**national accelerator laboratory**

# SPONTANEOUSLY BROKEN GAUGE SYMMETRIES

## Part I - Preliminaries

Benjamin W. Lee

National Accelerator Laboratory
P. O. Box 500, Batavia, Illinois 60510

and

Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, New York 11790


and


Jean Zinn-Justin

Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, New York 11790

## ABSTRACT

This is the first of a series of papers addressed to the renorma-
lizability question of spontaneous broken gauge theories. We give a
brief outline of the motiviation for such an investigation and describe
the manner in which the renormalizability of such theories will be
proved in the sequel. Put briefly, we will show that in an appropriate
gauge, ultraviolet divergences of a spontaneously broken gauge theory
are removed completely by the gauge invariant counter terms in the
Lagrangian which would make Green's functions of the corresponding
unbroken gauge theory finite; that the S-matrix computed in this gauge
is unitary, and that the S-matrix is independent of the gauge chosen.

In this paper, the renormalizability question of the unbroken
gauge theory is considered. We derive the Ward-Takahashi identities
of the theory. We discuss several ways of regulating divergent Feyn-
man integrals of the theory without destroying gauge invariance.
Infrared divergences are avoided by the device of intermediate
renormalization, wherein we choose as subtraction points some points
where external momenta are Euclidean. This suffices to establish that
the BPH renormalization will give renormalized Green's functions
which satisfy the Ward-Takahashi identities.

The existence of finite, renormalized Green's functions satisfying the Ward-Takahashi identities provides us with the means of proving the renormalizability of the spontaneous broken symmetry case. The Ward-Takahashi identities were previously derived for the gauge bosons by Slavnov. We present here a new derivation. The discussions on regularization methods and intermediate renormalization procedure and the renormalization conditions for matter fields, we believe, are new contributions of the present paper.

CONTENTS:

## INTRODUCTION

This is the first of a series of papers which will deal with the renormalizability of spontaneously broken gauge symmetries. The intriguing possibilities of unifying electromagnetic and weak interactions in terms of Yang-Mills gauge bosons,[1-9] whose masses are generated by spontaneous breakdown of gauge invariance of the second kind,[3, 4] and of constructing a finite theory of weak interactions[4, 7-9] prompt a closer examination of the quantization and renormalization questions of theories of this genre.

In the sequel of this series, we wish to examine the following questions: (1) We will discuss both the group theoretic and field theoretic problems associated with the Higgs phenomenon.[10, 11, 12] This entails a careful study of the stability of the physical system which possesses the freedom associated with the gauge invariance; (2) We will also study the perturbative treatment of such theories. Here our aim is to show that, in an appropriate gauge, ultraviolet divergences of a spontaneous broken gauge theory are removed completely by the counter terms in the Lagrangian which would make Green's functions of the corresponding unbroken gauge theory finite. Thus the renormalizability of the unbroken Yang-Mills theory (to be defined below) implies the same for the spontaneously broken gauge theory. The philosophy and methodology we shall follow are the same

as those we employed in the study of the $\sigma$-model;[13] (3) In the gauge

in which the renormalizability can be proven, the unitarity of the

S-matrix is not manifest, since the quantization in that gauge implies

the use of an indefinite metric Hilbert space for the construction of

Green's functions. We will show that the physical S-matrix is none-

theless unitarity; (4) We shall also discuss the equivalence of the S-

matrix constructed in this gauge and in the gauge in which the unitariry

of the S-matrix is manifest (but not the renormalization).

In this paper, we shall give a discussion of the renormalization

problem of the (unbroken) Yang-Mills field theory. It is not attempted

in the present paper to establish that a renormalized Yang-Mills theory

exists as a physically satisfactory theory of massless particles. Due

to the infrared problem associated with massless quanta, such a

theory may very well not exist at all. What we wish to demonstrate is

that renormalized Green's functions of the theory exist (without imply-

ing the same for the S-matrix), which satisfy the Ward-Takahashi

identities which will be derived. The existence of renormalized Green's

functions will prove to be a sufficient foundation for the discussion of

the renormalizability of the spontaneously broken symmetry theory,

which we shall discuss in the sequal.

We will proceed in the following manner. After a brief review

of the quantization of the nonabelian gauge theories, we shall derive

the Ward-Takahashi identities. We will then discuss ways of regular-

izing divergent Feynman integrals in a gauge invariant manner. The

regularized Feynman amplitude then satisfies the identities automati-

cally. The Bogoliubov-Parasiuk-Hepp-Zimmerman renormalization

procedure[14-17] requires specifying the values of primitively diver-

gent vertices at some subtraction points. When these values are

chosen in accordance with the Ward-Takahashi identities, and the cut-

off parameters associated with the regularization are let go to infinity,

the renormalized amplitudes are obtained which satisfy the Ward-

Takahashi identities. Because of the infrared divergence, it is prudent

to choose as subtraction points some points other then where all exter-

nal momenta vanish. We shall describe in some detail this "intermed-

iate" renormalization procedure.

The Ward-Takahashi identities were previously derived by

Slavnov[18] for the gauge bosons. The deriviation we shall present is

somewhat different from his. The discussions on regularization meth-

ods and intermediate renormalization procedure, and renormalization

conditions for matter fields, we believe, are new contributions of the

present paper.

## II. QUANTIZATION

Following the works of Feynman,[19] de Witt,[20] Popov and

Fadeev,[21] and 't Hooft[22] we define the generating functional

$Z[J_\mu]$ of connected Green's functions by

$$e^{i Z[J_\mu]} = \int [dA] \, \Delta_L [A] \, exp \, i \left\{ \mathcal{L}(x) \right.$$
$$\left. - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 - J_\mu(x) \cdot A^\mu(x) \right\} , \tag{2.1}$$

where $[dA]$ is the canonical functional metric for the vector fields

$$[dA] = \prod_{a,\mu,x} dA_\mu^a(x) , \tag{2.2}$$

where a is the internal symmetry label. We shall assume the internal

symmetry is to be SU(2), so that a = 1, 2, 3, but the generalization

to other groups are trivial and immediate. $\mathcal{L}(x)$ is the Lagrangian

for the Yang-Mills fields

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} , \tag{2.3}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g A_\mu \times A_\nu , \tag{2.4}$$

which is invariant under local gauge transformations, the infinitesimal

version of which is

$$A_\mu^a \to A_\mu^a + g (\omega \times A_\mu)^a + \partial_\mu \omega^a \tag{2.5}$$
$$= A_\mu^a + (D_\mu)^{ab} \omega^b$$

$D_\mu$ being the covariant dirivative :

$$D_\mu [A]^{ab} = \delta^{ab} \partial_\mu - g(t \cdot A_\mu)^{ab}, \qquad (2.6)$$

$$(t_c)^{ab} = \epsilon^{acb} \qquad (2.7)$$

The Jacobian $\Delta_L [A_\mu]$ is essentially the determinant of the operator

$\partial_\mu D^\mu$, and may be expressed as

$$\Delta_L [A_\mu] = exp \; Tr \; \ln \left(1 - g \frac{1}{\partial^2} \partial^\mu A_\mu \cdot t \right)$$

$$= exp \left\{ - \sum_{n=1}^{\infty} \frac{(-g)^n}{n} \int dx_1 \cdots dx_n \; tr \right.$$

$$\left. \tilde{D}_F(x_n - x_1) \partial_{x_1}^{\mu_1} A_{\mu_1}(x_1) \cdot t \cdots \tilde{D}_F(x_{n-1} - x_n) \partial_{x_n}^{\mu_n} A_{\mu_n}(x_n) \cdot t \right\} \qquad (2.8)$$

$$= exp \; Tr \; \ln \left(1 - g \; t \cdot A^\mu \partial_\mu \frac{1}{\partial^2} \right)$$

where we have used the Feynman propagator $D_F(x-y)$ defined as

$$\tilde{D}_F(x-y) = \langle x | (-\partial^2 + i\epsilon)^{-1} | y \rangle. \qquad (2.9)$$

The Feynman rules for this theory we obtained in the usual manner

if we regard

$$\int d^4x \left[ \mathcal{L}(x) - \frac{1}{2\alpha} \left(\partial_\mu A^\mu(x)\right)^2 \right] - i \; Tr \; \ln \left(1 - g \; t \cdot A^\mu \partial_\mu \frac{1}{\partial^2} \right) \qquad (2.10)$$

as the effective Lagrangian. The bare vector boson propagator is

$$- i \Delta_{\mu\nu}(k^2; \alpha) = -i \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 + i\epsilon} (1 - \alpha) \right) \frac{1}{k^2 + i\epsilon} \qquad (2.11)$$

In Eq. (2.8) and (2.11) the $i\epsilon$ prescription is dictated by the

unitarity considerations which we shall discuss in the sequel. The

term $(1/2\alpha) (\partial_\mu A^\mu)^2$ specifies the gauge one is employing and depends

on a parameter which can vary from $-\infty$ to $\infty$. For $\alpha = 0$, we obtain

the transverse or Landau gauge, and for $\alpha = 1$, the Feynman gauge.

The last, nonlocal term in Eq. (2.10) is the new feature of nonabelian

gauge theories. It may be viewed [19] diagrammatically as the sum

of closed loop contributions from fictitious complex massless scalar

fields (ghosts) obeying Fermi statistics which are coupled to the

gauge fields through the interaction

$$g \, \bar{c}(x) \, \partial^\mu [ \, \underset{\sim}{t} \cdot \underset{\sim}{A}_\mu(x) \, c(x) ] \, , \tag{2.12}$$

The connected Green's functions of $\underset{\sim}{A}_\mu$'s are obtained as the func-

tional derivatives of $Z[\underset{\sim}{J}_\mu]$ :

$$i \, \partial^n Z[\underset{\sim}{J}_\mu] / \partial J_\mu^a(x) \, \partial J_\nu^b(y) \cdots = (-i)^n \langle T^*( A_\mu^a(x) A_\nu^b(y) \cdots ) \rangle_c \tag{2.13}$$

The Feynman rules are summarized in Figure 1. In addition,

the following rules should be kept in mind: The ghost-ghost-vector

vertex is "dotted", the dot indicating which ghost line is differentiated;

a ghost line cannot be dotted at both ends; a ghost loop carries an extra

minus sign.

### III. WARD-TAKAHASHI IDENTITIES - I

The invariance of the Lagrangian under the local gauge transfor-

mation (2.5) gives rise to a hierachy of identities amoung the Green's

functions (2.13). Alternatively, these relations may be expressed

globally as an equation satisfied by the generating functional $Z[\underset{\sim}{J}_\mu]$ .

