with a double summation index is encountered. This function belongs to the class of generalized Lauricella series [17] (see Sec. A.3 below for the definition), and the questions arise how the derivatives of this function look like and to which class of special functions they belong.

In high-energy physics, one has to calculate higher-order Feynman diagrams for quantum corrections to electroweak and QCD processes. These are expressible in the form of Mellin-Barnes integrals [18–21], which depend on the external kinematic invariants, the dimension D of space-time, and the powers of the propagators. Upon application of Cauchy's theorem, the Feynman integrals can be converted into linear combinations of Horn hypergeometric series,

$$\sum_{k_1,\cdots,k_{r+m}=0}^{\infty} \prod_{a,b} \frac{\Gamma(\sum_{i=1}^{m} A_{ai}k_i + B_a)}{\Gamma(\sum_{j=1}^{r} C_{bj}k_j + D_b)} x_1^{k_1} \cdots x_{r+m}^{k_{r+m}}, \tag{4}$$

where x_i are some rational functions of the external kinematic invariants (e.g., Mandelstam variables) and A_{ai}, B_a, C_{bj}, D_b are linear functions of the space-time dimension and the propagator powers. The parameters A_{ai}, C_{bj} do not belong to the set of natural numbers $\mathbb{N}$ as in the case of generalized Lauricella functions, but can take any integer value. Within the framework of dimensional regularization, i.e., taking the space-time parameter to be $D = 4 - 2\varepsilon$, one has to construct the so-called ε expansion of Eq. (4) in the parameter of dimensional regularization, or just the derivatives of Eq. (4) with respect to the B_a, D_b parameters. It is very interesting to find explicit formulae for such derivatives and the class of functions to which they belong. These questions provide the motivation for the present paper.

This paper is organized as follows. We begin in Sec. 2 by considering derivatives with respect to one-summation-index parameters. As an example, the first-order derivatives of generalized hypergeometric functions and the Appell function are presented. Next, Sec. 3 is devoted to derivatives with respect to multiple-summation-index parameters, and arbitrary derivatives of the well-known generalized hypergeometric functions and Appell series with respect to their parameters are discussed. In Sec. 4, derivatives with respect to parameters with summation index $2n$ are considered, while the more involved cases of parameters with summation index qn, $q \in \mathbb{N}$, are discussed in Secs. 5 and 6. As applications, the derivative of the Horn hypergeometric function $H_3(a, b, c; x, y)$ with respect to its upper parameter a is calculated, and derivatives of generalized Lauricella hypergeometric functions with respect to their parameters are discussed. Subsequently, Secs. 7 and 8 are devoted to the case of parameters with summation index qn, where q is negative. In Sec. 9, the regions of convergence of the series representing derivatives of the hypergeometric functions considered here are discussed. The main results are collected in Sec. 10, where we present compact equations for derivatives of hypergeometric functions with respect to their parameters for the general case of summation indices. We conclude in Sec. 11, where we also discuss possible applications to the calculation of Feynman diagrams. Appendix A summarizes the definitions of hypergeometric series used in this paper.

2. Derivatives with respect to parameters with one summation index

2.1. Upper-parameter derivatives

As a first step, we consider the derivative of a hypergeometric function with respect to the parameter a in the case when the Pochhammer symbol contains only one index of summation, $(a)_n$. As mentioned above, our calculations in this section are similar to Refs. [1–8].

The main trick is to consider the derivative of the Pochhammer symbol $(a)_n$. By using the definition of the digamma function in Eq. (1), which is the logarithmic derivative of the gamma function $\Gamma(z)$, and the difference equation,

$$\Psi(z+n) - \Psi(z) = \sum_{k=0}^{n-1} \frac{1}{z+k}, \tag{5}$$

we can write the derivative of a Pochhammer symbol in the form

$$\frac{\mathrm{d}(a)_n}{\mathrm{d}a} = (a)_n \left[\Psi(a+n) - \Psi(a) \right] = (a)_n \sum_{k=0}^{n-1} \frac{1}{a+k} = (a)_n \frac{1}{a} \sum_{k=0}^{n-1} \frac{(a)_k}{(a+1)_k} . \tag{6}$$

For convenience, let us write the hypergeometric function in the form

$$F(a) = \sum_{n=0}^{\infty} B(n)(a)_n \frac{x^n}{n!} . \tag{7}$$

Here, we explicitly write the parameter a to be differentiated and connect it with the summation index n of variable x. The summation over the index n is then explicitly displayed, but any number of additional summation indices and Pochhammer symbols are summarized in the coefficient

$$B(n) = \sum_{m_1,\dots,m_l=0}^{\infty} \frac{x_1^{m_1}\dots x_l^{m_l}}{m_1!\dots m_l!} \prod_j \frac{(a_j)_{q_{j0}n+\sum_{i=1}^{l} q_{ji}m_i}}{(b_j)_{q_{j0}n+\sum_{i=1}^{l} q_{ji}m_i}} . \tag{8}$$

We will use this short-hand notation throughout the article.

With the help of Eq. (6) and a shift of the summation index $n \to n+1$, we can write for the derivative of the function $F(a)$ with respect to the upper parameter

$$\begin{aligned} \frac{\mathrm{d}F(a)}{\mathrm{d}a} &= \sum_{n=1}^{\infty} B(n)(a)_n \frac{x^n}{n!} \frac{1}{a} \sum_{k=0}^{n-1} \frac{(a)_k}{(a+1)_k} \\ &= \sum_{n=0}^{\infty} B(n+1)(a+1)_n \frac{x^{n+1}}{(n+1)!} \frac{1}{a} \sum_{k=0}^{n} \frac{(a)_k}{(a+1)_k} . \end{aligned} \tag{9}$$

By using the rearrangement formula of the summation indices in Eq. (3), we obtain the first derivative with respect to an upper one-index parameter,

$$\frac{\mathrm{d}F(a)}{\mathrm{d}a} = x \sum_{n,k=0}^{\infty} B(n+k+1) \frac{x^n}{n!} \frac{x^k}{k!} \frac{(1)_k (1)_n}{(2)_{n+k}} \frac{(a+1)_{n+k}(a)_k}{(a+1)_k} . \tag{10}$$

If the initial hypergeometric series in Eq. (7) converges in some region A, then the series of its derivative in Eq. (9) converges in the same region A, due to the fact that the ratio of the additional Pochhammer symbols tends to unity in the large-n limit,

$$\lim_{n\to\infty} \frac{(a+i)_n}{(a+j)_n} = 1, \quad i, j \in \mathbb{R} . \tag{11}$$

As a consequence, the operation of resummation in Eq. (10) is mathematically rigorous, and the convergence region of the new series is the same as for the initial function in Eq. (7). For a general consideration of the convergence, see Sec. 9 and Ref. [17].

For the Gauss hypergeometric function ${}_2F_1(a,b,c;x)$, we obtain [4]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}a}\, {}_2F_1\left(\left. \begin{matrix} a, b \\ c \end{matrix} \right| x \right) &= \frac{bx}{c} \sum_{k=0}^{\infty} \frac{(a)_k}{(a+1)_k} \sum_{n=0}^{\infty} \frac{(a+1)_{n+k}(b+1)_{n+k}}{(c+1)_{n+k}} \frac{x^{n+k}}{(n+k+1)!} \\ &= \frac{bx}{c} \sum_{k=0}^{\infty} \frac{(1)_k (a)_k}{(a+1)_k} \sum_{n=0}^{\infty} \frac{(1)_n (a+1)_{n+k}(b+1)_{n+k}}{(2)_{n+k}(c+1)_{n+k}} \frac{x^n x^k}{n!k!} . \end{aligned} \tag{12}$$

This hypergeometric series can be understood as a generalized Kampé de Fériet hypergeometric function (see Sec. A.2 below for the definition),

$$\frac{\mathrm{d}}{\mathrm{d}a}\,{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right|x\right)=\frac{bx}{c}F^{2:2;1}_{2:1;0}\left[\begin{matrix}(a+1,b+1):(1,a);(1)\\(c+1,2):(a+1);(-)\end{matrix}\;x,x\right]. \tag{13}$$

It can easily be seen that Eq. (10) is suitable for computing the derivative of any Horn hypergeometric function with respect to a one-summation-index parameter. For example, one can calculate the derivative of the Appell function $F_1(a,b_1,b_2,c;x,y)$ with respect to the parameter b_2 as [6]

$$\frac{\mathrm{d}F_1}{\mathrm{d}b_2}=\frac{ay}{c}\sum_{k=0}^{\infty}\frac{(1)_k(b_2)_k}{(b_2+1)_k}\frac{y^k}{k!}\sum_{n=0}^{\infty}\frac{(1)_n(b_2+1)_{n+k}}{(2)_{n+k}}\frac{y^n}{n!}\sum_{m=0}^{\infty}\frac{(b_1)_m(a+1)_{m+n+k}}{(c+1)_{m+n+k}}\frac{x^m}{m!}, \tag{14}$$

where the derivative of F_1 is now expressed in terms of the generalized Lauricella hypergeometric function (see Sec. A.3 below for the definition) as

$$\frac{\mathrm{d}F_1}{\mathrm{d}b_2}=\frac{ay}{c}F^{2:1;1;2}_{2:0;0;1}\left(\begin{matrix}\text{[a+1:1,1,1; }b_2+1\text{:0,1,1]: [}b_1\text{:1];[1:1];[1:1;}b_2\text{;1]}\\\text{[c+1:1,1,1; 2:0,1,1]: [-];[-];[}b_2\text{+1,1]}\end{matrix}\;x,y,y\right). \tag{15}$$

The corresponding expression for the derivative of F_1 with respect to the parameter b_1 can then be obtained via the exchange rule,

$$\frac{\mathrm{d}F_1(a,b_1,b_2,c;x,y)}{\mathrm{d}b_1}=\left.\frac{\mathrm{d}F_1(a,b_1,b_2,c;x,y)}{\mathrm{d}b_2}\right|_{b_1\leftrightarrow b_2,x\leftrightarrow y}. \tag{16}$$

With similar manipulations, one can also compute the derivatives of the Appell functions $F_2(a,b_1,b_2,c_1,c_2;x,y)$ and $F_3(a_1,a_2,b_1,b_2,c;x,y)$ with respect to an upper one-index parameter,

$$\frac{\mathrm{d}F_2}{\mathrm{d}b_2}=\frac{ay}{c_2}\sum_{k=0}^{\infty}\frac{(1)_k(b_2)_k}{(b_2+1)_k}\frac{y^k}{k!}\sum_{n=0}^{\infty}\frac{(1)_n(b_2+1)_{n+k}}{(2)_{n+k}(c_2+1)_{n+k}}\frac{y^n}{n!}\sum_{m=0}^{\infty}\frac{(b_1)_m(a+1)_{m+n+k}}{(c_1)_m}\frac{x^m}{m!}, \tag{17}$$

$$\frac{\mathrm{d}F_3}{\mathrm{d}b_2}=\frac{a_2y}{c}\sum_{k=0}^{\infty}\frac{(1)_k(b_2)_k}{(b_2+1)_k}\frac{y^k}{k!}\sum_{n=0}^{\infty}\frac{(1)_n(b_2+1)_{n+k}(a_2+1)_{n+k}}{(2)_{n+k}}\frac{y^n}{n!}\sum_{m=0}^{\infty}\frac{(b_1)_m(a_1)_m}{(c+1)_{m+n+k}}\frac{x^m}{m!}. \tag{18}$$

