THE EFFECT OF RADIATION ON THE MOTION OF RELATIVISTIC ELECTRONS IN A SYNCHROTRON

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(presented by A. A. Kolomenski)

Protons with energies of several billion electron volts have recently been obtained in a number of cyclic accelerators. Although the production of electrons with energies of the same order of magnitude is undoubtedly of the greatest interest, the solution of this problem as compared with the case of protons leads to considerable difficulties due essentially to the fact that the motion of electrons on circular orbits is accompanied by powerful electromagnetic radiation exhibiting certain peculiar featurs. The ensuing need of compensating the radiation losses imposes serious demands on the accelerating system.

The electromagnetic radiation exerts a considerable effect on the dynamics of motion of accelerated electrons, as well as on the synchrotron and betatron oscillations. We here present a physical picture of the motion of relativistic electrons in the accelerator taking into account the radiation (including quantum effects), and derive formulas for obtaining the necessary quantitative evaluations.

1. Fundamental properties of radiation

The radiation properties of a relativistic electron moving in a magnetic field has been investigated by several authors [1,3,4]. The effect of radiation on the accelerator is essentially characterized by the rapid increase of its power W with increasing energy E of the electron

$$W = \frac{2}{3}\frac{e^2c}{r_c^2}\left(\frac{E}{mc^2}\right)^4 \tag{1.1}$$

where r_c is the radius of curvature of the orbit. The major part of the power of the accelerating system is thus expended at already relatively low energies in compensating for radiation losses. At 10^9 eV, for example, the losses per turn are $\sim$ 20 Kev. An increase of energies by a factor of ten brings the radiation losses to about 20-30 Mev per turn. Finally, at energies $\sim 10^{11}$ ev the loss per turn would be comparable with the total energy of the particle, which would result in a substantial modification of the character of its motion.

The radiation of relativistic particles is further characterized by its sharp peaking: practically all the intensity is concentrated within a narrow cone having an aperture $\sim mc^2/E$ about the direction of instantaneous velocity of the electron. Since this angle is verv small (at $E = 5{,}10^9$ et only $\sim 1'$) we may consider all the radiation to be tangental to the orbit. Calculations show that this assumption introduces errors of an order of $(mc^2/E)^2$. This sharply asymmetrical angular distribution plays an important part in processes within the accelerator.

The radiation spectrum is very rich in the higher harmonics of the basic frequency c/r. In contrast to the non-relativistic case, the maximum of the radiation spectrum occurs at the frequency

$$\omega_{max} = \frac{c}{r}\left(\frac{E}{mc^2}\right)^3 \tag{1.2}$$

which at high energies ($\gtrsim$ 1 Bev) corresponds to the soft X-rays region.

A further point to be noted is that the radiation is polarized to a considerable degree, chiefly in the orbit plane.

The foregoing considerations, as well as the formulae (1.1) and (1.2), are derived from the classical theory of radiation. To determine their limits of applicability, the following simple criterion may be used [5]: the quantum corrections for the fundamental characteristics of the radiation are small if its wave-length considerably exceeds the De Broglie wave of the electron. Several equally obvious arguments might be adduced which, however, may be formally summed up in the following condition:

$$E \ll E_{1/2} = mc^2 \, (rmc/\hbar)^{1/2} \tag{1.3}$$

This criterion has been confirmed by rigourous calculations. Since $E_{1/2} \approx 10^{15}$eV, quantum corrections for the intensities and radiation spectrum are practically negligible in all cases. As will be shown below however, the effect of the quantum statistical nature of the radiation on electron orbits cannot be neglected.

2. The mouvement of the electron in an axially symmetrical field in the presence of radiation

Consider the movement of a relativistic electron in an axially symmetrical magnetic field $H(r) = H_0 (1 - n \rho/R)$ (where $\rho = r - R$) under the action of an external accelerating field taking radiation into account. The Lagrange equation in the coordinates r, θ, z for this case takes the form * :

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - \frac{er}{c}\dot{\theta}H_z + \frac{\partial F}{\partial r} - \frac{\partial Q}{\partial \dot{r}} ;$$
$$\frac{d}{dt}\left(mr^2\dot{\theta} - \frac{e\Phi}{2\pi c}\right) = \frac{\partial F}{\partial \theta} - \frac{\partial Q}{\partial \dot{\theta}} \qquad (2.1)$$
$$\frac{dt}{d}(m\dot{z}) = \frac{e}{c}r\dot{\theta}H_r + \frac{\partial F}{\partial z} - \frac{\partial Q}{\partial \dot{z}},$$

where Φ is the magnetic field flux within the orbit of radius r, F is the function characterizing the accelerating field and Q is the dissipative function, taking the presence of radiation into account.

Assuming the radiation to be directed along the velocity of the electron, which in our case may be taken as $|V| \approx C$ and taking Q to express the intensity of the losses, we may write :

$$\partial Q/\partial V = W/c^2\, V \qquad (2.2)$$

Hence, for the generalized friction,

$$\partial Q/\partial \dot{r} = W\dot{\rho}/c^2 \; ; \; \partial Q/\partial \dot{z} = W\dot{z}/c^2 \qquad (2.3)$$
$$\partial Q/\partial \dot{\theta} = Wr/c$$

In the synchrotron an AC voltage of frequency $q \cdot c/R$ is applied to several accelerating gaps. The external field may, as is well-known [6], be represented in the following form :

$$\partial F/\partial r = \partial F/\partial z = 0$$
$$\partial F/\partial \theta = eV_0 \cos \varphi/2\pi \; ; \; \dot{\varphi} = -q(\dot{\theta} - c/R) \qquad (2.4)$$

where V_0 is the total voltage amplitude at all gaps and φ is the phase of the particle relative to voltage.

