Anomalous Transport in Chiral Systems

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Abstract of the Dissertation

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The experimental realization of Dirac and Weyl semimetals in 2014 and 2015 respectively has increased the interest in the topic. Similarly to graphene, the discovered materials are characterized by massless quasiparticles. In three dimensions these quasiparticles can be described by the Weyl Hamiltonian which exhibits so-called chiral anomaly at low energies. The chiral anomaly has a transport signature, namely, the enhancement of longitudinal conductivity along the direction of external magnetic field. This effect in new materials is the condensed matter version of the chiral magnetic effect (CME) predicted to happen in heavy ion collisions. Due to its topological nature the chiral anomaly it is believed to be robust with respect to the interaction strength and anomalous contribution to transport is believed to be universal and independent of the interaction.

This thesis is devoted to the study of magnetotransport in Dirac and Weyl metals. For that, we use the chiral kinetic theory to describe within the same framework both the negative magnetoresistance caused by chiral magnetic effect and quantum oscillations

in the magnetoresistance due to the existence of the Fermi surface. In the second part, we refer to the hydrodynamics with gauge anomaly and study the non-dissipative transport using variational principle as a main tool. In the last part of the Thesis we also apply variational approach to study the Hall viscosity in two-dimensional systems.

Publications

1. Gustavo M. Monteiro and Alexander G. Abanov. Variational Principle for Hall Fluids
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2. Gustavo M. Monteiro, Alexander G. Abanov, and Dmitri E. Kharzeev. Magnetotransport in Dirac metals: Chiral magnetic effect and quantum oscillations. *Phys. Rev. B*, 92 (165109), Oct 2015.

3. Gustavo M. Monteiro, Alexander G Abanov, and V. P. Nair. Hydrodynamics with gauge anomaly: Variational principle and hamiltonian formulation. *Phys Rev D*, 91(125033), 2015.

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Chapter 1

Introduction

1.1 Thesis Outline

This thesis is organized in a way that one chapter is completely independent of each other. The underlying connection between them is traced in the introduction, where the different topics are tied together, even though the actual pieces of the work may seem distinct.

At chapter 2, we will study the longitudinal magnetoconductivity of Dirac and Weyl metals by analytically computing the Boltzmann equation. Motivated by the experiments on Cd_3As_2, we will study the interplay between the Shubnikov-de Haas effect with the negative magnetoresistance due to the chiral anomaly at low energies. This chapter allows for a direct comparison to the experimental data and is based on the published work [3] of the PhD candidate.

Chapter 3 is more formal and has no direct connection to experiments. We will construct the variational principle for the hydrodynamic equations with gauge anomaly, focusing on the symmetry properties and the mathematical structure of the problem. This chapter corresponds to the work done by the PhD candidate in [6]. At chapter 4, we will study Hall viscosity terms in the hydrodynamics from a variation principle. We will fix any ambiguity in defining conserved currents by introducing background fields from Newton-Cartan geometry. Both chapters 3 and 4 rely on the so-called Clebsch parametrization, which will be explained in the section 1.5.

In the last chapter, about other ongoing projects, we will present a convergence point between both published pieces of work [3] and [6]. Such connection rely on the transport properties of topologically protected surface states in both kinetic theory and hydrodynamics. In the section 5.2, we will discuss the existence of hydrodynamic surface modes, which can have some implication on the study of Fermi arcs in Weyl and Dirac semimetals.

1.2 Weyl and Dirac Semimetals

Throughout the last decade, we have succeeded in including the adjective topological to various states of matter. Two ground states are topologically equivalent if and only if we can adiabatically deform one into the other without breaking any of the underlying symmetries. As an example, any conventional insulator can be continuously connected to non-interacting atoms arranged in a lattice. In a particular tight-binding model, this can be done by setting the hopping parameter to zero. This definition relies on validity of the adiabatic theorem and is thus restricted to gapped systems. In general, gapless phases occur in the quantum phase transition between two distinct topological phases, where the adiabatic theorem breaks down. They are said to be unstable when the gap closes only at the quantum critical point and stable when gap remains closed for a finite interval in the parameter space [7]. Dirac semimetals can be seen as an intermediate phase in between the trivial and the topological insulator phases of the same material, where the gap closes at isolated points in the Brillouin zone. The most well-known example is the compound $ZrTe_5$ which was predicted to be a 3D quantum spin Hall insulator [8], though transport and ARPES measurements have shown that it behaves instead as a Dirac semimetal [9]. The effective low-energy Hamiltonian for Dirac semimetals is invariant under time-reversal and inversion symmetry, and its quasiparticle excitations form a Kramer's doublet. Weyl semimetals can be obtained from Dirac semimetals by breaking either time reversal or inversion symmetry, which splits the Dirac point into two band-touching (Weyl) points with opposite chiralities.

Dirac semimetals were only experimentally realized in 2014, and the first synthesized compounds were Cd_3As_2 [10–12], Na_3Bi [13] and $ZrTe_5$ [9]. Experimental data have shown that these materials are characterized by large mobility and high magnetoresistance, which are mostly but not fully understood. Although proposed before the Dirac semimetals, Weyl semimetals have proven to be more challenging to be realized experimentally, being synthesized only in 2015. There are only four Weyl semimetals known to date, these are TaAs [14, 15], TaP [16], NbAs [17] and NbP. In addition to that, Weyl semimetals are characterized by topologically protected surface states, called Fermi arcs [16–19]. Fermi arcs connect two disjoint pieces of Fermi surface with opposite chirality. In the case of Dirac semimetals, such Fermi arcs are not topologically protected and can hybridize into a closed Fermi loop on the surface [20].

For the sake of simplicity, we will describe Weyl semimetals in the next subsection in terms of a two-band model system. Dirac semimetals can be understood as two copies of Weyl semimetals connected by time-reversal symmetry.

1.2.1 Two-band System: An Example

The most general two-band system Hamiltonian can be written as:

$$H(\boldsymbol{k}) = b_0(\boldsymbol{k})\, I_{2\times 2} + \boldsymbol{b}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}, \tag{1.1}$$

where $(b_0, \boldsymbol{b})$ are real functions of the crystal quasimomentum $\boldsymbol{k}$ and $\boldsymbol{\sigma}$ is the vector whose components are the Pauli matrices. The symmetries of such Bloch Hamiltonian can be extracted from the microscopics[1]:

$$\mathbb{T}: \quad H^*(\boldsymbol{k}) = H(-\boldsymbol{k}), \tag{1.2}$$

$$\mathbb{I}: \quad H(\boldsymbol{k}) = H(-\boldsymbol{k}), \tag{1.3}$$

where $\mathbb{T}$ is the time-reversal operator and $\mathbb{I}$ is the lattice inversion. A two-band system is only $\mathbb{IT}$-invariant when $b_2(\boldsymbol{k}) = 0$, and the other real functions are even in $\boldsymbol{k}$. Band-touching points occur when $b_1(\boldsymbol{k_0}) = b_3(\boldsymbol{k_0}) = 0$, for some $\boldsymbol{k_0}$ in the Brillouin zone. Since $b_{1,3}(\boldsymbol{k}) = 0$ correspond to surfaces on the three-dimensional Brillouin zone, their intersection defines either lines or kissing points. Therefore, one cannot obtain a Weyl semimetal from a $\mathbb{IT}$-invariant system[2]. For systems with broken $\mathbb{IT}$, the Bloch Hamiltonian (1.1) becomes non-real and the Weyl points occur when the equation

$$b_1(\boldsymbol{k_a}) = b_2(\boldsymbol{k_a}) = b_3(\boldsymbol{k_a}) = 0$$

admits solutions for a discrete set of points $\{\boldsymbol{k_a}\}$ in the Brillouin zone. When the Fermi energy lies near the energy of a Weyl point $\boldsymbol{k_0} \in \{\boldsymbol{k_a}\}$, we can linearize the Hamiltonian around such point. If we set the Weyl point energy to be zero, the effective Hamiltonian can be written as:

$$H(\boldsymbol{k}) = (I_{2\times 2}\, \nu_j + \sigma_i v_{ij})\, (\delta k)_j, \tag{1.4}$$

[1] The equation (1.2) is only true when the quasiparticle does not belong to a Kramer's doublet, i.e., it is a singlet under $\mathbb{T}$.

[2] The quasiparticle dispersion relation will be flat in at least one direction. More systematically, one can show that Berry phase of Bloch states away from the band-touching points vanishes for a $\mathbb{IT}$-invariant system.

where

$$\nu_j = \left.\frac{\partial b_0}{\partial k_j}\right|_{\boldsymbol{k_0}}, \tag{1.5}$$

$$v_{ij} = \left.\frac{\partial b_i}{\partial k_j}\right|_{\boldsymbol{k_0}}, \tag{1.6}$$

$$\boldsymbol{\delta k} = \boldsymbol{k} - \boldsymbol{k_0}. \tag{1.7}$$

The zero-energy solutions of the Hamiltonian (1.4) fall into two classes: a single Weyl point at $\boldsymbol{k} = \boldsymbol{k_0}$; or a band-touching point at $\boldsymbol{k} = \boldsymbol{k_0}$ connecting two disjoint pieces of Fermi surface, one filled by hole states and the other by electron quasiparticles. The former is a characteristic of type-I Weyl semimetals and the latter describes the recently discovered type-II Weyl semimetals [1]. The dispersion relations for both cases is represented in Fig. 1.1.

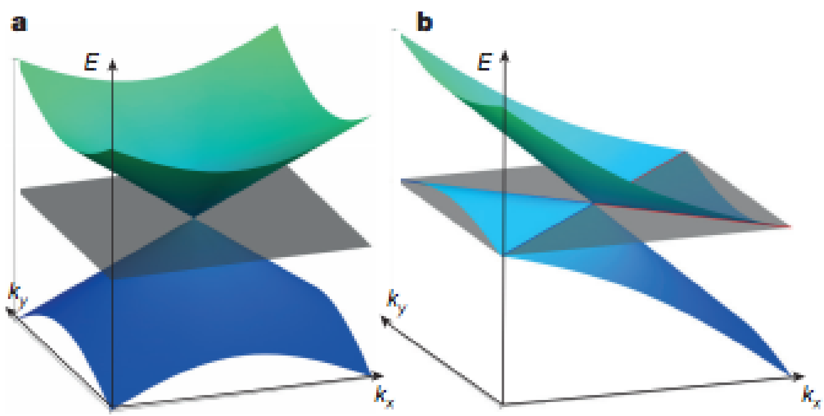

Figure 1.1: Examples of dispersion relations for **a)** type-I and **b)** type-II Weyl semimetals. These plots were extracted from [1].

Let us ignore the time-reversal symmetry for now and impose only the inversion symmetry. If $\boldsymbol{b}(\boldsymbol{k})$ vanishes for some $\boldsymbol{k_0}$ in the Brillouin zone, equation (1.3) imposes that necessarily $\boldsymbol{b}(-\boldsymbol{k_0}) = \boldsymbol{0}$ and that both Weyl points are at the same energy. Assuming that there are only two band touching points, the low-energy effective Hamiltonian becomes:

$$H_{eff}(\boldsymbol{k}) = H_+(\boldsymbol{k}) \oplus H_-(\boldsymbol{k}), \tag{1.8}$$

where $H_\pm(\boldsymbol{k})$ are the linearized Hamiltonians expanded at $\pm\boldsymbol{k_0}$. They can be expressed as:

$$H_\pm(\boldsymbol{k}) = \pm\left(I_{2\times 2}\,\nu_j + \sigma_i v_{ij}\right)[k_j \mp (k_0)_j]. \tag{1.9}$$

We have again set the Weyl points to be at zero energy. The chirality of the Weyl quasiparticle is defined by the sign of det $\left(\frac{\partial b_i}{\partial k_j}\right)$ evaluated at the Weyl point. Invariance over $\mathbb{I}$ imposes that the Weyl points have opposite chirality, since:

$$\det\left(-v_{ij}\right) = -\det v_{ij}.$$

Let us repeat the analysis for a system with broken inversion symmetry, yet invariant under time-reversal. If there is a Weyl point at $\boldsymbol{k_0}$, so will there be another one at $-\boldsymbol{k_0}$. However, equation (1.2) show us that the Weyl points have the same chirality. Nielsen-Ninomiya theorem states that the existence of only Weyl points of the same chirality is inconsistent with lattice symmetries [21]. In other words, Weyl point must come in pairs of opposite chirality as it will be shown in the next section. Therefore the smallest number of Weyl points in a $\mathbb{T}$-invariant system with broken $\mathbb{I}$ is four.

1.2.2 Nielsen-Ninomiya Theorem

In this section, we will review the Nielsen-Ninomiya theorem, also known as lattice doubling theorem. It was first introduced in [21], where the authors considered a system of Weyl fermions on a lattice coupled to a gauge field. The Nielsen-Ninomiya theorem states that Weyl points must come in pairs and with opposite chiralities. In a more modern jargon, the total Berry flux across all disjoint pieces of Fermi surface must vanish.

Let us consider the Hamiltonian (1.1) and assume that band-touching points form discrete set of points $\{\boldsymbol{k_a}\}$ in the Brillouin zone, $\mathcal{BZ}$. For all other points in the Brillouin zone, there is a gap between bands[3]. Neither the gap nor the positions of the Weyl points depend on the the function $b_0(\boldsymbol{k})$. Therefore, the bands can be labeled by eigenstates of:

$$h(\boldsymbol{k}) = \boldsymbol{\sigma} \cdot \frac{\boldsymbol{b}(\boldsymbol{k})}{|\boldsymbol{b}(\boldsymbol{k})|}\,, \tag{1.10}$$

Obviously, this operator is not defined for the points $\{\boldsymbol{k_a}\}$. Away from these points the eigenvalues of such operator are ± 1, where -1 corresponds to the "valence band" and $+1$ labels the "conduction band". The domain of such operation is the reduced Brillouin zone $\mathcal{BZ}'$ with such bad points removed.

[3] One can view this two-band system as a two-level system at each point in the Brillouin zone.

Formally, this reduced Brillouin zone is defined as:

$$\mathcal{BZ}' = \mathcal{BZ} \setminus \bigcup_{\alpha} \mathcal{U}_\alpha,$$

where each $\mathcal{U}_\alpha$ is a small open set around the Weyl point $\boldsymbol{k}_\alpha$. The equation

$$h(\boldsymbol{k})|u^{\pm}_{\boldsymbol{k}}\rangle = \pm|u^{\pm}_{\boldsymbol{k}}\rangle, \tag{1.11}$$

defines what we call line bundles. In principle, a general state can be viewed as a $\mathbb{C}^2$-vector at each point in $\mathcal{BZ}'$. The Hilbert space[4] for states that satisfy equation (1.11) are split into a direct sum $\mathcal{H}_{\boldsymbol{k}} \cong \mathbb{C} \oplus \mathbb{C}$. If we restrict ourselves to positive eigenvalue states of (1.11), we define a line bundle[5] over $\mathcal{BZ}'$, since $\alpha|u^{\pm}_{\boldsymbol{k}}\rangle$, with $\alpha \in \mathbb{C}$, is still a positive eigenvalue state of $h(\boldsymbol{k})$.

The normalization condition fixes the modulus of $|u^{+}_{\boldsymbol{k}}\rangle$, however its phase cannot be fixed by any of these arguments. This reduces the line bundle to a $U(1)$ principal bundle over the Brillouin zone. In physical terms, this means that we can always multiply the state by a local phase, that is,

$$|u^{+}_{\boldsymbol{k}}\rangle \to e^{i\varphi(\boldsymbol{k})}|u^{+}_{\boldsymbol{k}}\rangle. \tag{1.12}$$

Let us consider now a path on the Brillouin zone. The normalization condition imposes that along the path:

$$\langle u^{+}_{\boldsymbol{k}}|\frac{d}{ds}u^{+}_{\boldsymbol{k}}\rangle = 0. \tag{1.13}$$

This defines how the phase is parallel transported on the $U(1)$ principle bundle. In other words,

$$\frac{d}{ds}|u^{+}_{\boldsymbol{k}}\rangle \equiv \frac{d\boldsymbol{k}}{ds} \cdot \left[\boldsymbol{\nabla}_{\boldsymbol{k}} + i\boldsymbol{\mathcal{A}}(\boldsymbol{k})\right]|u^{+}_{\boldsymbol{k}}\rangle, \tag{1.14}$$

where $\boldsymbol{\mathcal{A}}(\boldsymbol{k})$ is the Berry connection over the $U(1)$-bundle. Contracting this equation with $\langle u^{+}_{\boldsymbol{k}}|$, we obtain:

$$\boldsymbol{\mathcal{A}}(\boldsymbol{k}) = i\langle u^{+}_{\boldsymbol{k}}|\boldsymbol{\nabla}_{\boldsymbol{k}} u^{+}_{\boldsymbol{k}}\rangle. \tag{1.15}$$

Under the "gauge" transformation (1.12), the Berry connection transforms as:

$$\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}} + \boldsymbol{\nabla}_{\boldsymbol{k}}\varphi\,. \tag{1.16}$$

[4]The total Hilbert space should be understood as $\mathcal{H} \cong \int^{\oplus}_{\mathcal{BZ}} \mathcal{H}_{\boldsymbol{k}}\, \mathrm{d}\mu(\boldsymbol{k})$.

[5]A manifold which is locally isomorphic to $\mathbb{C} \times \mathcal{U}$, where $\mathcal{U}$ is an open set of $\mathcal{BZ}'$

In analogy with the electromagnetism, the gauge invariant quantity is the Berry curvature, defined as:

$$\boldsymbol{\Omega} \equiv \boldsymbol{\nabla_k} \times \boldsymbol{\mathcal{A}}. \tag{1.17}$$

Thus,

$$\boldsymbol{\nabla_k} \cdot \boldsymbol{\Omega} = 0, \qquad \forall \boldsymbol{k} \in \mathcal{BZ}'. \tag{1.18}$$

Integrating (1.18) over the whole domain and using the Stokes theorem, we find that:

$$\sum_\alpha \int_{\partial \mathcal{U}_\alpha} \boldsymbol{\Omega} \cdot \mathrm{d}\boldsymbol{S} = \sum_\alpha c_1(\partial \mathcal{U}_\alpha) = 0, \tag{1.19}$$

Therefore, Weyl points correspond to monopole solutions of the Berry curvature and the total monopole charge (Chern number) must vanish. The general proof for a many-band system can be found at [22].

1.2.3 Fermi Arcs

Fermi arcs are fingerprints of Weyl semimetals. They consist of topologically protected surface states that connect two disjoint pieces of Fermi surface with opposite Chern numbers. In this section, we will review on how Fermi arcs occur in a simple two-band system.