We will first rewrite Eq. (2.1) as

$$W[J_\mu] = \exp i Z[J_\mu] = \Delta_L[i\delta/\delta J_\mu] W_o[J_\mu], \qquad (3.1)$$

where

$$W_o[J_\mu] = \int [dA] \exp i \left\{ \mathcal{L}(x) - \frac{1}{2x}[\partial^\mu A_\mu(x)]^2 - J^\mu(x) \cdot A_\mu(x) \right\} \qquad (3.2)$$

We perform the gauge transformation (2.5) on the variables of

integration $A_\mu(x)$. Due to the invariance of the Lagrangian and the

metric [dA], this transformation will affect only the source term

and the gauge defining term:

$$\delta \int d^4x \left\{ \frac{1}{2x}[\partial_\mu A^\mu]^2 + J^\mu \cdot A_\mu \right\} = \int d^4x \, \delta\omega \cdot [\frac{1}{x} D^\lambda \partial_\lambda \partial^\mu A_\mu - D^\mu J_\mu]. \qquad (3.3)$$

Since a transformation of integration variables does not change the

value of an integral, we may put the variation of $W_0$ with respect

to $\delta\omega(x)$ equal to zero. We obtain thereby

$$\frac{1}{x} D^\lambda[i\delta/\delta J]^{ab} \partial_\lambda \partial_\mu i \delta W/\delta J_\mu^b(x) - D^\mu[i\delta/\delta J]^{ab} J_\mu^b(x) W_o = 0. \qquad (3.4)$$

We note that (see Appendix A)

$$\Delta_L[i\delta/\delta J] J_\lambda^a(x) \Delta_L^{-1}[i\delta/\delta J] = J_\lambda^a(x) - ig \left( x \, t^a[\partial_\lambda H(x,y;i\delta/\delta J)] \right)_{x=y} \qquad (3.5)$$

where H is the solution of the equation

$$D_\lambda[A]^{ab} \partial^\lambda H^{bc}(x,y;A) = \delta^{ac} \delta^4(x-y), \qquad (3.6)$$

satisfying the outgoing boundary condition; it has the representation

$$H^{ab}(x,y;A) = -\langle x,a | [-\partial^2 + i\epsilon + g \, t \cdot A_\mu \partial^\mu]^{-1} | y,b \rangle \qquad (3.7)$$

Combining Eqs. (3.4) and (3.5), and recalling Eq. (3.1), we obtain

$$D_\lambda^\lambda [i\delta/\delta J]^{ab} \left\{ \frac{i}{\alpha} \partial_\lambda \partial_\mu \frac{\delta}{\delta J_\mu^b(x)} - J_\lambda^b(x) \right. \tag{3.8}$$

$$\left. + ig\, t_\lambda\, t^b [\partial_\lambda H(x,y\,;\,i\delta/\delta J)]_{y=x} \right\} W = 0$$

The last term on the left is equal to (see Appendix A)

$$ig\, D_\lambda^\lambda [i\delta/\delta J]^{ab}\, t^{cbd} [\partial_\lambda H^{dc}(x,y\,;\,i\delta/\delta J)]_{y=x} \tag{3.9}$$

$$= -ig \int d^4y\, tr\, t^c [\partial_\mu H(x,y\,;\,i\delta/\delta J)]\, D_y^\mu [i\delta/\delta J]\, \delta^4(x-y).$$

In showing this, one makes use of Eq. (3.6) and the Jacobi identity

of the matrices $t^a$. Equation (3.8) may now be written as

$$D^\lambda [i\delta/\delta J]^{ab} \partial_\lambda \left\{ \frac{i}{\alpha} \partial_\mu \frac{\delta}{\delta J_\mu^b(x)} - \int d^4y : H^{bc}(x,y\,;\,i\delta/\delta J)\, D_y^\mu [i\delta/\delta J]^{cd} J_\lambda^d(y) : \right\} W = 0 \tag{3.10}$$

where the symbol :: denotes the normal product prescription that the

$\delta/\delta J$ must stand to the right of the J.

We now define G by

$$D_\lambda [A]^{ab} G^{bc}(x,y\,;\,A) = \delta^{ac} \delta^4(x-y), \tag{3.11}$$

with the outgoing boundary condition, so that it may be represented as

$$G^{ab}(x,y\,;\,A) = -\langle x,a| [-\partial^2 + i\epsilon + g\, t \cdot A_\mu \partial^\mu]^{-1} |y,b\rangle = H^{ba}(y,x,\,A) \tag{3.12}$$

In terms of G, Eq. (3.10) may be considerably simplified. We finally

obtain the desired identity:

$$\frac{i}{\alpha} \partial_\mu \frac{\delta W}{\delta J_\mu^c(x)} - \int d^4y\, J_\lambda^c(y)\, D_y^\lambda [i\delta/\delta J]^{cb}\, G^{ba}(y,x,\,i\delta/\delta J) \tag{3.13}$$

$$\times W = 0$$

The above Ward-Takahashi identity was previously derived by

Slavnov.[18] He consideres a restricted class of gauge transforma-

tions which satisfy $\partial_\mu D^\mu \omega = \chi$ where $\chi$ is an arbitrary function.

He shows that the product $[dA]\Delta_L [A_\mu]$ remains invariant under

the nonlinear gauge transformation generated by $\omega = \omega[A_\mu, \chi]$.

The point of the above derivation is to show that Slavnov's form of

the Ward-Takahashi identities is the most general form of the con-

straint on $W[J_\mu]$ that follows from the gauge invariance of the

Lagrangian.

For the purpose of renormalization, it is usually much more

convenient to study the Ward-Takahashi identities connecting single

particle irreducible (proper) vertices, as was done for the $\sigma$-

model[13, 23] and the spontaneously broken abelian gauge theories.[8]

However, in the present instance, the Ward-Takahashi identities

for the proper vertices are extremely complicated, being non-

linear relations in them. The Ward-Takahashi identity satisfied by

the generating functional of the proper vertices is nevertheless

derived and analyzed in Appendix C. The renormalization conditions

will be analyzed on the basis of Eq. (3.13) in the following Sections.

## IV. WARD-TAKAHASHI IDENTITIES - II

We shall study the implications of Eq. (3.13) on the primitively

divergent vertices.

## A.  Two Point Functions

Differentiating Eq. (3.13) with respect to $J_\nu^b(y)$ and then letting

$J_\mu = 0$, we obtain

$$\frac{i}{\alpha}\frac{\partial}{\partial x_\mu}\frac{\delta^2 W}{\delta J_\mu^a(x)\delta J_\nu^b(y)}\bigg|_{J=0} - D_j^\nu[i\delta/\delta J]^{bc} G^{ca}(y,x,i\delta/\delta J)W\bigg|_{J=0} = 0 \quad (4.1)$$

Taking the divergence of the above equation with respect to y, and

remembering the equation satisfied by G, we see that

$$\frac{i}{\alpha}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\frac{\delta^2 W}{\delta J_\mu^a(x)\delta J_\nu^b(y)}\bigg|_{J_\mu=0} = \delta^{ab}\,\delta^4(x-y) \quad (4.2)$$

which shows that the longitudinal part of the propagator $\Delta\mu\nu$,

$$\frac{\delta^2 Z[J_\mu]}{\delta J_\mu^a(x)\delta J_\nu^b(y)}\bigg|_{J_\mu=0} = \delta^{ab}\tilde{\Delta}_{\mu\nu}(x-y) \quad (4.3)$$

is not renormalized:  The vector propagator has the form

$$\tilde{\Delta}_{\mu\nu}(x-y) = (g_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2)\tilde{D}(x-y) + \alpha(\partial_\mu\partial_\nu/\partial^2)\tilde{D}_F(x-y) \quad (4.4)$$

In the momentum space, the inverse of the vector propagator,

therefore, takes the form

$$[\Delta^{-1}(k)]_{\mu\nu} = (k^2 g_{\mu\nu} - k_\mu k_\nu)J(k^2) + \frac{1}{\alpha}k_\mu k_\nu \quad (4.5)$$

Were it not for the n particle thresholds at $k^2 = 0$, $J(k^2)$ would be

regular at $k^2 = 0$ (at least in perturbation theory), and the transverse

part of the vector meson propagator would have a simple pole at

$k^2 = 0$.

Equation (4.1), when combined with Eq. (4.4) gives

$$\left\{ \delta^{ab} \partial_\mu \tilde{D}_F (x-y) + \partial_\mu G^{ab}(x,y; i\delta/\delta J) \right.$$

$$\left. - iq \left[ t \cdot \frac{\delta}{\delta J^\mu(x)} \right]^{ac} G^{cb}(x,y; i\delta/\delta J) \right\} W \bigg|_{J_\mu = 0} = 0 \qquad (4.5)$$

or

$$\tilde{\mathcal{G}}^{ab}(x-y) = \delta^{ab} \tilde{D}_F(x-y) + iq \int d^4 z \, \tilde{D}_F(x-z) \frac{\partial}{\partial z_\mu} \left[ t \cdot \frac{\delta}{\delta J^\mu(z)} \right]^{ac}$$

$$\times W^{-1}[J_\mu] G^{cb}(z,y; i\delta/\delta J) W[J_\mu] \bigg|_{J_\mu = 0} = 0 \qquad (4.6)$$

where $\tilde{\mathcal{G}}$ is the ghost propagator:

$$\tilde{\mathcal{G}}^{ab}(x-y) = - \left\{ W^{-1}[J_\mu] G^{ab}(x,y; i\delta/\delta J) W[J_\mu] \right\} \bigg|_{J_\mu = 0}$$

We may define the self-energy part $\Sigma_g$ of the ghost by

$$\delta^{ab} \mathcal{G}(k) = \int d^4 z \, e^{ik \cdot x} \, \tilde{\mathcal{G}}^{ab}(z),$$

$$\mathcal{G}(k) = (k^2 + i\epsilon)^{-1} \left[ 1 - \Sigma_g(k, \mathcal{G}(k^2)) \right] \qquad (4.7)$$

The structure of Eq. (4.6) implies that $\Sigma_g(k^2)$ is of the form

$-k_\mu \Sigma_g^\mu(k)$.  Again, were it not for the fact that $k^2 = 0$ is the onsets of

n particle intermediate states, $\Sigma_g(k^2)$ would behave like $k^2$ near

$k^2 = 0$, so that $\tilde{\mathcal{G}}(k^2)$ would have just a simple pole there.