Allowing, as usual, for small deviations from the equilibrium orbit R, we obtain the equations of motion from (2.1) in the following form :

$$(a)\quad \frac{1}{m}\frac{d}{dt}(m\dot{\rho}) + \frac{W}{E}\dot{\rho} + \frac{c^2}{R^2}(1-n)\rho = \frac{E - eH_0R}{eH_0}\frac{c^2}{R^2}$$

$$(b)\quad r\dot{E} = \frac{ceV_0}{2\pi}\cos\varphi - W(r)r + e\dot{H}_0\rho R \qquad (2.5)$$

$$(c)\quad \frac{1}{m}\frac{d}{dt}(m\dot{z}) + \frac{c^2}{R^2}nz + \frac{W}{E}\dot{z} = 0$$

It must be noted that $rW = RW(R)[1 + (1-2n)\rho/R + 2(E - eH_0R)/eH_0R]$. Considering equations (2.5a) and (2.5b) together, we may obtain the following equation in linear approximation

$$\rho^{V} + \frac{W_t + 2W}{E_s}\rho^{IV} + \omega^2(1-n)\rho^{III} + \omega^2(3-4n)\frac{W}{E_s}\ddot{\rho}$$
$$+ \omega^2\Omega^2(1-n)\dot{\rho} + 2\omega^2\Omega^2(1-n)\left[\frac{\dot{\Omega}}{\Omega} + \frac{\dot{V}}{2V} - \frac{\dot{E}_s}{E_s}\right] = 0 \qquad (2.6)$$

where

$$\omega = \frac{c}{R};\quad \Omega^2 = \frac{q\omega^2 eV}{(1-n)2\pi E_s};\quad V = V_0 \sin\varphi_0;\quad E_s = eH_0R.$$

$eV_0c \cos \varphi_0/2\pi R = W_t = e\dot{H}_0R + W(R)$ is the total power spent by the accelerating system on the radiation losses compensation and electron energy increase due to the magnetic field increase.

Similar equations are obtained for $\varphi - \varphi_0$ and $E - E_s$.

For comparison, consider the usual equation of the betatron oscillations (in the absence of radiation)

$$d(E_s\dot{\rho})/dt + E_s\omega^2(1-n)\rho = 0 \qquad (2.6a)$$

It follows from (2.6) and (2.5b) that in the presence of radiation the oscillation damping will be determined by W and not only by E_s.

It follows from (2.6) and (2.5b), also, that the values ρ, $\Delta E = E - E_s$ and $\Delta\varphi = \varphi - \varphi_0$ may be expressed as a superposition of rapid and slow oscillations with the frequencies $\omega\sqrt{1-n}$ and Ω. These oscillations may be described as betatron and synchrotron ones.

Up to now, there have been various authors who have considered the effect of radiation damping on synchrotron oscillations only. We have found [7] that a powerful radiation damping (positive or negative) also exists in betatron oscillations. As a result, the change of amplitudes with time in the presence of a compensated radiation takes the following form :

$$(a)\quad (a_B)_r \sim E_s^{-1/2} \exp\left[-\frac{n}{2(1-n)}\int_0^t \frac{W}{E_s}dx\right];$$
$$(b)\quad (a_B)_z \sim E_s^{-1/2} \exp\left[-\frac{1}{2}\int_0^t \frac{W}{E_s}dx\right] \qquad (2.7)$$

instead of $a_B \sim E_s^{1/2}$ for radial and vertical betatron oscillations,

* The use of the Lorentz-Dirac equation leads to the same final results.

$$a_{ph} \sim E_s^{-1/4} V^{-1/4} \exp\left[-\frac{3-4n}{2(1-n)}\int_0^t \frac{W}{E_s}dx\right] \quad (2.8)$$

instead of $a_{ph} \sim E_s^{-1/4}$ for phase synchrotron oscillations, and

$$a_c \sim E_s^{-3/4} V^{1/4} \exp\left[-\frac{3-4n}{2(1-n)}\int_0^t \frac{W}{E_s}dx\right] \quad (2.9)$$

instead of $a_c \sim E^{-3/4}$ for radial synchrotron oscillations.

Formulae similar to (2.7) – (2.9) may be derived for other types of radiation losses, e.g. for bremsstrahlung losses on residual gas nuclei.

Let us now consider the influence of quantum effects on an electron motion. This problem was first discussed by Sokolov and Ternov [8]. We have shown [9] that all similar effects may be derived on the basis of the classical theory of betatron oscillations accounting for the fluctuations of radiation due to its statistical character. In the $E \ll E_{1/2}$ regions, such an approach seems entirely justified on the basis of general electrodynamic considerations.

In order to account for the statistical character of the losses, the second equation in (2.5) may be rewritten in the form * :

$$r\dot{E} = \frac{ceV_0 \cos\varphi}{2\pi} - RW(R)\left[1 + (1-2n)\rho + 2\frac{\Delta E}{E_s} + \frac{\sum_i \varepsilon_i \delta(t-t_i) - W}{W}\right]; \quad (2.10)$$

where ε_i is the emitted photon energy.

We make now the following assumptions :

1. The emission of quanta occurs independently and this applies for $E \ll E_{1/2}$. Under this condition, the quantum may be regarded as being instantly emitted.