In the case of Na_3Bi and Cd_3As_2, the single-particle Hamiltonian near the Γ-point can be described by the Kane model [23, 24]. The most general Hamiltonian respecting crystal symmetries reads:

$$H_\Gamma(\boldsymbol{k}) = C(\boldsymbol{k}) + \begin{pmatrix} M(\boldsymbol{k}) & Ak_+ & 0 & B^*(\boldsymbol{k}) \\ Ak_- & -M(\boldsymbol{k}) & B^*(\boldsymbol{k}) & 0 \\ 0 & B(\boldsymbol{k}) & M(\boldsymbol{k}) & -Ak_- \\ B(\boldsymbol{k}) & 0 & -Ak_+ & -M(\boldsymbol{k}) \end{pmatrix}, \tag{1.20}$$

where $k_\pm = k_x \pm ik_y$ and

$$M(\boldsymbol{k}) = -m_0 + m_1 k_z^2 + m_2 k_+ k_-, \tag{1.21}$$

$$C(\boldsymbol{k}) = c_0 k_z^2 + c_1 k_+ k_-, \tag{1.22}$$

$$B(\boldsymbol{k}) = D k_z k_+^2. \tag{1.23}$$

Dirac points occur at $k_D = (0, 0, \pm\sqrt{m_0/m_1})$. If we neglect $B(\boldsymbol{k})$, the Hamiltonian exhibits an emergent $\mathbb{Z}_2$-symmetry and can be split into two copies of two-band Hamiltonians:

$$H_\Gamma = H_\Gamma^+ \oplus H_\Gamma^-, \tag{1.24}$$

where,

$$H_\Gamma^\pm = C(\boldsymbol{k}) I_{2\times 2} \pm A k_x \sigma_x - A k_y \sigma_y + M(\boldsymbol{k})\sigma_z. \tag{1.25}$$

According to Peierls substitution, the crystal momentum should be replaced by $\boldsymbol{k} \to -i\boldsymbol{\nabla} + \frac{e}{\hbar}\boldsymbol{A}(\boldsymbol{x}, t)$ in the effective Hamiltonian. For simplicity, let us consider the material to be confined in the half space, with $y \in [0, \infty)$, and $c_1 = m_2 = 0$, so that higher orders of the k_y can be neglected at low energies. The self-adjoint condition on the Hamiltonian operator requires that:

$$\langle \Phi | H \Psi \rangle - \langle H \Phi | \Psi \rangle = iA \int_{\mathbb{R}^2} \mathrm{d}x \, \mathrm{d}z \left(\Phi^\dagger \sigma_y \Psi \right) |_{y=0} = 0. \tag{1.26}$$

Let us set the vector potential to zero for simplicity and Fourier transform the eigenstate equation in (x, z)-directions. The equation for the Fourier components is given by:

$$H_\Gamma^\pm \left[k_x, -i\frac{d}{dy}, k_z \right] u_{\boldsymbol{k}}^\pm(y) = \varepsilon \, u_{\boldsymbol{k}}^\pm(y). \tag{1.27}$$

The boundary condition (1.26) can be imposed by assuming that, for given Hermitian projectors $\mathcal{P}_\pm$, such that

$$\mathcal{P}_- + \mathcal{P}_+ = I_{2\times 2} \qquad \text{and} \qquad \mathcal{P}_- \, \sigma_y = \sigma_y \, \mathcal{P}_+ \, ,$$

the boundary conditions can be written as $\mathcal{P}_- \Psi|_{y=0} = 0$. As an example, let us consider the following projectors:

$$\mathcal{P}_\pm = \frac{1}{2}[I_{2\times 2} \pm \sigma_x].$$

The corresponding boundary condition becomes:

$$u_{\boldsymbol{k}}^\pm(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} f_{\boldsymbol{k}}^\pm, \tag{1.28}$$

where $f_{\boldsymbol{k}}^\pm$ must be determined by the normalization condition. Boundary states exist for $M(k_z) > 0$,[6] that is, when

$$u_{\boldsymbol{k}}^{\pm}(y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sqrt{\frac{A}{M(k_z)}}\, e^{-\frac{M(k_z)y}{A}} \tag{1.29}$$

is normalizable. The energies of surface states are given by:

$$\varepsilon^{\pm} = C(k_z) \pm Ak_x. \tag{1.30}$$

Hence it easy to see that surface states start in one Weyl Fermi surface and terminate at the opposite chirality Weyl Fermi surface, forming an arc instead of a loop in the surface Brillouin zone.

1.3 Anomalous Hydrodynamics

So far, we have neglected the Coulomb interaction between electrons. All the previous discussion is restricted to single-particle Hamiltonians or to weakly-interacting systems that can be perturbatively connected to non-interacting particles. In fact, to account for the electron-electron interaction, one may consider the Fermi liquid theory developed by Landau. One the subtleties of the Fermi liquid theory is the concept of quasiparticles. Roughly speaking, quasiparticles are approximate eigenstates of the full electron Hamiltonian. For excitations near the Fermi surface, the quasiparticle decay rate is given by the inelastic scattering rate between electrons, which can be estimated solely from kinematic arguments as:

$$\frac{1}{\tau_{ee}} \sim \frac{k_B^2 T^2}{\hbar \mu}.$$

Hence, the Fermi liquid theory is justified for systems which $\mu \gg k_B T$. Coulomb interactions in metals are effectively short-ranged due to screening. In low-disorder Dirac and type-I Weyl semimetals, the Fermi energy lies near the band-touching points throughout the whole sample, what makes the Coulomb interaction much less screened and essentially long-ranged. Therefore, the quasiparticle picture becomes meaningless and one may hope that the electronic transport can still be described by a phenomenological hydrodynamic theory which captures the non-trivial topology of Weyl/Dirac semimet-

[6] This is valid for $m_0, m_1 < 0$, otherwise we should have chosen the boundary conditions to be $\mathcal{P}_+\Psi|_{y=0} = 0$. For $m_0, m_1 > 0$, and $\mathcal{P}_-\Psi|_{y=0} = 0$, the Fermi arcs connect the Weyl point through the Brillouin zone periodicity and the approximation (1.25) is not valid anymore.

als. If justified, this collective behavior can also give some insights about the dynamics of quark-gluon plasma (QGP).

Since quarks up and down are states of approximately massless particles, one should expect that at certain conditions[7] they behave similarly to quasi-particles in Dirac semimetals. However, such conditions are inaccessible in collision experiments. On the other hand, QGP is a state of matter which quarks and gluons coexist in a strongly interacting "soup" and can be produced for very short time in highly energetic hadronic collisions such as in RHIC and LHC. Experimental data from RHIC have shown that QGP is best described as a fluid. Therefore, strongly interacting Weyl/Dirac semimetals might be condensed matter analogs of quark-gluon plasma under the conditions previously discussed.

Hydrodynamics is a long-wavelength effective description of interacting systems based on the assumption of local equilibrium. Hydrodynamic equations are essentially local conservation laws supplemented by the constitutive relations between conserved densities. In the case of massless quark matter, there are two species of fluid particles to each flavor, which are labeled by their chirality. Although the charge of each fluid component is not conserved separately, the total fluid charge is indeed conserved. In a quantum field theory jargon it means that the chiral current is anomalous. One of the signatures of the chiral anomaly in hydrodynamics is the chiral magnetic effect (CME) [25], which corresponds to a non-dissipative current along and external magnetic for a chirality unbalanced system.

1.3.1 The Chiral Magnetic Effect (CME)

In this section, we will review some of the arguments in [25] about the chiral magnetic effect in a condensed matter context. Although the hydrodynamic behavior of Weyl/Dirac materials could potentially appear only at high temperatures and small chemical potential, one can still obtain the linear response transport from hydrostatics. Let us consider a two-component Dirac/Weyl fluid, such that the total fluid charge is conserved, but the charge of each specie is not. The conservation laws in the presence of an external electromag-

[7] For extremely high temperatures, the QCD coupling becomes small and quarks behave almost as free particles, which is called asymptotic freedom.

netic field is given by:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0, \tag{1.31}$$

$$\frac{\partial P_i}{\partial t} + \nabla_j T^j{}_i = nE_i + (\boldsymbol{j} \times \boldsymbol{B})_i\,, \tag{1.32}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J_E} = \boldsymbol{j} \cdot \boldsymbol{E}, \tag{1.33}$$

$$\frac{\partial n_5}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j_5} = \frac{e^3}{2\pi^2}\, \boldsymbol{E} \cdot \boldsymbol{B}. \tag{1.34}$$

From the first law of thermodynamics in its local form, one obtains:

$$\frac{\partial \mathcal{E}}{\partial t} = T\frac{\partial s}{\partial t} + \mu\frac{\partial n}{\partial t} + \mu_5\frac{\partial n_5}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \boldsymbol{P}}{\partial t}. \tag{1.35}$$

Assuming that the external fields are homogenous, let us seek for homogeneous solutions of eqs. (1.31) to (1.34):

$$T\frac{\partial s}{\partial t} + \frac{e^3\mu_5}{2\pi^2}\, \boldsymbol{E} \cdot \boldsymbol{B} + \boldsymbol{v} \cdot (n\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}) = \boldsymbol{j} \cdot \boldsymbol{E}. \tag{1.36}$$

At zero temperature, it implies that:

$$\boldsymbol{j} = n\boldsymbol{v} + \frac{e^3\mu_5}{2\pi^2}\boldsymbol{B}. \tag{1.37}$$

The second term in (1.37) is the CME current. Such term is non-dissipative and vanishes at equilibrium, since $\mu_5 = 0$ in absence of electric field. Let us now introduce both momentum and chirality dissipation to eqs. (1.31) to (1.34). The origin of this dissipation can be either impurity scattering at low temperature or phonon scattering at high temperature. Let us denote the momentum and chirality characteristic relaxation times by τ and τ_v respectively. For homogenous and stationary field configurations, we are left with the following equations:

$$n\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} - \frac{\boldsymbol{P}}{\tau} = \boldsymbol{0}, \tag{1.38}$$

$$\frac{e^2}{2\pi^2}\, \boldsymbol{E} \cdot \boldsymbol{B} - \frac{n_5}{\tau_v} = 0. \tag{1.39}$$

The chiral density for an isotropic type-I Weyl/Dirac semimetal at zero temperature and finite chemical potential can be written as:

$$n_5 = \frac{e\mu_5}{\pi^2 v_F^3}\left(\mu^2 + \frac{\mu_5^2}{3}\right) \approx \frac{e\mu^2}{\pi^2 v_F^3}\mu_5$$

By definition, we can express the momentum density[8] as $\frac{e}{m^*}\boldsymbol{P} \equiv n\boldsymbol{v}$. Therefore, the equation (1.38) becomes:

$$n\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B} - \frac{m^*}{e\tau}\left[\boldsymbol{j} - \frac{e^4 v_F^3 \tau_v}{4\pi^2\mu^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)\boldsymbol{B}\right] = \boldsymbol{0}. \tag{1.40}$$

Solving for the resistivity tensor, we find:

$$\rho_{ij} = \rho_0\,\delta_{ij} - \frac{\epsilon_{ijk}B^k}{n} - \frac{\mathcal{C}\rho_0^2}{1+\mathcal{C}\rho_0 B^2}B_i B_j, \tag{1.41}$$

where $\rho_0 = \frac{m^*}{ne\tau}$ is the Drude conductivity and $\mathcal{C}$ is CME coefficient, given by:

$$\mathcal{C} = \frac{e^4 v_F^3 \tau_v}{4\pi^2\mu^2}. \tag{1.42}$$

For uniform magnetic field in z-direction, that is, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, we obtain an enhancement of the conductivity along the magnetic field.

$$\rho_{zz} = \frac{\rho_0}{1+\mathcal{C}\rho_0 B^2}. \tag{1.43}$$

As we will see in the next chapter, this result agrees with the one calculated from kinetic theory by solving the Boltzmann equation.

1.4 Hall Fluid

As already mentioned, hydrodynamics is a powerful tool to study strongly interacting systems, including for example the fractional quantum Hall effect (FQHE). One of the attempts to model fractional quantum Hall states relies on the Landau-Ginsburg theory, also referred as Chern-Simons-Landau-Ginzburg theory. Such theory can be rewritten in terms of hydrodynamic-type equations describing the dynamics of FQH liquid [26]. The main features of this approach are the incompressibility of the electron flow, due to the gap separation between the FQH ground state and the excited ones[9], and the relation

[8] The quantity m^* is called effective mass, which is a function of μ, μ_5 and T.

[9] The gap refers to FQH states with the same filling factor.

between the density of the fluid and its vorticity. Although the hydrodynamic model in [26] captures many features of FQHE states, it fails to give the correct value for the Hall viscosity. A refinement of this model which accounts for the correct value of Hall viscosity for Laughlin states was proposed in [27].

Throughout this thesis, we will reserve the term Hall viscosity for fluids which the density-vorticity constraint is imposed. In the absence of such constraint, we will adopt the term odd viscosity instead. Odd viscosity is the dissipationless and parity-odd part of response to strain and shear. It is part of the viscosity tensor, though it performs no work on the fluid. From elasticity theory, first-order gradient corrections to the stress tensor can be written as

$$\tau^{ij} = \frac{1}{2}\lambda^{ijkl}\left(\partial_k u_l + \partial_l u_k\right) + \frac{1}{2}\eta^{ijkl}\left(\partial_k \dot{u}_l + \partial_l \dot{u}_k\right), \tag{1.44}$$

where u_i is the displacement field. The coefficients λ^{ijkl} form the elastic modulus tensor and η^{ijkl} is the viscosity tensor, since $\dot{u}_i$ for a fluid gives the flow velocity v_i. Usually viscosity is associated to dissipation, however only the symmetric part of the viscosity tensor, i.e., $\eta^{ijkl} = \eta^{klij}$ contributes to it.

By Onsager relation, the antisymmetric part of η^{ijkl} must vanish in a time reversal system. It must also vanish in three dimensions if the tensor is isotropic. Nevertheless, in two dimensions the odd viscosity is compatible with isotropy [28]:

$$\eta^{ijkl} = \eta_H\left(\epsilon^{ik}\delta^{jl} + \epsilon^{jl}\delta^{ik}\right). \tag{1.45}$$

The odd viscosity part of the stress tensor can be written as:

$$\tau_H^{ik} = \eta_H\left(\epsilon^{ij}\partial_j v^k + \epsilon^{kj}\partial_j v^i + \delta^{ik}\epsilon^{jl}\partial_j v_l\right), \tag{1.46}$$

$$= \eta_H\left(\epsilon^{ij}\partial_j v^k + \epsilon^{kl}\delta^{ij}\partial_j v_l\right). \tag{1.47}$$

For FQH states [29, 30], $\eta_H = \frac{1}{2}\bar{n}\bar{s}\hbar$, where $\bar{n}$ average particle density and $\bar{s}$ is the average orbital spin per particle. In fact, as we will describe in the chapter 4, the odd viscosity is also closely related to the existence of the fluid intrinsic angular momentum (spin).

1.5 Clebsch Parametrization

In this section, we will present the variational formalism for hydrodynamic equations of a perfect fluid. It contains the basic tools we will use in chapters 3 and 4. To write the hydrodynamic action, it requires the parametrization of hydrodynamic variables in terms of unphysical auxiliary variables, called Clebsch potentials. They were first introduced in 1859 by Clebsch himself. He has shown that velocity flows which satisfy the non-relativistic Euler equation in 3 spatial dimensions can be parametrized by 3 scalar functions, that is,

$$\boldsymbol{v} = \boldsymbol{\nabla}\theta + \alpha\boldsymbol{\nabla}\beta. \tag{1.48}$$

The use of the Clebsch parametrization enlarges the phase space and removes the degeneracy of the Poisson algebra. The latter degeneracy of the Poisson structure makes impossible to write a symplectic form and consequently the full action only in terms of hydrodynamic quantities. To illustrate this fact, let us construct the Poisson algebra for hydrodynamic variables only by symmetry arguments.

From classical mechanics, the total momentum of a system is the canonical generator of translations. The same way, the momentum density $\boldsymbol{P}$ of a fluid corresponds to the canonical generator of local translations (spatial diffeomorphisms). This fixes the Poisson bracket between components of the momentum density to be:

$$\{P_i(\boldsymbol{x}), P_k(\boldsymbol{x}')\} = [P_k(\boldsymbol{x})\partial_i + P_i(\boldsymbol{x}')\partial_k]\,\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{1.49}$$

This can be derived by imposing that:

$$\int \mathrm{d}^D x\, \zeta^i(\boldsymbol{x})\{P_i(\boldsymbol{x}), P_k(\boldsymbol{x}')\} \equiv \mathcal{L}_{\boldsymbol{\zeta}'} P_k(\boldsymbol{x}'),$$

where $\mathcal{L}_{\boldsymbol{\zeta}}$ denotes the Lie derivative with respect to the vector field $\boldsymbol{\zeta}$[10]. From the same argument, the particle density should transform as a tensor density under differmorphisms, what gives us:

$$\{P_i(\boldsymbol{x}), \rho(\boldsymbol{x}')\} = \rho(\boldsymbol{x})\partial_i\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{1.50}$$

The last bracket to be determined is between the particle density with itself. Since the particle number conservation follows from the gauge invariance, we can view the particle density as the generator of local gauge transformations. Because it does not transform under gauge transformation, we find this last bracket to be:

$$\{\rho(\boldsymbol{x}), \rho(\boldsymbol{x}')\} = 0. \tag{1.51}$$

The algebra (1.49 - 1.51) is known as Lie-Poisson algebra. It is important to point out that this algebra is based only symmetry analysis and does not rely on the form of the fluid Hamiltonian.

Given the Poisson brackets and the Hamiltonian, we can only write down

[10] Notice that the bracket (1.49) does not depend on the number of dimensions

an action if and only if the Poisson structure admits inverse[11]. The Poisson algebra is only invertible if it admits no Casimirs, that is, if there exist no functional F such that:

$$\{F, P_i(\boldsymbol{x})\} = \{F, \rho(\boldsymbol{x})\} = 0. \tag{1.52}$$

However, the algebra (1.49-1.51) admits Casimirs in all dimensions, making the variational principle solely in terms of hydrodynamic variables impossible. As mentioned in the beginning of this section, we can view this algebra as some sort of reduction from a canonical Poisson algebra, which can be inverted. The goal of the next sections is to obtain the hydrodynamic action in terms of these canonical variables.

1.5.1 Non-relativistic Hydrodynamic Action

In this section, we will construct the variational principle for the non-relativistic hydrodynamics in 2 and 3 spatial dimensions at zero temperature. We will discuss the finite temperature case in the following section. Intuitively, let us start from the following action:

$$S = \int \left\{ \frac{1}{2}\rho \boldsymbol{v}^2 - \varepsilon(\rho) + \theta \Big[\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \Big] \right\} \mathrm{d}^D x \, \mathrm{d}t, \tag{1.53}$$

where the first term is the kinetic energy density of the fluid and the second term is the internal energy density[12]. In the last term, we have imposed the continuity equation as a constraint in the action.

In order to obtain the equations of motion, let us vary the action with respect to ρ, θ and $\boldsymbol{v}$:

$$\theta : \qquad \partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0\,, \tag{1.54}$$

$$\boldsymbol{v} : \qquad \boldsymbol{v} - \boldsymbol{\nabla}\theta = 0\,, \tag{1.55}$$

$$\rho : \qquad \partial_t \theta - \boldsymbol{v} \cdot \left(\frac{\boldsymbol{v}}{2} - \boldsymbol{\nabla}\theta \right) + \varepsilon'(\rho) = 0\,. \tag{1.56}$$

We can combine equations (1.54 - 1.56) into the form of momentum conservation:

$$\partial_t(\rho v_i) = -\partial_k \left[\rho\, \delta^{jk} v_i v_j + \delta_i^k \left(\varepsilon - \rho \varepsilon' \right) \right], \tag{1.57}$$

where v_i is given in (1.55). From thermodynamic identities, we find that $\varepsilon - \rho\varepsilon'$ gives the fluid pressure $P(\rho)$.