By using the symmetries of hypergeometric functions, we can then find the other derivatives with respect to upper one-index parameters,

$$\begin{aligned}
\frac{\mathrm{d}F_2(a,b_1,b_2,c_1,c_2;x,y)}{\mathrm{d}b_1}&=\left.\frac{\mathrm{d}F_2(a,b_1,b_2,c_1,c_2;x,y)}{\mathrm{d}b_2}\right|_{b_1\leftrightarrow b_2,c_1\leftrightarrow c_2,x\leftrightarrow y},\\
\frac{\mathrm{d}F_3(a_1,a_2,b_1,b_2,c;x,y)}{\mathrm{d}b_1}&=\left.\frac{\mathrm{d}F_3(a_1,a_2,b_1,b_2,c;x,y)}{\mathrm{d}b_2}\right|_{a_1\leftrightarrow a_2,b_1\leftrightarrow b_2,x\leftrightarrow y},\\
\frac{\mathrm{d}F_3(a_1,a_2,b_1,b_2,c;x,y)}{\mathrm{d}a_2}&=\left.\frac{\mathrm{d}F_3(a_1,a_2,b_1,b_2,c;x,y)}{\mathrm{d}b_2}\right|_{a_1\leftrightarrow b_1,a_2\leftrightarrow b_2},\\
\frac{\mathrm{d}F_3(a_1,a_2,b_1,b_2,c;x,y)}{\mathrm{d}a_1}&=\left.\frac{\mathrm{d}F_3(a_1,a_2,b_1,b_2,c;x,y)}{\mathrm{d}b_2}\right|_{a_1\leftrightarrow b_2,b_1\leftrightarrow a_2,x\leftrightarrow y}.
\end{aligned} \tag{19}$$

2.2. Lower-parameter derivatives

If the derivative is acting on a lower parameter, Eq. (6) changes to the derivative of a reciprocal Pochhammer symbol,

$$\frac{\mathrm{d}}{\mathrm{d}b}\frac{1}{(b)_n}=\frac{1}{(b)_n}\Big[\Psi(b)-\Psi(b+n)\Big]=-\frac{1}{(b)_n}\sum_{k=0}^{n-1}\frac{1}{b+k}=-\frac{1}{(b)_n}\frac{1}{b}\sum_{k=0}^{n-1}\frac{(b)_k}{(b+1)_k}. \tag{20}$$

Then, using a short-hand notation for the hypergeometric function similar to Eq. (7), i.e.,

$$F(b)=\sum_{n=0}^{\infty}B(n)\frac{1}{(b)_n}\frac{x^n}{n!}, \tag{21}$$

one can express the derivative with respect to the lower one-index parameter b as

$$\begin{aligned}\frac{\mathrm{d}F(b)}{\mathrm{d}b}&=-\sum_{n=1}^{\infty}D(n)\frac{x^n}{n!}\frac{1}{(b)_n}\frac{1}{b}\sum_{k=0}^{n-1}\frac{(b)_k}{(b+1)_k}\\&=-\sum_{n=0}^{\infty}D(n+1)\frac{x^{n+1}}{(n+1)!}\frac{1}{(b)_{n+1}}\frac{1}{b}\sum_{k=0}^{n}\frac{(b)_k}{(b+1)_k}.\end{aligned}$$

Upon rearrangement of the sums, one obtains the following formula:

$$\frac{\mathrm{d}F(b)}{\mathrm{d}b}=-\frac{x}{b^2}\sum_{n,k=0}^{\infty}D(n+k+1)\frac{x^nx^k}{n!k!}\frac{(1)_n(1)_k}{(2)_{n+k}}\frac{1}{(b+1)_{n+k}}\frac{(b)_k}{(b+1)_k}. \tag{22}$$

As an example, we obtain for the Gauss hypergeometric function ${}_2F_1(a,b,c;x)$ the following result [4]:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}c}\,{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right|x\right)&=-\frac{abx}{c^2}\sum_{k=0}^{\infty}\frac{(c)_k}{(c+1)_k}\sum_{n=0}^{\infty}\frac{(a+1)_{n+k}(b+1)_{n+k}}{(c+1)_{n+k}}\frac{x^{n+k}}{(n+k+1)!}\\&=-\frac{abx}{c^2}\sum_{k=0}^{\infty}\frac{(1)_k(c)_k}{(c+1)_k}\sum_{n=0}^{\infty}\frac{(1)_n(a+1)_{n+k}(b+1)_{n+k}}{(2)_{n+k}(c+1)_{n+k}}\frac{x^nx^k}{n!k!},\end{aligned} \tag{23}$$

which is a series that can be expressed in terms of the generalized Kampé de Fériet-type function,

$$\frac{\mathrm{d}}{\mathrm{d}c}\,{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right|x\right)=-\frac{abx}{c^2}F^{2:1;1}_{2:1;1}\left[\begin{matrix}(a+1,b+1):(1,c);(1)\\(c+1,2):(c+1);(-)\end{matrix}\;x,x\right]. \tag{24}$$

One can use Eq. (22) also for differentiating any hypergeometric function with respect to a lower parameter with one summation index only. For the derivatives of the Appell functions $F_2(a,b_1,b_2,c_1,c_2;x,y)$ and $F_4(a,b,c_1,c_2,c;x,y)$ with respect to the lower parameters c_1 and c_2, we find

$$\frac{\mathrm{d}F_2}{\mathrm{d}c_2}=-\frac{yab_2}{c_2^2}\sum_{k=0}^{\infty}\frac{(1)_k(c_2)_k}{(c_2+1)_k}\frac{y^k}{k!}\sum_{n=0}^{\infty}\frac{(1)_n(b_2+1)_{n+k}}{(2)_{n+k}(c_2+1)_{n+k}}\frac{y^n}{n!}\sum_{m=0}^{\infty}\frac{(b_1)_m(a+1)_{m+n+k}}{(c_1)_m}\frac{x^m}{m!},$$

$$\frac{\mathrm{d}F_4}{\mathrm{d}c_2} = -\frac{yab}{c_2^2}\sum_{k=0}^{\infty}\frac{(1)_k(c_2)_k}{(c_2+1)_k}\frac{y^k}{k!}\sum_{n=0}^{\infty}\frac{(1)_n}{(2)_{n+k}(c_2+1)_{n+k}}\frac{y^n}{n!}$$
$$\times\sum_{m=0}^{\infty}\frac{(a+1)_{m+n+k}(b+1)_{m+n+k}}{(c_1)_m}\frac{x^m}{m!},$$
$$\frac{\mathrm{d}F_2(a,b_1,b_2,c_1,c_2;x,y)}{\mathrm{d}c_1} = \left.\frac{\mathrm{d}F_2(a,b_1,b_2,c_1,c_2;x,y)}{\mathrm{d}c_2}\right|_{b_1\leftrightarrow b_2, c_1\leftrightarrow c_2, x\leftrightarrow y},$$
$$\frac{\mathrm{d}F_4(a,b,c_1,c_2,c;x,y)}{\mathrm{d}c_1} = \left.\frac{\mathrm{d}F_4(a,b,c_1,c_2,c;x,y)}{\mathrm{d}c_2}\right|_{c_1\leftrightarrow c_2, x\leftrightarrow y}. \tag{25}$$

Also these derivatives are then expressible in terms of generalized Lauricella hypergeometric functions,

$$\frac{\mathrm{d}F_2}{\mathrm{d}c_2} = -\frac{yab_2}{c_2^2}F^{2:1;1;2}_{2:1;0;1}\left(\begin{array}{c}\text{[a+1:1,1,1; } b_2\text{+1:0,1,1]: [}b_1\text{:1];[1:1];[1:1;}c_2\text{:1]}\\ \text{[}c_2\text{+1:0,1,1; 2:0,1,1]: [}c_1\text{:1];[-];[}c_2\text{+1:1]}\end{array}\; x,y,y\right), \tag{26}$$

$$\frac{\mathrm{d}F_4}{\mathrm{d}c_2} = -\frac{yab}{c_2^2}F^{2:0;1;2}_{2:1;0;1}\left(\begin{array}{c}\text{[a+1:1,1,1; } b\text{+1:1,1,1]: [-];[1:1];[1:1;}c_2\text{:1]}\\ \text{[}c_2\text{+1:0,1,1; 2:0,1,1]: [c:1];[-];[}c_2\text{+1,1]}\end{array}\; x,y,y\right). \tag{27}$$

From Eqs. (10) and (22), and the definition of the generalized Lauricella series in Sec. A.3, one can see that only the first derivatives of generalized hypergeometric functions with respect to a lower parameter can be written in terms of generalized Kampé de Fériet functions of two variables. On the other hand, mixed or higher-order derivatives of generalized hypergeometric functions as well as Appell hypergeometric series can only be expressed in terms of generalized Lauricella series.

3. Derivatives with respect to parameters with multiple summation indices

3.1. Upper-parameter derivatives

Next, we consider the derivative in the case of multiple summation indices in the Pochhammer symbol $(a)_{n_1+n_2+\ldots}$. Such sums arise in the calculation of mixed derivatives of ${}_pF_q$ and the first derivative of an Appell function with respect to a parameter with two summation indices $(a)_{n_1+n_2}$. In that case, we factorize the Pochhammer symbol as a product of two terms with one summation index each,

$$(a)_{n_1+n_2} = (a+n_1)_{n_2}(a)_{n_1}, \tag{28}$$

or, in the case of multiple summation indices,

$$(a)_{\sum_{\lambda=1}^{\phi}n_\lambda} = (a+\sum_{\lambda=1}^{\phi-1}n_\lambda)_{n_\phi}(a+\sum_{\lambda=1}^{\phi-2}n_\lambda)_{n_{\phi-1}}\ldots(a)_{n_1} = \prod_{r=1}^{\phi}(a+\sum_{\lambda=1}^{r-1}n_\lambda)_{n_r}. \tag{29}$$

Upon expressing the hypergeometric function in the form

$$F(a) = \sum_{m,n=0}^{\infty}B(n,m)(a)_{m+n}\frac{x^n y^m}{n!m!}, \tag{30}$$

and applying Eq. (28) to factorize the Pochhammer symbol together with the results for the derivatives with respect to one-index parameters from Sec. 2.1, one obtains

$$\frac{\mathrm{d}F(a)}{\mathrm{d}a} = y\sum_{k,n,m=0}^{\infty} B(n,m+k+1)\frac{(1)_k(1)_m}{(2)_{m+k}}\frac{(a)_k(a+1)_{m+n+k}}{(a+1)_k}\frac{x^n y^m y^k}{n!m!k!}$$
$$+x\sum_{k,n,m=0}^{\infty} B(n+k+1,m)\frac{(1)_k(1)_n}{(2)_{n+k}}\frac{(a)_{m+k}(a+1)_{m+n+k}}{(a+1)_{m+k}}\frac{x^n y^m x^k}{n!m!k!}. \quad (31)$$