2. The radiation reaction force is small compared with the main forces acting on an electron; this is correct for E up to about 10^{11} eV.

3. Spectrum and intensity equations are derived on the basis of the classical theory. As noted in § 1, this applies for $E \ll E_{1/2}$.

Proceeding from these assumptions and the equations (2.5) – (2.10) and taking into account damping (2.7), (2.9), we find that radiation fluctuations lead to the excitation of betatron and synchrotron oscillations characterized by the following expressions (for weak focusing)

$$\overline{\rho_c^2} \leqslant \frac{55\sqrt{3}}{96}\,\frac{\Lambda R}{(3-4n)(1-n)}\left(\frac{E}{mc^2}\right)^2$$

$$\overline{\rho_b^2} \leqslant \frac{55\sqrt{3}}{96}\,\frac{\Lambda R}{n(1-n)}\left(\frac{E}{mc^2}\right)^2$$

$$\overline{z^2} \leqslant \frac{13\sqrt{3}}{96}\,\frac{\Lambda R}{n}\,; \quad \Lambda = \frac{\hbar}{mc} \quad (2.11)$$

A similar expression for $\overline{\rho_c^2}$ has been derived in [11].

Let us now consider the physical picture of the effect. The emission of the quantum is followed by a sharp reduction in radius. It is easily seen that during half the oscillation period this leads to a decrease in amplitude and during the other half to an increase. It is known, however, that the mean increase of amplitude is proportional to the square root of the time owing to the statistical character of the process. The same is true for phase oscillations, with the difference that in this case the velocity $\dot{\varphi}$ is abruptly changed.

This process of excitation could be prolonged ad infinitum but for the occurrence of friction (2.7) – (2.9). Owing to the latter, the excitation process is limited by the time of "relaxation" which, as can be easily seen from (2.6), is $[(1-n)/n]\cdot E_s/W$ for betatron oscillation and $[(1-n)/(3-4n)]\cdot E_s/W$ for phase oscillations. The established oscillation value is given by equation (2.11).

It may be noted that expression (2.11) for $\overline{z^2}$ is not derived directly from (2.5) and (2.10), as the vertical oscillations are not excited in the same way as the radial ones. This problem is discussed in section 2.

It is interesting to consider in detail the damping mechanism when the discrete emission of separate quanta is taken into account (for the simpler case of the vertical motion). Since the emission of a quantum ε_i takes place in the direction of the instantaneous electron velocity, it carries off a momentum the vertical component of which is $\varepsilon_i \dot{z}/c^2$. Hence, the charge from the quasi-elastic magnetic forces is $d(m\dot{z})/dt = -\varepsilon_i \dot{z}\delta(t-t_i)/c^2$. On the other hand $d(m\dot{z})/dt = \dot{m}\dot{z} + m\ddot{z}$. The change of mass may be considered as consisting of two parts radiation losses and increase due to the accelerating field

$$\dot{m} = -\frac{\varepsilon_i}{c^2}\delta(t-t_i) + \dot{m}_{acc.} = -\frac{\varepsilon_i}{c^2}\delta(t-t_i) + \frac{W}{c^2} + \frac{\dot{E}_s}{c^2}$$

It can easily be seen that the increase in oscillation amplitude due to the decrease in mass at the moment of emission is fully compensated by radiation damping. Hence the amplitude of betatron oscillations is influenced only by the increase in mass $\dot{m}_{acc}$ due to the presence of a compensating field ((2.7) – (2.9)).

The damping of oscillation of the closed orbit (2.8) at $n < 3/4$ means that when the orbit is shifted radially the radiation of the particle is such that the change of its energy leads to the return of the closed orbit to the equilibrium position.

* See, for example [10]

More general formulae which also describe the process of relaxation of oscillations may be derived from these equations. For instance, for ρ_B we have

$$\bar{\rho}_B^2 = \frac{55}{48\sqrt{3}}\,\frac{e^2\Lambda c}{R(1-n)^2E_s}\int_0^t \exp\left[\frac{-n}{1-n}\int_x^t \frac{W}{E_s}\,d\xi\right]\left(\frac{E_s}{mc^2}\right)^6 dx \qquad (2.12)$$

It can easily be seen that in the limiting case of small energies or short duration of the process, when the radiation damping may be neglected, (2.12) gives:

$$\bar{\rho}_B^2 = \frac{55}{48\sqrt{3}}\,\frac{e^2\Lambda c}{R(1-n)^2E_s}\int_0^t \left(\frac{Es}{mc^2}\right)^6 dx \qquad (2.13)$$