[11] The inverse of the Poisson structure is called symplectic form.
[12] The potential energy of a fluid is given by its internal energy.

Although the action (1.53) reproduces the continuity equation and the momentum conservation, equation (1.55) imposes that the flow is irrotational. We already know from (1.48) that it is necessary 3 scalar fields to parametrize a general flow in 3 dimensions. We could impose this condition directly into the action, however a more consistent way to do so is to introduce a passive scalar field β, that is, a scalar field which is transported by the flow:

$$\partial_t \beta + \boldsymbol{v} \cdot \boldsymbol{\nabla} \beta = 0.$$

Therefore, we can rewrite the hydrodynamic action as:

$$S = \int \left\{ \frac{1}{2} \rho \boldsymbol{v}^2 - \varepsilon(\rho) - \rho \Big[\partial_t \theta + \alpha\, \partial_t \beta + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \theta + \alpha \boldsymbol{\nabla} \beta) \Big] \right\} \mathrm{d}^D x \, \mathrm{d}t. \quad (1.58)$$

The introduction of the new passive scalar β automatically provide us (1.48) as the equation of motion for $\boldsymbol{v}$. Equation (1.54) is unchanged and the other equations of motion are given by:

$$\beta: \qquad \partial_t(\alpha\rho) + \boldsymbol{\nabla} \cdot (\alpha \rho \boldsymbol{v}) = 0\,, \quad (1.59)$$

$$\alpha: \qquad \partial_t \beta + \boldsymbol{v} \cdot \boldsymbol{\nabla} \beta = 0\,, \quad (1.60)$$

$$\rho: \qquad \partial_t \theta + \alpha\, \partial_t \beta - \boldsymbol{v} \cdot \left(\frac{\boldsymbol{v}}{2} - \boldsymbol{\nabla} \theta + \alpha \boldsymbol{\nabla} \beta \right) + \varepsilon'(\rho) = 0\,. \quad (1.61)$$

One can check that equation (1.57) is still valid if we write the velocity field as

$$\boldsymbol{v} = \boldsymbol{\nabla} \theta + \alpha \boldsymbol{\nabla} \beta.$$

In the action (1.58), $\boldsymbol{v}$ is a Lagrange multiplier and can be "integrated out", giving us an action that depends only on ρ, θ, α and β. These are the canonical variables for hydrodynamics. We recover the Poisson algebra (1.49-1.51) as a reduction of the canonical Poisson structure, given in terms of ρ, θ, α and β.

1.5.2 Entropy Conservation

In the previous section, we have considered the hydrodynamic action given in terms of 4 variables. It turns out that the action (1.58) is valid for both 2 and 3 dimensions at zero temperature. The reason is that the total number of conservation laws is given by $D + 1$, that is, D equations for (1.57) and one for (1.54). It is not hard to show that for the action (1.58), the energy conservation follows from the other $D + 1$ equations. However, this is not true for flows at finite temperature, since the energy conservation must also account for the entropy conservation. Therefore, for finite temperature, the total number of conservation laws is $D + 2$.

For $D = 2$, the number of equations match the number of variables, and we need not add another passive scalar to the problem. However, for $D = 3$, the number of equations exceeds the number of variables and we must add another pair of Clebsch potentials. The entropy flow can be introduced in 2 dimensions by promoting the passive scalar β to be the entropy per particle σ. On the other hand, in 3 dimension, we must add the entropy per particle σ in the same way we have added β in the action (1.58). The energy density now becomes a function of ρ and σ. The action of a perfect fluid at finite temperature can be written as:

$$S = \int \left[\frac{1}{2}\rho \boldsymbol{v}^2 - \varepsilon(\rho, \sigma) - \rho \left(\xi_0 + v^i \xi_i \right) \right] \mathrm{d}^D x \, \mathrm{d}t, \tag{1.62}$$

where ξ_0 and ξ_i are defined as:

$$\xi_0 = \partial_t \theta + \alpha \, \partial_t \beta + \gamma \, \partial_t \sigma, \tag{1.63}$$

$$\xi_i = \partial_i \theta + \alpha \, \partial_i \beta + \gamma \, \partial_i \sigma, \tag{1.64}$$

in 3 dimensions and in 2 dimensions as:

$$\xi_0 = \partial_t \theta + \alpha \, \partial_t \sigma, \tag{1.65}$$

$$\xi_i = \partial_i \theta + \alpha \, \partial_i \sigma. \tag{1.66}$$

1.5.3 Relativistic Hydrodynamic Action

Let us now consider the variation principle for relativistic hydrodynamics. We will discuss only the zero temperature case, however the generalization for finite temperatures is straightforward. We will also restrict ourselves to $3 + 1$ dimensions. Let us define $\xi_\nu = \partial_\nu \theta + \alpha \, \partial_\nu \beta$ and denote the components of the charge current by J^ν. Here we have used the covariant notation with ν runs for 0 to 3.

The charge density at the rest frame is given by:

$$n = \sqrt{-g_{\nu\lambda} J^\nu J^\lambda},$$

where $g_{\nu\lambda} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. At zero temperature, the energy density at the rest frame, ε, is function of n only. Thus, we can write the perfect fluid action as:

$$S = -\int \left[J^\nu \xi_\nu + \varepsilon(n) \right] \mathrm{d}^4 x. \tag{1.67}$$

The full set of variational equations is obtained by varying (1.67) over J^λ, θ, α and β:

$$\frac{\delta S}{\delta J^\lambda} = \varepsilon'(n) \frac{J_\lambda}{n} - \xi_\lambda = 0\,, \tag{1.68}$$

$$\frac{\delta S}{\delta \theta} = \partial_\lambda J^\lambda = 0\,, \tag{1.69}$$

$$\frac{\delta S}{\delta \alpha} = J^\lambda \partial_\lambda \beta = 0\,, \tag{1.70}$$

$$\frac{\delta S}{\delta \beta} = \partial_\lambda \left(\alpha J^\lambda \right) = J^\lambda \partial_\lambda \alpha = 0\,. \tag{1.71}$$

In order to derive the energy-momentum conservation, let us consider the following identity:

$$J^\lambda \left[\partial_\lambda \left(\frac{\delta S}{\delta J^\nu} \right) - \partial_\nu \left(\frac{\delta S}{\delta J^\lambda} \right) \right] = 0. \tag{1.72}$$

After some manipulations, equation (1.72) becomes:

$$\partial_\lambda \left(\varepsilon' \frac{J^\lambda J_\nu}{n} \right) - n^2 \partial_\nu \left(\frac{\varepsilon'}{n} \right) - \frac{\varepsilon'}{2n} \partial_\nu (n^2) = J^\lambda \left(\partial_\lambda \alpha \, \partial_\nu \beta - \partial_\nu \alpha \, \partial_\lambda \beta \right). \tag{1.73}$$

Using equations (1.70) and (1.71), we can see that the right hand side of the equation above vanishes. Here, it is convenient to rewrite the 4-current J^λ as:

$$J^\lambda \equiv n u^\lambda\,, \quad \text{such that} \quad u^\lambda u_\lambda = -1\,.$$

In addition to that, from thermodynamics, the chemical potential is defined as $\varepsilon'(n) \equiv \mu(n)$ and the pressure variation is given by $dP = n d\mu$. Therefore, we can write the energy-momentum conservation as:

$$\partial_\lambda \left(\mu n \, u_\nu u^\lambda + \delta^\lambda_\nu P \right) = 0. \tag{1.74}$$

The coupling with an external gauge field is given by

$$S \to S + \int J^\nu A_\nu \, \mathrm{d}^4 x.$$

Chapter 2

Magnetotransport in Dirac and Weyl Metals

The CME has been observed in the Dirac semimetals Na_3Bi [13], $ZrTe_5$ [9] and in the type-I Weyl semimetals TaAs [31], NbP [32] and TaP [4, 5]. A weak, yet not conclusive, indication of negative magnetoresistance was found in [2] for Cd_3As_2, though no signature of such has been observed in [33]. Both experiments however have observed Shubnikov-de Haas (SdH) oscillations, inferring the presence of a large Fermi surface[1]. Therefore, this metallic behavior allow us to study transport within the semiclassical regime. The framework to study responses in Weyl/Dirac *metals* is the chiral kinetic theory, developed in [34, 35]. In this chapter, we will consider the interplay between CME and SdH effect by analytically computing the longitudinal magnetoresistance [3].

2.1 Semiclassical Dynamics

Kinetic theory is a powerful tool to study metallic transport. It relies on the semiclassical approximation to the dynamics of quasiparticle wave-packets near the Fermi surface[2]. In this section, we will study the dynamics of Weyl quasiparticles by focusing on its phase space structure. This allows for the introduction of quantum effects such as the discreteness of the density of states in the presence of magnetic field (Landau levels), which is responsible for the SdH effect.

[1]Large Fermi surface in comparison to other energy scales of the problem, namely temperature and the gap between Landau levels.

[2]Quasiparticles away from the Fermi surface decay too fast and cannot be captured by the semiclassical approach. However, in the Fermi liquid theory only states near Fermi surface contribute to transport.

The semiclassical action that accounts for the Berry curvature contribution in the electron quasiparticle trajectory was derived in [36].

$$S = \int_{t_i}^{t_f} \Big[(\hbar \boldsymbol{k} - e\boldsymbol{A}) \cdot \dot{\boldsymbol{x}} + \boldsymbol{\mathcal{A}} \cdot \hbar \dot{\boldsymbol{k}} - \varepsilon_0(\boldsymbol{k}) + \boldsymbol{m}(\boldsymbol{k}) \cdot \boldsymbol{B} + e\Phi \Big] \, \mathrm{d}t, \tag{2.1}$$

where $\boldsymbol{\mathcal{A}}$ is the Berry connection, $\boldsymbol{A}$ the vector potential, Φ the electric potential, $\varepsilon_0(\boldsymbol{k})$ is the band dispersion relation and $\boldsymbol{m}(\boldsymbol{k})$ is the wave-packet magnetization given by:

$$\boldsymbol{m}(\boldsymbol{k}) = -\frac{ie}{2\hbar} \langle \boldsymbol{\nabla}_{\boldsymbol{k}} u_{\boldsymbol{k}} | \times [H(\boldsymbol{k}) - \varepsilon_0(\boldsymbol{k})] | \boldsymbol{\nabla}_{\boldsymbol{k}} u_{\boldsymbol{k}} \rangle. \tag{2.2}$$

For the two-band system (1.1), the magnetization simplifies:

$$\boldsymbol{m}(\boldsymbol{k}) = \frac{e}{\hbar} |\boldsymbol{b}(\boldsymbol{k})| \boldsymbol{\Omega}(\boldsymbol{k}). \tag{2.3}$$

From now on, let us denote $\varepsilon = \varepsilon_0 - \boldsymbol{m} \cdot \boldsymbol{B}$ for short. The equations of motion for the quasiparticle trajectories are given by[3]:

$$\dot{\boldsymbol{x}} = \boldsymbol{v}_{\boldsymbol{k}} - \dot{\boldsymbol{k}} \times \boldsymbol{\Omega}(\boldsymbol{k}), \tag{2.4}$$

$$\hbar \dot{\boldsymbol{k}} = -e\boldsymbol{E} - e \, \dot{\boldsymbol{x}} \times \boldsymbol{B}, \tag{2.5}$$

In the equation (2.4), we have introduced the group velocity vector $\boldsymbol{v}_{\boldsymbol{k}} = \frac{1}{\hbar} \boldsymbol{\nabla}_{\boldsymbol{k}} \, \varepsilon$. These equations decouple and can be rewritten as:

$$\Big(1 + \frac{e}{\hbar} \boldsymbol{B} \cdot \boldsymbol{\Omega}\Big) \, \dot{\boldsymbol{x}} = \boldsymbol{v}_{\boldsymbol{k}} + \frac{e}{\hbar} \boldsymbol{B} \, (\boldsymbol{v}_{\boldsymbol{k}} \cdot \boldsymbol{\Omega}) + \frac{e}{\hbar} \boldsymbol{E} \times \boldsymbol{\Omega}, \tag{2.6}$$

$$\Big(1 + \frac{e}{\hbar} \boldsymbol{B} \cdot \boldsymbol{\Omega}\Big) \, \dot{\boldsymbol{k}} = -\frac{e}{\hbar} \boldsymbol{E} - \frac{e}{\hbar} \, \boldsymbol{v}_{\boldsymbol{k}} \times \boldsymbol{B} - \frac{e^2}{\hbar^2} \boldsymbol{\Omega} \, (\boldsymbol{E} \cdot \boldsymbol{B}). \tag{2.7}$$

The presence of a non-vanishing Berry curvature modifies the phase space volume. To illustrate this, let us introduce the phase-space coordinates $\xi^A = (k_a, x^b)$. In this new notation, the action (2.1) has the following general form:

$$S = \int_{t_i}^{t_f} \Big[\zeta_A \, (\boldsymbol{\xi}, t) \, \dot{\xi}^A - H(\boldsymbol{\xi}, t) \Big] \, \mathrm{d}t. \tag{2.8}$$

[3]The last term in (2.4) is sometimes referred as anomalous velocity in allusion to the work of Karplus and Luttinger in 1954.

Equations of motion are:

$$\omega_{AB}\dot{\xi}^B + \frac{\partial \zeta_A}{\partial t} + \partial_A H = 0, \tag{2.9}$$

where $\omega_{AB} = \partial_A \zeta_B - \partial_B \zeta_A$ are the symplectic form components. The inverse of the symplectic matrix defines the Poisson structure, that is,

$$\left(\omega^{-1}\right)^{AB} \equiv \{\xi^A, \xi^B\}\,.$$

Hence, equation (2.9) becomes:

$$\dot{\xi}^A = \{\xi^A, \xi^B\}\left(\frac{\partial \zeta_B}{\partial t} + \frac{\partial H}{\partial \xi^B}\right). \tag{2.10}$$

Poisson brackets for this system are given by:

$$\{x^a, x^b\} = \frac{\epsilon^{abc}\,\Omega_c}{\hbar \Upsilon}\,, \tag{2.11}$$

$$\{k_a, k_b\} = -\frac{e\,\epsilon_{abc}\,B^c}{\hbar^2 \Upsilon}\,, \tag{2.12}$$

$$\{x^a, k_b\} = \frac{\hbar\,\delta^a_b + eB^a\,\Omega_b}{\hbar^2 \Upsilon}\,, \tag{2.13}$$

with

$$\Upsilon = 1 + \frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}(\boldsymbol{k}). \tag{2.14}$$

From equations (2.11-2.13), one can notice that the semiclassical approximation is only justified for

$$\frac{eB}{\hbar}|\boldsymbol{\Omega}| \ll 1. \tag{2.15}$$

Since the Berry curvature is a function of the crystal quasimomentum, the inequality (2.15) can be viewed as defining the values of the quasiparticle momenta which the semiclassical approximation is valid. The region $\mathcal{Q}$, where

$$\{(\boldsymbol{x}, \boldsymbol{k}) \in \mathcal{Q},\ \text{if}\ \frac{eB(\boldsymbol{x}, t)}{\hbar}|\boldsymbol{\Omega}(\boldsymbol{k})| > 1\},$$

is called quantum region and reflects the existence of chiral modes in the lowest Landau level. Nevertheless, only quasiparticles near the Fermi surface contribute to transport and such region of the phase space will be inaccessible for most of transport quantities.

The phase space measure is defined as:

$$\sqrt{|\det \omega_{AB}|}\,\frac{\mathrm{d}^3x\mathrm{d}^3k}{(2\pi)^3} = |\Upsilon|\,\frac{\mathrm{d}^3x\mathrm{d}^3k}{(2\pi)^3}\,.$$

We are only interested in the region of the phase space where semiclassical regime holds, therefore we can drop the modulus sign of Υ. Formally this can be done by removing these quantum regions from the phase space:

$$\{(\boldsymbol{x},\boldsymbol{k}) \in \mathbb{R}^3 \times \mathcal{BZ} \setminus \mathcal{Q}\}.$$

For uniform magnetic field, the reduced phase space becomes simply $\mathbb{R}^3 \times \mathcal{BZ}'$, where the radius of each open set is defined by the implicit equation $|\boldsymbol{\Omega}(\boldsymbol{k})| = \ell_B^2$. In the presence of magnetic field, not every point in the phase space correspond to an allowed state. In fact, the density of states becomes discrete due to the existence of Landau levels. We can introduce quantum effects to this framework by accounting for the discreteness of Landau levels. To do so, we will use the Bohr-Sommerfeld quantization condition. The prescription here is the same one used in the old quantum theory; given a classical system, we introduce quantum effects by imposing that canonical variables satisfy:

$$\oint_\gamma p_i\,\mathrm{d}q_i = 2\pi\hbar\,(n_i + \tfrac{1}{4}\mathrm{ind}\gamma),$$

where γ is a loop in phase space in which the Hamiltonian is constant, and $\mathrm{ind}\gamma$ is the Maslov index of γ.

However, in the presence of a nonvanishing Berry curvature the perpendicular components of $\boldsymbol{k}$, with respect to $\boldsymbol{B}$, fail to be canonically conjugated, vide (2.12). Following the same recipe and assuming a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, the discreteness of Landau levels can be imposed by setting:

$$\frac{1}{2}\oint_\gamma \left(1 + \frac{e}{\hbar}\boldsymbol{\Omega}\cdot\boldsymbol{B}\right)\hat{\boldsymbol{z}}\cdot\boldsymbol{k}\times\mathrm{d}\boldsymbol{k} = \frac{2\pi}{\ell_B^2}(\nu + \tfrac{1}{4}\mathrm{ind}\gamma)\,. \tag{2.16}$$

Equation (2.16) implies the area quantization for the section of the Brillouin zone with k_z constant, in terms of the magnetic length

$$\ell_B = \sqrt{\frac{\hbar}{eB}}. \tag{2.17}$$

As an example, let us consider the isotropic version of the effective Hamiltonian (1.8), with $\nu_i = 0$ and $v_{ij} = v_F\delta_{ij}$. For clean systems, we can treat each chirality independently. Therefore, the Berry curvature for each chirality

χ has the monopole form:

$$\boldsymbol{\Omega}_\chi(\boldsymbol{k}) = \chi \frac{\hat{\boldsymbol{k}}}{2k^2}. \tag{2.18}$$

The semiclassical assumption (2.15) translates into $k^2\ell_B^2 \gg \frac{1}{2}$. This condition is equivalent to the weak field limit where many Landau level are filled. In this limit one can still think about Fermi sphere albeit stratified into Landau level "cylinders", see Figure 2.1.