## B. Three Point Vertices

From Eq. (3.13) we obtain

$$\left(\frac{i}{\alpha}\right)^2 \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \frac{\delta^3 W}{\delta J_\mu^a(x) \delta J_\nu^b(y) \delta J_\lambda^c(z)}$$

$$+ D_z^\gamma [i\delta/\delta J]^{cd} G^{da}(z,x; i\delta/\delta J) \frac{i}{\alpha} \frac{\partial}{\partial y^\nu} \frac{\delta W}{\delta J_\nu^b(y)} \bigg|_{J_\mu = 0} = 0 \qquad (4.8)$$

We note that

$$q\,W^{-1}\,[t \cdot \partial/\delta J_\lambda(z)]^{cd}\,G^{da}(z,x\,;\,i\,\partial/\delta J)\,\delta W/\delta J^b_\nu(y)\,\Big|_{J_\mu = 0}$$

$$= q\int d^4z'\,d^4y'\,\widetilde{\gamma}^{cab}_{\lambda\rho}(z,x';y')\,\widetilde{\mathcal{G}}(x'-x)\,\widetilde{\Delta}^{\rho\nu}(y'-y)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.9)$$

$$+ q\int d^4z'\,d^4z''\,d^4x'\,d^4y'\,\widetilde{\Sigma}^\lambda_g(z-z')\,\widetilde{\mathcal{G}}(z'-z'')\,\widetilde{\gamma}^{cab}_\rho(z'';x';y')\,\widetilde{\mathcal{G}}(x'-x)\,\widetilde{\Delta}^{\rho\nu}(y'-y)$$

where
$$\widetilde{\gamma}^{abc}_\rho(x,y;z) = -i\,\frac{\partial}{\partial x_\lambda}\,\widetilde{\gamma}^{abc}_{\lambda\rho}(x,y;z),$$

and $\gamma^{abc}_\rho(p,q;r)$ defined by

$$\gamma^{abc}_\rho(p,q;r)\,(2\pi)^4\,\delta^4(p+q+r) = \int d^4x\,d^4y\,d^4z\,e^{i(p\cdot x+q\cdot y+r\cdot z)}\,\widetilde{\gamma}^{abc}_\rho(x,y;z)$$

is the proper vertex for the coupling of a vector meson of momentum

r, polarization ρ and isospin index c, with two ghosts, of momenta

p and q and isospin indices a and b, respectively (we define the

momenta outward from the vertex). See Figure 1. We have

$$\gamma^{abc}_\rho(p,q;r) = p^\lambda\,\gamma^{abc}_{\lambda\rho}(p,q;r) \qquad\qquad (4.10)$$

The quantity $\widetilde{\Sigma}^\lambda_g$ is defined by the equation

$$\widetilde{\Sigma}^\lambda_g(x-y) = -i\,\frac{\partial}{\partial x_\mu}\,\widetilde{\Sigma}^{\prime\mu}_g(x-y) \qquad\qquad (4.11)$$

where

$$\Sigma_g(k^2) = \int d^4x\,e^{-ik\cdot x}\,\widetilde{\Sigma}_g(x) \qquad\qquad (4.12)$$

and satisfies the unrenormalized equation

$$i\,\delta^{ab}\,\widetilde{\Sigma}^\lambda_g(x-y) = q\,\epsilon^{adc}\int (-i)\,\widetilde{\Delta}^{\lambda\mu}(x-z)\,i\,\widetilde{\mathcal{G}}(x-w)\,i\,\widetilde{\gamma}^{dbc}_\mu(w,y;z) \qquad (4.13)$$

$$\times\,d^4w\,d^4z$$

We consider the part which is transverse with respect to the index

λ of Eq. (4.8). Noting that

$$W^{-1}[J_{\mu}] \frac{\delta^3 W[J_{\mu}]}{\delta J_{\mu}^a(x) \delta J_{\nu}^b(y) \delta J_{\lambda}^c(z)}\Big|_{J=0}$$

$$= -ig \int d^4x' d^4y' d^4z' \tilde{\Delta}^{\mu\mu'}(x-x') \tilde{\Delta}^{\nu\nu'}(y-y') \tilde{\Delta}^{\lambda\lambda'}(z-z') \tilde{\Gamma}_{\mu'\nu'\lambda'}^{abc}(x',y',z')$$

where $\tilde{\Gamma}_{\mu\nu}^{abc}(x,y,z)$ is the proper three-point vertex of vector mesons,

and taking the Fourier tramsform we obtain

$$(p^{\lambda}/p^2)(q^{\mu}/q^2)[k^2 J(k^2)]^{-1}(g^{\rho\nu} - k^{\rho}k^{\nu}/k^2) \Gamma_{\lambda\mu\nu}^{abc}(p,q,k) \qquad (4.14)$$

$$= (q^{\mu}/q^2)\mathcal{G}(p^{\nu})(q^{\rho} - k^{\lambda}k^{\rho}/k^2) \tilde{\gamma}_{\lambda\mu}^{cab}(k,p;q), \qquad p+q+k=0.$$

Equation (4.14) is a constraint amongst the propagators and three-

point proper vertices.   Equation (4.14) was first derived by Slavnov.[18]

## C. Four Point Vertices

It follows from Eq. (3.13) that

$$\frac{\partial}{\partial x^{\lambda}} \frac{\partial}{\partial y^{\mu}} \frac{\partial}{\partial z^{\nu}} \frac{\partial}{\partial w^{\rho}} \frac{\delta^4 Z[J_{\mu}]}{\delta J_{\lambda}^a(x) \delta J_{\mu}^b(y) \delta J_{\nu}^c(z) \delta J_{\rho}^d(w)}\Big|_{J_{\mu}=0} = 0 \qquad (4.15)$$

from which one obtains a constraint on the four point vertex $\Gamma_{\lambda\mu\nu\rho}^{abc}$:

$$p^{\lambda}q^{\mu}k^{\nu}s^{\rho}\left\{ \Gamma_{\lambda\mu\nu\rho}^{abcd}(p,q,k,s) \right.$$

$$+ \left[ \Gamma_{\lambda\mu\sigma}^{abe}(p,q,-p-q) \Delta^{\sigma\tau}(p+q) \Gamma_{\nu\rho\tau}^{cde}(k,s,p+q) \right.$$

$$\left. + \text{ two more terms } \right]\Big\} = 0,$$

$$p+q+k+s = 0$$

where $\Gamma^{abcd}_{\lambda\mu\nu\rho}$ is defined as

$$i \frac{\delta^4 Z [J_\mu]}{\delta J^a_\lambda(x)\, \delta J^b_\mu(y)\, \delta J^c_\nu(z)\, \delta J^d_\rho(w)} = (-i)^4 i g \int d^4x'\, d^4y'\, d^4z'\, d^4w' \qquad (4.17)$$

$$\tilde{\Delta}^{\lambda\lambda'}(x-x')\, \tilde{\Delta}^{\mu\mu'}(y-y')\, \tilde{\Delta}^{\nu\nu'}(z-z')\, \tilde{\Delta}^{\rho\rho'}(w-w')\, \tilde{\Gamma}^{abcd}_{\lambda'\mu'\nu'\rho'}(x',y',z',w')$$

$$+ \text{reducible parts}$$

and

$$\int d^4x\, d^4y\, d^4z\, d^4w\ e^{i(p\cdot x + q\cdot y + r\cdot z + s\cdot w)}\, \tilde{\Gamma}^{abcd}_{\lambda\mu\nu\rho}(x,y,z,w) \qquad (4.18)$$

$$= (2\pi)^4\, \delta^4(p+q+r+s)\, \Gamma^{abcd}_{\lambda\mu\nu\rho}(p,q,r,s)$$

The ghost-ghost-vector-vector vertex is superficially convergent and requires no discussion.

# V. REGULARIZATION

They Feynman amplitudes constructed from the expression (2.1) would automatically satisfy the Ward-Takahashi identities discussed in the last two sections, if it were not for the ultraviolet divergences in their construction. A standard procedure of constructing the renormalized amplitudes satisfying the Ward-Takahashi identities is to regularize the Feynman integrals in a gauge invariant manner, and then perform the R-operation[14-17] of Bogoliubov, Parasuik and Hepp (BPH). The resulting amplitudes are cutoff independent, and if the values of primitively divergent vertices at subtraction points are chosen in accordance with the Ward-Takahashi identities, then the full amplitudes satisfy them too. Furthermore, under such circumstances, the R-operation may be formally implemented by a gauge invariant set of counter terms in the Lagrangian.

In this section we will discuss a few gauge invariant regularization methods which can be implemented by adding gauge invariant terms in the Lagrangian (e. g. , Pauli-Villors regularization). 't Hooft[22] discussed a method which works for one loop diagrams, but which does not appear to be implementable by modifying the Lagrangian.

We choose as regulator fields both scalar and spinor fields. They have all positive masses. They may belong to arbitrary representations (in general reducible) of the symmetry group. They are coupled

to the gauge fields by the minimal gauge invariant coupling. They

may, however, be quantized by the wrong spin-statistics connections

(i. e., some scalar field multiplets may be quantized by the anti-

commutation relation). Let us show that the addition of these regulator

fields to the Lagrangian renders the divergent Feynman integrals

with one loop finite.

Let us first consider the self energy of the gauge boson in the

one loop approximation. We will carry out the computations in the

Feynman gauge. There are three diagrams (one is a ghost loop),

and the sum of the contributions from these is

$$
\Sigma_{\mu\nu}(p) = -4\, g_{\mu\nu}\, \frac{g^2}{16\pi^2}\, \int_0^\infty \frac{dz_1\, dz_2}{(z_1+z_2)^3}\, e^{i\frac{z_1 z_2}{z_1+z_2} p^2}\left(-1 + i\frac{z_1 z_2}{z_1+z_2} p^2\right)
$$

$$
+ \frac{g^2}{16\pi^2}\, i\left(g_{\mu\nu} p^2 - p_\mu p_\nu\right)\int_0^\infty \frac{dz_1\, dz_2}{(z_1+z_2)^2}\, e^{i\frac{z_1 z_2}{z_1+z_2} p^2}\, \frac{z_1^2 + 6 z_1 z_2 + z_2^2}{(z_1+z_2)^2}
$$

(5.1)

The first term on the right is gauge non-invariant and quadratically

divergent; the second term, which is gauge invariant, is logarithmi-

cally divergent. We shall regulate $\Sigma_{\mu\nu}$ by the replacement

$$
\Sigma_{\mu\nu} \to \Sigma_{\mu\nu}^{(reg)} = \Sigma_{\mu\nu} + \Sigma_{\mu\nu}^{(1)} + \Sigma_{\mu\nu}^{(2)}
$$