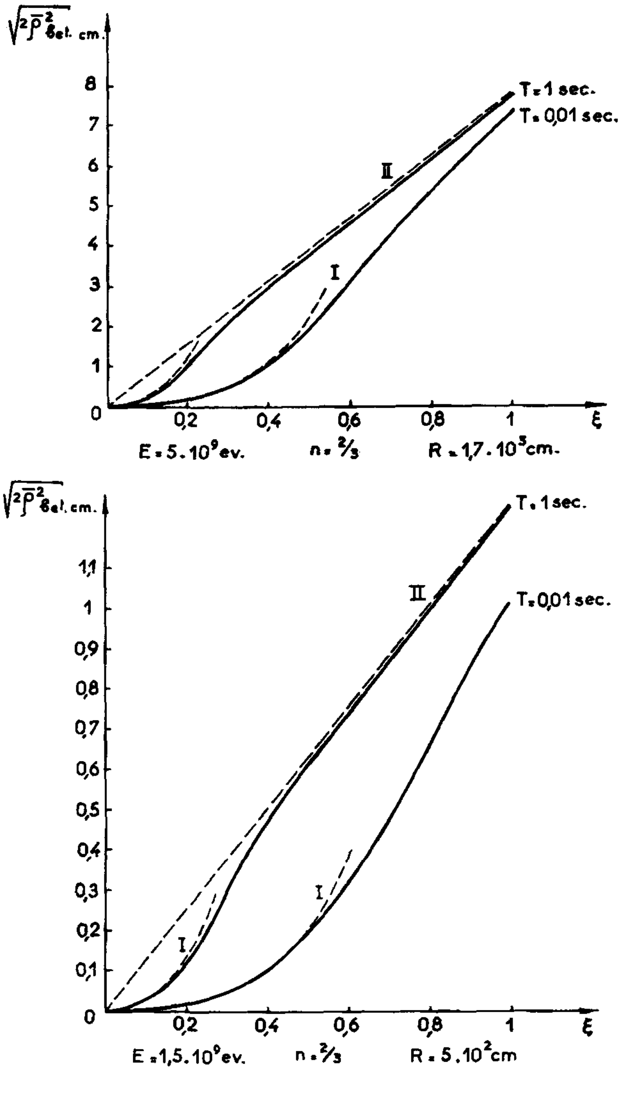

Fig. 1.

Fig. 1 shows the curves $\sqrt{2\bar{\rho}_B^2}\,(\xi)$; $\xi = t/T$ where T is the acceleration time; curves I and II correspond to the two limiting cases (2.13) and (2.11).

It should be noted that fluctuations can be taken into account by means of another more obvious method. This method, permitting the elucidation of the mechanism of the excitation of fluctuation, will be applied to periodic magnetic systems used in accelerators.

3. Betatron oscillations in the presence of radiations in a strong-focusing accelerator

Consider a system consisting of successive sectors with magnetic fields and sectors free of field forming a closed ring and including N identical periodic elements. The element of periodicity in a race-track consists of a magnetic sector and a straight section free of field. In a strong-focusing system, the element of periodicity might consist of two sectors, with a positive and a negative radial field gradient respectively and straight sections between each sector. Other more complex elements of periodicity may also be suggested.

Let us consider the motion in a plane of symmetry $z = 0$, described by the variable ρ equal to the distance separating a given point from the fundamental (equilibrium) orbit:

$$\rho(\theta) = \rho_c(\theta) + \rho_b(\theta) \qquad (3.1)$$

The betatron oscillations ρ_B are determined from (2.6), where n is the periodic function of azimuth θ with a period θ_0 (assuming $W = o$)

$$n(\theta + \theta_0) = n(\theta) \qquad (3.2)$$

The solution of equation (2.6) under condition (3.2) takes the form

$$\rho_b(\theta) = ae^{i\mu m}f(\theta) + a^*e^{-i\mu m}f^*(\theta);\ o < \theta < \theta_0 \qquad (3.3)$$

where m is the number of the successive elements of periodicity. The deviation of the closed orbit $\rho_c(\theta)$ may be expressed in the form

$$\rho_c = \frac{R}{\varkappa^2}\,\psi(\theta)\,\frac{\Delta E}{E_s}\ ;\ 0 < \theta < \theta_0 \qquad (3.4)$$

where the function $\psi(\theta)$ as well as $f(\theta)$ and μ depends on the parameters of the system and is independent of initial conditions; $\varkappa^2 = 1 - n$ in a weak-focusing magnet and $\varkappa^2 = |n|$ in a strong-focusing magnet ($|n| = \text{const}$); while ΔE is the deviation E from the equilibrium value eH_0R. It follows from (3.3) that the amplitude of betatron oscillations at different azimuths is described by the function $2|f||a|$. The largest amplitude, which we designate by A, is

$$A = 2|f|_{max}|a| \qquad (3.5)$$

According to (2.7), the value a changes in the presence of radiation in the following way :

$$a \sim E_s^{-1/2} \exp\left[\frac{\zeta}{2}\int_0^t \frac{W}{E_s} dt\right] \tag{3.6}$$

The parameter ζ depends on the magnet structure and may be positive or negative.

Let us assume a quantum of energy $\varepsilon = \hbar\omega$ emitted at a certain azimuth (m, θ). According to (3.4), this leads to an abrupt change in the slope and position of the closed orbit and to simultaneous changes in the same characteristics of betatron oscillations in the same place

$$\Delta\rho_c(\theta) = -\Delta\rho_b(\theta) = -\frac{R}{\varkappa^2}\frac{\varepsilon}{E}\psi$$
$$\Delta\rho_c'(\theta) = -\Delta\rho_c'(\theta) = -\frac{R}{\varkappa^2}\frac{\varepsilon}{E}\psi' \tag{3.7}$$

Owing to the increments of (3.7), the value of $|a|$ in (3.3) likewise experiences a certain increase $\Delta|a|$. Let us compute the value of $\overline{\Delta|a|^2}$ averaged over the azimuths of emission and the spectrum of radiation. According to (3.3) and (3.7) and assuming the normalization

$$ff^{*\prime} - f^*f' = -2i \tag{3.8}$$

we obtain

$$\overline{\Delta|a|^2} = \frac{1}{4}[\overline{|f'|^2(\Delta\rho)^2} + \overline{|f|^2(\Delta\rho')^2}] \tag{3.9}$$

According to (3.5) and (3.9) and taking into account the fact that the total change of the value $|a|^2$ involves the systematic variation (3.6) and the stochastic part (3.9), we obtain

$$\frac{d}{dt}\overline{A^2} = \overline{\Delta_1 A^2} = |f|^2_{max}[\overline{|f'|^2(\Delta_1\rho)^2} + \overline{|f|^2(\Delta_1\rho')^2}] + \frac{\zeta W - \dot{E}_s}{E_s}\overline{A^2}. \tag{3.10}$$

where the index 1 always means that the corresponding value is referred to unity of time.