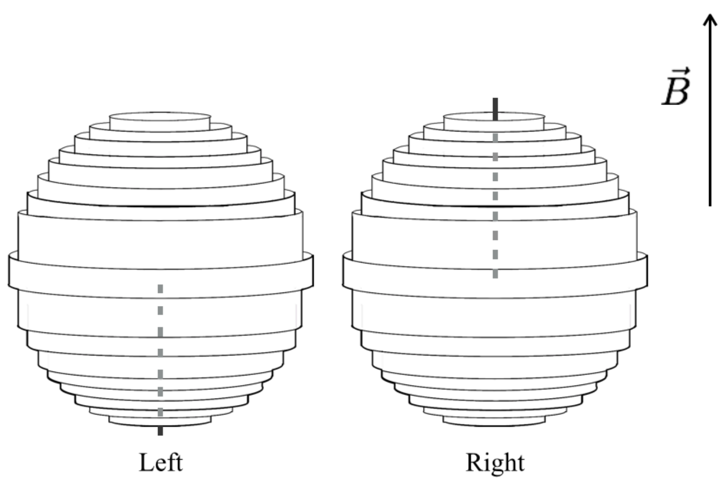

Figure 2.1: Semiclassical picture of Fermi surface for right and left chiral modes is shown in the presence of the magnetic field. For the linear dispersion, the support of Hamiltonian eigenstates is a collection of cylinders corresponding to eigenvalues $\varepsilon_\nu(k_z) = \hbar v_F \sqrt{k_z^2 + k_\perp^2}$, where $k_\perp^2 = 2eB\nu$ with $\nu \in \mathbb{Z}_+$. The state with $\nu = 0$ is chiral and exists only for $k_z > 0$ (parallel to B) for the right chirality and for $k_z < 0$ for the left one.

We can find the surfaces with constant ν in k-space by solving equation (2.16):

$$\nu + \frac{1}{4}\mathrm{ind}\gamma = \frac{\ell_B^2}{2}\left(k_\perp^2 - \frac{\chi k_z}{\ell_B^2\sqrt{k_\perp^2 + k_z^2}}\right), \tag{2.19}$$

$$= \frac{1}{2}\left(k^2\ell_B^2 \sin^2\theta - \chi\cos\theta\right). \tag{2.20}$$

If we impose that $\nu = 0$ is the smallest possible integer solution of (2.20) and use the fact that ind$\gamma \in \mathbb{Z}$; the only possible values of the Maslov index are $\{-2, -1, 0\}$. In addition to that, the area of any cross-section in the $\mathcal{BZ}$ must be positive. These conditions necessarily fix ind$\gamma = 0$.

2.2 Chiral Kinetic Theory

In this section, we will review the chiral kinetic theory developed in [34, 35]. Let us restrict ourselves to the reduced phase space. For uniform magnetic field, the phase space measure is transported by the Hamiltonian flow[4], that is:

$$\frac{\partial \Upsilon}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot (\Upsilon \dot{\boldsymbol{x}}) + \boldsymbol{\nabla}_{\boldsymbol{k}} \cdot \left(\Upsilon \dot{\boldsymbol{k}}\right) = 0, \tag{2.21}$$

Here, we have used that $\boldsymbol{\nabla}_{\boldsymbol{k}} \cdot \boldsymbol{\Omega} = 0$ to $(\boldsymbol{x}, \boldsymbol{k}) \in \mathbb{R}^3 \times \mathcal{BZ}'$. The phase space measure refers to the density of states on a region in phase space. On the other hand, the distribution function $f(\boldsymbol{x}, \boldsymbol{k}, t)$ indicates the probability of such state to be occupied. The distribution function satisfies the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f + \dot{\boldsymbol{k}} \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} f = \mathcal{I}[f], \tag{2.22}$$

where $\mathcal{I}[f]$ is the collision integral. At low temperatures, the impurity scattering is the leading contribution to conductivity tensor. In this and in the following sections, we will restrict ourselves to collision integrals that correspond impurity scattering. Given the transition rate $w_{\boldsymbol{k}' \to \boldsymbol{k}}$ from an initial state $\boldsymbol{k}'$ to a final state $\boldsymbol{k}$, the collision integral can be written as:

$$\mathcal{I}[f] = \int_{\mathcal{BZ}} \left[f(\boldsymbol{k}') - f(\boldsymbol{k})\right] w_{\boldsymbol{k}' \to \boldsymbol{k}}\, \Upsilon' \frac{\mathrm{d}^3 k'}{(2\pi)^3}\,. \tag{2.23}$$

We have assumed the elastic scattering probability to be invariant under time reversal, i.e. $w_{\boldsymbol{k}' \to \boldsymbol{k}} = w_{\boldsymbol{k} \to \boldsymbol{k}'}$. In addition to that, we have used that

$$\left[f'\,(1-f) - f\,(1-f')\right] = f(\boldsymbol{k}') - f(\boldsymbol{k}).$$

The prime quantities denote functions of momentum $\boldsymbol{k}'$. Multiplying equation

[4]This is not true for non-uniform magnetic field. For that, equation (2.21) does not equate to zero and one must be careful on the choice of the phase space domain, either by defining $\mathcal{Q}$ which satisfies the Liouville theorem, or by accounting for the discreteness of Landau levels.

(2.22) by Υ and integrating over the $\mathcal{BZ}'$, we end up with:

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{j}=-\frac{e^2}{\hbar}\sum_{\alpha}\int_{\partial\mathcal{U}_\alpha} f\left[\boldsymbol{E}+\boldsymbol{v}_{\boldsymbol{k}}\times\boldsymbol{B}+\frac{e}{\hbar}\boldsymbol{\Omega}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)\right]\cdot\frac{\mathrm{d}\boldsymbol{S}}{(2\pi)^3},\quad(2.24)$$

where we have defined:

$$n(\boldsymbol{x},t)=-e\int_{\mathcal{BZ}} f\left(1+\frac{e}{\hbar}\boldsymbol{\Omega}\cdot\boldsymbol{B}\right)\frac{\mathrm{d}^3k}{(2\pi)^3},\quad(2.25)$$

$$\boldsymbol{j}(\boldsymbol{x},t)=-e\int_{\mathcal{BZ}} f\left[\boldsymbol{v}_{\boldsymbol{k}}+\frac{e}{\hbar}\left(\boldsymbol{v}_{\boldsymbol{k}}\cdot\boldsymbol{\Omega}\right)\boldsymbol{B}+\frac{e}{\hbar}\boldsymbol{E}\times\boldsymbol{\Omega}\right]\frac{\mathrm{d}^3k}{(2\pi)^3}.\quad(2.26)$$

The distribution function is assumed to vary slowly inside the open sets $\mathcal{U}_\alpha$ (small quantum region). Therefore, we can approximate $f(\boldsymbol{x},\boldsymbol{k},t)$ in the right hand side of equation (2.24) to its value at the Weyl point:

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{j}=-\frac{e^3}{4\pi^2\hbar^2}\boldsymbol{E}\cdot\boldsymbol{B}\sum_{\alpha} f(\boldsymbol{x},\boldsymbol{k}_\alpha,t)c_1(\partial\mathcal{U}_\alpha).\quad(2.27)$$

Charge conservation imposes that

$$\sum_{\alpha} f(\boldsymbol{x},\boldsymbol{k}_\alpha,t)c_1(\partial\mathcal{U}_\alpha)=0.$$

If the Weyl/Dirac metal is composed by several disjoint pieces of Fermi surface and if we neglect the scattering between them, equation (2.27) shows that the charge of each piece of Fermi surface is not conserved. Let us consider again the effective Hamiltonian (1.8), with $\nu_i=0$ and $v_{ij}=v_F\delta_{ij}$. The total density is given by

$$n=\sum_{\chi=\pm} n_\chi,$$

where

$$n_\chi(\boldsymbol{x},t)=-e\int_{\mathcal{BZ}} f_\chi\left(1+\frac{\chi e}{2\hbar k^2}\hat{\boldsymbol{k}}\cdot\boldsymbol{B}\right)\frac{\mathrm{d}^3k}{(2\pi)^3}.\quad(2.28)$$

Neglecting the inter-chirality scattering (clean samples), the charge density for each chirality satisfies the following equation:

$$\frac{\partial n_\chi}{\partial t}+\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{j}_\chi=-\chi\frac{e^3}{4\pi^2\hbar^2}\boldsymbol{E}\cdot\boldsymbol{B}\,f_\chi(\boldsymbol{x},\chi\boldsymbol{k}_0,t).\quad(2.29)$$

If we also account for the holes, the Fermi-Dirac distribution at each Weyl point is given by:

$$f_\chi(\boldsymbol{x}, \chi\boldsymbol{k}_0, t) = \frac{1}{e^{\beta[\varepsilon(\chi\boldsymbol{k}_0)-\mu]}+1} - \frac{1}{e^{\beta[\varepsilon(\chi\boldsymbol{k}_0)+\mu]}+1} = 1,$$

since $\varepsilon(\chi\boldsymbol{k}_0) = 0$. In fact, by imposing that $f(\boldsymbol{x}, \boldsymbol{k}_\alpha, t) = 1$ for all Weyl points[5] $\boldsymbol{k}_\alpha$, the charge conservation (2.27) is automatically satisfied. Let us define the axial density to be

$$n_5 = -\sum_{\chi=\pm} \chi\, n_\chi.$$

This way, we will recover the equations (1.31) and (1.34).

2.3 Boltzmann Equation

In this section, we will solve the Boltzmann equation for a system with Weyl quasiparticles in the regime when we can treat each chirality independently. Although Dirac/Weyl metals are usually characterized by linear dispersion of quasiparticles $\varepsilon(\boldsymbol{k}) = \hbar v_F|\boldsymbol{k} - \boldsymbol{k}_0|$, the assumption of linear spectrum will be absent in this section. Yet, we will restrict ourselves to an isotropic system and neglect magnetization effects. The chemical potential or Fermi energy define the size of Fermi surface $\varepsilon_F = \varepsilon(k_F)$, where the Fermi momentum is related to the density of conduction electrons (per chirality) by standard formula $k_F = (6\pi^2 n_\chi)^{1/3}$.

The impurity scattering introduces another scale into the problem, the scattering rate. The system is called clean when the quasiparticle performs many cyclotron orbits before colliding or, equivalently, when the mean-free-path is much larger than the cyclotron radius, $v_F\tau \gg k_F\ell_B^2$. We will restrict ourselves to single-impurity scattering approximation and neglect interference and localization effects. This approximation is valid when the density of impurities is low. Thus, the regime of interest in this work is defined by

$$1 \ll (k_F\ell_B)^2 \ll k_F v_F\tau\,, \tag{2.30}$$

where for the linear spectrum the last term correspond to $\varepsilon_F\tau/\hbar$.

The distribution function is obtained by solving the Boltzmann equation. Since we are interested in linear response, we must expand $f(\boldsymbol{x}, \boldsymbol{k}, t)$ around the equilibrium (Fermi-Dirac) distribution function $f_0(\varepsilon)$:

$$f(\boldsymbol{x}, \boldsymbol{k}, t) = f_0(\varepsilon) + e\frac{\partial f_0}{\partial\varepsilon}\boldsymbol{E}\cdot\boldsymbol{g} + \mathcal{O}(\boldsymbol{E}^2)\,. \tag{2.31}$$

[5]This can be viewed as a particular boundary condition on the distribution function.

Having $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{k}, t)$, the solution of the linearized Boltzmann equation, and substituting the ansatz (2.31) into (2.26), the conductivity tensor reads:

$$\begin{aligned}\sigma_{ab} = &- e^2 \int \frac{\partial f_0}{\partial \varepsilon} g_b \left(\boldsymbol{v}_{\boldsymbol{k}} + \frac{e}{\hbar} \left(\boldsymbol{v}_{\boldsymbol{k}} \cdot \boldsymbol{\Omega} \right) \boldsymbol{B} \right)_a \frac{\mathrm{d}^3 k}{(2\pi)^3} + \\ &+ \frac{e^2}{\hbar} \varepsilon_{abc} \int \Omega_c(\boldsymbol{k}) f_0(\varepsilon) \frac{\mathrm{d}^3 k}{(2\pi)^3} , \qquad (2.32)\\ = &- e^2 \sum_{\chi=\pm} \int \frac{\partial f_0}{\partial \varepsilon} g_b \, v_k \left(\hat{\boldsymbol{k}} + \chi \zeta_k \hat{\boldsymbol{z}} \right)_a \frac{\mathrm{d}^3 k}{(2\pi)^3} . \qquad (2.33)\end{aligned}$$

Here and in the following all the expressions will refer to a single pair of Weyl quasiparticles with opposite chirality. The assumption that the Weyl points can be treated independently is valid when they are far apart in the Brillouin zone[6], so that the quasiparticle scattering from one Weyl cone to the other requires a large momentum transfer. The last term in equation (2.32) vanishes for isotropic dispersion relations. The last equality is obtained with the use of (2.18) assuming that the system is isotropic and that the integral is dominated by a vicinity to the Fermi surface due to the factor $\partial f_0 / \partial \varepsilon$. We have also introduced the small parameter $\zeta_k = 1/(2k^2 \ell_B^2)$ and considered that the magnetic field is along the z-direction.

For Dirac metals, the $\mathbb{Z}_2$-symmetry holds at low energies and interaction terms that break this symmetry are sub-leading in comparison to the Chern number-preserving ones. In this limit, the Boltzmann equations for different chiralities decouple and the collision integral accounts only for intra-chirality scattering. Using the equations of motions (2.6) and (2.7), the Boltzmann equation for $\boldsymbol{g}(t, \boldsymbol{k})$ in the linearized regime becomes:

$$\begin{aligned}&\left[\Upsilon(\partial_t + i\omega) - \frac{e}{\hbar} (\boldsymbol{v}_{\boldsymbol{k}} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} \right] \boldsymbol{g} = \qquad (2.34)\\ &= \boldsymbol{v}_{\boldsymbol{k}} + \frac{e}{\hbar} (\boldsymbol{v}_{\boldsymbol{k}} \cdot \boldsymbol{\Omega}) \boldsymbol{B} + \int\limits_{\mathcal{BZ}} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \left(\Upsilon' w_{\boldsymbol{k}' \to \boldsymbol{k}} \Upsilon \right) \left[\boldsymbol{g}' - \boldsymbol{g} \right] .\end{aligned}$$

In equation (2.34), we have assumed that the system is uniform and the electric field oscillates with the frequency ω, i.e., $\boldsymbol{E} = \boldsymbol{E}_0 \, e^{i\omega t}$. It is straightforward to observe that this equation does not admit any stationary solution[7] when $\omega = 0$. This is the manifestation of the chiral anomaly in kinetic theory, the constant parallel electric and magnetic field continue to pump chirality into the system. However, a stationary solution does exist in the presence of

[6] In comparison to the Fermi momentum of each disjoint piece of Fermi surface.

[7] This can be seen by integrating (2.34) over the solid angle.

a chirality relaxation mechanism.

To determine $w_{\boldsymbol{k}'\to\boldsymbol{k}}$, we assume that the elastic scattering occurs on weak and short-range impurity potential and the concentration of impurities is very dilute. We thus model the single-impurity scattering by

$$w_{\boldsymbol{k}'\to\boldsymbol{k}} = \frac{3}{2\nu(\varepsilon)\tau(\varepsilon)}(1+\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}})\,\delta(\varepsilon-\varepsilon')\,, \tag{2.35}$$

where $\nu(\varepsilon)$ is the density of states – in the absence of magnetic field – at the energy ε. We assumed that the scattering is elastic and averaged over impurity positions. All microscopic details are absorbed into the transport scattering time τ. One must notice that although we focused on the small wave vector limit, the scattering rate from equation (2.35) is not isotropic. This is because the Weyl-particle spins are always polarized along their momenta, producing a universal factor $(1+\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}})$, which suppresses the backscattering of particles by impurities. For example, for massless Dirac quasiparticles one can find at leading order in the partial-wave expansion of scattering amplitude[8]:

$$\frac{1}{\tau} = n_{imp}\frac{2v_F}{3\pi^2k^2}\sin^2\delta_1\,.$$

The scattering phase δ_1 in the general case should also depend on the magnitude of magnetic field since the screening of the impurity potential might be modified by B.

Rewriting equation (2.34) in spherical coordinates and plugging the formula for scattering rate (2.35) into it, we obtain:

$$\begin{aligned}\left(i\omega\Upsilon+2k\zeta_k v_k\frac{\partial}{\partial\phi}\right)\boldsymbol{g}(\boldsymbol{k})-v_k\hat{\boldsymbol{k}} = \\ = v_k\chi\zeta_k\hat{\boldsymbol{z}}+\frac{3\Upsilon}{16\pi^3}\int \mathrm{d}^3k'\Upsilon'(\boldsymbol{g}'-\boldsymbol{g})\frac{(1+\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}})}{\nu(\varepsilon)\tau(\varepsilon)}\delta(\varepsilon-\varepsilon').\end{aligned} \tag{2.36}$$

Integrating equation (2.36) over the solid angle, we obtain:

$$\int_{\mathbb{S}^2}\mathrm{d}\phi\,\mathrm{d}(\cos\theta)\,\Upsilon(\boldsymbol{k})\boldsymbol{g}(\boldsymbol{k}) = \frac{4\pi\chi\zeta_k\hat{\boldsymbol{z}}}{i\omega}. \tag{2.37}$$

[8]Although the magnetic field breaks the 3D rotation invariance, the assumption of adiabatic evolution allows us to write the eigenbasis in terms of Bloch functions or plane waves. The effect of magnetic field is absorbed into the trajectory in k-space and in the measure. A solution of the Dirac scattering problem can be found, e.g., in [37] and gives for scattering amplitude $A(\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}}) = \frac{\hbar v_F}{2i\varepsilon}\sum_{l=1}^{\infty} l\left(e^{2i\delta_l}-1\right)\left[P_l(\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}})+P_{l-1}(\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}})\right]$.

Since $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, the azimuthal symmetry along the z-direction allows us to find solutions to (2.36) that are independent of ϕ. After the integration over (k', ϕ'), we end up with:

$$i\omega \Upsilon g_z = v_k(\cos\theta + \chi\zeta_k) + \int_{-1}^{1} \mathrm{d}(\cos\theta')\Upsilon'(g'_z - g_z)\Upsilon \\ \times \frac{3}{4\tau}(1 + \cos\theta\cos\theta') \,. \tag{2.38}$$

The easiest way to solve this equation is to expand Υg_z in terms of Legendre polynomials and use their orthogonality conditions. Thus,

$$\Upsilon g_z = \sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta), \\ = \frac{\chi\zeta_k v_k}{i\omega} + \left[\frac{\zeta_k^2}{i\omega} + \frac{(1-\zeta_k^2)}{i\omega + \tau^{-1}}\right] v_k \cos\theta, \tag{2.39}$$

where a_0 is obtained through (2.37). Therefore, the expression for $g_z(k, \theta)$ becomes:

$$g_z(k,\theta) = \frac{\chi\zeta_k v_k}{i\omega + \eta} + \frac{1-\zeta_k^2}{1 + \chi\zeta_k\cos\theta}\,\frac{v_k\cos\theta}{i\omega + 1/\tau}\,, \tag{2.40}$$

where $\eta \to +0$ in the absence of chirality flipping and will be replaced by $1/\tau_v$ if the chirality flipping processes are taken into account.