(5.2)

where $\Sigma_{\mu\nu}^{(1)}$ is the sum of the scalar regulator contributions:

$$
\Sigma_{\mu\nu}^{(1)} = \sum_i c_i \left\{ -\frac{2g^2}{16\pi^2}\, g_{\mu\nu} \int_0^\infty \frac{dz_1\, dz_2}{(z_1+z_2)^2}\, e^{i\left[\frac{z_1 z_2}{z_1+z_2} p^2 - (z_1+z_2) m_i^2\right]} \right.
$$

$$\times \left[ -1 + i \left( \frac{z_1 z_2}{z_1 + z_2} p^2 - (z_1 + z_2) m_i^2 \right) \right]$$

$$- \frac{q^2}{16\pi^2} i (q^2 g_{\mu\nu} - p_\mu p_\nu) \int_0^\infty \frac{dz_1 dz_2}{(z_1 + z_2)^2} e^{i \frac{z_1 z_2}{z_1 + z_2} p^2 - i(z_1 + z_2) m_i^2} \left( \frac{z_1 - z_2}{z_1 + z_2} \right)^2$$

and $\Sigma_{\mu\nu}^{(2)}$ is the sum of the spinor regulator contributions:

$$\Sigma_{\mu\nu}^{(2)} = - \sum_i D_i \int \frac{dz_1 dz_2}{(z_1 + z_2)^2} e^{i \frac{z_1 z_2}{z_1 + z_2} p^2 - i(z_1 + z_2) \mu_i^2}$$

$$\left\{ -\frac{2 q^2}{16\pi^2} g_{\mu\nu} \left[ \frac{-1}{z_1 + z_2} + i \left( \frac{z_1 z_2}{(z_1 + z_2)^2} p^2 - \mu_i^2 \right) \right] - \frac{4 q^2}{16\pi^2} i (g_{\mu\nu} p^2 - p_\mu p_\nu) \frac{z_1 z_2}{z_1 + z_2} \right\}$$

The coefficients $C_i$ and $D_i$ depend on the representations to which

the regulators belong, and also on whether they obey the normal or

abnormal statistics. In any case, if we choose

$$2 + \sum_i C_i - \sum_j D_j = 0 \tag{5.3}$$

and

$$\sum_i C_i m_i^2 - \sum_j D_j \mu_j^2 = 0 \tag{5.4}$$

then the gauge noninvariant term vanishes identically. Furthermore

if we choose

$$10 - \sum_i C_i + 2 \sum_j D_j = 0 \tag{5.5}$$

the logarithmic divergence in the gauge invariant part may be elimin-

ated. The introduction of two kinds of regulators is necessitated by

the requirement that both quadratic and logarithmic divergences be

eliminated.

Next, we consider the three-point vertex $\Gamma_{\lambda\mu\nu}(p, q, r)$ of three

gauge bosons in the one loop approximation. The integral is linearly

divergent and has the asymptotic structure, in the Feynman gauge,

$$\sim \int \frac{d^4 \ell}{(2\pi)^4} \frac{\ell_\lambda \ell_\mu \ell_\nu}{(\ell^2)^3} , \tag{5.6}$$

when all diagrams, including the one with a ghost loop, are added.

Again, by taking a suitable combination of scalar and spinor regulators,

it is possible to eliminate all divergences from the Feynman integral

for the three-point vertex.

The four point vertex is logarithmically divergent and offers no

special difficulty. We have not verified that this method works for

higher order diagrams. In any case, when there are matter fields

present in the Lagrangian, the method presented above is insufficient

and it becomes necessary to dampen the high energy behavior of the

gauge boson propagator itself. The method described below will do

just this, and when combined with the spinor-scalar regulators, will

render all Feynman integrals finite.

We will add gauge invariant higher derivative terms to the Lagran-

gian. Consider for example the Lagrangian

$$\mathcal{L}_A = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} - \frac{\alpha}{4\Lambda^2} (D^\tau F^{\mu\nu}) \cdot (D_\sigma F_{\mu\nu})$$

$$- \frac{\beta}{4\Lambda^4} (D^\tau D_\sigma F_{\mu\nu}) \cdot (D^\rho D_\beta F^{\mu\nu})$$

(5.7)

The vector boson propagator is now

$$\Delta_{\mu\nu}(k; \Lambda^2) = (q_{\mu\nu} - k_\mu k_\nu/k^2)(k^2 + i\epsilon)^{-1} \left[ 1 + \alpha \frac{k^2}{\Lambda^2} + \beta \left(\frac{k^2}{\Lambda^2}\right)^2 \right]$$

+ gauge dependent term

and behaves like $(p^2)^{-3}$ asymptotically. The maximum dimension

of various new couplings ( in powers of mass ) is eight. A power

counting argument (see Appendix D) shows that in this case only the

two-, three- and four-point vertices with <u>one loop</u> are primitively

divergent (quadratically, linearly, and logarithmically, respectively).

Other proper vertices, including two-, three- and four point vertices

with more than one loop are at least superficially convergent. As one

adds still higher gauge covariant derivatives, the propagator becomes

more convergent at larger momentum, but the maximum dimensions of

the interaction terms increase also, in such as way that the two-, three-

and four-point vertices with one loop remain always divergent (see

Appendix D). Note also that ghost loops for these vertices remain

divergent.

Therefore, by the addition of the last two terms to the Lagrangian

(5.7), the divergences of the theory have now been isolated to those

diagrams for which the spinor-scalar regularization was shown to

work.

The BPH R-operation is to be applied to the entire two-, three-

and four-point proper vertices. The resulting vertices are cutoff

independent, in the sense that the limits $\Lambda^2 \to \infty$ of these amplitudes

are finite and independent of $\alpha$ and $\beta$ of Eq. (5.7). This can be seen

as follows. A proper amplitude with two, three or four external

lines which is proportional to some powers of $\alpha$ and $\beta$ has in general

an ambiguous limit as $\Lambda^2 \to \infty$. However, the finite part of such an

integral vanishes like $\Lambda^{-2}(\ell n \Lambda^2)^m$ as $\Lambda^2 \to \infty$. A proper Feynman

diagram with n external lines, n > 4, with one of more vertices pro-
portional to $\alpha$ or $\beta$ which are not contained in any subdiagrams with
two, three or four external lines vanishes at least as fast as
$\Lambda^{-(n-4)}(\ln\Lambda^2)^m$ as $\Lambda^2 \to \infty$, after the R-operations are applied to the
subdiagrams. (The above is a summary of a rather lengthy analysis.)

The results of regulating the Feynman integral by the method
described above, applying the R-operation and then letting the cut-
off $\Lambda^2$ go to infinity is identical to applying the R-operation directly
to the Feynman integral. This shows that the BPH R-operation is in
fact a gauge invariant procedure.

A similar regularization procedure has been applied to nonlinear
chiral Lagrangians by Slavnov.[23] We understand from R. Jackiev
and L. D. Fadeev that Slavnov has considered the regularization meth
method of Eq. (5.7) for the gauge fields also. (After the completion
of this work we received a preprint of Slavnov.[24]) This possibility
has also been known to K. Johnson.[25]

<center>VI. RENORMALIZATION CONDITIONS</center>

<center>AND INFRARED DIVERGENCES</center>

Let us first discuss briefly how the values of primitively divergent
vertices are determined from the considerations of Sec. IV, ignoring
the problems associated with infrared divergences. Under such cir-
cumstances, we may choose as subtraction points the points at which

all external momenta vanish. Later we will discuss the nature of

infrared divergences in gauge theories and give a set of renormali-

zation conditions which avoid the infrared difficulties.

We may by convention choose, in Eq. (4.5),

$$J(0) = 1$$

which amounts to

$$\lim_{k \to 0} \Delta_{\mu\nu}(k) = \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2} + \alpha \frac{k_\mu k_\nu}{(k^2)^2} \qquad (6.1)$$

The normalization of the ghost propagator is arbitrary. The ghost

propagator has a simple pole at $k^2 = 0$, [ see the discussion following

Eq. (4.7)] , and we write

$$\lim_{k^2 \to 0} \mathcal{G}(k^2) = \frac{Z_g}{k^2} \qquad (6.2)$$

where $Z_g$ is an arbitrary (finite) constant.

In the limit p, q and r = -p-q all go to zero, the three point vertex
$\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)$ has the form

$$\lim_{p,q,r \to 0} \Gamma^{abc}_{\lambda\mu\nu}(p,q,r) = -i\, G\, \epsilon^{abc} \left[ (p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} \right. \qquad (6.3)$$
$$\left. + (r-p)_\mu g_{\nu\lambda} \right]$$

as follows from Lorentz covariance, isospin conservation and Bose

symmetry.   Likewise, the low energy form of the vertex $\gamma_{\lambda\mu}^{cab}(r,p;q)$

is given by

$$\lim_{\substack{(p,q,r) \\ \to 0}} \gamma_{\lambda\mu}^{cab}(r,p;q) = -i\,G'\epsilon^{abc}\,g_{\lambda\mu}\,,$$

so that

$$\lim_{\substack{(p,q,r) \\ \to 0}} \gamma_{\mu}^{cab}(r,p;q) = -i\,G'\epsilon^{abc}\,\lambda_{\mu}\,.$$

Equation (4.14) then tells us that

$$G = G'Z_g \tag{6.4}$$

In the Green's functions, only the combination $G'Z_g$ enters, because

the ghost never appears as an external line, so that it is convenient to

set $Z_g = 1$ and $G = G'$.

The low energy form of the four point vertex is given by

$$\lim_{\substack{(p,q,r,s) \\ \to 0}} i\,\Gamma_{\rho\mu\nu\lambda}^{abcd}(p,q,r,s) = -i\,F\left[\epsilon^{abe}\epsilon^{cde}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\rho}g_{\mu\nu})\right.$$
$$+\epsilon^{ace}\epsilon^{bde}(g_{\lambda\mu}g_{\nu\rho}-g_{\lambda\rho}g_{\mu\nu})+\epsilon^{ade}\epsilon^{cbe}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\mu}g_{\rho\nu})\left.\right]$$
$$+F'\left[\delta^{ab}\delta^{cd}g_{\lambda\mu}g_{\nu\rho}+\delta^{ac}\delta^{bd}g_{\lambda\nu}g_{\mu\rho}+\delta^{ad}\delta^{bc}g_{\lambda\rho}g_{\mu\nu}\right].$$

$$\tag{6.5}$$

Equation (4.6) tells us that

$$F = G^2 \text{ and } F' = 0 \tag{6.6}$$

The conditions (6.1), (6.4) and (6.6) allow us to express all primitive-

ly divergent vertices in terms of only one constant G.

As we have stated before, the foregoing discussion is of heuristic value only because of the infrared divergences that the Feynman integrals experience when all external momenta are set equal to zero. More precisely, a Feynman integral of the theory becomes divergent if two or more external lines are set on the mass shell. We assert that a Feynman integral suffers no infrared catastrophe if all of the external lines are kept off the mass shell. One can easily see this for diagrams with one loop. As long as all of the external lines are off the mass shell, infrared divergences in any subintegrations can occur only in the measure zero of the space of all integration variables where the rest of the integrand is nonsingular. Therefore, by choosing subtraction points to be somewhere other than where all external lines are on the mass shell, we can circumbent the infrared difficulties in the construction of Green's functions (but not the S-matrix! ) altogether.

A convenient convention for the subtraction points is given by Symanzik.[26] We choose as such the points where the squares of external momenta are all equal to a negative number, say $-a^2$. Defining all external momenta outward from the vertex, we have at the subtraction point $p_i^2 = -a^2$, $p_i \cdot p_j = \frac{1}{n-1} a^2$.