According to (3.7), the values $\overline{(\Delta_1\rho)^2}$, are equal to

$$\overline{(\Delta_1\rho)^2} = \frac{R^2\overline{\psi}^2}{\varkappa^4E^2}\int_0^\infty \varepsilon\omega(\varepsilon)d\varepsilon$$
$$\overline{(\Delta_1\rho')^2} = \frac{R^2\overline{\psi'^2}}{\varkappa^4E^2}\int_0^\infty \varepsilon\omega(\varepsilon)d\varepsilon \tag{3.11}$$

where $\omega(\varepsilon)d\varepsilon$ is the probability per second of emission of a photon with the energy ε. To compute the integral in (3.11) we can use the expression for the spectral density of a relativistic electron moving on a circular orbit (see [4])

$$\omega(\varepsilon)d\varepsilon = \frac{3\sqrt{3}}{4\pi}\frac{e^2c}{R^2}\left(\frac{E}{mc^2}\right)^4 ydy\int_y^\infty K_{5/3}(x)dx;$$
$$y = \omega/\omega_{max}; \tag{3.12}$$

where ω_{max} is given by (1.2).

Thus the following expression for the value $\overline{A^2}$ is obtained :

$$\frac{d}{dt}\overline{A^2} = \frac{55}{24\sqrt{3}}\frac{e^2}{mcR}\frac{\Lambda F}{\varkappa^4}\left(\frac{E}{mc^2}\right)^5 + \frac{\zeta W - \dot{E}_s}{E_s}\overline{A^2} \tag{3.13}$$

where F is a factor depending on the paprameters and configuration of the system :

$$F = |f|^2_{max}[\overline{|f'|^2\psi^2} + \overline{|f|^2\psi'^2}] \tag{3.14}$$

For the case of an axially symmetrical magnetic field the expression for f(θ), ψ(θ) and F may be re-written in simplified form :

$$|f|^2 = |f|^2_{max} = 1/\varkappa;\ \psi = 1;\ F = 1.$$

It must be noted that in averaging the functions f(θ) and ψ(θ) and their derivatives in (3.14) the magnetic field free regions may be neglected, while the functions themselves are dependent on the length and positions of those regions.

By integrating (3.13) we obtain the expression for the mean square amplitude of betatron oscillations.

$$\overline{A^2} = \overline{A_0^2}\exp\left[\int_0^t \frac{\zeta W - \dot{E}_s}{E_s}dt\right] + \frac{55}{24\sqrt{3}}\frac{e^2c}{R}\frac{\Lambda F}{\varkappa^4 E_s}\int_0^t \exp\left[\int_0^t \frac{\zeta W d\xi}{E_s}\right]\left(\frac{E_s}{mc^2}\right)^6 dx \tag{3.15}$$

It follows that under the condition $\zeta > 0$ both parts of the betatron oscillation—the classical part and the stochastic (quantum) part—are subject to the action of anti-damping effects. It is easy to see that the nature of this anti-damping is completely classical.

The increase of oscillation amplitudes under the condition $\zeta > 0$ may limit the possible duration of the accelerating cycle.

When the radiation damping is small (corresponding to a short duration of the process), we have, from (3.15) :

$$\overline{A^2} = \overline{A_0^2}\frac{E_{inj}}{E} + \frac{55}{24\sqrt{3}}\frac{e^2c\Lambda F}{R\varkappa^4E_s}\int_0^t\left(\frac{E_s}{mc^2}\right)^6 dx \tag{3.16}$$

Evaluations have been made which show that in this case the "quantum" part of (3.16) does not increase enough to be dangerous.

If the duration of the process and the energy of the electron are large enough, the oscillation damping of the type $\exp\left(-\int_0^t \frac{W}{E}\,dx\right)$ can be found by means of a corresponding choice of ζ (see p. 453).

The established value of $\bar{A}^2$ identical to (2.II) may be obtained in this case :

$$\bar{A}^2 = \bar{A}_0^2 \frac{E_{inj}}{E_s} + \frac{55\sqrt{3}}{48} \frac{\Lambda RF}{\varkappa^4|\zeta|} \left(\frac{E_s}{mc^2}\right)^2 \qquad (3.17)$$

Only radial betatron oscillations have been considered, in the foregoing. However, the quantum character of the radiation also leads to the excitation of vertical oscillation. This is due to the fact that the direction of the emergent photon does not always lie in the plane $z = 0$ but deviates by an angle $\sim mc^2/E$. Accordingly, each act of electron emission is accompanied by a recoil momentum in the z-direction which in the average gives rise to oscillations. Any increase in oscillations, however, is limited by damping (2.7). The established value of these oscillations is independent of energy and very small in size. For the mean square of a maximum deviation B, we have :

$$\bar{B}^2 \leqslant \frac{13\sqrt{3}}{48} \Lambda R\,|f|^2_{max}\,\overline{|f|^2} \qquad (3.18)$$

The value, even for weak focusing at $R = 3.10^3$ cm. is

$$\sqrt{\bar{B}^2} \approx 10^{-3}\,\text{cm}.$$

The physical meaning of this independence of energy of the established value is that although the rise of energy is accompanied by an increase in recoil momentum and the number of photons, the angle of flight correspondingly decreases and thus the two factors offset each other.