2.4 Interplay Between CME and SdH Effect

In this section, we will present the expression for σ_{zz} which accounts for both CME and SdH oscillation. If we take into account the discreteness of Landau levels given by (2.20) into (2.33), the conductivity per chirality becomes:

$$\sigma_{zz}^{(\chi)} = -\frac{e^2}{4\pi^2}\sum_{\nu=0}^{\infty}\int \mathrm{d}k \int_{-1}^{1} \mathrm{d}(\cos\theta)\, k^2 \frac{\partial f_0}{\partial\varepsilon} v_k(\cos\theta + \chi\zeta_k) \\ \times \delta\left(\nu - \frac{1 - \cos^2\theta - 2\chi\zeta_k\cos\theta}{4\zeta_k}\right) g_z(k,\theta)\,. \tag{2.41}$$

Since the argument of the delta function has no real roots when $\nu \in \mathbb{Z}_-$, we can consider the sum starting from $\nu = -\infty$ and use the Poisson summation formula. Thus,

$$\sigma_{zz}^{(\chi)} = \sigma_{zz}^{(0)} + 2\sum_{l=1}^{\infty} \sigma_{zz}^{(l)} \cos\left(\frac{\pi l}{2\zeta_F} + \frac{\pi}{4}\right), \tag{2.42}$$

where we have used that

$$\zeta_F \equiv \zeta_k|_{k=k_F} = \frac{1}{2k_F^2 l_B^2} = \frac{eB}{2\hbar k_F^2}. \tag{2.43}$$

For details of the calculation, vide appendices A and B. The non-oscillating part of (2.42) is given by

$$\sigma_{zz}^{(0)} = \frac{n_\chi e^2 v_F}{\hbar k_F}\left(\frac{1 - \frac{12}{5}\zeta_F^2}{i\omega + 1/\tau} + \frac{3\zeta_F^2}{i\omega + \eta}\right), \tag{2.44}$$

where $n_\chi = k_F^3/(6\pi^2)$ is the total density of electrons per chirality. And, for the oscillating part we have

$$\sigma_{zz}^{(l)} = \frac{n_\chi e^2 v_F}{\hbar k_F}\frac{1}{i\omega + 1/\tau}\frac{3}{2\pi}\frac{\lambda l}{\sinh \lambda l}\left(\frac{2\zeta_F}{l}\right)^{3/2}, \tag{2.45}$$

where $\lambda = \pi^2 T/(\hbar k_F v_F \zeta_F)$. In the DC limit and in the absence of magnetic field, $\zeta_F = 0$, equations (2.42-2.45) are reduced to a standard Drude formula appropriately modified for Dirac spectrum:

$$\sigma_0 = \frac{n_\chi e^2 \tau}{\hbar k_F / v_F}. \tag{2.46}$$

In finite magnetic field the second term of (2.44) describes an ideal conductivity. In the absence of chirality flipping this conductivity diverges in static limit $\omega \to 0$. In more realistic models, processes of chirality flipping are always present and one should replace $\eta \to 1/\tau_v$, where τ_v is a mean chirality lifetime. As the scattering with and without changes of chirality are due to very different processes one should expect the ratio τ_v/τ to be significant. Both τ and τ_v can in principle be extracted from optical conductivity measurements.

There are two small parameters in the regime of interest of this work. One is ζ_F, i.e., the weakness of the magnetic field compared to the Fermi scale. The other is the smallness of temperature compared to the Fermi energy. We do not, however, make any assumptions on the relative size λ of these small parameters. In deriving (2.42-2.45) we kept the leading (B-independent) and next to the leading terms of the expansion in ζ_F but restricted the expansion only to the leading term in T/ε_F. This is why the only temperature dependence in (2.42-2.45) is through the parameter λ. This means that we omitted all corrections proportional to T/ε_F which could be comparable to the ones proportional to ζ_F. The former corrections, however, are not universal and do not affect the magnetic field dependence of the conductivity.

A very convenient way to exclude the non-universal temperature corrections is to study the ratio $\sigma_{zz}(B)/\sigma_0$. In DC limit ($\omega \to 0$), it is given by

$$\frac{\sigma_{zz}(B)}{\sigma_0} = 1 + 3\left(\frac{\tau_v}{\tau} - \frac{4}{5}\right)\zeta_F^2 + \frac{3}{\pi}\sum_{l=1}^{\infty} e^{-\lambda_D l}\frac{\lambda l}{\sinh \lambda l}\left(\frac{2\zeta_F}{l}\right)^{\frac{3}{2}} \cos\left(\frac{\pi l}{2\zeta_F} + \frac{\pi}{4}\right). \tag{2.47}$$

In the last equation, we introduced the Dingle factor $\lambda_D = \pi\Gamma/(\hbar v_F k_F \zeta_F)$, which accounts for the smearing of LLs due to impurities. In the case of a homogeneous sample, $\Gamma = \hbar/\tau_Q$, with the "quantum time" τ_Q determined by impurity scattering and equal to the quasiparticle lifetime. If either $\lambda \gg 1$ or $\lambda_D \gg 1$, i.e., the temperature or smearing of Landau levels is larger than the gap between Landau levels, the oscillations in (2.47) disappear and the conductivity is given by the first two terms in (2.47). For smaller temperatures and Landau level smearing, oscillations appear and become less and less harmonic with a further decrease of both λ and λ_D.

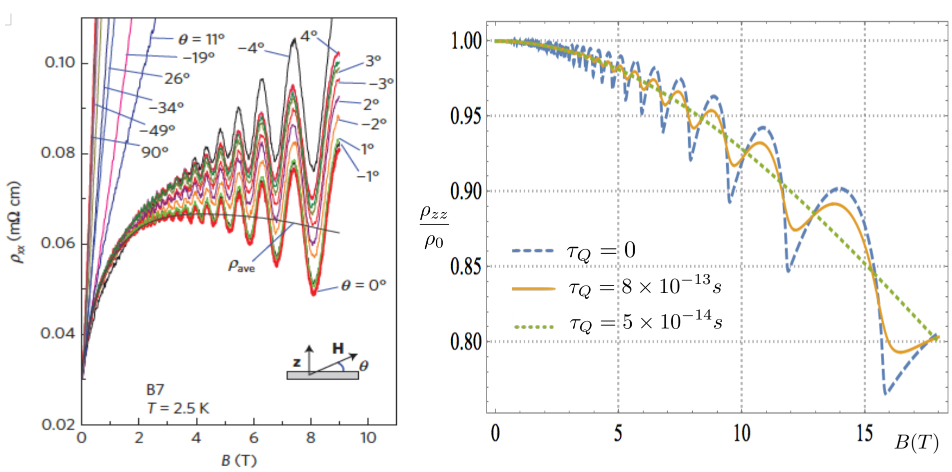

Figure 2.2: Longitudinal magnetoresistance as a function of the magnetic field. On the left: the results of experiment [2] on Cd_3As_2. On the right: the results of computation [3] with numerical parameters $k_F = 3.8 \times 10^8 m^{-1}$, $v_F = 9.3 \times 10^5 m/s$, $\tau = 8 \times 10^{-13} s$, $T = 2.5K$. The plots are made for three values of the quantum lifetime $\tau_Q/\tau = 0, 1, 16$ and for the chirality relation time $\tau_v = 10\tau$.

On the right panel of Fig. 2.2 we have plotted the magnetoresisitivity given by the inverse of expression in equation (2.47) for parameters consistent with the recent experiment on Cd_3As_2 [2]. Comparison with the experimental plot on the left panel shows that the approach to magnetotransport in Dirac semimetals developed here describes qualitatively the emergence of quantum SdH oscillations and the tendency to negative magnetoresistance at strong magnetic fields (but still small ζ_F) observed experimentally in Cd_3As_2 [2]. Unfortunately, the direct comparison with experimental data of [2] is difficult due to the large positive magnetoresistance (MR) typical for Cd_3As_2. The latter has a very complicated unit cell structure and is prone to various defects and the cause of positive MR is still unknown. The more thorough comparison of our theory with experimental data requires an explanation of the positive magnetoresistance. Magnetic field dependent Coulomb screening might be one of the reasons for positive MR (see, e.g., [38]) as well as the influence of Zeeman effects on band structure and possible spatial inhomogeneity of samples. On the other hand, our the computation shows a good qualitative agreement with the data from TaP [4], vide Fig. 2.3.

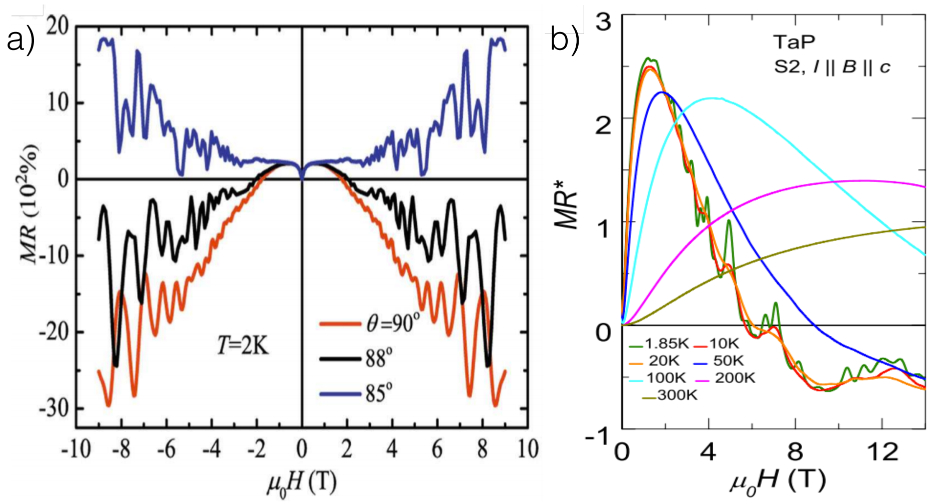

Figure 2.3: Magnetoresistence in TaP, the plot in **a)** was extracted from [4], whereas the plot in **b)** was taken from [5].

A much weaker positive MR has also been observed in the weak magnetic field region in Dirac semimetals $ZrTe_5$ and Na_3Bi and it is believed to be due to the weak antilocalization.

Chapter 3

Hydrodynamics with Gauge Anomaly

The goal of this chapter is to develop variational and Hamiltonian formulations of the hydrodynamics with gauge anomaly[1]. Differently from the previous chapter, this one will be more formal, without a direct connection to experiments. Yet, one may hope to use some of these ideas to model surface states in strongly interacting Weyl materials.

The possibility of an universal hydrodynamic description with additional hydrodynamic terms taking anomalies into account was noticed initially in AdS/CFT systems [39, 40], and then in genuine relativistic hydrodynamic formulation in [41], for a particular case of Abelian gauge anomaly. Together with the CME, the constitutive relations obtained in [41] predict the existence of the chiral vortical effect (CVE), where the current acquires a term proportional to the flow vorticity.

The Hamiltonian formalism is appropriate to study wavelike excitations and instabilities near the fixed point, through the linear analysis of the eigenmodes, and provides the most appropriate framework to study perturbation theory and symmetries of the system. We will show how the quantum anomaly affects the canonical generators of gauge transformations and diffeomorphisms as well as their semidirect product algebra. Our approach will be entirely 3+1 dimensional, providing a minimal generalization of the standard action principle for fluid dynamics to accommodate anomalies.

The variational problem for hydrodynamics with gauge anomaly in 1+1 dimensions was successfully developed in [42], however it cannot be trivially generalized to 3+1 dimensions. The most successful attempt so far in finding

[1] The set of equations (1.31-1.34) can be viewed as two copies of a system with gauge anomaly.

an effective action for equations (3.1-3.4) was given in [43], but the obtained action contained unphysical hydrodynamic excitations propagating in a fourth auxiliary spatial dimension. All these approaches rely on an effective action for the Lagrangian specification of fluid variables [44, 45]. On the other hand, the action principle for non-abelian hydrodynamics was presented in [46], where the authors introduced the idea of coarse graining the coadjoint orbit action. A similar approach to fluid dynamics for spinning particles has been recently developed in [47]. An action that includes anomalies in the standard model of particle physics within the framework of the coadjoint orbit method was given in [48]. The anomaly structure in the standard model is different from what is given in (3.1-3.4) and so the effective action for anomalies in [48] is not immediately applicable to the present problem.

In the next sections, we will use the so-called Clebsch potentials to parametrize the Eulerian variables and to write down a variational principle that produces the anomalous hydrodynamic equations at zero temperature. We will restrict ourselves to the flat Minkowski spacetime, though the generalization to more general geometric backgrounds is straightforward. Unless otherwise specified, we will use the Cartesian orthonormal frame, where the pseudo-metric can be chosen as $g_{\lambda\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

The variational principle and the symmetries are analyzed in sections 3.2 and 3.3. Using the obtained action, we will derive the corresponding Hamiltonian formulation specifying the form of the relativistic Hamiltonian and the Poisson brackets. We will emphasize the symmetries of the system and their manifestations in Hamiltonian formalism, pointing out the special feature of one of the Clebsch potentials appearing separately and not via the combination in the dynamic velocity field. This feature is commented on in section 3.6. This chapter refers to the work [6]. In the following sections, we will use the covariant notation and we will set $\hbar = c = 1$ for simplicity.

3.1 Constitutive Relations

Let us start with equations of anomalous hydrodynamics of [41]. The current and energy-momentum conservation laws for anomalous QFT in the background gauge field can be written as:

$$\partial_\lambda j^\lambda = -\frac{C}{8}\epsilon^{\lambda\nu\sigma\tau}F_{\lambda\nu}F_{\sigma\tau}\,, \tag{3.1}$$

$$\partial_\lambda T^{\lambda\nu} = F^{\nu\sigma}j_\sigma\,. \tag{3.2}$$

The right hand side of the equation (3.2) is the Lorentz force, while the right hand side of (3.1) is the gauge anomaly term, fully characterized by a single dimensionless constant C. Here and in the following we will drop the angular brackets denoting expectation values, e.g., $\langle j \rangle \to j$, so that j^λ and $T^{\lambda\nu}$ are classical fields representing the current and the energy-momentum tensor.

Assuming local equilibrium and imposing the local form of the second law of thermodynamics, the authors in [41] were able to constrain the form of constitutive relations. In this chapter we are interested in the case of zero temperature and absence of dissipation. Thus, we will use a particular form of these constitutive relations, which is given by:

$$j^\lambda = nu^\lambda + \frac{C}{12}\epsilon^{\lambda\nu\sigma\tau}\,\mu u_\nu\,(2\mu\,\partial_\sigma u_\tau + 3F_{\sigma\tau})\,, \tag{3.3}$$

$$T^{\lambda\nu} = n\mu\, u^\lambda u^\nu + P(\mu)\, g^{\lambda\nu}. \tag{3.4}$$

We have introduced the equation of state of the fluid $P(\mu)$ which gives the fluid pressure P as a function of the chemical potential μ. The charge density in the fluid rest frame is given by $n = P'(\mu)$. The fluid 4-velocity u^λ satisfies $u^\lambda u_\lambda = -1$ and, therefore, has only three independent components. In this case, the zeroth component of the equation (3.2) – the energy conservation – is not independent, but can be viewed as a consequence of the other four equations (3.1) and (3.2). The latter four independent equations fully determine the evolution of n and three independent components of 4-velocity u^λ.

Equations (3.1-3.4) constitute the first-order hydrodynamics equations written in Landau frame. Namely, the constitutive relations (3.3) and (3.4) are first order in derivatives and the ambiguity in the definition of 4-velocity is resolved by defining it as an eigenvector of the energy-momentum tensor. Landau frame was used in [49] and was adopted in [41] to construct the hydrodynamics with gauge anomaly.

3.2 Hydrodynamic Action

The variational principle for perfect relativistic fluid dynamics is well known and goes back to [50, 51]. The key point in finding a hydrodynamic action is the introduction of a set of variables appropriate to the canonical framework, the so-called Clebsch potentials. For a review on the Clebsch parametrization and variational principle for relativistic as well as non-relativistic hydrodynamics, vide section 1.5.

The field content of the hydrodynamic action is given by 4 components of the 4-current J^λ and 3 scalar Clebsch potentials (θ, α, β) parametrizing dynamic velocity ξ_λ:

$$\xi_\lambda = \partial_\lambda \theta + \alpha \partial_\lambda \beta \,. \tag{3.5}$$

One of the main results of this work is that the action generating equations (3.1-3.4) is given by:

$$S = -\int \left[J^\lambda \left(\xi_\lambda - A_\lambda\right) + \varepsilon(n) - \frac{C}{6}\, \epsilon^{\lambda\nu\eta\sigma} A_\lambda \, \xi_\nu \, \partial_\eta \left(\xi_\sigma + A_\sigma\right) \right] \mathrm{d}^4 x. \tag{3.6}$$

Here, $\varepsilon(n)$ is the proper energy density of the fluid which is assumed to be a known function of the proper charge density n. The latter is given by an absolute value of the 4-current J^λ as $n \equiv \sqrt{-g_{\lambda\nu} J^\lambda J^\nu}$. The second term on the right hand side of (3.6) describes the anomaly. Taking $C = 0$ in (3.6) we recover the action for a relativistic perfect fluid without anomaly [50, 51].

The full set of variational equations is obtained by varying (3.6) over $J^\lambda, \theta, \alpha, \beta$. Let us start with equations of motion for J^λ:

$$\frac{\delta S}{\delta J^\lambda} = -\left(\xi_\lambda - A_\lambda\right) + \varepsilon'(n) \frac{J_\lambda}{n} = 0 \,. \tag{3.7}$$

It is convenient to introduce a complete parametrization of the 4-current J^λ in terms of its absolute value n and its direction given by 4-velocity u^λ as:

$$J^\lambda \equiv n u^\lambda \,, \qquad u^\lambda u_\lambda = -1 \,. \tag{3.8}$$

Then equation (3.7) can be viewed as a relation between the dynamic velocity, density and the 4-velocity[2]:

$$\xi_\lambda - A_\lambda = \mu \, u_\lambda \,, \tag{3.9}$$

where the chemical potential $\mu(n)$ is given by the derivative of the energy density as:

$$\mu(n) \equiv \varepsilon'(n) \,. \tag{3.10}$$

Clebsch potentials θ, α, β enter (3.6) only through ξ_λ given by (3.5). The

[2]For the case of irrotational flows, such as superfluids, the dynamic velocity can be fully characterized by $\partial_\lambda \theta$ and equation (3.9) corresponds to Josephson condition.

corresponding variations give the following equations of motion:

$$\frac{\delta S}{\delta \theta} = \partial_\lambda \left(\frac{\delta S}{\delta \xi_\lambda} \right) = 0 \,, \tag{3.11}$$

$$\frac{\delta S}{\delta \alpha} = \frac{\delta S}{\delta \xi_\lambda} \partial_\lambda \beta = 0 \,, \tag{3.12}$$

$$\frac{\delta S}{\delta \beta} = \partial_\lambda \left(\alpha \frac{\delta S}{\delta \xi_\lambda} \right) = \frac{\delta S}{\delta \xi_\lambda} \partial_\lambda \alpha = 0 \,, \tag{3.13}$$

with

$$-\frac{\delta S}{\delta \xi_\lambda} = n u^\lambda + \frac{C}{6} \epsilon^{\lambda\nu\eta\sigma} \left[2 A_\nu \partial_\eta \xi_\sigma - (\xi_\nu - A_\nu) \partial_\eta A_\sigma \right] \,. \tag{3.14}$$

Introducing the charge current:

$$j^\lambda = -\frac{\delta S}{\delta \xi_\lambda} + \frac{C}{6} \epsilon^{\lambda\nu\eta\sigma} \left[3\, \partial_\nu (A_\eta \xi_\sigma) - 3\, A_\nu \partial_\eta A_\sigma + \xi_\nu \partial_\eta \xi_\sigma \right] \,, \tag{3.15}$$

we obtain (3.1) from (3.11) and (3.5). The relations (3.15) and (3.14) give the constitutive relation (3.3).

Defining the energy-momentum tensor by (3.4), one can derive the conservation law (3.2) from (3.9) and (3.11-3.13) after some tedious but straightforward manipulations[3]. We will not go through this derivation in more detail, since, in the section 3.3, we will derive equations (3.1-3.4) more straightforwardly from symmetries of the action (3.6).