As an example, the derivative of the Appell hypergeometric function $F_1(a,b_1,b_2,c;x,y)$ with respect to the parameter a with two summation indices $m+n$ reads [6]

$$\frac{\mathrm{d}F_1}{\mathrm{d}a} = \frac{yb_1}{c}\sum_{k,n,m=0}^{\infty}\frac{(b_1+1)_{m+k}(b_2)_n}{(c+1)_{m+n+k}}\frac{(1)_k(1)_m}{(2)_{m+k}}\frac{(a)_k(a+1)_{m+n+k}}{(a+1)_k}\frac{x^n y^m y^k}{n!m!k!}$$
$$+\frac{xb_2}{c}\sum_{k,n,m=0}^{\infty}\frac{(b_1)_m(b_2+1)_{n+k}}{(c+1)_{m+n+k}}\frac{(1)_k(1)_n}{(2)_{n+k}}\frac{(a)_{m+k}(a+1)_{m+n+k}}{(a+1)_{m+k}}\frac{x^n y^m x^k}{n!m!k!}, \quad (32)$$

or, alternatively, in terms of the generalized Lauricella series,

$$\frac{\mathrm{d}F_1}{\mathrm{d}a} = \frac{yb_1}{c}F^{2:1;1;2}_{2:0;0;1}\left(\begin{array}{c}[\mathrm{a}{+}1{:}1,1,1;\ b_1{+}1{:}1,0,1]{:}\ [1{:}1];[b_2{:}1];[1{:}1;a{:}1]\\ [c{+}1{:}1,1,1;\ 2{:}1,0,1]{:}\ [\text{-}];[\text{-}];[\mathrm{a}{+}1{:}1]\end{array}\ y,x,y\right)$$
$$+\frac{xb_2}{c}F^{3:1;1;1}_{3:0;0;0}\left(\begin{array}{c}[\mathrm{a}{+}1{:}1,1,1;\ a{:}1,0,1;\ b_2{+}1{:}0,1,1]{:}\ [b_1{:}1];[1{:}1];[1{:}1]\\ [c{+}1{:}1,1,1;\ 2{:}0,1,1;\ \mathrm{a}{+}1{:}1,0,1]{:}\ [\text{-}];[\text{-}];[\text{-}]\end{array}\ y,x,x\right). \quad (33)$$

In the case of hypergeometric function with multiple summation indices,

$$F(a) = \sum_{n_1,\dots,n_\phi=0}^{\infty} B(n_1,\dots,n_\phi)(a)_{\sum_{\lambda=1}^{\phi} n_\lambda}\prod_{r=1}^{\phi}\frac{x^{n_r}}{n_r!}, \quad (34)$$

using Eq. (29) and the parametric derivative with multiple summation indices,

$$\frac{\mathrm{d}}{\mathrm{d}a}(a+\sum_{\lambda=1}^{\xi-1}n_\lambda)_{n_\xi} = (a+\sum_{\lambda=1}^{\xi-1}n_\lambda)_{n_\xi}\sum_{k=0}^{n_\xi-1}\frac{1}{a+\sum_{\lambda=1}^{\xi-1}n_\lambda+k}, \quad (35)$$

one obtains for the case of multiple summation indices

$$\frac{\mathrm{d}F(a)}{\mathrm{d}a} = \sum_{k,n_1,\dots,n_\phi=0}^{\infty}\sum_{\xi=1}^{\phi}x_\xi B(n_1,\dots,n_\xi+k+1,\dots,n_\phi)\frac{(1)_k(1)_{n_\xi}}{(2)_{n_\xi+k}}\prod_{r=1}^{\phi}\frac{x_r^{n_r}x_\xi^k}{n_r!k!}$$
$$\times\prod_{r=1}^{\xi-1}\frac{(a)_{\sum_{\lambda=1}^{r}n_\lambda}}{(a)_{\sum_{\lambda=1}^{r-1}n_\lambda}}\prod_{r=\xi+1}^{\phi}\frac{(a+1)_{\sum_{\lambda=1}^{r}n_\lambda+k}}{(a+1)_{\sum_{\lambda=1}^{r-1}n_\lambda+k}}\frac{(a+1)_{\sum_{\lambda=1}^{\xi}n_\lambda+k}}{(a+1)_{\sum_{\lambda=1}^{\xi-1}n_\lambda+k}}\frac{(a)_{\sum_{\lambda=1}^{\xi-1}n_\lambda+k}}{(a)_{\sum_{\lambda=1}^{\xi-1}n_\lambda}}. \quad (36)$$

3.2. Lower-parameter derivatives

The same trick can be applied to the derivative acting on a lower parameter with multiple summation indices, i.e., the case

$$F(b) = \sum_{m,n=0}^{\infty} B(n,m) \frac{1}{(b)_{m+n}} \frac{x^n y^m}{n!m!} . \tag{37}$$

Upon factorizing the reciprocal Pochhammer symbol similarly to Eq. (28) as

$$\frac{1}{(b)_{m+n}} = \frac{1}{(b+m)_n} \frac{1}{(b)_m} , \tag{38}$$

or, in the case of multiple summation indices, as

$$\frac{1}{(b)_{\sum_{\lambda=1}^{\phi} n_\lambda}} = \prod_{r=1}^{\phi} \frac{1}{(b+\sum_{\lambda=1}^{r-1} n_\lambda)_{n_r}} , \tag{39}$$

one can express the derivative of a hypergeometric function with respect to a lower double-summation-index parameter in terms of hypergeometric series,

$$\begin{aligned}\frac{\mathrm{d}F(b)}{\mathrm{d}b} =& -y \sum_{k,n,m=0}^{\infty} B(n,m+k+1) \frac{(1)_k (1)_m}{(2)_{m+k}} \frac{1}{b^2} \frac{(b)_k}{(b+1)_{m+n+k}(b+1)_k} \frac{x^n y^m y^k}{n!m!k!} \\ & -x \sum_{k,n,m=0}^{\infty} B(n+k+1,m) \frac{(1)_k (1)_n}{(2)_{n+k}} \frac{1}{b^2} \frac{(b)_{m+k}}{(b+1)_{m+n+k}(b+1)_{m+k}} \frac{x^n y^m x^k}{n!m!k!} .\end{aligned} \tag{40}$$

As an illustrative example, we present the derivative of the Appell function $F_3(a_1, a_2, b_1, b_2, c; x, y)$ with respect to its lower parameter c with summation indices $m+n$ (see Ref. [6]),

$$\begin{aligned}\frac{\mathrm{d}F_3}{\mathrm{d}c} =& -\frac{y a_1 b_1}{c^2} \sum_{k,n,m=0}^{\infty} \frac{(a_1+1)_{m+k}(a_2)_n(b_1+1)_{m+k}(b_2)_n(c)_k}{(c+1)_{m+n+k}(c+1)_k} \frac{(1)_k(1)_m}{(2)_{m+k}} \frac{x^n y^m y^k}{n!m!k!} \\ & -\frac{x a_2 b_2}{c^2} \sum_{k,n,m=0}^{\infty} \frac{(a_1)_m(a_2+1)_{n+k}(b_1)_m(b_2+1)_{n+k}(c)_{m+k}}{(c+1)_{m+n+k}(c+1)_{m+k}} \frac{(1)_k(1)_n}{(2)_{n+k}} \frac{x^n y^m x^k}{n!m!k!} ,\end{aligned} \tag{41}$$

or, alternatively, in terms of the generalized Lauricella series,

$$\begin{aligned}\frac{\mathrm{d}F_3}{\mathrm{d}c} =& -\frac{y a_1 b_1}{c^2} F^{2:1;2;2}_{2:0;0;1} \left(\begin{array}{c} [a_1{+}1{:}1{,}0{,}1;\ b_1{+}1{:}1{,}0{,}1]{:}\ [1{:}1];[a_2{:}1;b_2{:}1];[1{:}1;c{:}1] \\ [c{+}1{:}1{,}1{,}1;\ 2{:}1{,}0{,}1]{:}\ [\text{-}];[\text{-}];[c{+}1{:}1] \end{array} \ y, x, y \right) \\ & -\frac{x a_2 b_2}{c^2} \\ & \times F^{3:2;1;1}_{3:0;0;0} \left(\begin{array}{c} [a_2{+}1{:}0{,}1{,}1;\ b_2{+}1{:}0{,}1{,}1;\ c{:}1{,}0{,}1]{:}\ [a_1{:}1;b_1{:}1];[1{:}1];[1{:}1] \\ [c{+}1{:}1{,}1{,}1;\ 2{:}0{,}1{,}1;\ c{+}1{:}1{,}0{,}1]{:}\ [\text{-}];[\text{-}];[\text{-}] \end{array} \ y, x, x \right) .\end{aligned}$$

In complete analogy to the case of the previous subsection, for hypergeometric series with a lower parameter containing multiple summation indices,

$$F(b) = \sum_{n_1,\dots,n_\phi=0}^{\infty} B(n_1,\dots,n_\phi) \frac{1}{(b)_{\sum_{\lambda=1}^{\phi} n_\lambda}} \prod_{r=1}^{\phi} \frac{x^{n_r}}{n_r!}, \tag{42}$$

using Eq. (39) and the derivative with respect to a lower parameter with multiple summation indices,

$$\frac{\mathrm{d}}{\mathrm{d}b} \frac{1}{(b+\sum_{\lambda=1}^{\xi-1} n_\lambda)_{n_\xi}} = -\frac{1}{(b+\sum_{\lambda=1}^{\xi-1} n_\lambda)_{n_\xi}} \frac{1}{b+\sum_{\lambda=1}^{\xi-1} n_\lambda} \sum_{k=0}^{n_\xi-1} \frac{(b+\sum_{\lambda=1}^{\xi-1} n_\lambda)_k}{(b+\sum_{\lambda=1}^{\xi-1} n_\lambda+1)_k}, \tag{43}$$

one obtains for the derivative with respect to a lower parameter

$$\begin{aligned} \frac{\mathrm{d}F(b)}{\mathrm{d}b} = &-\frac{1}{b^2} \sum_{k,n_1,\dots,n_\phi=0}^{\infty} \sum_{\xi=1}^{\phi} x_\xi B(n_1,\dots,n_\xi+k+1,\dots,n_\phi) \frac{(1)_k (1)_{n_\xi}}{(2)_{n_\xi+k}} \prod_{r=1}^{\phi} \frac{x_r^{n_r} x_\xi^k}{n_r! k!} \\ &\times \prod_{r=1}^{\xi-1} \frac{(b)_{\sum_{\lambda=1}^{r-1} n_\lambda}}{(b)_{\sum_{\lambda=1}^{r} n_\lambda}} \prod_{r=\xi+1}^{\phi} \frac{(b+1)_{\sum_{\lambda=1}^{r-1} n_\lambda+k}}{(b+1)_{\sum_{\lambda=1}^{r} n_\lambda+k}} \frac{(b)_{\sum_{\lambda=1}^{\xi-1} n_\lambda}}{(b+1)_{\sum_{\lambda=1}^{\xi} n_\lambda+k}} \frac{(b)_{\sum_{\lambda=1}^{\xi-1} n_\lambda+k}}{(b+1)_{\sum_{\lambda=1}^{\xi-1} n_\lambda+k}}. \end{aligned} \tag{44}$$

With the relations presented thus far, well-known hypergeometric functions such as the generalized hypergeometric, Appell, and Lauricella series can be written in terms of generalized Lauricella series with summation coefficients $\theta_1^{(1)}, \psi_1^{(1)}, \phi_1^{(1)}, \delta_1^{(1)}, \dots, \theta_A^{(n)}, \psi_C^{(n)}, \phi_{B^{(n)}}^{(n)}, \delta_{D^{(n)}}^{(n)}$ (see Sec. A.3 for the definitions) taking the values 0 or 1. Moreover, from Eqs. (36) and (44), one can see that any derivative of these functions with respect to one of their parameters can also be expressed in terms of generalized Lauricella series with the summation coefficients taking the values 0 or 1. As a more general statement, we note that, if the initial function can be expressed as a generalized Lauricella series with summation coefficients from the alphabet {0, 1}, then any n-th derivative of this initial function can be expressed within the class of the same functions. The number of variables of an n-th derivative of an initial function with m variables is $n+m$.