As seem from (3.15) and (3.17), the effect of quantum fluctuations of radiation on the excitation of betatron oscillation depends on the factor F, determined by the properties of the periodic system.

Consider now two typical examples of magnetic periodic systems—a magnet with an alternating field gradient (strong-focusing magnet) and a race-track.

(a) *Magnet with an alternating field gradient (strong-focusing magnet)*

We now discuss the most familiar case in which the element of periodicity consists of a converging (c) and a diverging (d) sector having the same characteristics (angular aperture ν, radius R, absolute value of the field gradient index n)

$$\nu_1 = \nu_2 = \nu;\ R_1 = R_2 = R;\ |n_1| = n_2 = n = \varkappa^2 \qquad (3.19)$$

Between the successive magnetic sectors straight field-free sections, also of equal size, are incorporated

$$L_1 = L_2 = L \qquad (3.20)$$

Instead of deriving the expression for the Floquet function $f(\theta)$ and the function of the closed orbit $\psi(\theta)$, we shall give a table for the values of F as a function of the relative length of the straight sections $L/R\nu$ provided that the parameters are in each case selected in such a way that $\cos\mu \approx 0$ (corresponding to the centre of the region of stability).

Table I

$L/R\nu$	0	0,5	1
$\varkappa\nu$	$\pi/2$	1,14	0,93
$L\varkappa/R$	0	0.57	0.93
F	214	100	80

It is readily apparent that the factor F depends rather strongly on the relative magnitude of the straight sections.

In the generally considered case, the relative length $L/R\nu$ is small (< 0.5) and it therefore seems advisable to consider also the simpler case where the straight sections are absent. In this case, we obtain the following expression for factor F, corresponding to the variant (3.19) and (3.20) :

$$F = \frac{sh + sch}{2(ch - c - ssh)\sin^2\mu} \Big\{ (sch - csh)(ch - c - ssh) + 2[sch(ch-1) - csh(1-c)] + \frac{2ssh}{\varkappa\nu}\Big[2(ch-1)(1-c) - \frac{1}{\varkappa\nu}\Big(s(ch-1) - sh(1-c)\Big)\Big]\Big\}; \quad \begin{array}{l} s = \sin\varkappa\nu;\ c = \cos\varkappa\nu \\ ch = ch\,\varkappa\nu;\ sh = sh\,\varkappa\nu \end{array} \qquad (3.21)$$

The variations of F with μ, viz., with the position of the representative point on the stability diagram, are shown in Table 2.

As could be expected, the factor F increases infinitely on the stability region boundaries ($\cos\mu = \pm 1$) and passes through a minimum in the centre of the stability region ($\cos\mu \approx 0$).

TABLE 2

μ	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$3\pi/4$	$7\pi/8$	π
$\varkappa\nu$	0	0.82	1.15	1.39	1.57	1.71	1.80	1.85	1.87
F	∞	1,465	375	240	215	260	450	1,545	∞

It should be noted that oscillation damping of the type $\sim \exp\left[-\int^t \frac{W}{E_s} d\xi\right]$ can be obtained in the strong-focusing synchrotron also. For this purpose, we can, for instance, include a number of small magnets with $n \leqslant 1$ in the magnet system.

The damping of the radial betatron oscillations can also be obtained by appropriate use of the coupling between radial and vertical oscillations. The specially modified form of accelerating gaps may also be useful. The methods providing for the damping of radial betatron oscillations in alternating gradient systems are developed in detail in our other papers (see, for example [13]).

(*b*) *The race-track*

In this simplest system, the element of periodicity consists of a sector with a magnetic field (angular aperture ν, radius of curvature of the orbit R) and a field-free section of length L. Calculations give the following expression for the factor F:

$$F = \frac{\left(1 + p\frac{1+c}{s}\right)\left[\left(1 - p\frac{1}{\varkappa\nu} - \frac{c}{s}\right)\right]}{1 + p\left(2\frac{c}{s} - p\right)}\ ;\quad p = \frac{\varkappa L}{2R} \tag{3.22}$$

In more familiar cases, the rectilinear sections are relatively small, the parameter $p \ll 1$ and hence the relative value of the correction on F in the race-track is small in comparison with a continuous magnet. According to (3.22), this correction amounts to about $p\varkappa\nu/6$, not more than a few per cent. Here we also observe the effect of radiation damping, leading to results similar to (2,7a) and (2,11b).