In the absence of the gauge field background $A_\nu = 0$ the action (3.6) becomes the conventional action for relativistic perfect fluid dynamics [50, 51]. The only manifestation of the gauge anomaly in this case is the non-conventional relation between current and 4-velocity. Namely, the relation (3.3) becomes $j^\lambda = n u^\lambda + \frac{C}{3} \mu^2 \omega^\lambda$ with relativistic vorticity defined as $\omega^\lambda = \frac{1}{2} \epsilon^{\lambda\nu\sigma\tau} u_\nu \partial_\sigma u_\tau$. This current is conserved $\partial_\lambda j^\lambda = 0$ because both relations $\partial_\lambda (n u^\lambda) = 0$ and $\partial_\lambda (\mu^2 \omega^\lambda) = 0$ follow from (3.6) in the absence of the gauge background[4] — this consequence can be observed directly from [41] by setting the temperature and the external fields to zero. Such "removal" of the anomaly responses by current redefinition is not possible though when a non-trivial gauge field background is present.

[3]Technical remark: it is convenient to start this derivation with an obvious equation $\frac{\delta S}{\delta \xi_\lambda} \left[\partial_\lambda \left(\frac{\delta S}{\delta J^\nu} \right) - \partial_\nu \left(\frac{\delta S}{\delta J^\lambda} \right) \right] = 0$.

[4]One can think of the second relation as a consequence of (3.1) and (3.2). The second conserved quantity $\mu^2 \omega^\lambda$ can be identified as a density of the Casimir (helicity) of the relativistic perfect fluid dynamics.

3.3 Symmetries

In this section we will show explicitly that the equations (3.1) and (3.2) can be obtained as consequences of (anomalous) gauge symmetry and space-time translational symmetry of the action (3.6), respectively.

The first two terms in (3.6) are symmetric with respect to the gauge transformation with the gauge parameter $\Lambda(x)$

$$\delta_\Lambda A_\lambda = \partial_\lambda \Lambda \,, \qquad \delta_\Lambda \theta = \Lambda \,. \tag{3.16}$$

From (3.5) and (3.16), we obtain $\delta_\Lambda \xi_\lambda = \partial_\lambda \Lambda$, such that, the combination $\xi_\lambda - A_\lambda$ in (3.6) is gauge invariant. This gauge invariance, however, is broken by the anomalous part of the action. Up to boundary terms, the gauge transformation of the action is given by

$$\delta_\Lambda S = \int \partial_\lambda \Lambda \left(\frac{\delta S}{\delta \xi_\lambda} + \frac{\delta S}{\delta A_\lambda} \right) \mathrm{d}^4 x = \frac{C}{6} \int \Lambda \, \epsilon^{\lambda\nu\eta\sigma} \, \partial_\lambda A_\nu \, \partial_\eta A_\sigma \mathrm{d}^4 x. \tag{3.17}$$

Unlike the case of a general breaking of a symmetry, the loss of symmetry due to anomalies is rather special. The gauge variation of the action depends only on the background gauge field and has a very specific form, the latter being determined by the densities of certain topological invariants[5]. For field configurations that satisfy the equation of motion (3.11), the variation of (3.17) over Λ gives the charge conservation law modulo the anomaly as

$$\partial_\lambda \left(\frac{\delta S}{\delta A_\lambda} \right) = -\frac{C}{24} \epsilon^{\lambda\nu\sigma\tau} F_{\lambda\nu} F_{\sigma\tau}. \tag{3.18}$$

The quantity $\delta S / \delta A_\lambda$ is known as the *consistent current* versus the *covariant current* j^λ defined in (3.3). A quick calculation shows that

$$j^\lambda = \frac{\delta S}{\delta A_\lambda} - \frac{C}{6} \epsilon^{\lambda\nu\sigma\tau} A_\nu F_{\sigma\tau} \,. \tag{3.19}$$

Taking the divergence of (3.19), we obtain (3.1). We will now turn to the energy-momentum conservation (3.4). The standard way of deriving this law is to gauge space-time translational symmetries by introducing the background

---

[5] It is easy to see that the action can be made fully gauge invariant by supplementing it with the Chern-Simons term $-\frac{C}{6} \int_{M_5} A \wedge dA \wedge dA$. The integral in this term is taken over an auxiliary 5-dimensional space M_5 which boundary coincides with the physical space-time. This gives an elegant interpretation of the anomaly of the 4-dimensional theory as being due to the inflow of charge from the fifth dimension, a set-up known as *anomaly inflow*; this is standard and well known in QFT with quantum anomalies.

metric and study the invariance of the action under diffeomorphisms $x^\lambda \to x^\lambda + \zeta^\lambda(x)$.

Let us consider (3.6) in an arbitrary background metric by replacing the measure d^4x by the invariant one $\sqrt{-g}\,\mathrm{d}^4x$ and by introducing the metric into all scalar products. Notice that ξ_λ is naturally a covariant vector, being derivatives of the scalar Clebsch potentials, and thus $J^\lambda\,\xi_\lambda$ being an invariant scalar product does not require additional metric factors. However, a scalar product like J^2 will become $J^\mu J^\nu\,g_{\mu\nu}$. The resulting action is invariant under diffeomorphisms, i.e., $\delta_\zeta S = 0$, and on equations of motion we have

$$\int \left[(\mathcal{L}_\zeta \boldsymbol{g})_{\nu\lambda} \frac{\delta S}{\delta g_{\nu\lambda}} + (\mathcal{L}_\zeta A)_\lambda \frac{\delta S}{\delta A_\lambda} \right] \mathrm{d}^4x = 0\,, \tag{3.20}$$

since the terms corresponding to the variations of the fields vanish by the equations of motion. Here $\mathcal{L}_\zeta$ denotes the Lie derivative with respect to the vector field ζ. Explicitly

$$(\mathcal{L}_\zeta \boldsymbol{g})_{\nu\lambda} = \partial_\nu \zeta_\lambda + \partial_\lambda \zeta_\nu\,, \tag{3.21}$$

$$(\mathcal{L}_\zeta A)_\lambda = \zeta^\nu F_{\nu\lambda} + \partial_\lambda(\zeta^\nu A_\nu)\,. \tag{3.22}$$

Using these formulas and setting the coefficient of ζ^ν in (3.20) to zero we obtain[6]

$$\partial_\lambda T^\lambda{}_\nu = F_{\nu\lambda} \frac{\delta S}{\delta A_\lambda} - \frac{C}{6} F_{\nu\lambda}\,\epsilon^{\lambda\eta\sigma\tau} A_\eta F_{\sigma\tau}\,, \tag{3.23}$$

with

$$T^{\lambda\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\lambda\nu}}\,, \tag{3.24}$$

A quick calculation shows that the energy-momentum tensor (3.24) is the same as (3.4). This is expected since the last term of (3.6) is the integral of a 4-form — which is metric-independent — and gives no contribution to the energy-momentum tensor. Therefore, (3.4) is identical in form to the energy-momentum tensor for conventional perfect fluid dynamics. The metric independence of the anomalous contribution to (3.6) is an essential feature of the analysis in the hydrodynamic Landau frame where the energy-momentum tensor is not modified by corrections which are of the first order in gradients of the velocity.

Finally, it is easy to see that the equation (3.23) with the relation (3.19) is equivalent to (3.2). This completes the demonstration that the action (3.6) does indeed reproduce equations (3.1-3.4).

[6]The identity $A_\nu\,\epsilon^{\lambda\eta\sigma\tau} F_{\lambda\eta} F_{\sigma\tau} = -4 F_{\nu\lambda}\,\epsilon^{\lambda\eta\sigma\tau} A_\eta F_{\sigma\tau}$ can be useful.

3.4 Hamiltonian Formalism

In this section we will set up the Hamiltonian formulation of equations (3.1-3.4) starting with the action (3.6). Let us start by reducing the 7 independent variational fields of (3.6) to 4, given by J^0 and by the Clebsch parameters θ, α, β. The spatial components of (3.8) and (3.9) combine to

$$J_i = \frac{n}{\mu}(\xi_i - A_i). \tag{3.25}$$

We can thus eliminate the spatial components of the current J^i using (3.25). The definition of n, namely $(J^0)^2 - (J^i)^2 = n^2$, gives us

$$J^0 \equiv \rho = \frac{n}{\mu}\sqrt{\mu^2 + (\xi_i - A_i)^2}\,. \tag{3.26}$$

Here and in the following we will use ρ to denote J^0. We can regard ρ as the independent variable, with n given implicitly as a function of ρ by (3.26)[7]. Substituting (3.25) and (3.26) into (3.6) we obtain the action which is linear in the time-derivatives and depends only on fields $\rho, \theta, \alpha, \beta$. After some integrations by parts, it can be brought to the following form:

$$S = \int \Big(\langle \pi_\theta \, , \dot{\theta} \rangle + \langle \pi_\beta \, , \dot{\beta} \rangle - H \Big) \, dt \,, \tag{3.27}$$

where $\langle f, g \rangle \equiv \int f(x) g(x) \, \mathrm{d}^3 x$ denotes the L^2-inner product in the space of real functions, H is the Hamiltonian, π_θ and π_β are the canonical field momenta conjugate to θ and β, respectively. The explicit formulas for the canonical momenta are:

$$\pi_\theta \;=\; -\left[\rho + \frac{C}{6} \left(A_i + \alpha \, \partial_i \beta \right) B^i \right] , \tag{3.28}$$

$$\pi_\beta \;=\; -\alpha \left[\rho + \frac{C}{6} \left(A_i - \partial_i \theta \right) B^i \right] . \tag{3.29}$$

The Hamiltonian H in (3.27) is given by

$$\begin{aligned} H \;&=\; \int \left[\rho \sqrt{\mu^2 + (\xi_i - A_i)^2} - P(\mu) - \rho A_0 \right] d^3 x \\ &-\; \frac{C}{6} \int \left[\xi_i B^i A_0 + \epsilon^{ijk} (\partial_i \theta - A_i) \, \xi_j E_k \right] d^3 x \,. \end{aligned} \tag{3.30}$$

[7]As $\mu(n)$ is assumed to be a known function of n (3.10) the equation (3.26) can in principle be solved to obtain $n(\rho, \xi_i)$, $\mu(\rho, \xi_i)$ etc.

The pressure $P(\mu)$ is related to the energy density by the Legendre transform $\varepsilon(n) = n\mu - P(\mu)$, with $P'(\mu) = n$ and we have also introduced the magnetic and electric fields $B^i = \epsilon^{ijk}\partial_j A_k$ and $E_i = \partial_i A_0 - \partial_0 A_i$ with $\epsilon^{ijk} \equiv \epsilon^{0ijk}$.

Although the equations of motion (3.1) and (3.4) still do not contain the Clebsch potentials explicitly, the Hamiltonian (3.30) depends on Clebsch potentials not only through ξ_i anymore[8]. We shall comment on the the meaning of this explicit dependence on θ in the following sections. Here we will just point out that the coefficient of E_k in the last term of (3.30) may be interpreted as an intrinsic fluid electric polarization. It is worth recalling that one of the main predictions of the anomaly for fluids is the chiral magnetic effect which leads to charge separation in a magnetic field. An electric dipole moment obviously suggest a charge separation and we may regard the last term of equation (3.30) as a reflection of this feature in the Hamiltonian framework.

So far we have considered the background gauge field as space and time-dependent. An interesting special case is when the magnetic field is time-independent. It is then possible to choose a vector potential A_i which is independent of time as well. Then the last term of (3.30) can be integrated by parts and the Hamiltonian takes the form

$$
\begin{aligned}
H &= \int \left[\rho\sqrt{\mu^2 + (\xi_i - A_i)^2} - P(\mu) - A_0\rho\right] d^3x \\
&- \frac{C}{6}\int A_0 \left[2\,\xi_i B^i + \epsilon^{ijk}(\xi_i - A_i)\partial_j\xi_k\right] d^3x
\end{aligned}
\tag{3.31}
$$

In this case, the explicit dependence on θ has disappeared and the Clebsch potentials only appear in the combination ξ_i. It is straightforward to observe that, for a time-independent gauge field, the potential term is simply $\int A_0 j^0 d^3x$, as one should expect.

3.5 Poisson Brackets

In the this section we will discuss the effect of the anomaly on the Poisson structure of the Hamiltonian formulation derived in this section. The variational principle (3.27) which is linear in time-derivatives immediately provides us with the canonically conjugate pairs θ, π_θ and β, π_β. The Poisson brackets of all fields follow then from the canonical ones for the above fields

$$
\{\theta, \pi'_\theta\} = \{\beta, \pi'_\beta\} = \delta(\boldsymbol{x} - \boldsymbol{x}')\,, \tag{3.32}
$$

[8]This reduction on equations of motion happens because the Hamiltonian (3.30) is still invariant under transformations on the Clebsch potentials which leave ξ_i unchanged.

where we have listed only the non-vanishing Poisson brackets. We have used a concise notation omitting the spatial arguments of the fields so that, e.g., β means $\beta(\boldsymbol{x})$, π'_θ means $\pi_\theta(\boldsymbol{x}')$ etc.

The hydrodynamic equations of motion (3.1-3.4) can be formulated as equations written entirely in terms of ρ and ξ_i without an explicit dependence on the Clebsch parameters. Therefore, we shall look for the possible Hamiltonian reduction of (3.30) and (3.32). The reduction consists of the dynamic reduction, i.e., the Hamiltonian should be expressible only in terms of the density ρ and dynamic velocity ξ_i, and the kinematic reduction, i.e., the closure of Poisson brackets of ρ and ξ_i without the use of the Clebsch parameters.

As pointed out before, with the inclusion of the anomaly, the dynamic reduction is only partially successful. Namely, the Hamiltonian (3.30) does depend on $\partial_i\theta$ in the case of general time-dependent gauge field background. In the case of time-independent background the dynamic reduction is complete and the Hamiltonian (3.31) depends on the Clebsch parameters only through ξ_i.

Remarkably, the Poisson algebra of ρ and ξ_i is closed for any gauge field background so that the kinematic reduction is achieved. Indeed, after some straightforward calculations, we derive from (3.32) and the definition (3.5) the following set of Poisson brackets closed with respect to the fields ρ and ξ_i,

$$\left\{\rho_+, \rho'_+\right\} = \frac{C}{3} B^i \partial_i \, \delta(\boldsymbol{x} - \boldsymbol{x}')\,, \tag{3.33}$$

$$\left\{\widetilde{\xi}_i, \rho'_+\right\} = \partial_i \, \delta(\boldsymbol{x} - \boldsymbol{x}')\,, \tag{3.34}$$

$$\left\{\widetilde{\xi}_i, \widetilde{\xi}'_j\right\} = -\frac{\partial_i\widetilde{\xi}_j - \partial_j\widetilde{\xi}_i - \epsilon_{jik}B^k}{\rho_-} \, \delta(\boldsymbol{x} - \boldsymbol{x}')\,. \tag{3.35}$$

For the sake of brevity, we have introduced the following compact notation,

$$\widetilde{\xi}_i \equiv \xi_i - A_i\,, \tag{3.36}$$

$$\rho_\pm \equiv \rho \pm \frac{C}{6}\, \widetilde{\xi}_i \, B^i\,. \tag{3.37}$$

A comment on the first of these equations, namely (3.33), is appropriate at this point. It is well known that the $[j^0, j^{0\prime}]$ commutator will be modified by a Schwinger term in the presence of an anomaly for the corresponding symmetry. This can be shown by explicit computation of the corrections to commutators via Feynman diagrams, the triangle diagram leading to the specific form given.[9] It can also be seen from a 2-cocycle constructed in terms

[9] The computation of modified commutators follows a procedure known as the Bjorken-

of the descent equations which lead to the anomalies. Our action effectively reproduces this in the Poisson brackets. We may also point out that an expression analogous to (3.33) has appeared in [52].

The dynamic velocity $\widetilde{\xi}_i$ and the modified densities $\rho_\pm$ are invariant under the transformations (3.16), therefore, the Poisson algebra (3.33-3.35) is written in terms of explicitly gauge-invariant quantities. However, as it is well known generator of gauge transformations cannot be realized canonically in the presence of anomaly (see Sec. 3.6).

The algebra (3.33-3.35) is obtained as a result of Hamiltonian reduction and is degenerate. It admits two Casimirs — the quantities having vanishing Poisson brackets with fields entering Poisson algebra. They are given by

$$C_1 = \int \rho_+ \, \mathrm{d}^3 x \,, \tag{3.38}$$

$$C_2 = \int \epsilon^{ijk} \, \widetilde{\xi}_i \, \partial_j \left(\widetilde{\xi}_k + 2A_k \right) \mathrm{d}^3 x \,. \tag{3.39}$$

The charge density j^0 defined in (3.3) is given by

$$j^0 = \rho_+ + \frac{C}{6} \epsilon^{ijk} \, \widetilde{\xi}_i \, \partial_j \left(\widetilde{\xi}_k + 2A_k \right) \,. \tag{3.40}$$

It is a combination of densities of two Casimirs of the algebra. It is worth to point out that when the gauge field is time-independent the anomaly term can be written as a total derivative, i.e., $E_i B^i = \partial_i (A_0 B^i)$, what automatically implies that the total charge is indeed conserved.

In the absence of anomaly $C = 0$, all expressions (3.30, 3.33-3.40) become the known formulas for perfect fluid dynamics [46, 53]. Even when the anomaly is present, i.e., $C \neq 0$, if we consider the case of the background gauge field being absent, we obtain again the formulas of anomaly-free hydrodynamics with a single exception. Namely, the definition of the charge density (3.40) still differs from ρ by the density of Casimir (3.39). The latter is known as the helicity of the hydrodynamic flow.

Having Hamiltonian and Poisson brackets, one can obtain equations of motion for any quantity Q as $\dot{Q} = \partial Q / \partial t + \{H, Q\}$, where $\partial Q / \partial t$ denotes the "explicit" time-derivative. In our case this explicit derivative acts only on the time varying external gauge field. The dynamical fields ξ_i and ρ do not depend on time explicitly. For example, the equation of motion for ξ_i will read $\dot{\widetilde{\xi}}_i = -\partial_t A_i + \{H, \widetilde{\xi}_i\}$, etc.

Johnson-Low method where correlators of currents at slightly unequal times are calculated and a suitable equal-time limit is taken.

While the Clebsch variables appear in the algebra (3.33-3.35) only via ξ_i, we should note that, in the presence of the time-dependent gauge field background, the Hamiltonian (3.30) contains $\partial_i\theta$ in addition to the density and the dynamic velocity fields. Thus the algebra (3.33-3.35) is not adequate for a complete Hamiltonian description, and it should be supplemented by Poisson brackets involving the θ field. We will list these brackets here for completeness:

$$\{\rho_+, \partial_k\theta'\} = \partial_k\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{3.41}$$

$$\{\widetilde{\xi}_i, \partial_k\theta'\} = \frac{\widetilde{\xi}_i + A_i - \partial_i\theta}{\rho_-}\partial_k\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{3.42}$$

3.6 Symmetry generators

The Poisson algebra (3.33-3.35) is closed and, in the case of the time-independent background, produces the hydrodynamic equations with the use of the Hamiltonian (3.31). However, the brackets (3.33-3.35) are nonlinear and therefore do not have the Lie-Poisson form. For the symmetry analysis it is preferable to find an equivalent set of Poisson brackets corresponding to the algebra of symmetry generators of the system.