As an example, we will work out the renormalization conditions for two and three point vertices.

We shall normalize the fields $A_\mu$ and $c$ so that

$$J(-a^2) = 1 \; ,$$

and

$$\Sigma_g(-a^2) = 0 \; .$$

Then we have

$$\lim_{k^2 \to -a^2} \Delta_{\mu\nu}(k) = \left(g_{\mu\nu} - k_\mu k_\nu / k^2\right) \frac{1}{-a^2} + \text{gauge dependent terms,}$$

$$G(a^2) = \frac{1}{-a^2}$$

At the symmetric point $p^2 = q^2 = r^2 = -a^2$, the three point vertices $\Gamma^{abc}_{\lambda\mu\nu}(p, q, r)$ and $\gamma^{cab}_{\lambda\mu}(r, p; q)$ have the structures:

$$\lim_{p^2 = q^2 = r^2 = -a^2} i\, \Gamma^{abc}_{\lambda\mu\nu}(p, q, r) = \Big\{ G\left[(p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda}\right]$$

$$+ H\left[(r-q)_\lambda r_\mu q_\nu + (p-r)_\mu p_\nu r_\lambda + (q-p)_\nu q_\lambda p_\mu\right]$$

$$+ J\left[r^\lambda p^\mu q^\nu - q^\lambda r^\mu p^\nu\right]\Big\} \, \epsilon^{abc} \, , \qquad (6.7)$$

$$\lim_{p^2 = q^2 = r^2 = -a^2} i\, \gamma^{cab}_{\lambda\mu}(r, p; q) = \epsilon^{abc}$$

$$\times \Big\{ G' g_{\lambda\mu} + K_1 p_\lambda r_\mu + K_2 p_\lambda p_\mu + K_3 p_\lambda q_\mu$$

$$+ L_1 q_\lambda r_\mu + L_2 q_\lambda p_\mu + L_3 q_\lambda q_\mu \qquad (6.8)$$

$$+ r_\lambda (\; --- \;) \Big\}$$

where the omitted terms (---) are of no interest to the present problem. The quantities G and G' require subtractions whereas the other form factors J, J, $K_i$ --- are superficially convergnet. Substituting Eqs. (6.7) and (6.8) into Eq. (4.14) and taking the limit $p^2 = q^2 = r^2 = -a^2$, we obtain

$$G - \frac{a^2}{2} ( H + J ) = G' + \frac{a^2}{2} \left( L_1 + L_2 - 2 L_3 - K_1 - K_2 + 2 K_3 \right) \quad (6.9)$$

which gives G' in terms of G.

## VII. RENORMALIZATION OF MATTER FIELDS

So far our discussion was based on the Lagrangian (2.3) which contains only the Yang-Mills quanta. As an illustration of the renormalization procedure in the presence of matter fields, we consider the case in which a **triplet** of real scalar fields $\phi^a$ is added to the Lagrangian by the minimal gauge invariant coupling.

Let $K^a(x)$ be the sources of the scalar fields $\phi^a$. It is not difficult to derive the generalization of Eq. (3.13). It is

$$\frac{i}{\alpha} \partial_\mu \frac{\delta W'}{\delta J_\mu^a(x)} - \int d^4y \; J_\lambda^c(y) \, D_y^\lambda \, [i \delta/\delta J]^{cb} \, G^{ba}(y,x; i \delta/\delta J) \, W$$

$$+ ig \int d^4y \; K^c(y) \, t^{cdb} \frac{\delta}{\delta K^b(y)} \, G^{da}(y,x; i\delta/\delta J) \, W$$

$$= 0 \; . \tag{7.1}$$

In this theory the $A_\mu \phi^2$ - , $A_\mu^2 \phi^2$ - , $\phi^2$ - and $\phi^4$ - vertices are primi-

tively divergent. The renormalization of the $\phi^4$ - vertex has nothing

to do with the local gauge invariance and presents no problem. In order

to regularize Feynman integrals for these vertices, it becomes

necessary to regularize the gauge boson propagators in a gauge invar-

iant manner, for example, by the use of the Lagrangian (5.7). The

renormalization conditions for the $\phi^2$ -, $A_\mu \phi^2$ - and $A_\mu^2 \phi^2$ - vertices

are obtained from Eq. (7.1). First, though, let us ignore the infrared

complications. Later, we will detail the renormalization conditions

which take due account of the infrared difficulties.

From Eq. (7.1) we obtain

$$\left(\frac{i}{\alpha}\right)^2 \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \cdot \frac{\delta^4 \; i Z[\underset{\sim}{J_\mu}, \underset{\sim}{K}]}{\delta J_\mu^a(x) \delta J_\nu^b(y) \, \delta K^c(z) \, \delta K^d(w)} \bigg|_{\underset{\sim}{J_\mu} = \underset{\sim}{K} = 0}$$

$$= -ig \, \epsilon^{cfe} \, W^{-1} \frac{\delta}{\delta K^e(z)} \, G^{fa}(z,z; i\delta/\delta \underset{\sim}{J}) \frac{\delta}{\delta K^d(w)} \frac{i}{\alpha} \frac{\partial}{\partial y^\nu} \frac{\delta W}{\delta J_\nu^b(y)} \bigg|_{\underset{\sim}{J_\mu} = \underset{\sim}{K} = 0}$$

$$+ \left( c \leftrightarrow d, \; z \leftrightarrow w \right). \qquad\qquad (7.2)$$

If we amputate the scalar propagators and go to the mass shells of two scalar particles $q_1^2 = q_2^2 = \mu^2$ in Eq. (7.2), the right hand side of the equation vanishes. Let us define the primitively divergent vertices:

$$\frac{\delta^2 Z}{\delta K^a(x)\, \delta K^b(y)}\bigg|_{\underset{\sim}{J_\mu} = \underset{\sim}{K} = 0} = -\widetilde{\Delta}^{ab}(x-y) = -\delta^{ab}\, \widetilde{\Delta}(x-y),$$

$$\frac{\delta^3 i Z}{\delta J_\mu^a(x)\, \delta K^b(y)\, \delta K^c(z)}\bigg|_{\underset{\sim}{J_\mu} = \underset{\sim}{K} = 0} = -g \int d^4x'\, d^4y'\, d^4z'\ \widetilde{\Delta}_{\mu\nu}(x-x')\, \widetilde{\Delta}(y-y')\, \widetilde{\Delta}(z-z')\, \widetilde{C}_\nu^{abc}(x';y',z')$$

$$\frac{\delta^4 i Z}{\delta J_\mu^a(x)\, \delta J_\nu^b(y)\, \delta K^c(z)\, \delta K^d(w)}\bigg|_{\underset{\sim}{J_\mu} = \underset{\sim}{K} = 0} = i g^2 \int d^4x'\, d^4y'\, d^4z'\, d^4w'$$

$$\times\ \widetilde{\Delta}_{\mu\rho}(x-x')\, \widetilde{\Delta}_{\nu\sigma}(y-y')\, \widetilde{\Delta}(z-z')\, \widetilde{\Delta}(w-w')\, \widetilde{C}_{\rho\sigma}^{abcd}(x',y';z',w'),$$

and

$$\int d^4x\, d^4y\, d^4z\ e^{i(p\cdot x + q_1\cdot y + q_2\cdot z)}\ \widetilde{C}_\nu^{abc}(x';y',z')$$

$$= C_\nu^{abc}(p;q_1,q_2)\,(2\pi)^4\, \delta^4(p+q_1+q_2),$$

$$C_\nu^{abc}(p;q_1,q_2) = \epsilon^{abc}\, C_\nu(p;q_1,q_2),$$

and similarly for $C_{\mu\nu}^{abcd}$. Then we have from Eq. (7.2) and the

subsequent discussion that

$$\lim_{\substack{q_1^2 = q_2^2 = \mu^2}} p_1^\mu p_2^\nu \left\{ C_{\mu\nu}^{abcd}(p_1, p_2; q_1, q_2) \right.$$

$$+ \Gamma_{\mu\nu\lambda}^{abf}(p_1, p_2, -p_1-p_2) \Delta^{\lambda\rho}(q_1 \cdot q_2) C_\rho^{fcd}(p_1, p_2; q_1, q_2)$$

$$- \left[ C_\mu^{acf}(p_1; q_1, p_2-q_2) \Delta(p_2-q_2) C_\nu^{bdf}(p_2; q_2, p_1-q_1) \right.$$

$$\left. \left. + (c \leftrightarrow d, q_1 \leftrightarrow q_2) \right] \right\} = 0 \qquad (7.3)$$

We shall now consider the limit $p_1$, $p_2 \to 0$ while $Q = q_1 - q_2$ is kept

finite. The low energy forms of the vertices above are

$$\lim_{\substack{q_1^2 = q_2^2 = \mu^2}} C_\mu(p; q_1, q_2) = -i\, C\, Q_\mu \qquad (7.4)$$

$$\lim_{p_1, p_2 \to 0} \Gamma_{\mu\nu\lambda}^{abc}(p_1, p_2, r) = -i\, G\left[(p_1-p_2)_\lambda\, g_{\mu\nu} + (p_2-r)_\mu\, g_{\nu\lambda} + (r-p_1)_\nu\, g_{\mu\lambda}\right] \qquad (7.5)$$

where G is defined previously and $r + p_1 + p_2 = 0$;

$$\lim_{\substack{q_1^2 = q_2^2 = \mu^2 \\ p_1, p_2 \to 0}} C_{\mu\nu}^{abcd}(p_1, p_2; q_1, q_2) = \xi^{ab} \xi^{cd} (A\, g_{\mu\nu} + B\, Q_\mu Q_\nu)$$

$$+ (\xi^{ac}\xi^{cd} + \xi^{ad}\xi^{cb})(A'\, g_{\mu\nu} + B'\, Q_\mu Q_\nu) \qquad (7.6)$$

The factors C, A and A' require subtractions. We may renormalize

the $\phi$ fields so that

$$\lim_{p^2 \to \mu^2} \Delta(p^2) = (p^2 - \mu^2)^{-1} \qquad (7.7)$$

Substituting Eqs. (7.4) - (7.7) in Eq. (7.3) and isolating the part

antisymmetric in a and b, we get

$$GC = C^2$$

or

$$G = C \tag{7.8}$$

Next looking at the part symmetric in a and b, we obtain

$$B = B' = 0$$

and

$$A = -2G^2$$
$$A' = G^2 \tag{7.9}$$

The above treatment is careless, since the vertices $C_\mu \ C_{\mu\nu}$ exhibit infrared divergence in the limit $q_1^2$ or $q_2^2 = \mu^2$. We must, therefore, determine the renormalization conditions $\Delta$, $C_\mu$ and $C_{\mu\nu}$ without going to the mass shells of the scalar fields. For this purpose, we will return to Eq. (7.1). From Eq. (7.1) follows the relation:

$$x \frac{\partial}{\partial x^\mu} \left. \frac{\delta^3 iZ}{\delta J_\mu^a(x) \delta K^b(y) \delta K^c(z)} \right|_{J_\mu = K = 0} =$$

$$-ig \left[ W^{-1} \epsilon^{bfe} \frac{\delta}{\delta K^e(y)} \frac{\delta}{\delta K^c(z)} G^{fa}(y, x; i\delta/\delta J_\mu) W \right.$$

$$\left. + (c \leftrightarrow d, \ y \leftrightarrow z) \right]_{J_\mu = K = 0} \tag{7.10}$$