4. Derivatives with respect to parameters with summation index 2*n*

4.1. Upper-parameter derivatives

In some hypergeometric series, one encounters summation indices with a factor of two, i.e., $(a)_{2n}$,

$$F(a) = \sum_{n=0}^{\infty} B(m,n)(a)_{2n} \frac{x^n}{n!}. \tag{45}$$

The simplest realizations are Horn hypergeometric functions of two variables (see Sec. A.4 for some examples). In this case, one can use Eq. (6) for the derivative of the Pochhammer symbol,

$$\frac{\mathrm{d}(a)_{2n}}{\mathrm{d}a} = (a)_{2n} \sum_{k=0}^{2n-1} \frac{1}{a+k} = (a)_{2n} \frac{1}{a} \sum_{k=0}^{2n-1} \frac{(a)_k}{(a+1)_k}, \tag{46}$$

together with a rearrangement of the summation formula in Eq. (3), which splits into two terms due to the upper summation limit at $2n+1$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(A(2k,n+k) + A(2k+1,n+k))\,. \tag{47}$$

For the hypergeometric function containing a parameter a with summation index $2n$,

$$F(a) = \sum_{m,n=0}^{\infty} B(n)(a)_{2n}\frac{x^n}{n!}\,, \tag{48}$$

one obtains for the derivative

$$\begin{aligned}\frac{\mathrm{d}F(a)}{\mathrm{d}a} &= x\sum_{k,n=0}^{\infty} B(n+k+1)\frac{(1)_k(1)_n}{(2)_{n+k}}(a+1)(a+2)_{2n+2k} \\ &\quad\times\left(\frac{a}{a+1}\frac{(a+1)_{2n}}{(a+2)_{2n}} + \frac{(a)_{2n}}{(a+1)_{2n}}\right)\frac{x^n x^k}{n!k!}\,. \end{aligned} \tag{49}$$

4.2. Lower-parameter derivatives

With the same procedure as in Sec. 4.1, for the derivative with respect to a lower parameter of the type $(b)_{2n}$, one obtains

$$\begin{aligned}\frac{\mathrm{d}F(b)}{\mathrm{d}b} &= -x\sum_{k,n=0}^{\infty} B(n+k+1)\frac{(1)_k(1)_n}{(2)_{n+k}}\frac{1}{b^2(b+1)(b+2)_{2n+2k}} \\ &\quad\times\left(\frac{b}{b+1}\frac{(b+1)_{2k}}{(b+2)_{2k}} + \frac{(b)_{2k}}{(b+1)_{2k}}\right)\frac{x^n x^k}{n!k!}\,. \end{aligned} \tag{50}$$

5. Derivatives with respect to parameters with summation index qn, $q \in \mathbb{N}$

5.1. Upper-parameter derivatives

Here, we consider the case when the summation index has a positive integer coefficient qn with $q \in \mathbb{N}$,

$$F(a) = \sum_{n=0}^{\infty} B(n)(a)_{qn}\frac{x^n}{n!}\,. \tag{51}$$

The particular case of $q = 2$ has been dealt with previously in Sec. 4. The derivative of the Pochhammer symbol in this case reads

$$\frac{\mathrm{d}(a)_{qn}}{\mathrm{d}a} = (a)_{qn}\sum_{k=0}^{qn-1}\frac{1}{a+k} = (a)_{qn}\frac{1}{a}\sum_{k=0}^{qn-1}\frac{(a)_k}{(a+1)_k}\,. \tag{52}$$

With the help of an extension of the resummation formula in Eq. (47) to the case of positive integer factors q,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{qn+q-1} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\lambda=0}^{q-1} A(qk+\lambda, n+k)\,, \tag{53}$$

one obtains the derivative with respect to an upper parameter with a summation index of the type qn as

$$\frac{\mathrm{d}F(a)}{\mathrm{d}a} = x\sum_{\lambda=0}^{q-1}\frac{1}{a+\lambda}\frac{\Gamma(a+q)}{\Gamma(a)}\sum_{k,n=0}^{\infty}B(n+k+1)\frac{(1)_k(1)_n}{(2)_{n+k}} \times \frac{(a+q)_{qn+qk}(a+\lambda)_{qk}}{(a+\lambda+1)_{qk}}\frac{x^n x^k}{n!k!}\,. \tag{54}$$

5.2. Lower-parameter derivatives

For the same summation index qn, but related to a lower parameter, we obtain for the derivative with respect to this parameter

$$\frac{\mathrm{d}F(b)}{\mathrm{d}b} = -x\sum_{\lambda=0}^{q-1}\frac{1}{b+\lambda}\frac{\Gamma(b)}{\Gamma(b+q)}\sum_{k,n=0}^{\infty}B(n+k+1)\frac{(1)_k(1)_n}{(2)_{n+k}} \times \frac{(b+\lambda)_{qk}}{(b+q)_{qn+qk}(b+\lambda+1)_{qk}}\frac{x^n x^k}{n!k!}\,. \tag{55}$$

6. Derivatives with respect to parameters with multiple summation indices $q_\lambda n_\lambda$, $q_\lambda \in \mathbb{N}$

6.1. Upper-parameter derivatives

By exploiting the previous results, in particular Eqs. (36) and (54), one can obtain the derivatives of hypergeometric functions with respect to a parameter with multiple summation indices $q_\lambda n_\lambda$, where $q_\lambda \in \mathbb{N}$,

$$F(a) = \sum_{n_1,\dots,n_\phi=0}^{\infty} B(n_1,\dots,n_\phi)(a)_{\sum_{\lambda=1}^{\phi} q_\lambda n_\lambda}\prod_{r=1}^{\phi}\frac{x^{n_r}}{n_r!}\,, \tag{56}$$

in the form

$$\frac{\mathrm{d}F(a)}{\mathrm{d}a} = \sum_{k,n_1,\dots,n_\phi=0}^{\infty}\sum_{\xi=1}^{\phi}\sum_{\gamma=0}^{q_\xi-1} x_\xi B(n_1,\dots,n_\xi+k+1,\dots,n_\phi)\frac{(1)_k(1)_{n_\xi}}{(2)_{n_\xi+k}}\prod_{r=1}^{\phi}\frac{x_r^{n_r}x_\xi^k}{n_r!k!} \times \frac{\Gamma(a+q_\xi)}{\Gamma(a)(a+\gamma)}\prod_{r=1}^{\xi-1}\frac{(a)_{\sum_{\lambda=1}^{r} q_\lambda n_\lambda}}{(a)_{\sum_{\lambda=1}^{r-1} q_\lambda n_\lambda}}\prod_{r=\xi+1}^{\phi}\frac{(a+q_\xi)_{\sum_{\lambda=1}^{r} q_\lambda n_\lambda+q_\xi k}}{(a+q_\xi)_{\sum_{\lambda=1}^{r-1} q_\lambda n_\lambda+q_\xi k}} \times \frac{(a+q_\xi)_{\sum_{\lambda=1}^{\xi} q_\lambda n_\lambda+q_\xi k}}{(a)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda}}\frac{(a+\gamma)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda+q_\xi k}}{(a+\gamma+1)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda+q_\xi k}}\,. \tag{57}$$

As an example, we calculate the derivative of the Horn function $H_3(a,b,c;x,y)$ (see Sec. A.4 for the definition) with respect to its upper parameter a with summation indices $2m+n$. This proceeds as

$$\frac{\mathrm{d}H_3(a)}{\mathrm{d}a} = \frac{yb}{c}\sum_{m,n,k=0}^{\infty}\frac{(1)_k(1)_n}{(2)_{n+k}}\frac{x^m y^n y^k}{m!n!k!}\frac{(a)_{2m+k}(1+a)_{2m+n+k}(b+1)_{n+k}}{(1+a)_{2m+k}(c+1)_{m+n+k}}$$
$$+\frac{x(1+a)}{c}\sum_{m,n,k=0}^{\infty}\frac{(1)_k(1)_m}{(2)_{m+k}}\frac{x^m y^n x^k}{m!n!k!}\frac{(a)_{2k}(2+a)_{2m+n+2k}(b)_n}{(1+a)_{2k}(c+1)_{m+n+k}}$$
$$+\frac{xa}{c}\sum_{m,n,k=0}^{\infty}\frac{(1)_k(1)_m}{(2)_{m+k}}\frac{x^m y^n x^k}{m!n!k!}\frac{(1+a)_{2k}(2+a)_{2m+n+2k}(b)_n}{(2+a)_{2k}(c+1)_{m+n+k}}, \tag{58}$$

which can be written as a sum of three generalized Lauricella series,

$$\frac{\mathrm{d}H_3(a)}{\mathrm{d}a} = \frac{yb}{c}F^{3:0;1;1}_{3:0;0;0}\left(\begin{matrix}\text{[a+1:2,1,1; b+1:0,1,1;a:2,0,1]: [-];[1:1];[1:1]}\\ \text{[c+1:1,1,1; a+1:2,0,1; 2:0,1,1]: [-];[-];[-]}\end{matrix}\; x, y, y\right)$$
$$+\frac{x(1+a)}{c}F^{1:1;1;2}_{2:0;0;1}\left(\begin{matrix}\text{[a+2:2,1,2]: [1:1];[b:1];[1:1;a:2]}\\ \text{[c+1:1,1,1; 2:1,0,1]: [-]; [-];[a+1:2]}\end{matrix}\; x, y, x\right)$$
$$+\frac{xa}{c}F^{2:1;1;1}_{2:0;0;1}\left(\begin{matrix}\text{[a+2:2,1,2]: [1:1];[b:1];[1,1]}\\ \text{[c+1:1,1,1; 2:1,0,1]: [-];[-];[a+2:2]}\end{matrix}\; x, y, x\right). \tag{59}$$