It is easy to see from (3.27) that the momentum densities can be defined as:

$$\Theta_{0i} = -\pi_\theta\partial_i\theta - \pi_\beta\partial_i\beta\,. \tag{3.43}$$

The momentum densities Θ_{0i} satisfy the diffeomosphism algebra and act as local translations in the absence of background field. However, one cannot express (3.43) only in terms of the density ρ and the dynamic velocity in the background of nonvanishing magnetic field. More precisely, the canonical energy-momentum tensor acquires an explicit θ dependence:

$$\Theta_{0i} = \left(\rho + \frac{C}{6}\,A_k B^k\right)\,\xi_i + \frac{C}{6}\,B^k(\xi_k\partial_i\theta - \xi_i\partial_k\theta)\,. \tag{3.44}$$

Let us now turn to gauge transformations which can be viewed as shifts in the field θ. The naive canonical gauge generator for this symmetry is $-\pi_\theta$. Using (3.28,3.5) we can write it as

$$-\pi_\theta = \rho + \frac{C}{6}\,(A_i + \xi_i - \partial_i\theta)\,B^i \tag{3.45}$$

and notice that it also depends explicitly on $\partial_i\theta$.

It is straightforward to check that the Poisson structure (3.33-3.35) can be put in a semidirect product Lie-Poisson algebra [54] in terms of (3.44) and (3.45). The gauge transformation of an arbitrary functional F of basic fields generated by $-\pi_\theta$ is given by:

$$\delta_\Lambda F \equiv \int \left(-\Lambda(\boldsymbol{x}')\{\pi'_\theta, F\} + \frac{\delta F}{\delta A_i(\boldsymbol{x}')}\partial'_i\Lambda \right) \mathrm{d}^3x' \,, \tag{3.46}$$

where the transformation of the gauge potential has also been added.

However, it is easy to see that (3.46) gives $\delta_\Lambda\alpha \neq 0$, as well as $\delta_\Lambda\rho \neq 0$ in obvious discrepancy with gauge invariance of α and ρ. In fact, we will prove by contradiction that the gauge symmetry (3.16) cannot be canonically realizable. Let us assume that there exists a generator Q_Λ, such that,

$$\{Q_\Lambda, \theta(\boldsymbol{x})\} = \Lambda(\boldsymbol{x}), \tag{3.47}$$

$$\{Q_\Lambda, \alpha(\boldsymbol{x})\} = \{Q_\Lambda, \rho(\boldsymbol{x})\} = \{Q_\Lambda, \beta(\boldsymbol{x})\} = 0. \tag{3.48}$$

From the Jacobi identity, we obtain:

$$\{\rho(\boldsymbol{x}'), \{Q_\Lambda, \alpha(\boldsymbol{x})\}\} - \{\alpha(\boldsymbol{x}), \{Q_\Lambda, \rho(\boldsymbol{x}')\}\} = \{Q_\Lambda, \{\alpha(\boldsymbol{x}), \rho(\boldsymbol{x}')\}\}. \tag{3.49}$$

The right hand side can be evaluated using equation (C.9):

$$\{Q_\Lambda, \{\alpha(\boldsymbol{x}), \rho(\boldsymbol{x}')\}\} = -\frac{C^2}{36}\frac{B^i\partial_i\alpha}{\rho_-^2}(B^k\partial_k\Lambda)\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{3.50}$$

On the other hand, using the definition (3.48), we find that the left hand side vanishes, which concludes the proof.

Let us now consider ρ_+ given by (3.37) as a generator of gauge transformations instead of $-\pi_\theta$. We easily check that $\delta_\Lambda\alpha = \delta_\Lambda\beta = 0$ and $\delta_\Lambda\theta \equiv \Lambda$. Moreover, under the modified gauge transformations generated by ρ_+ the density ρ transforms as:

$$\delta_\Lambda\rho = -\frac{C}{6}B^i\partial_i\Lambda \,, \tag{3.51}$$

and there exists the gauge invariant quantity $\rho + \frac{C}{6}B^iA_i$.

While ρ_+ can be considered as a modified generator of gauge transformations two subsequent gauge transformations generated by ρ_+ do not commute and the commutative algebra of gauge transformations has acquired a central extension (3.33). This is, of course, a classical manifestation of a well known phenomenon in studies of quantum anomalies. At this point it is not clear whether similar modifications can be made for diffeomorphism generators

(3.43)[10].

[10]The variables α and ρ do not transform nicely under these generators.

Chapter 4

Odd Viscosity from Variational Principle

Odd viscosity effects in incompressible fluids cannot be observed in flat space without the introduction of a boundary. For the compressible case, on the other hand, we can always get rid of the odd viscosity term by some momentum redefinition. It would be convenient to have a frame independent way to define odd viscosity without such ambiguities. Since the odd viscosity term is dissipationless, one may hope to capture it in the hydrodynamic action. Once we have the variational principle, coupling with a geometrical background uniquely defines the momentum density and stress tensor. Thus, we will be able to lift the ambiguity in the odd viscosity definition.

The work presented in this chapter has not been published yet.

4.1 Odd Viscosity Ambiguity

In this section, we will show that, in a compressible fluid with conserved charge, the odd viscosity term (1.47) can be introduced or removed by a momentum redefinition. The possibility of such redefinition is a direct consequence of the continuity equation on the density of the conserved charge.

In the case of FQH fluid, the odd viscosity η_H is called Hall viscosity 1.4 and it is proportional to the average orbital spin per particle [29, 30, 55]. As we will show in this section, this relation also holds for general two-dimensional fluids. Let us consider a compressible fluid on the plane with intrinsic angular momentum. If $\bar{s}$ is the average spin per particle, the expression for the total angular momentum can be written as:

$$L = \int \mathrm{d}^2x \left(\epsilon^{ik} x_i P_k + \bar{s}\rho\right) , \tag{4.1}$$

where P_i is the momentum density. Let us assume for simplicity that total intrinsic angular momentum of the fluid is extensive, that is, $\bar{s}$ is a constant. Thus, we can rewrite the equation (4.1) as:

$$L = \int \mathrm{d}^2x \, \epsilon^{ij} x_i \left(P_j + \frac{\bar{s}}{2} \delta_{jk} \epsilon^{kl} \partial_l \rho \right) = \int \mathrm{d}^2x \, \epsilon^{ij} x_i \Pi_j, \tag{4.2}$$

where Π_i is the modified momentum density. Equations (4.1) and (4.2) differ by boundary terms. Let us consider for simplicity the system with Galilean symmetry, that is, momentum and current are the same[1]. As a consequence of the continuity equation, we find:

$$\partial_t \Big(\epsilon^{ij} \partial_j \rho \Big) = -\partial_k \Big(\epsilon^{ij} \partial_j \Pi^k + \delta^{ij} \epsilon^{kl} \delta_{lm} \partial_j \Pi^m \Big) + \partial_k \Big(\delta^{ik} \epsilon^{jl} \partial_j \Pi_l \Big). \tag{4.3}$$

This identity will play a central role here and in the following section. To derive the equation (4.3), we have used that $\partial_k j^k = \partial_k P^k = \partial_k \Pi^k$ and that the last two terms on the right hand side add up to zero. If we define the velocity flow as $v_i = \Pi_i / \rho$, equation (4.3) becomes:

$$\partial_t \Big(\epsilon^{ij} \partial_j \rho \Big) = -\partial_k \Big[\rho \, \delta_{ij} \eta_H^{-1} \tau_H^{jk} + \left(\delta_{il} \epsilon^{lj} v^k + \delta_i^j \epsilon^{kl} v_l \right) \partial_j \rho - \delta_i^k \epsilon^{jl} \partial_j (\rho v_l) \Big], \tag{4.4}$$

where τ_H^{ik} is the odd viscosity term defined in (1.47), i.e.,

$$\tau_H^{ik} = \eta_H \left(\epsilon^{ij} \partial_j v^k + \epsilon^{kl} \delta^{ij} \partial_j v_l \right).$$

Therefore, the addition of $\frac{1}{2} \bar{s} \, \delta_{jk} \epsilon^{kl} \partial_l \rho$ to the perfect fluid momentum density will automatically introduce the odd viscosity to the system, with:

$$\eta_H = \frac{1}{2} \rho \bar{s} \, . \tag{4.5}$$

Equivalently, if we start with a system with non-vanishing odd viscosity term, the momentum redefinition:

$$P_i \to P_i + \lambda \delta_{jk} \epsilon^{kl} \partial_l \rho,$$

with always shift the value of the odd viscosity by

$$\eta_H \to \eta_H + \lambda \rho.$$

From this argument, we obtain that the Hall viscosity is only defined mod-

[1]This assumption is not necessary though.

ulo a real constant times the fluid density. Since λ can be associated to the average internal angular momentum per fluid particle, we could fix such ambiguity by knowing about the microscopic details of system. In terms of a phenomenological hydrodynamic model, the parameter λ can only be determined for a charged fluid. For a charged fluid, the spin density couples to the magnetic field and its effect can be probed through non-uniform configurations of the external field. However, for a neutral fluid in flat space, the spin density can only be obtained by knowing the microscopic detail of the system.

The question we will address throughout this chapter is whether we can lift this ambiguity by fixing λ in a phenomenological hydrodynamic model.

4.2 Gradient Corrections to the Perfect Fluid Action

In this section, we will formulate the variational principle for the hydrodynamics with odd-parity terms. We will show how the ambiguity in the definition of the odd viscosity is manifested in this formulation and propose a phenomenological way to resolve this ambiguity. The odd viscosity term in the stress tensor is dissipationless and is of first order in derivative expansion. Let us thus assume that it can be reproduced by adding first order derivative terms in the hydrodynamic action:

$$S = \int \left[\rho\, \xi_0 + \frac{\rho}{2}\delta^{ik}\xi_i\xi_k - \varepsilon(\rho) + \epsilon^{ik} f(\rho)\, \partial_i \xi_k\right] \mathrm{d}t\, \mathrm{d}^2x, \tag{4.6}$$

where ξ_0 and ξ_i are defined in (3.5), $\varepsilon(\rho)$ is the internal energy density and $f(\rho)$ is a general function of the particle density ρ. Differently from the previous chapter, ξ_i now denotes the flow velocity v_i and not the dynamic velocity μu_i.

Notice, that we can add another parity odd term to the hydrodynamic action:

$$\int g(\rho)\epsilon^{ik}\xi_i\partial_j\rho\, \mathrm{d}t\, \mathrm{d}^2x.$$

However, integrating this term by parts, one sees that it only redefines the function $f(\rho)$ by:

$$f(\rho) \to f(\rho) + \int g(\rho)\mathrm{d}\rho.$$

Equations of motion for the action (4.6) are given by:

$$\theta: \qquad \partial_t \rho + \partial_i \left(\rho \xi^i\right) = 0\,, \tag{4.7}$$

$$\beta: \qquad \partial_t \alpha + \left(\xi^i + \frac{\epsilon^{ik}}{\rho}\partial_k f\right)\partial_i \alpha = 0\,, \tag{4.8}$$

$$\alpha: \qquad \partial_t \beta + \left(\xi^i + \frac{\epsilon^{ik}}{\rho}\partial_k f\right)\partial_i \beta = 0\,, \tag{4.9}$$

$$\rho: \qquad \xi_0 + \frac{\xi^i \xi_i}{2} - \varepsilon'(\rho) + \epsilon^{ik} f'(\rho)\,\partial_i \xi_k = 0\,. \tag{4.10}$$

Momentum conservation[2] follows from equations (4.7-4.10):

$$\partial_t \left(\rho \xi_i\right) = -\partial_k \left(\rho \xi^k \xi_i + \delta_i^k \mathcal{P}\right), \tag{4.11}$$

where we have defined the modified pressure $\mathcal{P}$ as:

$$\mathcal{P} \equiv \varepsilon - \rho\varepsilon' + \rho f' \epsilon^{jl} \partial_j \xi_l\,. \tag{4.12}$$

The expression (4.11) does not have any odd viscosity contribution, but instead has a parity odd term in the expression for pressure (4.12). As we have seen in the previous section, by redefining momentum and using the identity (4.3) one can trade the odd contribution to the pressure for odd viscosity. Here we will present a heuristic way to resolve the ambiguity by requiring the odd contribution to the pressure vanish. Let us consider a fluid configuration with constant density $\rho = \bar{\rho}$ slowly rotating as a rigid body with fixed angular velocity. On one hand the corresponding displacement field does not change any interparticle distances in a fluid. On the other hand the odd correction to the pressure is non-vanishing. We argue that this is unnatural and means that our definition of the velocity of the fluid (more precisely the relation of the velocity with the linear momentum) is incorrect and should be corrected so that the odd term in the pressure disappears.

We can then modify the momentum density in the same way as described in the last section, i.e.,

$$\Pi_i = \rho \xi_i + \lambda \delta_{ij} \epsilon^{jk} \partial_k \rho.$$

[2] In the equation (4.11), we have used the following identity:

$$\epsilon^{jk}\left(\partial_i f \partial_j \xi_k + \partial_j f \partial_k \xi_i - \partial_j f \partial_i \xi_k\right) = 0.$$

.

The new momentum density satisfies the following conservation equation:

$$\partial_t \Pi_i = -\partial_k \left[\frac{\Pi_i \Pi^k}{\rho} + \delta_i^k \, \mathcal{P} + \rho\lambda \left(\delta_{il} \epsilon^{lj} \partial_j \xi^k + \epsilon^{kj} \partial_i \xi_j \right) + \mathcal{O}(\partial^2) \right], \qquad (4.13)$$

where

$$\mathcal{P} = \varepsilon - \rho\varepsilon' + \rho(f' - \lambda)\epsilon^{jl} \partial_j \xi_l \,. \qquad (4.14)$$

In principle, we can assign any real value to λ, given that it cannot be fixed by any symmetry arguments. Therefore, there is a family of stress tensors labeled by a parameter λ. Although in terms of equations all values of λ are possible, its value should be fixed by the microscopic theory. The only information about the microscopic details of the system is encoded into the phenomenological function $f(\rho)$. Using the argument which odd contribution to pressure must vanish at constant density, we obtain:

$$\lambda = f'(\bar{\rho}). \qquad (4.15)$$

And the odd viscosity is given by $\eta_H = f(\bar{\rho})\rho$. It would be ideal to have a more precise mathematical formulation of this argument. In the following sections, we will show that the knowledge of how the fluid couples to an external geometric background (c.f. section 3.3) allows one to fix definitions of stress and momentum unambiguously. As the considered hydrodynamic system is non-relativistic, the proper geometric background is the so-called Newton-Cartan geometry, which will be discussed in the following section.

4.3 Newton-Cartan Geometry

In this section, we will review the main features of the Newton-Cartan (NC) geometry. For reviews, check [56–58]. We will restrict our discussion to (2+1)-dimensional spacetimes. In Newtonian physics, time and space are separate concepts, all observers measure the same time. Thus, the non-relativistic or Galilean spacetime is simply given by the cartesian product[3] $\mathbb{R} \times \mathbb{E}^2$, where $\mathbb{R}$ "labels the time" and $\mathbb{E}^2 \cong (\mathbb{R}^2, \delta)$ is the Euclidean plane, i.e., the plane $\mathbb{R}^2$ endowed with the Euclidean metric δ.

In complete analogy to the Riemannian geometry, a NC manifold $\mathcal{M}$ is a spacetime that locally resembles the Galilean spacetime $\mathbb{R} \times \mathbb{E}^2$. In fact, the tangent space $T_x\mathcal{M}$ at any point $x \in \mathcal{M}$ is isomorphic to $\mathbb{R} \times \mathbb{E}^2$. The development of the NC geometry in $(2 + 1)$ dimensions is out of the scope

[3] In mathematical jargon, this defines a trivial bundle, where time is base manifold and the space is the fiber.

of this section. Therefore, we will assume some knowledge about differential geometry. Let us introduce the set of vielbeins $\{E^\mu_a\}$ and their inverses $\{e^a_\mu\}$, such that,

$$E^\mu_a e^b_\mu = \delta^b_a, \tag{4.16}$$
$$E^\mu_a e^a_\nu = \delta^\mu_\nu. \tag{4.17}$$

Here, all indices run from 0 to 2 and one can think of Latin indices living on the tangent space and Greek ones on the spacetime $\mathcal{M}$. The metric structure on each tangent space induces the metric structure on the corresponding point on spacetime. Since the tangent-space metric is defined solely to spatial sections, it induces a degenerate metric on $\mathcal{M}$, given by:

$$\delta_{AB}\, e^A_\mu e^B_\nu = h_{\mu\nu}. \tag{4.18}$$

Indices A and B in equation (4.18) label spatial components on $T_x\mathcal{M}$ and assume only values 1 and 2. The lower-index metric $h_{\mu\nu}$ is degenerate and admits no inverse. However, we can still define an upper-index metric tensor as:

$$\delta^{AB} E^\mu_A E^\nu_B = h^{\mu\nu}. \tag{4.19}$$

Since the time direction is special in $T_x\mathcal{M}$, let us denote E^μ_0 and e^0_μ by v^μ and n_μ respectively. Using the definitions (4.16 - 4.19), we obtain the following constraints among NC background fields:

$$n_\mu v^\mu = 1, \tag{4.20}$$
$$n_\mu h^{\mu\nu} = n_\nu h^{\mu\nu} = 0, \tag{4.21}$$
$$v^\mu h_{\mu\nu} = v^\nu h_{\mu\nu} = 0, \tag{4.22}$$
$$h_{\lambda\mu} h^{\mu\nu} = \delta^\nu_\lambda - n_\lambda v^\nu. \tag{4.23}$$

Using the symmetry of $h_{\mu\nu}$ and $h^{\mu\nu}$, we find the total number of geometrical variables[4] to be 18. However, not all of them are independent; in fact, equation (4.23) states for example that $h_{\mu\nu}$ can be expressed in terms of $h^{\mu\nu}$, v^μ and n_μ. Out of the 12 components left, equations (4.21) and (4.20) reduces the total number of independent ones by 8. This will be important in the next section where we will study the invariance of the hydrodynamic action under diffeomorphisms.

The spin connection and the affine connection are defined by imposing that

[4]They are 6 from $h_{\mu\nu}$, 6 from $h^{\mu\nu}$, 3 from v^μ and 3 from n_μ.

n_μ and e^A_μ are covariantly constant[5], that is:

$$\nabla_\mu e^A_\nu \equiv \partial_\mu e^A_\nu + \delta^{AB}\epsilon_{BC}\,\omega_\mu e^C_\nu - \Gamma^\kappa{}_{\mu\nu}\, e^A_\kappa = 0, \tag{4.24}$$

$$\nabla_\mu n_\nu \equiv \partial_\mu n_\nu - \Gamma^\kappa{}_{\mu\nu} n_\kappa = 0. \tag{4.25}$$

Notice that in equation (4.24) we have completely stripped off the dependence on tangent-space indices from the spin connection, i.e.,

$$(\omega_\mu)^A{}_B = \delta^{AC}\epsilon_{CB}\,\omega_\mu.$$

Since the spin connection ω_μ depends only on spacetime indices, it can appear explicitly on the hydrodynamic action as part of the NC geometrical background. This is only possible in $(2+1)$ dimensions and will be important in the definition of odd viscosity.