4. Synchrotron oscillations in the presence of radiation in the case of a strong-focusing accelerator

In this paragraph, we consider separately the slow synchrotron oscillations of radius, phase and energy which may be considered to be inter-related by the well-known expressions:

$$\rho_c = \frac{R\psi}{1-n}\frac{\Delta E}{E},\quad \dot{\varphi} = q\omega\frac{\rho_c}{R}$$

The radius of curvature R_c (corresponding to the closed orbit) may be represented as follows:

$$R_c = R\left[1 + \frac{|n| + n(\theta)\psi(\theta)}{\psi(\theta)}\frac{\rho_c(\theta)}{R}\right] \tag{4.1}$$

Taking into account (4.1) the value $W(R + \rho_c)$ may be re-written in the form

$$W(R + \rho_c) = W(R)\left[1 + \left(Q - \frac{<\psi>_H}{|n|\sigma}\right)\frac{\Delta E}{E_s}\right] \tag{4.2}$$

where

$$Q = 2 + <\psi>_c - <\psi>_d + \frac{<\psi>_H}{|n|} \tag{4.3}$$

$<\psi>_c$ and $<\psi>_d$ are the mean values of the function $\psi(\theta)$ in the converging and diverging sectors respectively and the value $<\psi>_H$ is the function $\psi(\theta)$ averaged over sectors with magnetic field. The value of σ takes into account the increase in the time of revolution due to the presence of straight sections:

$$\sigma = 1 + L/R\nu. \tag{4.4}$$

Using the expression

$$\frac{<\psi>_H}{\sigma|n|}\frac{\Delta E}{E_s} = \frac{R}{qc}\dot{\eta};\ \eta = \varphi = \varphi_0 \tag{4.5}$$

and calculating the ratio between the energy imparted to the particle by the accelerating field and the energy lost by it in radiation (taking into account the fluctuations), we obtain a linearized phase equation for slow oscillations

$$\ddot{\eta} + \dot{\eta}\left(\frac{\dot{E}_s}{E_s} + \frac{W(R)Q}{\sigma E_s}\right) + \frac{q<\psi>_H c}{\sigma^2|n|RE_s}\left(\dot{E}_s + \frac{W}{\sigma}\right)tg\varphi_0\eta = \frac{q<\psi>_H c}{|n|\sigma^3 E_s R}\left(\sum_i \delta(t-t_i) - \bar{a}_1\right)\bar{\varepsilon} \tag{4.6}$$

where $\bar{\varepsilon}$ is the average energy of the radiated quanta and $\bar{a}$, the average number of quanta $\bar{\varepsilon}$ per unit of time. Solving equation (4.6) and obtaining the mean square value $\bar{\eta}^2$, we have

$$\bar{\eta}^2 = \frac{55}{48\sqrt{3}}\frac{q\hbar e^2 c\,ctg\varphi_0 <\psi>_H}{R^4 m(E_s + W/\sigma)\sigma^3|n|} \times \int_0^t e^{-\int_x^t \frac{WQ}{\sigma E_s} d\xi}\left(\frac{E_s}{mc^2}\right)^6 dx \tag{4.7}$$

Hence with $\int_0^t \frac{WQ}{\sigma E_s} d\xi \gg 1$, the established value of $\bar{\eta}^2$

may be obtained without difficulty:

$$\bar{\eta}^2_{est} = \frac{55\sqrt{3}}{64}\, q\, \frac{\hbar c}{e^2\sigma} \frac{<\psi>_H ctg\varphi_0\,(mc^2/E_s)}{(2 + <\psi>_c - <\psi>_d)|n| + <\psi>_H} \tag{4.8}$$

The factor Q for the centre of the stability region is ≈ 4.

In the case of oscillations where the radiation friction is small (corresponding to $\int_0^t \frac{WQ}{\sigma E_s} d\xi \ll 1$), we obtain from (4.8) the following expression

$$\bar{\eta}^2 = \frac{55}{48\sqrt{3}} \frac{q\hbar e^2 c\, ctg\varphi_0 <\psi>_H}{|n| R^4 m \dot{E}_s \sigma^3} \int_0^t \left(\frac{E_s}{mc^2}\right)^6 dx \tag{4.9}$$

For a weak-focusing synchrotron, (4.8) and (4.9) remain unchanged, and it only has to be considered that in that case:

$$\frac{<\psi>_H}{Q|n|} \rightarrow \frac{1}{3-4n}\,; \qquad \frac{<\psi>_H}{|n|} \rightarrow \frac{1}{1-n}$$

$$Q_{w.f.} = \frac{3-4n}{1-n} \tag{4.10}$$

Deriving the values $<\psi>_c$, $<\psi>_d$ and $<\psi>_H$, we obtain, for the strong-focusing case

$$<\psi>_{c,d} = \pm 1 + \frac{4c_{1/2}\, sh_{1/2}}{[s_{1/2}ch_{1/2} - c_{1/2}sh_{1/2} + g s_{1/2} sh_{1/2}]}$$

$$<\psi>_H = \frac{4}{\varkappa\nu\,[cth_{1/2} - ctg_{1/2} + g]}; \qquad g = \frac{L\varkappa}{R} \tag{4.11}$$

$$s_{1/2} = \sin\frac{\varkappa\nu}{2}; \quad ch_{1/2} = ch\,\frac{\varkappa\nu}{2}; \quad ctg_{1/2} = ctg\,\frac{\varkappa\nu}{2}. \text{ etc.}$$

For example, for the centre of the stability region $<\psi>_H \approx 5$. Since $<\psi>_H/|n|$ is small, the expression for $Q_{s.f.}$ gives:

$$Q_{s.f.} \simeq 2 + <\psi>_c - <\psi>_d \simeq 4 \tag{4.12}$$

It must be noted that if we take $|n| \gg 1$, in a constant gradient accelerator, the factor $Q_{w.f.}$, according to (4.10), becomes independent of n and equal to $Q_{w.f.} \approx 4$.