6.2. Lower-parameter derivatives

In complete analogy to Sec. 6.1, the derivative of a hypergeometric function with respect to a lower parameter with multiple summation indices $q_\lambda n_\lambda$ reads

$$\frac{\mathrm{d}F(b)}{\mathrm{d}b} = -\sum_{k,n_1,\ldots,n_\phi=0}^{\infty}\sum_{\xi=1}^{\phi}\sum_{\gamma=0}^{q_\xi-1} x_\xi B(n_1,\ldots,n_\xi+k+1,\ldots,n_\phi)\frac{(1)_k(1)_{n_\xi}}{(2)_{n_\xi+k}}\prod_{r=1}^{\phi}\frac{x_r^{n_r}x_\xi^k}{n_r!k!}$$
$$\times\frac{\Gamma(b)}{\Gamma(b+q_\xi)(b+\gamma)}\prod_{r=1}^{\xi-1}\frac{(b)_{\sum_{\lambda=1}^{r-1}q_\lambda n_\lambda}}{(b)_{\sum_{\lambda=1}^{r}q_\lambda n_\lambda}}\prod_{r=\xi+1}^{\phi}\frac{(b+q_\xi)_{\sum_{\lambda=1}^{r-1}q_\lambda n_\lambda+q_\xi k}}{(b+q_\xi)_{\sum_{\lambda=1}^{r}q_\lambda n_\lambda+q_\xi k}}$$
$$\times\frac{(b)_{\sum_{\lambda=1}^{\xi-1}q_\lambda n_\lambda}}{(b+q_\xi)_{\sum_{\lambda=1}^{\xi-1}q_\lambda n_\lambda+q_\xi k}}\frac{(b+\gamma)_{\sum_{\lambda=1}^{\xi-1}q_\lambda n_\lambda+q_\xi k}}{(b+\gamma+1)_{\sum_{\lambda=1}^{\xi-1}q_\lambda n_\lambda+q_\xi k}}. \tag{60}$$

From Eqs. (57) and (60), one can then deduce that any derivative of a generalized Lauricella series with respect to one of its parameters with summation indices of the type $q_\lambda n_\lambda$, $q_\lambda \in \mathbb{N}$, is expressible in terms of functions in the same class. In particular, we note that the derivative of a generalized Lauricella function of m variables can be expressed as a finite sum of generalized Lauricella functions of $m+1$ variables.

7. Derivatives with respect to parameters with a negative summation index

7.1. Upper-parameter derivatives

If one of the parameters in a Pochhammer symbol is endowed with multiple summation indices one of which is negative, as in the case $(a)_{n_1-n_2}$, the derivative with respect to this parameter requires additional care. In a first step, we can factorize,

$$(a)_{n_1-n_2} = (a+n_1)_{-n_2}(a)_{n_1}. \tag{61}$$

Here, the negative summation index $(-n_2)$ appears, and Eq. (5) needs to be replaced by

$$\Psi(z-n) - \Psi(z) = -\sum_{k=0}^{n-1} \frac{1}{z-k-1} . \tag{62}$$

Then, the derivative of a Pochhammer symbol can be written as

$$\frac{\mathrm{d}(a)_{-n}}{\mathrm{d}a} = -(a)_{-n} \sum_{k=0}^{n-1} \frac{1}{a-1} \frac{(a-1)_{-k}}{(a)_{-k}} . \tag{63}$$

As an example, we consider the hypergeometric function with a summation index of the type $(a)_{m-n}$,

$$F(a) = \sum_{m,n=0}^{\infty} B(n,m)(a)_{m-n} \frac{x^n y^m}{n!m!} . \tag{64}$$

By using the splitting formula in Eq. (61) together with Eq. (62) for the differentiation of a Pochhammer symbol with a negative index and the interchange of the order of summations in Eq. (3), we obtain

$$\begin{aligned}\frac{\mathrm{d}F(a)}{\mathrm{d}a} &= y \sum_{k,n,m=0}^{\infty} B(n,m+k+1) \frac{(1)_k(1)_m}{(2)_{m+k}} \frac{(a)_k(a+1)_{m+k-n}}{(a+1)_k} \frac{x^n y^m y^k}{n!m!k!} \\ &\quad - x \sum_{k,n,m=0}^{\infty} B(n+k+1,m) \frac{(1)_k(1)_n}{(2)_{n+k}} \frac{(a-1)_{m-n-k}(a-1)_{m-k}}{(a)_{m-k}(a-1)^2} \frac{x^n y^m x^k}{n!m!k!} .\end{aligned} \tag{65}$$

As another illustrative example, we calculate the derivative of the Horn hypergeometric function of two variables $H_1(a,b,c,d;x,y)$ with respect to its upper parameter a with summation indices $m-n$,

$$\begin{aligned}\frac{\mathrm{d}H_1(a)}{\mathrm{d}a} &= -\frac{ybc}{(a-1)^2} \sum_{m,n,k=0}^{\infty} \frac{(1)_k(1)_n}{(2)_{n+k}} \frac{x^m y^n y^k}{m!n!k!} \\ &\quad \times \frac{(a-1)_{m-k}(a-1)_{m-n-k}(b+1)_{m+n+k}(c+1)_{n+k}}{(a)_{m-k}(d)_m} \\ &\quad + \frac{xb}{d} \sum_{m,n,k=0}^{\infty} \frac{(1)_k(1)_m}{(2)_{m+k}} \frac{x^m y^n x^k}{m!n!k!} \frac{(a)_k(1+a)_{m-n+k}(b+1)_{m+n+k}(c)_n}{(1+a)_k(d+1)_{m+k}} ,\end{aligned} \tag{66}$$

which can be written as a sum of two generalized Lauricella series,

$$\begin{aligned}&\frac{\mathrm{d}H_1(a)}{\mathrm{d}a} \\ &= -\frac{ybc}{(a-1)^2} \\ &\quad \times F^{4:0;1;1}_{2:1;0;0}\left(\begin{matrix}\text{[a-1:1,-1,-1; b+1:1,1,1;c+1:0,1,1;a-1:1,0,-1]: [-];[1:1];[1:1]} \\ \text{[a:1,0,-1; 2:0,1,1;]: [d,1];[-];[-]}\end{matrix} \quad x,y,y \right) \\ &\quad + \frac{xb}{d} F^{2:1;1;2}_{2:0;0;1}\left(\begin{matrix}\text{[a+1:1,-1,1;b+1:1,1,1]: [1:1];[c:1];[1:1;a:1]} \\ \text{[d+1:1,0,1; 2:1,0,1]: [-]; [-];[a+1:1]}\end{matrix} \quad x,y,x \right) .\end{aligned} \tag{67}$$

7.2. Lower-parameter derivatives

Following the same procedure as above, for the derivative acting on a lower parameter of a hypergeometric function, we arrive at the explicit relation for the case of a summation index of the type $1/(b)_{m-n}$,

$$F(b) = \sum_{m,n=0}^{\infty} B(n,m) \frac{1}{(b)_{m-n}} \frac{x^n y^m}{n!m!} . \tag{68}$$

By using Eq. (61) and the derivative of the occurring Pochhammer symbol,

$$\frac{\mathrm{d}}{\mathrm{d}b} \frac{1}{(b+m)_{-n}} = \frac{1}{(b+m)_{-n}} \sum_{k=0}^{n-1} \frac{1}{b-1} \frac{(b-1)_{m-k}}{(b)_{m-k}} , \tag{69}$$

and exchanging the order of summation, we obtain

$$\begin{aligned} \frac{\mathrm{d}F(b)}{\mathrm{d}b} &= -y \sum_{k,n,m=0}^{\infty} B(n,m+k+1) \frac{(1)_k (1)_m}{(2)_{m+k}} \frac{1}{b^2} \frac{(b)_k}{(b+1)_{m+k-n} (b+1)_k} \frac{x^n y^m y^k}{n!m!k!} \\ &\quad + x \sum_{k,n,m=0}^{\infty} B(n+k+1,m) \frac{(1)_k (1)_n}{(2)_{n+k}} \frac{(b-1)_{m-k}}{(b-1)_{m-n-k} (b)_{m-k}} \frac{x^n y^m x^k}{n!m!k!} . \end{aligned} \tag{70}$$

8. Derivatives with respect to parameters with summation index qn, $-q \in \mathbb{N}$

8.1. Upper-parameter derivatives

The final step in obtaining the full set of relations for the derivative of a hypergeometric series with respect to one of its parameters with any set of summation indices consists of elaborating the case of negative summation index qn, where $-q \in \mathbb{N}$,

$$F(a) = \sum_{n=0}^{\infty} B(n) (a)_{qn} \frac{x^n}{n!} . \tag{71}$$

The derivative of the Pochhammer symbol in this case reads

$$\frac{\mathrm{d}(a)_{qn}}{\mathrm{d}a} = -(a)_{qn} \sum_{k=0}^{-qn-1} \frac{1}{a-1} \frac{(a-1)_{-k}}{(a)_{-k}} , \tag{72}$$

and we note that the upper limit of summation is indeed positive, due to $q < 0$. Upon adapting the interchange of the order of summation in Eq. (53) to the case of negative q, we obtain for the derivative of a hypergeometric function with respect to an upper parameter with a summation index of the type qn with integer $q < 0$

$$\begin{aligned} \frac{\mathrm{d}F(a)}{\mathrm{d}a} &= -x \sum_{\lambda=0}^{-q-1} \frac{1}{a-1-\lambda} \frac{\Gamma(a+q)}{\Gamma(a)} \sum_{k,n=0}^{\infty} B(n+k+1) \frac{(1)_k (1)_n}{(2)_{n+k}} \\ &\quad \times \frac{(a+q)_{qn+qk} (a-1-\lambda)_{qk}}{(a-\lambda)_{qk}} \frac{x^n x^k}{n!k!} . \end{aligned} \tag{73}$$

8.2. Lower-parameter derivatives

For the same summation index related to a lower parameter, i.e., exchanging $(a)_{qn} \to 1/(b)_{qn}$ in Eq. (71), one obtains for the derivative of a hypergeometric function with respect to that parameter

$$\frac{\mathrm{d}F(b)}{\mathrm{d}b} = x \sum_{\lambda=0}^{-q-1} \frac{1}{b-\lambda-1} \frac{\Gamma(b)}{\Gamma(b+q)} \sum_{k,n=0}^{\infty} B(n+k+1) \frac{(1)_k (1)_n}{(2)_{n+k}} \times \frac{(b-\lambda-1)_{qk}}{(b+q)_{qn+qk}(b-\lambda)_{qk}} \frac{x^n x^k}{n!k!} . \tag{74}$$

9. Convergence of the series for derivatives of hypergeometric functions

In establishing the regions of convergence of the series for the derivatives of hypergeometric functions, we follow the same rule as in Ref. [17] (see, in particular, p. 56 thereof), namely, we exclude all exceptional values of the parameters, i.e., those values for which the series terminates, becomes meaningless or reduces to a finite sum of hypergeometric series of lower dimension.