In order to discuss this equation, it is necessary to define a new proper vertex $R^{abc}$. We write

$$\epsilon^{bfe} \ W^{-1} \frac{\delta^2}{\delta K^e(y) \delta K^c(z)} G^{fa}(y, x; i\delta/\delta J_\mu) W \Big|_{J_\mu = K = 0}$$

$$\tag{7.11}$$

$$= i \int d^4x' d^4y' \ \tilde{R}^{bac}(y, x', z') \ \tilde{\mathcal{G}}(x'-x) \ \tilde{\Delta}(z'-z)$$

$$+ i \ \tilde{\mathcal{G}}(y-z) \ \tilde{\Delta}(y-z) \ \epsilon^{bac}$$

$$R^{abc}(q,p,r)(2\pi)^4 \, \delta(p+q+r) = \int d^4x d^4y d^4z \, \tilde{R}^{abc}(x,y,z) \, e^{i(px+qy+rz)},$$

$$R^{abc}(p,q,r) = \epsilon^{abc} R(p,q,r)$$

The new vertex does not arise in the pertubative construction of the Green's functions. It is, however, relevant to our discussion of the renormalization of Eq. (7.2). The vertex $R^{abc}$ has the superficial degree of divergence equal to zero, and therefore, requires one sub-traction. The value of this vertex at the subtraction point is related to that of $C_\mu^{abc}$ through Eq. (7.11). Let us again choose as subtraction point the point where $p^2 = q^2 = r^2 = -a^2$, $p \cdot q = q \cdot r = r \cdot p = a^2/2$. The quantity $R^{abc}(q_1, p, q_2)$ may be written in the neighborhood of the subtraction point as

$$R^{bac}(q_1, p, q_2) = \epsilon^{bac} \left[ R + (q_1^2 + a^2) r_1 + (q_2^2 + a^2) r_2 \right.$$
$$\left. + (p^2 + a^2) r_3 + \cdots \right] \tag{7.12}$$

where R requires a subtraction and $r_1$, $r_2$, $r_3$, are finite.

We now transform Eq. (7.10) into the momentum space. It reads then

$$\left[ p^2 \mathcal{G}(p^2) \right]^{-1} (-i p^\mu) C_\mu (p; q_1, q_2) = \Delta^{-1}(q_1^2) - \Delta^{-1}(q_2^2)$$
$$+ \left[ R(q_1, p, q_2) \Delta^{-1}(q_1^2) - R(q_2, p, q_1) \Delta^{-1}(q_2^2) \right] \tag{7.13}$$

Adopting the field normalization conventions:

$$\left[ p^2 \mathcal{G}(p^2) \right] \Big|_{p^2 + a^2 = 0} = 1 \tag{7.14}$$

$$\tag{7.15}$$

$$\lim_{q_1^2 \to -a^2} \Delta^{-1}(q_1^2) = q_1^2 + a^2 - M^2$$

and defining C by

$$\lim_{p^2 = q_1^2 = q_2^2 = -a^2} G_\mu(p; q_1, q_2) = -i \, G_\mu \, C \, , \tag{7.16}$$

$$G_\mu = (q_1 - q_2)_\mu$$

we have from Eq. (7.13)

$$C = 1 + R^2 - M^2(r_1 - r_2), \tag{7.17}$$

which is the required relation.

Now we turn to Eq. (7.2). We need again to define a still new

proper vertex. We write

$$\epsilon^{cfa} W^{-1} \frac{\partial}{\partial K_e(z)} G^{fa}(z, x; i\delta/\delta I) \frac{\delta}{\delta K^d(\omega)} \frac{\partial W}{\partial J_\nu^b(y)} \Big|_{J_\mu = K = 0}$$

$$= -ig \, \epsilon^{cfd} \, \tilde{\Delta}(z-\omega) \int d^4z' \, d^4x' \, d^4y' \, \tilde{\mathcal{G}}(z-z') \, \tilde{f}_\lambda^{fab}(z', x'; y') \tilde{\Delta}_\nu^\lambda(y'-y) \tilde{\mathcal{G}}(x'-x)$$

$$-ig \int d^4z' \, d^4x'' \, d^4y' \, d^4\omega' \, d^4u \, \tilde{R}^{cfd}(z, x', \omega') \, \tilde{\Delta}(\omega'-\omega) \, \tilde{\mathcal{G}}(x'-x'') \tag{7.18}$$

$$\times \tilde{f}_\lambda^{fab}(x'', u; y') \, \tilde{\Delta}_\nu^\lambda(y'-y) \, \tilde{\mathcal{G}}(u-z)$$

$$+ ig \, \epsilon^{cae} \, \tilde{\mathcal{G}}(z-y) \int d^4y' \, d^4z' \, d^4\omega' \, \tilde{\Delta}(z-z') \, \tilde{\Delta}(\omega'-\omega) \, \tilde{\Delta}_\nu^\mu(y-y') \, \tilde{C}_\mu^{bed}(y'; z', \omega')$$

$$+ ig \int \tilde{R}^{cae}(z, x', u) \, \tilde{\mathcal{G}}(x'-x) \, \tilde{\Delta}(u-z') \, \tilde{\Delta}(\omega'-\omega) \, \tilde{\Delta}_\nu^\mu(y-y') \, \tilde{C}_\mu^{bed}(y'; z'\omega')$$

$$\times d^4x' \, d^4u \, d^4y' \, d^4z' \, d^4\omega' \quad + \quad g \int d^4\omega' \, d^4x' \, d^4y' \, \tilde{S}_\mu(z, \omega, x', y') \, \tilde{\Delta}(\omega'-\omega) \tilde{\mathcal{G}}(x'-z)$$

$$\times \tilde{\Delta}_\nu^\mu(y-y'),$$

which defines the vertex $S_\mu^{abcd}$;

$$S_\mu^{abcd}(q_1, q_2, p_1, p_2)(2\pi)^4 \, \delta^4(q_1 + q_2 + p_1 + p_2) \tag{7.19}$$

$$= \int d^4x \, d^4x' \, d^4y \, d^4y' \, \tilde{S}_\mu^{abcd}(y, y', x, x') \, \epsilon^{i(q_1 y + q_2 y' + p_1 x + p_2 x')}$$

The fact that $S_\mu^{abcd}$ is superficially convergent is of importance in the

ensuing discussion.

Now going to the momentum space, we rewrite Eq. (7.2) as

$$i\left[p^2 \mathcal{Y}(p^2)\right]^{-1} p_1^\mu p_2^\nu \left\{ C_{\mu\nu}^{abcd}(p_1,p_2;q_1,q_2) \right.$$

$$\left. + \cdots \right\}$$

$$= p_2^\nu \Delta^{-1}(q_1)\left\{ \epsilon^{abf}\epsilon^{cfd}\mathcal{Y}((p_1+p_2)^2)\left[1+R(q_1,p_1+p_2,q_2)\right]\gamma_\nu(q_1+q_2,p_1;p_2) \right.$$

$$+ \epsilon^{acf}\epsilon^{bfd}\Delta((p_1+q_1)^2)\left[1+R(q_1,p_1,p_2+q_2)\right]C_\nu(p_2,p_1+q_1,q_2)$$

$$\left. + i S_\nu^{cdab}(q_1,q_2;p_1,p_2) \right\}$$

$$+ p_2^\nu \Delta^{-1}(q_2)\left\{ c \leftrightarrow d,\ q_1 \leftrightarrow q_2 \right\} \qquad (7.20)$$

where the expression in the curly bracket on the left is identical to

that on the left of Eq. (7.3). Equation (7.20) is the generalization of

Eq. (7.3) and it allows us to determine the values of the vertices

$C_{\mu\nu}^{abcd}$ and $C_\mu^{abc}$ at some subtraction points, in terms of that of the ver-

tex $\Gamma_{\mu\nu\lambda}^{abc}$. It is so, because the structure constant for $\gamma_\nu$ is known

from Eq. (6.9); the subtraction constant for R is known in terms of

the value of $C_\mu$ at the subtraction point through Eq. (7.17); and $S_\mu$

is superficially convergent, so that the right hand side of the Eq. (7.20)

contains the value we seek and no other unknowns.

## APPENDIX A

### Derivations of Eqs. (3.5) and (3.9)

We recall the definition

$$\Delta_L \left[ i\delta/\delta \underset{\sim}{J}_\mu \right] = \exp \operatorname{Tr} \ln \left( 1 - ig \, \underline{t} \cdot \frac{\delta}{\delta \underset{\sim}{J}} \partial^\mu \frac{1}{\partial^2} \right) \tag{A.1}$$

In evaluating $\Delta_L \left[ i\delta/\delta \underset{\sim}{J}_\mu \right] \underset{\lambda}{\overset{a}{J}} (x) \Delta_L^{-1} \left[ i\delta/\delta \underset{\sim}{J}_\mu \right]$ it is convenient to

make the following mapping:

$$\delta/\delta \underset{\sim}{J}_\mu \;\; \rightarrow \;\; - \underset{\sim}{\xi}_\mu$$

$$\underset{\sim}{J}_\mu \;\; \rightarrow \;\; \delta/\delta \underset{\sim}{\xi}_\mu \tag{A.2}$$

which is canonical. We see that

$$\exp \operatorname{Tr} \ln \left( 1 + ig \, \underline{t} \cdot \underset{\sim}{\xi}^\mu \partial_\mu \frac{1}{\partial^2} \right) \frac{\delta}{\delta \xi_\lambda^a (x)} \exp \left[ -\operatorname{Tr} \ln \left( 1 + ig \, \underline{t} \cdot \underset{\sim}{\xi}^\mu \partial_\mu \frac{1}{\partial^2} \right) \right]$$

$$= \frac{\delta}{\delta \xi_\lambda^a (x)} - ig \int d^4y \, d^4z \sum_{b,c} \left\langle y, b \left| \frac{1}{1 + ig \, \underline{t} \cdot \underset{\sim}{\xi}^\mu \partial_\mu \frac{1}{\partial^2}} \right| z, c \right\rangle \tag{A.3}$$

$$(t^a)^{cb} \delta^4 (x-z) \, \partial_z^\lambda \left\langle z \left| \frac{1}{\partial^2} \right| y \right\rangle$$

$$= \frac{\delta}{\delta \xi_\lambda^a (x)} - ig \, \underline{t} \, t^a \left[ \partial_x^\lambda H(x,y; -i\underset{\sim}{\xi}) \right]_{y=x}$$

which is Eq. (3.5).