Actually, for a number of reasons, the value of n in this case must be taken as equal to 0.6 - 0.7 and the value of $Q_{w.f.}$ becomes about $Q_{w.f.} \approx 1$. Since this value is part of the exponent describing the damping (see (2.8)), it is clear that under equal conditions radiation in the case of a strong-focusing system must result in a much greater damping of the deviation of the closed orbit. The case of instability (corresponding to $n > 3/4$ for the weak-focusing system) does not occur in strong-focusing accelerators.

The total behaviour of the function $\bar{\eta}^2(\xi)$; $\xi = t/T$ is shown in graph 2 for both cases. The amplitude of the excited phase oscillations is seen to be rather large, with a slight decrease in the high energy range ($\sim E^{-1/2}$). The azimuthal dimension of the bunch of accelerated particles appears to be fairly large, and hence the coherent radiation is rather weak [11, 13].

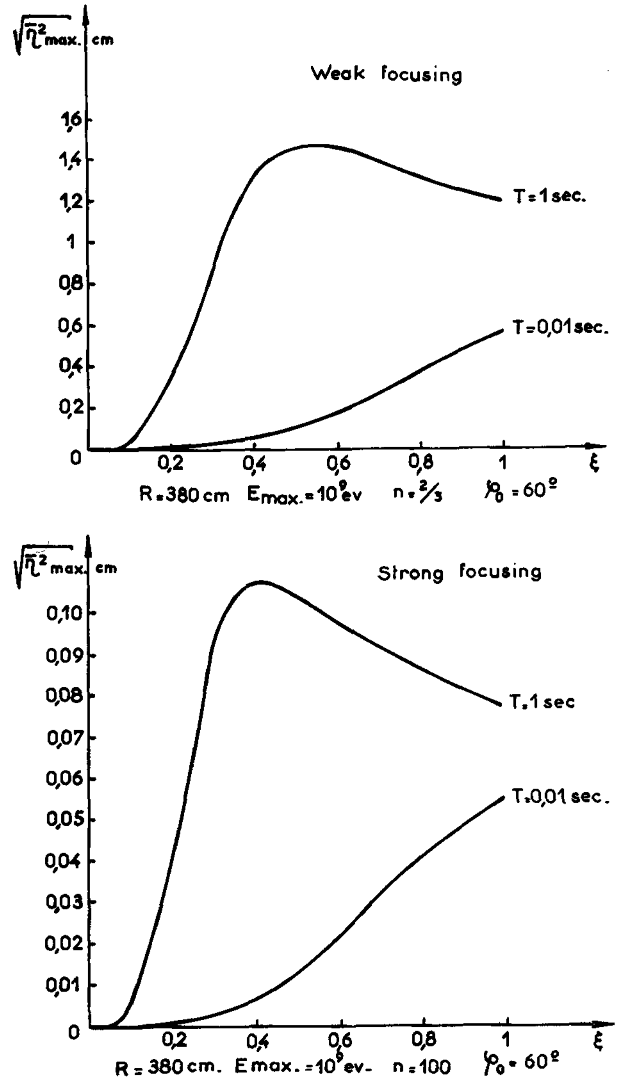

Fig. 2.

As seen from graph 2, the value $\bar{\eta}^2$ describing the excited phase oscillations starts increasing and then passes through a maximum. If the maximum phase value is comparable to the dimensions of the separatrix, this effect may result in the loss of a fraction of accelerated particles.

Finally, we derive formulae for the mean square value of the radial-phase oscillations. It follows from (4.6) that

$$\overline{(\dot{\eta})^2} = \Omega^2_{ph}\bar{\eta}^2 = \frac{q < \psi >_H c}{\sigma^2|n|E_s R}\left[\dot{E}_s + \frac{W}{\sigma}\right]tg\varphi_0\bar{\eta}^2. \tag{4.13}$$

Using (4.13) and the ratio

$$\rho_c^2 = \frac{R^4\sigma^4}{q^2c^2}(\dot{\eta})^2 \tag{4.14}$$

we obtain, from (4.8) and (4.9) respectively,

$$\overline{\rho c^2_{est}} = \frac{55\sqrt{3}}{96}\frac{\hbar R}{mc}\frac{< \psi >^2_H}{Q|n|^2}\left(\frac{E_s}{mc^2}\right)^2 \tag{4.15}$$

$$\overline{\rho_c^2} = \frac{55}{48\sqrt{3}}\frac{\hbar \cdot e^2}{\sigma m R E_s}\frac{< \psi >^2_H}{|n|^2}\int_0^t\left(\frac{E_s}{mc^2}\right)^6 dx \tag{4.15a}$$

For a weak-focusing system (see 4.10) the expression (4.15) based on an assumption similar to those adopted [11], coincides with formula (23) given in that paper.

The absolute value of the excited radial phase oscillations is very small. Thus from (4.15) (a formula always giving the maximum possible value for given E_s and t), we obtain:

$$\sqrt{\overline{\rho_c^2}} \leqslant \frac{E^{3/2}_{Bev}}{|n|}\text{ cm}$$

which gives only a small factor of a cm. for $n > 100$ and even for $E = 10$ Bev. It is interesting to note that while the radial dimensions of the bunch increase with increasing energy (due to fluctuations), its azimuthal dimensions (the value $\bar{\eta}^2_{est}$) decrease.

The procedure discussed in the present paper may with slight alterations be used for calculating other disturbing effects, such as bremsstrahlung on the residual gas nucleus, Coulomb scattering, disturbing effect of accelerating gaps etc.

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