To determine the regions of convergence for the series which have been obtained for derivatives of hypergeometric functions, we can utilize the parameter cancellation theorem (see p. 108 of Ref. [17]). This states that the region of convergence for a hypergeometric series is independent of the parameters, provided exceptional values of parameters are excluded. For example, the series

$$\sum_{n=0}^{\infty} B(m,n) \frac{(a)_{qn}}{(b)_{qn}} \frac{x^n}{n!} \tag{75}$$

and

$$\sum_{n=0}^{\infty} B(m,n) \frac{x^n}{n!} \tag{76}$$

have the same region of convergence. By using this theorem, we can exclude from the hypergeometric series all Pochhammer symbols with different parameters, but with the same summation index.

As an example, we consider here explicitly the regions of convergence for the derivatives in Sec. 5. The other cases can be dealt with in a similar way. It is easy to see that the region of convergence for the series in Eq. (54) after application of the convergence theorem is equivalent to the one of the series

$$\sum_{n,k=0}^{\infty} B(n+k+1)(a)_{qn+qk} \frac{x^n x^k}{(n+k)!} . \tag{77}$$

Then, applying the summation formula in Eq. (53), one finds that the region of convergence of the series in Eq. (77) is the same as the one of the expression

$$\sum_{n=0}^{\infty} \sum_{k=0}^{qn+q-1} B(n)(a)_{qn} \frac{x^n}{(n)!} = \sum_{n=0}^{\infty} (qn+q-1) B(n)(a)_{qn} \frac{x^n}{(n)!} . \tag{78}$$

Due to the convergence theorem, the region of convergence of the latter series coincides with the one of the original function before differentiation.

In this way, we prove the theorem that the expressions for the derivatives of hypergeometric series have the same regions of convergence as the initial hypergeometric functions.

The regions of convergence for hypergeometric series in two, three, and more variables can be found by using Horn's theorem of convergence [17] (see, in particular, p. 56 of Ref. [17]). This theorem implies an absolute region of convergence of these hypergeometric series, so that the use of the summation formulae in Eqs. (3), (47), and (53) in our calculations is mathematically rigorous.

10. Derivatives with respect to parameters for general case of summation indices

The combination of Eqs. (57) and (73) finally leads to an expression for the derivative of a hypergeometric function with respect to one of its upper parameters related to a summation index with any integer coefficient. The relevant equation reads

$$F(a) = \sum_{n_1,\ldots,n_\phi=0}^{\infty} B(n_1,\ldots,n_\phi)(a)_{\sum_{\lambda=1}^{\phi} q_\lambda n_\lambda} \prod_{r=1}^{\phi} \frac{x^{n_r}}{n_r!}, \quad q_\lambda \in \mathbb{Z}, \tag{79}$$

$$\begin{aligned}
\frac{\mathrm{d}F(a)}{\mathrm{d}a} &= \sum_{k,n_1,\ldots,n_\phi=0}^{\infty} \sum_{\xi=1}^{\phi} \sum_{\gamma=0}^{|q_\xi|-1} x_\xi B(n_1,\ldots,n_\xi+k+1,\ldots,n_\phi) \frac{(1)_k(1)_{n_\xi}}{(2)_{n_\xi+k}} \prod_{r=1}^{\phi} \frac{x_r^{n_r} x_\xi^k}{n_r!k!} \\
&\times \frac{\Gamma(a+q_\xi)}{\Gamma(a)} \prod_{r=1}^{\xi-1} \frac{(a)_{\sum_{\lambda=1}^{r} q_\lambda n_\lambda}}{(a)_{\sum_{\lambda=1}^{r-1} q_\lambda n_\lambda}} \prod_{r=\xi+1}^{\phi} \frac{(a+q_\xi)_{\sum_{\lambda=1}^{r} q_\lambda n_\lambda + q_\xi k}}{(a+q_\xi)_{\sum_{\lambda=1}^{r-1} q_\lambda n_\lambda + q_\xi k}} \\
&\times \frac{(a+q_\xi)_{\sum_{\lambda=1}^{\xi} q_\lambda n_\lambda + q_\xi k}}{(a)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda}} \beta, \\
\beta &= \frac{1}{a+\gamma} \frac{(a+\gamma)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}{(a+\gamma+1)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}, \qquad q_\xi > 0, \\
\beta &= -\frac{1}{a-\gamma-1} \frac{(a-\gamma-1)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}{(a-\gamma)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}, \qquad q_\xi < 0.
\end{aligned} \tag{80}$$

A similar equation holds for the derivative with respect to a lower parameter,

$$F(b) = \sum_{n_1,\ldots,n_\phi=0}^{\infty} B(n_1,\ldots,n_\phi) \frac{1}{(b)_{\sum_{\lambda=1}^{\phi} q_\lambda n_\lambda}} \prod_{r=1}^{\phi} \frac{x^{n_r}}{n_r!}, \quad q_\lambda \in \mathbb{Z}, \tag{81}$$

$$\begin{aligned}
\frac{\mathrm{d}F(b)}{\mathrm{d}b} &= \sum_{k,n_1,\ldots,n_\phi=0}^{\infty} \sum_{\xi=1}^{\phi} \sum_{\gamma=0}^{|q_\xi|-1} x_\xi B(n_1,\ldots,n_\xi+k+1,\ldots,n_\phi) \frac{(1)_k(1)_{n_\xi}}{(2)_{n_\xi+k}} \prod_{r=1}^{\phi} \frac{x_r^{n_r} x_\xi^k}{n_r!k!} \\
&\times \frac{\Gamma(b)}{\Gamma(b+q_\xi)} \prod_{r=1}^{\xi-1} \frac{(b)_{\sum_{\lambda=1}^{r-1} q_\lambda n_\lambda}}{(b)_{\sum_{\lambda=1}^{r} q_\lambda n_\lambda}} \prod_{r=\xi+1}^{\phi} \frac{(b+q_\xi)_{\sum_{\lambda=1}^{r-1} q_\lambda n_\lambda + q_\xi k}}{(b+q_\xi)_{\sum_{\lambda=1}^{r} q_\lambda n_\lambda + q_\xi k}} \\
&\times \frac{(b)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda}}{(b+q_\xi)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}} \beta,
\end{aligned}$$

$$\beta = -\frac{1}{b+\gamma} \frac{(b+\gamma)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}{(b+\gamma+1)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}, \qquad q_\xi > 0,$$
$$\beta = \frac{1}{b-\gamma-1} \frac{(b-\gamma-1)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}{(b-\gamma)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}, \qquad q_\xi < 0. \tag{82}$$

As an example, we present here the derivative of the function $G_3(a, b; x, y)$ in its upper parameter a with summation indices $2n - m$,

$$\begin{aligned}
&\frac{\mathrm{d}G_3(a)}{\mathrm{d}a} \\
&= \frac{y(a+1)}{(b-1)} \sum_{m,n,k=0}^{\infty} \frac{(1)_k (1)_n}{(2)_{n+k}} \frac{x^m y^n y^k}{m!n!k!} \frac{(a)_{-m+2k} (a+2)_{2n-m+2k} (b-1)_{2m-n-k}}{(a+1)_{-m+2k}} \\
&+ \frac{ya}{b-1} \sum_{m,n,k=0}^{\infty} \frac{(1)_k (1)_n}{(2)_{n+k}} \frac{x^m y^n y^k}{m!n!k!} \frac{(a+1)_{-m+2k} (a+2)_{2n-m+2k} (b-1)_{2m-n-k}}{(a+2)_{-m+2k}} \\
&- \frac{xb(b+1)}{(a-1)^2} \sum_{m,n,k=0}^{\infty} \frac{(1)_k (1)_m}{(2)_{m+k}} \frac{x^m y^n x^k}{m!n!k!} \frac{(a-1)_{-k} (a-1)_{2n-m-k} (b+2)_{2m-n+2k}}{(a)_{-k}},
\end{aligned}$$

which can be written as a sum of three generalized Lauricella series,

$$\begin{aligned}
&\frac{\mathrm{d}G_3(a)}{\mathrm{d}a} \\
&= \frac{y(a+1)}{b-1} \\
&\times F^{3:0;1;1}_{2:0;0;0} \left(\begin{matrix} \text{[a+2:-1,2,2; b-1:2,-1,-1;a:-1,0,2]:[-]; [1:1];[1:1]} \\ \text{[a+1:-1,0,2; 2:0,1,1;]: [-];[-];[-]} \end{matrix} \; x, y, y \right) \\
&+ \frac{ya}{b-1} F^{3:0;1;1}_{2:0;0;0} \left(\begin{matrix} \text{[a+2:-1,2,2; b-1:2,-1,-1;a+1:-1,0,2]:[-]; [1:1];[1:1]} \\ \text{[a+2:-1,0,2; 2:0,1,1;]: [-];[-];[-]} \end{matrix} \; x, y, y \right) \\
&- \frac{xb(b+1)}{(a-1)^2} F^{2:1;0;1}_{1:0;0;1} \left(\begin{matrix} \text{[a-1:-1,2,-1;b+2:2,-1,2]: [1:1];[-];[1:1;a-1:-1]} \\ \text{[2:1,0,1]: [-]; [-];[a:-1]} \end{matrix} \; x, y, x \right).
\end{aligned}$$

11. Conclusions

11.1. Derivatives of Horn hypergeometric functions with respect to parameters

With the results of the present article, the derivatives of the following Horn hypergeometric functions of multiple variables x_n,

$$\sum_{m_1,\ldots,m_l=0}^{\infty} \prod_{i,j} \frac{(a_j)_{\sum_{k=1}^{l} q_k m_k}}{(b_i)_{\sum_{k=1}^{l} q_k m_k}} \prod_{n=1}^{l} \frac{x_n^{m_n}}{m_n!}, \qquad q_k \in \mathbb{Z}, \tag{83}$$

with respect to one of their parameters, a_j and b_i, can be expressed with the help of Eqs. (80) and (82), or, alternatively, as finite sums of Horn hypergeometric functions, where the n-th derivative of a function of m variables is expressed as series with $n + m$ variables. The regions of convergence for these derivatives are the same as for the initial functions. Specifically, for the 34 distinct confluent and non-confluent Horn hypergeometric functions of two variables, the n-th derivatives are expressed as Horn hypergeometric functions of $n+2$ variables.

Derivatives of generalized Lauricella series, i.e., of Horn hypergeometric series with summation coefficients $q_k \in \mathbb{N}$, with respect to one of their (upper or lower) parameters can be expressed as finite sums of generalized Lauricella series, as explained in Sec. 6. In particular, the derivatives of generalized Lauricella series with respect to one of their parameters with summation coefficient $q_k \in \{0, 1\}$ can be written as finite sums of generalized Lauricella series with summation coefficients in the same alphabet $q_k \in \{0, 1\}$ as detailed in Sec. 3. Finally, all n-th derivatives of generalized Appell hypergeometric functions, generalized Kampé de Fériet functions, and generalized hypergeometric functions of one variable are expressible in terms of generalized Lauricella series with summation coefficients $q_k \in \{0, 1\}$.