As for Eq. (3.9), we begin by noting that

$$ig \, D_x^\lambda \left[ \underset{\sim}{A} \right]^{ab} t^{cbd} \left[ \partial_\lambda H^{dc} (x,y; \underset{\sim}{A}) \right]_{y=x} \tag{A.4}$$

$$= ig \, t^{cad} \frac{\partial^2}{\partial x^\mu \partial x_\mu} H^{dc} (x,y; \underset{\sim}{A}) \Big|_{y=x} + ig \, t^{cad} \frac{\partial^2}{\partial x^\mu \partial y_\mu} H^{dc} (x,y; \underset{\sim}{A}) \Big|_{x=y}$$

$$- ig^2 \, t^{aeb} t^{cbd} \frac{\partial}{\partial x_\mu} H^{dc} (x,y; \underset{\sim}{A}) \Big|_{x=y} A_\mu^e (x)$$

Since

$$\frac{\partial^2}{\partial x_\mu \partial x^\mu} H^{dc}(x,y;\underline{A}) = \delta^{dc}\delta^4(x-y) + g\, t^{def} A_\mu^e(x)\frac{\partial}{\partial x_\mu} H^{fc}(x,y;\underline{A}) \tag{A.5}$$

we can write the right hand side of Eq. (A.4) as

$$ig^2(t^{cad} t^{def} - t^{aed} t^{cdf})\, \partial H^{fc}(x,y;\underline{A})/\partial x_\mu\big|_{x=y}\, A_\mu^e(x)$$

$$+ ig\, t^{cad}\, \partial^2 H^{dc}(x,y;\underline{A})/\partial x^\mu \partial y_\mu\big|_{y=x} \tag{A.6}$$

$$= ig\, T_r\, t^a\left[\frac{\partial^2}{\partial x^\mu \partial y_\mu} H(x,y;\underline{A}) + \frac{\partial}{\partial x^\mu} H(x,y;\underline{A})\, g\, \underline{t}\cdot \underline{A}^\mu(x)\right]$$

We have used the Jacobi identity

$$t^{cad} t^{def} + t^{aed} t^{def} = t^{adf} t^{ecd}$$

Equation (A.6) is precisely the right hand side of Eq. (3.9).

## APPENDIX B

## Generating Functional of Proper Vertices

In the usual manner, the generating functional $\Gamma[\underset{\sim}{B}_\mu]$ may be

obtained by a Legendre transformation from $Z[\underset{\sim}{J}_\mu]$.

We define

$$-\underset{\sim}{B}_\mu(x) = i\, Z[\underset{\sim}{J}_\mu] / \delta \underset{\sim}{J}_\mu(x) \tag{B.1}$$

and

$$\Gamma[\underset{\sim}{B}_\mu] = Z[\underset{\sim}{J}_\mu] + \int d^4x \; \underset{\sim}{J}^\mu(x) \cdot \underset{\sim}{B}_\mu(x) . \tag{B.2}$$

It follows that

$$\underset{\sim}{J}_\mu(x) = \delta \Gamma[\underset{\sim}{B}_\mu] / i \underset{\sim}{B}_\mu \tag{B.3}$$

The expansion coefficients of $\Gamma[\underset{\sim}{B}_\mu]$ in terms of $B_\mu$ are the proper

vertices. The proof may be found in Jona-Lasinio.[27] It is possible to

construct $\Gamma[\underset{\sim}{B}_\mu]$ perturbatively by the functional integration technique.

First consider

$$\exp i\, Z[\underset{\sim}{J}_\mu] = \int [dA] \, e^{\,i\, S_\alpha[\underset{\sim}{A}_\mu] - i \int d^4x \, \underset{\sim}{A}_\mu(x) \cdot \underset{\sim}{J}^\mu(x) + T_\kappa \, \ell_n \left( 1 - g \underset{\sim}{A}^\mu \cdot t \, \partial_\mu \frac{1}{\partial^2} \right)}$$

where $S_\alpha[\underset{\sim}{A}_\mu]$ is the gauge-dependent action

$$S_\alpha[\underset{\sim}{A}_\mu] = \int d^4x \, [\mathcal{L}(x) - \frac{1}{2\alpha}(\partial^\mu \underset{\sim}{A}_\mu(x))^2 ]$$

We can perform the functional integration by the steepest descent

method. Let $\underset{\sim}{A}^0_\mu$ be defined by

$$\delta S_\alpha[\underset{\sim}{A}_\mu^c] / \delta \underset{\sim}{A}_\mu^0(x) = \underset{\sim}{J}_\mu(x)$$

i.e., $\underset{\sim}{A}^0_\mu$ is the solution of the classical equation of motion in the

presence of the external source, so that

$$A_\mu^i = A_\mu^c(x, J)$$

$$(20)$$

Now we may write, after de Witt,

$$\exp i Z[J_\mu] = \exp i \left\{ S_x[A_\mu^c] - \int d^4x \, A_\mu^c(x) \cdot \delta S_x[A_\mu^c]/\delta A_\mu^c(x) \right\}$$

$$\times \int [dA] \, \Delta_L[A_\mu] \exp i \left\{ S_x[A_\mu] - S_x[A_\mu^c] - \int d^4x \frac{\delta S_x[A_\mu^c]}{\delta A_\mu^c(x)} [A_\mu(x) - A_\mu^c(x)] \right\}$$

The tree approximation consists in approximating $Z[J_\mu]$ by

$$Z^{tree}[J_\mu] = S_x[A_\mu^c] - \int d^4x \, \delta S_x[A_\mu^c]/\delta A_\mu^c(x) \cdot A_\mu^c(x),$$

$$[B_\mu(x)]^{tree} \equiv \delta Z^{tree}[J_\mu]/\delta J_\mu(x) = A_\mu^c(x).$$

Therefore, in the tree approxmiation we have

$$\Gamma^{tree}[B_\mu] = S_x[B_\mu].$$

The one-loop approximation consists in evaluating the functional

integral by the steepest descent approxmiation. We then have

$$Z[J_\mu] \simeq Z^{tree}[J_\mu] + \frac{i}{2} Tr \ln \frac{\delta^2 S_x[A_\mu^c]}{\delta A_\mu \, \delta A_\nu} - i \, Tr \ln \left( 1 - g \underset{\sim}{t} \cdot A_\mu^c \, \partial^\mu \frac{1}{\partial^2} \right)$$

and

$$\Gamma[B_\mu] \simeq S_x[A_\mu^c] + \int d^4x \frac{\delta S_x[A_\mu^c]}{\delta A_\mu^c(x)} [B_\mu(x) - A_\mu^c(x)]$$

$$+ \frac{i}{2} Tr \ln \left\{ \delta^2 S_x[B_\mu]/\delta B_\mu \, \delta B_\nu \right\}$$

$$- i \, Tr \ln \left( 1 - g \underset{\sim}{t} \cdot B^\mu \, \partial_\mu / \partial^2 \right)$$

$$= S_x[B_\mu] + \frac{i}{2} Tr \ln \left\{ \delta^2 S_x[B_\mu]/\delta B_\mu \, \delta B_\nu \right\}$$

$$- i \, Tr \ln \left( 1 - g \underset{\sim}{t} \cdot B_\mu \, \partial^\mu / \partial^2 \right)$$

# APPENDIX C

## Ward-Takahashi Identity for the Generating Functional

## Of Proper Vertices

We begin by rewriting Eq. (3.8) as

$$\mathcal{D}_x^\lambda [i\delta/\delta \mathcal{I}]^{ab} \left\{ \frac{i}{\alpha} \partial_\lambda \partial_\mu \frac{\overset{\cdot}{\delta}}{\delta J_\mu^b(x)} - J_\lambda^b(x) \right\} W \tag{C.1}$$

$$-ig \int d\overset{\cdot}{y} \; \overline{t}_\kappa \; t^a \left[ \partial_\mu H(x,y \; ; i\delta/\delta\mathcal{I} ) \right] \mathcal{D}_y^\mu [i\delta/\delta\mathcal{I}] \overset{\cdot}{\delta}(x-y) W = 0$$

The last term on the left may be written as

$$-ig \; t^{bac} \left[ \partial_\mu H^{cd}(x,y \; ; i\delta/\delta\mathcal{I}) \right]_{x=y} (-ig) t^{deb} \frac{\delta W}{\delta J_\mu^e(x)} \tag{C.2}$$

$$+ig \; t^{bac} \left[ \partial^2 H^{cb}(x,y \; ; i\delta/\delta\mathcal{I})/\partial x_\mu \partial y^\mu \right]_{x=y} W$$

Noting that

$$\left( \partial^2 - g \, \partial^\nu A_\mu \cdot \underline{t} \right)_y^{ab} H^{cb}(x,y : \underline{A}) = \delta^{ac} \delta^4(x-y)$$

or

$$H^{cb}(x,y \; ; \underline{A}) = -\delta^{cb} \tilde{D}_F(x-y) \tag{C.3}$$

$$-g \, t^{bad} \int d^4 z \; \partial_y^\nu \tilde{D}_F(y-z) A_\nu^a(z) H^{cd}(x,y \; ; \underline{A})$$

we can cast the second term of Eq. (C.2) into

$$ig \; t^{bac} \left[ \partial^2 H^{cb}(x,y \; ; i\delta/\delta\mathcal{I})/\partial x_\mu \partial y^\mu \right]_{x=y} W$$

$$= -(ig)^2 \; t^{bac} t^{bed} \int d^4 z \; \partial_y^\mu \partial_y^\nu \tilde{D}_F(y-z) \partial_\mu H^{cd}(x,y \; ; \underline{A}) \Big|_{x=y} \frac{\delta W}{\delta J_\nu^e(z)}$$

Thus the last term on the left of Eq. (C.1) can be written as

$$-g^2 t^{bac} t^{deb} \int d^4 z \; \partial_x^\mu H^{cd}(x,y \; ; i\delta/\delta\mathcal{I}) g_{\mu\nu}^{t\kappa}(z-z) \cdot \frac{\delta W}{\delta J_\nu^e(z)} \Big|_{x=y}$$

where

$$g^{tr}_{\mu\nu}(x-y) = g_{\mu\nu}\tilde{\delta}^4(x-y) + \partial_\mu\partial_\nu \tilde{D}_F(x-y)$$