11.2. Applications in high-energy physics

The main motivation of the present research are calculations in quantum field theory, i.e., of Feynman diagrams and their series expansions. By applying the Mellin-Barnes method, Feynman diagrams can be written in the form of Eq. (4) as Horn multi-variable hypergeometric functions (see, e.g., Ref. [22]). For example, particular types of one-loop diagrams are expressible in terms of generalized hypergeometric functions or Appell series [23–27].

For the evaluation of the finite parts of dimensionally regularized Feynman integrals in $D = 4 - 2\varepsilon$ dimensional space time, one has to construct their expansions in ε. This has motivated us to seek a general method to obtain the all-order ε expansions of Horn hypergeometric functions. There exist a lot of analytic and numerical methods based on different algorithms which are appropriate for dealing with ε expansions of Feynman integrals [28–34] and are implemented in computer packages [15,35–40]; see, for example, the review in Ref. [41].

Feynman integrals written in the form of Horn hypergeometric functions as in Eq. (4) depend on the space-time parameter ε only through some parameters in the Pochhammer symbols, namely B_a and D_a, so that the construction of the ε expansion of a given Feynman integral is equivalent to taking the derivatives of the pertaining Horn hypergeometric functions with respect to their parameters.

From Sec. 11.1, we can conclude that the ε expansions of Feynman integrals at any order are expressible in terms of Horn hypergeometric functions. Specifically, the n-th term of the ε series can be expressed as a Horn hypergeometric function of $n+m$ variables, where m is the number of summations in the Horn representation of the Feynman integral. The region of convergence of any of these parameter derivatives, i.e., the coefficients in the ε expansion, and the initial Feynman integral are the same. The explicit formula for the n-th term of the ε expansion can be obtained with the help of Eqs. (80) and (82).

In the series of papers in Refs. [22,42–45], the method of differential reduction of hypergeometric function has been worked out. In particular, the so-called step-up and step-down differential operators have been introduced, which shift the parameters of hypergeometric functions by unity. By applying such differential operators to a hypergeometric function, the value of any parameter can be shifted by an arbitrary integer. The construction of differential operators allows us to define a set of exceptional parameters for a hypergeometric function and then to find the condition of reducibility of the monodromy group of the hypergeometric function.

By expressing the ε expansion of a Feynman diagram in terms of Horn hypergeometric functions and applying the above-mentioned method of differential reduction, one can reduce the corresponding integrals to some basic subset of hypergeometric functions and express them as series with the least number of infinite summations.

We plan to implement our procedure of taking derivatives of Horn hypergeometric functions with respect to their parameters in a `Mathematica`-based computer package along with the differential-reduction algorithm.

11.3. Example of application to Feynman integral calculation

In Ref. [46], the two-loop sunset diagram with two different masses and threshold kinematics was considered through $\mathcal{O}(1)$ accuracy; in Ref. [47], this result was extended to three- and four-point diagrams. In the hypergeometric-series representations of master integrals, derivatives of generalized hypergeometric functions with respect to parameters appeared, and some artificial parameter δ was introduced. As an example, let us consider the first equality in Eq. (2) of Ref. [47], which we copy here for the reader's convenience:

$$e^{2\varepsilon\gamma_E}\left(M^2\right)^{2\varepsilon+1} J^{mM}_{01022} = {}_4F_3\left(\begin{matrix}\frac{1}{2},1,1,1\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\end{matrix}\middle| -\frac{x^2}{4}\right)$$
$$+\frac{1}{4}\frac{d}{d\delta}\left[\frac{x^{1-2\delta}(1-2\delta)^2}{(1-\delta)(3-16\delta)}{}_4F_3\left(\begin{matrix}1,1-\delta,\frac{3}{2}-\delta,\frac{3}{2}-\delta\\ \frac{5}{4}-\delta,\frac{7}{4}-\delta,2-\delta\end{matrix}\middle| -\frac{x^2}{4}\right)\right]_{\delta=0}+\mathcal{O}(\varepsilon)\,. \qquad (84)$$

By application the results in Sec. 10, we find that the derivative of the generalized hypergeometric function in Eq. (84) can be expressed in terms of the well-known Kampé de Fériet series and that the artificial parameter δ can so be eliminated,

$$e^{2\varepsilon\gamma_E}\left(M^2\right)^{2\varepsilon+1} J^{mM}_{01022} = {}_4F_3\left(\begin{matrix}\frac{1}{2},1,1,1\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\end{matrix}\middle| -\frac{x^2}{4}\right)$$
$$+\frac{1}{4}\left(\frac{7}{9}x-\frac{2}{3}x\log x\right){}_4F_3\left(\begin{matrix}1,1,\frac{3}{2},\frac{3}{2}\\ \frac{5}{4},\frac{7}{4},2\end{matrix}\middle| -\frac{x^2}{4}\right)$$
$$-\frac{3x^3}{280}F^{3:2;1}_{3:1;0}\left[\begin{matrix}(2,\frac{5}{2},\frac{5}{2}):(1,1);(1)\\ (\frac{9}{4},\frac{11}{4},3):(2);(-)\end{matrix}\; -\frac{x^2}{4},-\frac{x^2}{4}\right]$$
$$-\frac{x^3}{70}F^{3:2;1}_{3:1;0}\left[\begin{matrix}(2,\frac{5}{2},\frac{5}{2}):(1,\frac{3}{2});(1)\\ (\frac{9}{4},\frac{11}{4},3):(\frac{5}{2});(-)\end{matrix}\; -\frac{x^2}{4},-\frac{x^2}{4}\right]$$
$$+\frac{3x^3}{350}F^{3:2;1}_{3:1;0}\left[\begin{matrix}(2,\frac{5}{2},\frac{5}{2}):(1,\frac{5}{4});(1)\\ (\frac{9}{4},\frac{11}{4},3):(\frac{9}{4});(-)\end{matrix}\; -\frac{x^2}{4},-\frac{x^2}{4}\right]$$
$$+\frac{3x^3}{490}F^{3:2;1}_{3:1;0}\left[\begin{matrix}(2,\frac{5}{2},\frac{5}{2}):(1,\frac{7}{4});(1)\\ (\frac{9}{4},\frac{11}{4},3):(\frac{11}{4});(-)\end{matrix}\; -\frac{x^2}{4},-\frac{x^2}{4}\right]$$
$$+\frac{3x^3}{560}F^{3:2;1}_{3:1;0}\left[\begin{matrix}(2,\frac{5}{2},\frac{5}{2}):(1,2);(1)\\ (\frac{9}{4},\frac{11}{4},3):(3);(-)\end{matrix}\; -\frac{x^2}{4},-\frac{x^2}{4}\right]+\mathcal{O}(\varepsilon)\,. \qquad (85)$$

The other six master integrals in Refs. [46,47] that contain derivatives of generalized hypergeometric functions with respect to the artificial parameter δ (see Eqs. (2) and (62) in Ref. [47]) may also be expressed in terms of Kampé de Fériet series by similar procedures.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We are grateful to M. Yu. Kalmykov and S.-O. Moch for fruitful discussions, useful remarks, and valuable contributions to this paper. The work of V.V.B. was supported in part by the Heisenberg–Landau Program. This work was supported in part by the German Federal Ministry for Education and Research BMBF through Grant No. 05H18GUCC1 and by the German Research Foundation DFG through Grant Nos. KN 365/12-1, KN 365/13-1, and KN 365/14-1.

Appendix A. Definitions of hypergeometric series

Here, we give the definitions of some hypergeometric series, whose derivatives have been considered in this article.

A.1. Appell hypergeometric functions of two variables

The Appell [48–50] hypergeometric functions F_1, F_2, F_3, and F_4 are defined as expansions about $x = y = 0$ as

$$\begin{aligned} F_1(a, b_1, b_2, c; x, y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \\ F_2(a, b_1, b_2, c_1, c_2; x, y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}, \\ F_3(a_1, a_2, b_1, b_2, c; x, y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \\ F_4(a, b, c_1, c_2; x, y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}. \end{aligned} \tag{A.1}$$

A.2. Multi-variable extensions of Kampé de Fériet series

The extensions of Kampé de Fériet functions of two variables to the multi-variable case read [17]

$$\begin{aligned} F^{p:q_1;\ldots;q_n}_{l:m_1;\ldots;m_n}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} &= F^{p:q_1;\ldots;q_n}_{l:m_1;\ldots;m_n}\left[\begin{matrix} (a)_p : (b^{(1)}_{q_1}); \ldots; (b^{(n)}_{q_n}) \\ (\alpha)_l : (\beta^{(1)}_{m_1}); \ldots; (\beta^{(n)}_{m_n}) \end{matrix} \; x_1, \ldots, x_n \right] \\ &= \sum_{s_1,\ldots,s_n=0}^{\infty} \Lambda(s_1, \ldots, s_n) \frac{x_1^{s_1}}{s_1!}\frac{x_n^{s_n}}{s_n!}, \end{aligned} \tag{A.2}$$

where

$$\Lambda(s_1,\dots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\dots+s_n}\prod_{j=1}^{q_1}(b_j^{(1)})_{s_1}\cdots\prod_{j=1}^{q_n}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{l}(\alpha_j)_{s_1+\dots+s_n}\prod_{j=1}^{m_1}(\beta_j^{(1)})_{s_1}\cdots\prod_{j=1}^{m_n}(\beta_j^{(n)})_{s_n}}. \tag{A.3}$$

A.3. Generalized Lauricella series

Series of this type have been introduced in Ref. [51]. Special cases of these functions are reduced to multi-variable extensions of Kampé de Fériet series. The latter include confluent and non-confluent Lauricella functions,

$$\begin{aligned}
&F^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\\
&=F^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta^{(1)},\dots,\theta^{(n)}]:[(b^1):\phi^{(1)}];\dots;[(b^n):\phi^{(n)}]\\ [(c):\psi^{(1)},\dots,\psi^{(n)}]:[(d^1):\delta^{(1)}];\dots;[(d^n):\delta^{(n)}]\end{matrix}\ x_1,\dots,x_n\right)\\
&=\sum_{s_1,\dots,s_n=0}^{\infty}\Omega(s_1,\dots,s_n)\frac{x_1^{s_1}}{s_1!}\frac{x_n^{s_n}}{s_n!},
\end{aligned} \tag{A.4}$$

where

$$\Omega(s_1,\dots,s_n)=\frac{\prod_{j=1}^{A}(a_j)_{s_1\theta_j^{(1)}+\dots+s_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{s_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{s_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{s_1\psi_j^{(1)}+\dots+s_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{s_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{s_n\delta_j^{(n)}}}, \tag{A.5}$$

and all parameters $\theta_1^{(1)},\psi_1^{(1)},\phi_1^{(1)},\delta_1^{(1)},\dots,\theta_A^{(n)},\psi_C^{(n)},\phi_{B^{(n)}}^{(n)},\delta_{D^{(n)}}^{(n)}$ are positive and real.

A.4. Horn series in two variables

Here, we recall the definitions of some Horn functions of two variables that have been used in this article,

$$H_1(a,b,c,d;x,y)=\sum_{n,m=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}c_n}{(d)_m}\frac{x^m y^n}{m!n!}, \tag{A.6}$$

$$H_3(a,b,c;x,y)=\sum_{n,m=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}}\frac{x^m y^n}{m!n!}, \tag{A.7}$$

$$G_3(a,b;x,y)=\sum_{n,m=0}^{\infty}(a)_{2n-m}(b)_{2m-n}\frac{x^m y^n}{m!n!}. \tag{A.8}$$

References

[1] A.R. Miller, Summations for certain series containing the digamma function, J. Phys. A, Math. Gen. 39 (2006) 3011.
[2] D. Cvijović, A.R. Miller, A reduction formula for the Kampé de Fériet function, Appl. Math. Lett. 23 (2010) 769.

[3] D. Cvijović, Closed-form summations of certain hypergeometric-type series containing the digamma function, J. Phys. A, Math. Theor. 41 (2008) 455205.

[4] L.U. Ancarani, G. Gasaneo, Derivatives of any order of the Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ with respect to the parameters a, b and c, J. Phys. A, Math. Theor. 42 (2009) 395208.

[5] L.U. Ancarani, G. Gasaneo, Derivatives of any order of the hypergeometric function ${}_pF_q(a_1, ..., a_p; b_1, .., b_q; z)$ with respect to the parameters a_i and b_i, J. Phys. A, Math. Theor. 43 (2010) 085210.

[6] L.U. Ancarani, J.A. Del Punta, G. Gasaneo, Derivatives of Horn hypergeometric functions with respect to their parameters, J. Math. Phys. 58 (2017) 073504.

[7] V. Sahai, A. Verma, Derivatives of Appell functions with respect to parameters, J. Inequal. Spec. Funct. 6 (3) (2015) 1.

[8] B.Xh. Fejzullahu, Parameter derivatives of the generalized hypergeometric function, Integral Transforms Spec. Funct. 28 (2017) 781.

[9] D. Greynat, J. Sesma, A new approach to the epsilon expansion of generalized hypergeometric functions, Comput. Phys. Commun. 185 (2014) 472.

[10] D. Greynat, J. Sesma, G. Vulvert, Derivatives of the Pochhammer and reciprocal Pochhammer symbols and their use in epsilon-expansions of Appell and Kampé de Fériet functions, J. Math. Phys. 55 (2014) 043501.

[11] E.R. Hansen, A Table of Series and Products, Prentice-Hall Series in Automatic Computation, vol. XVIII, Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[12] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables, Dover Publications Inc., New York, N.Y., 1974.

[13] Yu.A. Brychkov, Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas, Chapman and Hall/CRC Press, New York, 2008.

[14] J. Fröhlich, Parameter derivatives of the jacoby polynomials and the gaussian hypergeometric function, Integral Transforms Spec. Funct. 2 (1994) 253.

[15] S. Moch, P. Uwer, S. Weinzierl, Nested sums, expansion of transcendental functions, and multiscale multiloop integrals, J. Math. Phys. 43 (2002) 3363, arXiv:hep-ph/0110083.

[16] H.M. Srivastava, H.L. Manocha, A Treatise on Generating Functions, Ellis Horwood Series in Mathematics and Its Applications, Ellis Horwood Ltd., Chichester, 1984.

[17] H.M. Srivastava, P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Ellis Horwood Series in Mathematics and Its Applications, Ellis Horwood Ltd., Chichester, 1985.

[18] C. de Calan, A.P.C. Malbouisson, Complete Mellin representation and asymptotic behaviours of Feynman amplitudes, Ann. Inst. Henri Poincaré. Phys. Théor. 32 (1980) 91.

[19] C. de Calan, F. David, V. Rivasseau, Renormalization in the complete Mellin representation of Feynman amplitudes, Commun. Math. Phys. 78 (1981) 531.

[20] C. de Calan, A.P.C. Malbouisson, Infrared and ultraviolet dimensional meromorphy of Feynman amplitudes, Commun. Math. Phys. 90 (1983) 413.

[21] V.A. Smirnov, Analytic Tools for Feynman Integrals, Springer Tracts Mod. Phys., vol. 250, 2012, p. 1.

[22] V.V. Bytev, M.Yu. Kalmykov, B.A. Kniehl, Differential reduction of generalized hypergeometric functions from Feynman diagrams: One-variable case, Nucl. Phys. B 836 (2010) 129, arXiv:0904.0214 [hep-th].

[23] A.I. Davydychev, Some exact results for N-point massive Feynman integrals, J. Math. Phys. 32 (1991) 1052.

[24] A.I. Davydychev, General results for massive N-point Feynman diagrams with different masses, J. Math. Phys. 33 (1992) 358.

[25] J. Fleischer, F. Jegerlehner, O.V. Tarasov, A new hypergeometric representation of one-loop scalar integrals in d dimensions, Nucl. Phys. B 672 (2003) 303, arXiv:hep-ph/0307113.

[26] B.A. Kniehl, O.V. Tarasov, Functional equations for one-loop master integrals for heavy-quark production and Bhabha scattering, Nucl. Phys. B 820 (2009) 178, arXiv:0904.3729 [hep-ph].

[27] B.A. Kniehl, O.V. Tarasov, Analytic result for the one-loop scalar pentagon integral with massless propagators, Nucl. Phys. B 833 (2010) 298, arXiv:1001.3848 [hep-th].

[28] F. Jegerlehner, M.Yu. Kalmykov, O. Veretin, $\overline{\mathrm{MS}}$ vs. pole masses of gauge bosons II: two-loop electroweak fermion corrections, Nucl. Phys. B 658 (2003) 49, arXiv:hep-ph/0212319.

[29] A.I. Davydychev, M.Yu. Kalmykov, Massive Feynman diagrams and inverse binomial sums, Nucl. Phys. B 699 (2004) 3, arXiv:hep-th/0303162.

[30] M.Yu. Kalmykov, Gauss hypergeometric function: reduction, ϵ-expansion for integer/half-integer parameters and Feynman diagrams, J. High Energy Phys. 0604 (2006) 056, arXiv:hep-th/0602028.

[31] M.Yu. Kalmykov, B.F.L. Ward, S. Yost, All order ϵ-expansion of Gauss hypergeometric functions with integer and half/integer values of parameters, J. High Energy Phys. 0702 (2007) 040, arXiv:hep-th/0612240.

[32] M.Y. Kalmykov, B.F.L. Ward, S.A. Yost, On the all-order ϵ-expansion of generalized hypergeometric functions with integer values of parameters, J. High Energy Phys. 0711 (2007) 009, arXiv:0708.0803 [hep-th].

[33] M.Yu. Kalmykov, B.A. Kniehl, All-order ϵ expansions of hypergeometric functions of one variable, Phys. Part. Nucl. 41 (2010) 942, arXiv:1003.1965 [math-ph].

[34] S.A. Yost, V.V. Bytev, M.Yu. Kalmykov, B.A. Kniehl, B.F.L. Ward, The Epsilon Expansion of Feynman Diagrams via Hypergeometric Functions and Differential Reduction, arXiv:1110.0210 [math-ph].

[35] S. Weinzierl, Symbolic expansion of transcendental functions, Comput. Phys. Commun. 145 (2002) 357, arXiv:math-ph/0201011.

[36] S. Weinzierl, Expansion around half integer values, binomial sums and inverse binomial sums, J. Math. Phys. 45 (2004) 2656, arXiv:hep-ph/0402131.

[37] S. Moch, P. Uwer, -XSUMMER- Transcendental functions and symbolic summation in FORM, Comput. Phys. Commun. 174 (2006) 759, arXiv:math-ph/0508008.

[38] T. Huber, D. Maître, HypExp, a Mathematica package for expanding hypergeometric functions around integer-valued parameters, Comput. Phys. Commun. 175 (2006) 122, arXiv:hep-ph/0507094.

[39] T. Huber, D. Maître, HypExp 2, Expanding hypergeometric functions about half-integer parameters, Comput. Phys. Commun. 178 (2008) 755, arXiv:0708.2443 [hep-ph].

[40] Z.-W. Huang, J. Liu, NumExp: Numerical epsilon expansion of hypergeometric functions, Comput. Phys. Commun. 184 (2013) 1973, arXiv:1209.3971 [physics.comp-ph].

[41] J. Blümlein, C. Schneider, Analytic computing methods for precision calculations in quantum field theory, Int. J. Mod. Phys. A 33 (2018) 1830015, arXiv:1809.02889 [hep-ph].

[42] V.V. Bytev, M.Yu. Kalmykov, B.A. Kniehl, HYPERDIRE, HYPERgeometric functions DIfferential REduction: MATHEMATICA-based packages for differential reduction of generalized hypergeometric functions ${}_pF_{p-1}$, F_1, F_2, F_3, F_4, Comput. Phys. Commun. 184 (2013) 2332, arXiv:1105.3565 [math-ph].

[43] V.V. Bytev, B.A. Kniehl, HYPERDIRE HYPERgeometric functions DIfferential REduction: Mathematica-based packages for the differential reduction of generalized hypergeometric functions: Horn-type hypergeometric functions of two variables, Comput. Phys. Commun. 189 (2015) 128, arXiv:1309.2806 [math-ph].

[44] V.V. Bytev, M.Yu. Kalmykov, S.-O. Moch, HYPERgeometric functions DIfferential REduction (HYPERDIRE): MATHEMATICA based packages for differential reduction of generalized hypergeometric functions: F_D and F_S Horn-type hypergeometric functions of three variables, Comput. Phys. Commun. 185 (2014) 3041, arXiv:1312.5777 [math-ph].

[45] V.V. Bytev, B.A. Kniehl, HYPERDIRE—HYPERgeometric functions DIfferential REduction: Mathematica-based packages for the differential reduction of generalized hypergeometric functions: Lauricella function F_C of three variables, Comput. Phys. Commun. 206 (2016) 78, arXiv:1602.00917 [math-ph].

[46] B.A. Kniehl, A.V. Kotikov, A.I. Onishchenko, O.L. Veretin, Two-loop sunset diagrams with three massive lines, Nucl. Phys. B 738 (2006) 306, arXiv:hep-ph/0510235.

[47] B.A. Kniehl, A.V. Kotikov, A.I. Onishchenko, O.L. Veretin, Two-loop diagrams in nonrelativistic QCD with elliptics, Nucl. Phys. B 948 (2019) 114780, arXiv:1907.04638 [hep-ph].

[48] P. Appell, Sur les séries hypergéométriques de deux variables et sur des équations différentielles linéaires aux dérivées partielles, C. R. Hebd. Séances Acad. Sci. 90 (1880) 296, 731.

[49] P. Appell, J. Kampé de Fériet, Fonctions hypergéométriques et hypersphériques. Polynomes d'Hermite, Gauthier-Villars, Paris, 1926.

[50] M.J. Schlosser, Multiple Hypergeometric Series: Appell Series and Beyond, in: C. Schneider, J. Blümlein (Eds.), Computer Algebra in Quantum Field Theory, in: Texts & Monographs in Symbolic Computation, Springer, Vienna, 2013, p. 305.

[51] H.M. Srivastava, An infinite summation formula associated with Appell's function F_2, Math. Proc. Camb. Philos. Soc. 65 (1969) 679.