So finally, we obtain

$$\frac{i}{\alpha}\partial^2\partial_\mu \delta W/\delta J^a_\mu(x) + \frac{1}{\alpha}t^{abc}\frac{\delta}{\delta J^b_\lambda(x)}\partial_\lambda\partial_\mu\frac{\delta W}{\delta J^c_\mu(x)}$$

$$- \partial^\lambda J^a_\lambda(x)W - i\,t^{abc}\,J^b_\lambda(x)\,\delta W/\delta J^c_\lambda(x)$$

(C.4)

$$+ g^2 t^{abc}t^{bde}\int d^4z\,\partial_x^\mu H^{cd}(x,y;i\delta/\delta J)\,g^{tr}_{\mu\nu}(x-z)\,\delta W/\delta J^e_\nu(z)\Big|_{x=y} = 0$$

Equation (C.4) may be translated into an expression involving the

generating functional of the proper vertices. We recall that

$$W[J_\mu] = \exp i\,Z[J_\mu]$$

(B.1)

and

$$-B_\mu = \delta Z[J_\mu]/\delta J_\mu(x); \quad J_\mu = \delta\Gamma[B_\mu]/\delta B_\mu .$$

(B.3)

Thus

$$\frac{1}{\alpha}\partial^2\partial^\mu B^a_\mu(x) - \frac{1}{\alpha}t^{abc}B^b_\lambda(x)\partial^\lambda\partial^\mu B^c_\mu(x)$$

$$- \partial^\lambda\delta\Gamma[B_\mu]/\delta B^a_\lambda(x) + t^{abc}B^b_\lambda(x)\delta\Gamma[B_\mu]/\delta B^c_\lambda(x)$$

(C.5)

$$= \frac{1}{\alpha}t^{abc}\frac{\delta}{\delta J^b_\lambda(x)}\partial_\lambda\partial^\mu B^c_\mu(x)$$

$$+ ig^2 t^{abc}t^{bde}\int d^4z\,\partial_x^\mu H^{cd}(x,y;B-i\delta/\delta J)\,g^{tr}_{\mu\nu}(x-z)\,B^{e\nu}(z)\Big|_{x=y} \equiv -G^a$$

This equation is somewhat simplified if we define $\Gamma^0$ by

$$\Gamma^0[B_\mu] = \Gamma[B_\mu] + \frac{1}{2\alpha}\int d^4z\,[\partial^\mu B_\mu(z)]^2$$

(C.6)

which satisfies

$$\left[ \partial_\lambda \, \delta^{ac} - t^{abc} B_\lambda^{\ b} \right] \frac{\delta \Gamma^0 [\, B_\mu ]}{\delta B_\lambda^{\ c}(x)} \; = \; G^a(x) \qquad\qquad (C.7)$$

If $G^a(x)$ were identically zero, then Eq. (C.7) would imply that

$\Gamma^0 [\, \underset{\sim}{B}_\mu \, ]$ is invariant under the local gauge transformation

$$\underset{\sim}{B}_\mu \; \rightarrow \; \underset{\sim}{B}_\mu \; + \; D_\mu [\, B_\mu ] \; \underset{\sim}{\omega}$$

By an explicit computation of $\delta^2 G^a(x) / \delta B_b^{\ b}(y) \, \delta B_\nu^{\ c}(z)$, we have

verified that $G^a(x)$ cannot be identically zero, however.

## APPENDIX D

## Power Counting

Let $N_{n,m}$ be the number of vertices of the form $\partial^n \Phi^m$, i.e., m

point vertices with n derivatives, and let I and E be, respectively, the

number of internal and external lines in a proper diagram. By L we

denote the number of loops in the diagram.

We have two topological relations

$$E + 2I = \sum_{n,m}' m N_{n,m} \tag{D.1}$$

$$L = I + 1 - \sum_{n,m} N_{n,m} \tag{D.2}$$

The superficial degree of divergence D of the diagram is given by

$$D = \sum_{n,m}' n N_{n,m} + 4L - 2rI \tag{D.3}$$

where the propagator is assumed to have the asymptotic behavior

$(p^2)^{-r}$. Eliminating L and I in favor of E by the use of Eqs. (D.1)

and (D.2), we write Eq. (D.3) as

$$D = E(n-2) + 4 + \sum_{n,m}' N_{n,m} [n + (2-\lambda)m - 4] \tag{D.4}$$

For the gauge invariant terms discussed in Sec. V, we have generally

$$n + m = 2 + 2\lambda \tag{D.5}$$

We, therefore, have

$$D = E(\lambda-2) + 4 + (1-\lambda) \sum_{n,m}' N_{n,m} (m-2) \tag{D.6}$$

The two statements in Sec. V, for which we referred the reader

to this Appendix, can be justified on the basis of Eq. (D. 6).

REFERENCES

[1] J. Schwinger, Ann Phys. (N.Y.) 2, 407 (1957).

[2] S. L. Glashow, Nuclear Physics 22, 579 (1961).

[3] A. Salam and J. Ward, Phys. Letters 13, 168 (1964).

[4] S. Weinberg, Phys. Rev. Letters 13, 1264 (1967).

[5] A. Salam, "Elementary Particle Theory", edited by N. Svartholm (Almquist and Forlag A. B, Stockholdm, 1968).

[6] J. Schecter and Y. Ueda, Phys. Rev. D2, 736 (1970).

[7] G. 't Hooft, Nuclear Physics B35, 167 (1971).

[8] B. W. Lee, Phys. Rev. to be published.

[9] S. Weinberg, Phys. Rev. Letters 27, 1688 (1971).

[10] P. Higgs, Phys. 12, 132 (1966).

[11] T. W. B. Kibble, Phys, Rev. 155, 1554 (1967).

[12] Further references can be found in reference 8.

[13] B. W. Lee, Nuclear Physics B9, 649 (1969); J.-L. Gervain and B. W. Lee, ibid B12, 627 (1969); B. W. Lee "Chiral Dynamics" (Gordon and Breach, New York, to be published).

[14] N. N. Bogoliubov and O. S. Parasiuk, Acta Math. 97, 227 (1957).

[15] N. N. Bogoliubov and D. B. Shirkov, "Introduction to the Theory of Quantized Fields" (Interscience, New York, 1959).

[16] K. Hepp, Commun. Math. Phys. 1, 95 (1965); "Théories de la Renormalisation" (Springer, Berlin, 1969).

[17] W. Zimmermann, in "1970 Brandeis Lectures" (M.I.T. Press, Cambridge, Mass., 1970).

[18] A. Slavnov, to be published (Kiev ITP-71-83E).

[19] R. P. Feynman, Acta Phys. Polonica 26, 697 (1963), and to be published.

[20] B. de Witt, Phys. Rev., 162, 1195, 1239 (1967).

[21] L. D. Fadeev and V. N. Popov, Phys. Letters 25B, 29 (1967), and unpublished (Kiev ITP-67-36).

[22] G. 't Hooft, Nuclear Physics B33, 173 (1971).

[23] A. Slavnov, to be published.

[24] A. Slavnov, to be published.

[25] K. Johnson, private communication.

[26] K. Symanzik, Lett. Nuovo Cimenta 2, 10 (1969); Commun. Math. Phys. 16, 48 (1970).

[27] G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

FIGURE CAPTIONS
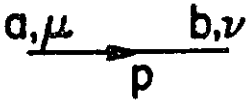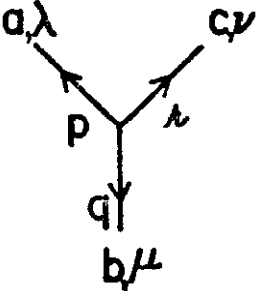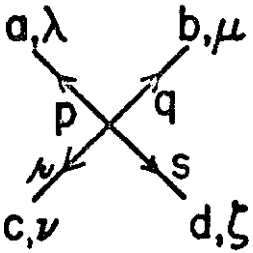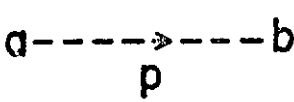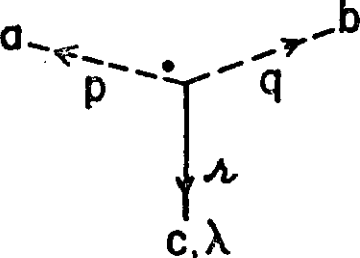
Figure 1.   Feynman rules of the Yang-Mills theory.


Figure 2.   Diagrammatic Representation of Eq. (7.11).


Figure 3.   Diagrammatic Representation of Eq. (7.18)

| VERTICES | | BARE VERTICES |
|---|---|---|

| | VERTICES | BARE VERTICES |
|---|---|---|
|  $a,\mu \xrightarrow{\quad p \quad} b,\nu$ | $-i\delta^{ab}\Delta_{\mu\nu}(p)$ | $-i\delta^{ab}\left[(g_{\mu\nu}-p_\mu p_\nu/p^2)1/p^2+\alpha p_\mu p_\nu (p^2)^{-2}\right]$ |
|  $p+q+\pitchfork=0$ | $i\Gamma^{abc}_{\lambda\mu\nu}(p,q,\pitchfork)$ | $\epsilon^{abc}\left[(p-q)_\nu\, g_{\lambda\mu}\right.$ $\left.+(q-\pitchfork)_\lambda\, g_{\mu\nu}+(\pitchfork-p)_\mu\, g_{\nu\lambda}\right]$ |
|  $p+q+\pitchfork+s=0$ | $i\Gamma^{abcd}_{\lambda\mu\nu\zeta}(p,q,\pitchfork,s)$ | $-i\epsilon^{abf}\epsilon^{cdf}(g_{\lambda\nu}g_{\mu\zeta}-g_{\lambda\zeta}g_{\mu\nu})$ $-i\epsilon^{acf}\epsilon^{bdf}(g_{\lambda\mu}g_{\nu\zeta}-g_{\lambda\zeta}g_{\mu\nu})$ $-\epsilon^{adf}\epsilon^{cbf}(g_{\lambda\nu}g_{\mu\zeta}-g_{\lambda\mu}g_{\zeta\nu})$ |
|  $a\ \text{-----}\xrightarrow{\ p\ }\text{-----}\ b$ | $\delta^{ab}i\mathscr{D}(p)$ | $i\delta^{ab}1/p^2$ |
|  | $i\gamma^{abc}_\lambda(p,q;\pitchfork)$ | $\epsilon^{abc}p_\lambda$ |

Diagrammatic rep of Eq. (7.11)

x,a
z,c
y,b

=

x,a
z,c
y,b

+

x,a
z,c
R
y,b

# FIG. 3

$$\frac{d^2\sigma}{dt\,dM^2}(a+b \to a+X) \cong$$

(a)

$$\left| \begin{array}{c} \text{Res.} \\ \end{array} \right|^2 + \left| \begin{array}{c} \text{Bkg.} \\ \end{array} \right|^2$$

(b)

$$\cong \text{Res} + \text{Bkg.}$$

(c)

$$\cong \text{Im} + \text{Im}$$