Under local frame rotation the spin connection transforms as:

$$\omega_\mu \to \omega_\mu + \partial_\mu \varphi, \tag{4.26}$$

where φ is the angle of rotation. The gauge invariant quantity made out from the spin connection is curvature 2-form, defined as:

$$\mathcal{R}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu. \tag{4.27}$$

The curvature 2-form is related to the Riemann curvature through the following expression:

$$\mathcal{R}_{\mu\nu} = n_\kappa \epsilon^{\kappa\lambda\sigma} h_{\lambda\tau} R^\tau{}_{\sigma\mu\nu} = n_\kappa \epsilon^{\kappa\lambda\sigma} h_{\mu\tau} R^\tau{}_{\nu\lambda\sigma}. \tag{4.28}$$

4.4 Hydrodynamics in Newton-Cartan Background

In section 3.3 we have derived momentum and energy conservations from the general covariance of the the action, that is, $\delta_\zeta S = 0$. However, the action (4.6) is not invariant under general coordinate transformation. Because time is a special coordinate in Newtonian physics, non-relativistic actions are usually written separating time and spatial coordinates. In order to write them

[5]In absence of torsion, the affine connection can be written as:

$$\Gamma^\mu{}_{\nu\rho} = v^\mu \partial_\rho n_\nu + \tfrac{1}{2} h^{\mu\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}).$$

covariantly, we need to introduce the NC geometric background discussed in the previous section. For example, let us focus on the action (4.6). Since ξ_0 corresponds to the time component of a spacetime one-form, we can rewrite it as the projection of ξ_μ onto a "unit vector" that points towards the direction of time:

$$\xi_0 \to v^\mu \xi_\mu \,.$$

Equivalently, the Euclidean metric and the spatial Levi-Civita tensor must be replaced by:

$$\delta^{ik} \to h^{\mu\nu},$$
$$\epsilon^{ik} \equiv \epsilon^{0ik} \to n_\mu \epsilon^{\mu\nu\lambda} \,.$$

To derive the covariant version of the volume form $\mathrm{d}^2x\,\mathrm{d}t$, we must use that the volume in 3 dimensions can be obtained from the triple product of the basis vectors, that is,

$$\mathrm{d}^2x\,\mathrm{d}t \to \epsilon^{\mu\nu\lambda}\, n_\mu e^1_\nu e^2_\lambda \,\mathrm{d}^2x\,\mathrm{d}t.$$

It is not hard to see that:

$$\epsilon^{\mu\nu\lambda}\, n_\mu e^1_\nu e^2_\lambda = \sqrt{\det\left(h_{\mu\nu} + n_\mu n_\nu\right)}. \tag{4.29}$$

Therefore, the naive covariant version of the action (4.6) can be written as:

$$S = \int_{\mathcal{M}} \left[\rho\, v^\mu \xi_\mu + \frac{1}{2}\rho\, h^{\mu\nu}\xi_\mu\xi_\nu - \varepsilon(\rho) + f(\rho)\epsilon^{\mu\nu\lambda}\, n_\mu \partial_\nu \xi_\lambda\right] \mathrm{d}V, \tag{4.30}$$

where $\mathrm{d}V \equiv \sqrt{\det\left(h_{\mu\nu} + n_\mu n_\nu\right)}\,\mathrm{d}^2x\,\mathrm{d}t$. However, we can always add to the action (4.30) terms which vanish when we set the NC background to be flat, such as terms proportional to the curvature of $\mathcal{M}$. In fact, there is no unique way to make the action (4.6) covariant.

As we pointed out in the previous section, the spin connection can appear explicitly on the hydrodynamic action. For a flat background we have $\mathcal{R}_{\mu\nu} = 0$, and ω_μ becomes pure gauge. The gauge transformation (4.26) restricts the possible appearance of the spin connection in the action. If the frame rotation affects no hydrodynamic variables, ω_μ can only appear[6] in the form $\mathcal{R}_{\mu\nu}$. However, $\mathcal{R}_{\mu\nu}$ can be fully expressed in terms of $h_{\mu\nu}$, $h^{\mu\nu}$, v^μ and n_μ, and the introduction of the ω_μ in the action is indeed spurious.

[6] Technically, it could also appear in terms of the form $\int \epsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu a_\lambda \mathrm{d}V$, however they can be expressed in terms of $\mathcal{R}_{\mu\nu}$ after integration by parts.

In analogy to (3.16), let us consider the case where θ transform under frame rotation:

$$\theta \to \theta - \bar{s}\varphi. \tag{4.31}$$

We have introduced the constant $\bar{s}$, so that the invariant quantity is given by $\xi_\mu + \bar{s}\,\omega_\mu$.

4.5 Symmetries in Newton-Cartan Geometry

Here, we will repeat the steps from section 3.3 to derive momentum, energy and spin conservation laws for a non-relativistic 2-dimensional fluid. Let us start from spin conservation. Since only θ and ω_μ transforms under frame rotation:

$$\delta_\varphi S = -\int_{\mathcal{M}} \partial_\lambda \varphi \left(\bar{s}\frac{\delta S}{\delta \xi_\lambda} - \frac{\delta S}{\delta \omega_\lambda} \right) \mathrm{d}^2x\,\mathrm{d}t. \tag{4.32}$$

If we restrict ourselves to field configurations that satisfy equation of motion, we obtain:

$$\partial_\mu \left(\frac{1}{\sqrt{h}} \frac{\delta S}{\delta \omega_\mu} \right) = 0. \tag{4.33}$$

In the last equation, we have denoted $h \equiv \det(h_{\mu\nu} + n_\mu n_\nu)$. Equation (4.33) shows that the spin current is conserved, as a consequence of local frame rotations.

Let us now turn our attention to diffeomorphisms. As we have already discussed, not all the NC fields are independent and the variation of $h_{\mu\nu}$ and n_ν can be expressed in terms of variation of $h^{\mu\nu}$ and v^μ:

$$\delta n_\mu = -n_\mu n_\nu \delta v^\nu - \frac{1}{2}\left(h_{\mu\lambda} n_\nu + h_{\nu\mu} n_\lambda\right) \delta h^{\nu\lambda}, \tag{4.34}$$

$$\delta h_{\mu\nu} = -\left(h_{\nu\lambda} n_\mu + h_{\mu\lambda} n_\nu\right) \delta v^\lambda - h_{\mu\kappa} h_{\nu\lambda}\, \delta h^{\kappa\lambda}. \tag{4.35}$$

Therefore, on solutions of the equations of motion, the variation of the action with respect to diffeomorphisms is given by:

$$\begin{aligned} \delta_\zeta S = \int_{\mathcal{M}} \Bigg\{ & \left[\frac{\delta S}{\delta v^\mu} - n_\mu n_\nu \frac{\delta S}{\delta n_\nu} - \left(h_{\nu\mu} n_\lambda + h_{\mu\lambda} n_\nu\right) \frac{\delta S}{\delta h_{\nu\lambda}} \right] (\mathcal{L}_\zeta v)^\mu + \frac{\delta S}{\delta \omega_\mu} (\mathcal{L}_\zeta \omega)_\mu + \\ & \left[\frac{\delta S}{\delta h^{\mu\nu}} - \frac{1}{2}\left(h_{\mu\lambda} n_\nu + h_{\nu\lambda} n_\mu\right) \frac{\delta S}{\delta n_\lambda} - h_{\mu\kappa} h_{\nu\lambda} \frac{\delta S}{\delta h_{\kappa\lambda}} \right] (\mathcal{L}_\zeta h)^{\mu\nu} \Bigg\} \mathrm{d}^2x\,\mathrm{d}t. \end{aligned} \tag{4.36}$$

The Lie derivative $\mathcal{L}_\zeta$ acting on the background fields gives us:

$$(\mathcal{L}_\zeta v)^\mu \equiv \zeta^\nu \nabla_\nu v^\mu - v^\nu \nabla_\nu \zeta^\mu = -v^\nu \nabla_\nu \zeta^\mu, \tag{4.37}$$

$$(\mathcal{L}_\zeta \omega)_\mu = \zeta^\nu \mathcal{R}_{\nu\mu} + \nabla_\mu (\zeta^\nu \omega_\nu), \tag{4.38}$$

$$(\mathcal{L}_\zeta h)^{\mu\nu} = -h^{\nu\lambda} \nabla_\lambda \zeta^\mu - h^{\mu\lambda} \nabla_\lambda \zeta^\nu. \tag{4.39}$$

Imposing that $\delta_\zeta S = 0$ and using equation (4.33), we find:

$$\int_{\mathcal{M}} \zeta^\mu \left(\nabla_\nu T^\nu{}_\mu + \frac{1}{\sqrt{h}} \mathcal{R}_{\mu\nu} \frac{\delta S}{\delta \omega_\nu} \right) \mathrm{d}V = 0, \tag{4.40}$$

where

$$T^\nu{}_\mu = \frac{1}{\sqrt{h}} \left(v^\nu \frac{\delta S}{\delta v^\mu} + 2h^{\kappa\nu} \frac{\delta S}{\delta h^{\kappa\mu}} - n_\mu \frac{\delta S}{\delta n_\nu} - 2h_{\mu\kappa} \frac{\delta S}{\delta h_{\nu\lambda}} \right). \tag{4.41}$$

The curvature 2-form appears in the equation for the momentum conservation in the same way as the Lorentz force for a charged fluid. However, differently from the latter, the second term in (4.40) can be rewritten as total divergence. For that, let us consider the equation (4.28):

$$\begin{aligned} \frac{1}{\sqrt{h}} \mathcal{R}_{\mu\nu} \frac{\delta S}{\delta \omega_\nu} &= n_\kappa \epsilon^{\kappa\lambda\sigma} h_{\mu\tau} R^\tau{}_{\nu\lambda\sigma} \frac{1}{\sqrt{h}} \frac{\delta S}{\delta \omega_\nu}, \\ \frac{1}{\sqrt{h}} \mathcal{R}_{\mu\nu} \frac{\delta S}{\delta \omega_\nu} &= n_\kappa \epsilon^{\kappa\lambda\sigma} h_{\mu\nu} \Big[\nabla_\lambda \, , \nabla_\sigma \Big] \left(\frac{1}{\sqrt{h}} \frac{\delta S}{\delta \omega_\nu} \right). \end{aligned} \tag{4.42}$$

In the last line, we have used the definition of the Riemann tensor. Therefore, equation (4.40) becomes:

$$\nabla_\nu \left[T^\nu{}_\mu + 2 n_\kappa \epsilon^{\kappa\nu\sigma} h_{\mu\lambda} \nabla_\sigma \left(\frac{1}{\sqrt{h}} \frac{\delta S}{\delta \omega_\lambda} \right) \right] = 0. \tag{4.43}$$

Chapter 5

Future Directions: Transport Properties of Topological Surface States

The fingerprint of topological materials is the existence of protected surface states. In addition to the fundamental interest in states from Fermi arcs, their signature in the magnetotransport appears in small size samples. For example, surface modes in the SdH oscillations were observed in [33]. The Fermi arc contribution to quantum oscillations depends on the sample size and was estimated in [59], using semiclassical arguments. The surface contribution to the conductivity was estimated as:

$$\delta\sigma \sim \cos\left[2k_F\left(\frac{k_0 eB}{\hbar\cos\theta} + L\right) \pm \gamma\pi\right], \tag{5.1}$$

where k_0 is the Fermi arc length in k-space, γ is a phase shift and θ is the angle between the magnetic field and the surface normal vector. In addition to that, in very clean samples the states at Fermi arcs may serve as the main chirality relaxation mechanism.

5.1 Chiral Kinetic Theory with Fermi Arcs

For usual metals, boundary effects are washed out by the bulk properties of the materials. Therefore, one should expect to see transport signatures of surface states only for very thin samples. In this spirit, the Fermi arc contribution for the SdH oscillations was proposed and estimated in [59]. Their prediction was qualitatively confirmed later in samples of Cd_3As_2 [33]. To develop a quantitative theory of an interplay between bulk anomaly in Weyl/Dirac metals and contributions from surface Fermi arc states the chiral kinetic theory should be modified by respective boundary terms. This generalized chiral kinetic theory with surface states should also account for the chirality relaxation due to Fermi arcs.

Our goal is to extend the analysis developed in the chapter 2 to account for the surface states discussed in the section 1.2.3. This extension will in highlight the connection between Fermi arcs and the Atiyah-Patodi-Singer index theorem, as suggested in [60], and will serve as a basis for the chiral kinetic theory with Fermi arcs.

5.2 Boundary Modes in Anomalous Hydrodynamics

As an alternative approach to kinetic theory we also propose to consider the regime of strong interactions where hydrodynamics can be applicable. The fundamental question is how to impose realistic boundary conditions on anomalous hydrodynamic equations. Due to the topological nature of such surface states, we expect that it is possible to impose boundary conditions consistently so that the strongly interacting physics corresponding to Fermi arcs would appear as hydrodynamic surface modes derivable from anomalous hydrodynamics in the bulk with correspondent boundary conditions.

In order to account for surface modes, it is convenient to start from an effective action for anomalous hydrodynamics. In the presence of a boundary, the gauge transformation of the action (3.17) acquires edge contributions:

$$\begin{aligned}\delta_\Lambda S \; &= \frac{C}{24}\int_{\mathcal{M}} \Lambda\, \epsilon^{\lambda\nu\eta\sigma}\, F_{\lambda\nu}\, F_{\eta\sigma}\, \mathrm{d}^4 x \\ &\quad + \frac{C}{6}\int_{\partial\mathcal{M}} \Lambda\, \epsilon^{\lambda\nu\eta\sigma}\, (\xi_\nu - A_\nu)\, \partial_\eta\, (\xi_\sigma + A_\sigma)\, \mathrm{d}\Sigma_\lambda .\end{aligned} \tag{5.2}$$

We notice that the field θ appears explicitly in the hydrodynamic action even for static (time-independent) background gauge fields. This means that this field becomes physical and correspond to physical degrees of freedom of the fluid. Therefore, terms containing ξ_i in second line of (5.2) have to be canceled by the gauge variation of a boundary action. This boundary action has to be of the form:

$$S_{\partial\mathcal{M}} = \frac{C}{6}\int_{\partial\mathcal{M}} \epsilon^{\lambda\nu\eta\sigma}\, \partial_\nu\theta\, A_\eta\, \xi_\sigma\, \mathrm{d}\Sigma_\lambda + \text{gauge invariant terms}, \tag{5.3}$$

since

$$\delta_\Lambda(S + S_{\partial\mathcal{M}}) = \frac{C}{24}\int_{\mathcal{M}} \Lambda\, \epsilon^{\lambda\nu\eta\sigma}\, F_{\lambda\nu}\, F_{\eta\sigma}\, \mathrm{d}^4x - \frac{C}{6}\int_{\partial\mathcal{M}} \Lambda\, \epsilon^{\lambda\nu\eta\sigma} A_\nu \partial_\eta A_\sigma \mathrm{d}\Sigma_\lambda. \quad (5.4)$$

The field θ becomes physical even for time-independent gauge fields and appears explicitly in the action. Nevertheless, the full action is still invariant to reparametrizations of Clebsch potentials which leave ξ_i invariant.

Let us focus on anomaly terms in the expression for the consistent current:

$$\begin{aligned} \frac{\delta S}{\delta A_\lambda} = \; & \ldots + \frac{C}{6}\epsilon^{\lambda\nu\eta\sigma}\left[\xi_\nu\, \partial_\eta \xi_\sigma + \frac{1}{2}\xi_\nu\, F_{\eta\sigma} + \partial_\nu \left(A_\eta\, \xi_\sigma\right)\right] \\ & + \delta_{\partial\mathcal{M}}(x)\, \frac{C}{6}\epsilon^{\lambda\nu\eta\sigma} n_\nu \left(\partial_\eta\theta - A_\eta\right) \xi_\sigma. \end{aligned} \quad (5.5)$$

In the second line of (5.5), $\delta_{\partial\mathcal{M}}(x)$ denotes the delta function on the manifold boundary. If we compare the consistent charge density, $\delta S/\delta A_0$, with the last term in (3.30), we find that the surface charge density is given by the normal component of the fluid electric polarization.

The extension of this model to two coexistent chiralities and surface hydrodynamic modes can in principle model the transport in type-I Weyl semimetals with Fermi arcs. Since the Coulomb interaction breaks the emergent Lorentz symmetry and hydrodynamic behavior is expected at large temperatures, the non-relativistic generalization of the effective action for finite temperature might become necessary to allow for comparisons with experiments[1].

[1]This is true for effects beyond linear response, where the non-linear nature of hydrodynamics becomes important.

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Appendix A

Poisson Summation Formula and SdH Effect

In this section, we will apply the Bohr-Sommerfeld quantization prescription to introduce quantum effects in the conductivity. Assuming that the only contribution to transport comes from the discrete levels, the conductivity per chirality becomes:

$$\begin{aligned}\sigma_{zz}^{(\chi)} = &-\frac{e^2}{4\pi^2}\sum_{\nu=-\infty}^{\infty}\int \mathrm{d}k \int_{-1}^{1} \mathrm{d}(\cos\theta)\, k^2 \frac{\partial f_0}{\partial \varepsilon}(\cos\theta + \chi\zeta_k) \\ &\times \delta[\nu - \tfrac{1}{4}\left(\sin^2\theta/\zeta_k - 2\chi\cos\theta\right)] v_k\, g_z, \end{aligned} \tag{A.1}$$

where g_z is given in (2.40). We have used that there is no real solution for (2.20) when $\nu \in \mathbb{Z}_-$. Therefore, all surfaces for negative integer ν are outside of the integration range. The integral over k is performed near Fermi surface.

Using the Poisson formula,

$$\sum_{\nu=-\infty}^{\infty} \delta(x-\nu) = \sum_{l=-\infty}^{\infty} e^{i2\pi l\nu},$$

the conductivity can be rewritten as:

$$\sigma_{zz}^{(\chi)} = -\frac{e^2}{4\pi^2}\sum_{l=-\infty}^{\infty}\left(\frac{\mathcal{I}_1^{(l)}}{i\omega + \tau_v^{-1}} + \frac{\mathcal{I}_2^{(l)}}{i\omega + \tau^{-1}}\right), \tag{A.2}$$

where,

$$\mathcal{I}_1^{(l)} = \int \mathrm{d}k\, k^2 v_k^2 \,\frac{\partial f_0}{\partial \varepsilon}\, \chi\zeta_k \, e^{i\frac{\pi l}{2}(\zeta_k^{-1}+\zeta_k)} \qquad \text{(A.3)}$$
$$\times \int_{-1}^{1} \mathrm{d}(\cos\theta)(\cos\theta + \chi\zeta_k)\, e^{-i\frac{\pi l}{2}(\cos\theta+\chi\zeta_k)^2},$$
$$\mathcal{I}_2^{(l)} = \int \mathrm{d}k\, k^2 v_k^2 \,\frac{\partial f_0}{\partial \varepsilon}\,(1-\zeta_k^2)\, e^{i\frac{\pi l}{2}(\zeta_k^{-1}+\zeta_k)} \qquad \text{(A.4)}$$
$$\times \int_{-1}^{1} \mathrm{d}(\cos\theta)\frac{\cos\theta + \chi\zeta_k}{1+\chi\zeta_k\cos\theta}\,\cos\theta\, e^{-i\frac{\pi l}{2}(\cos\theta+\chi\zeta_k)^2}.$$

The integral in equation (A.3) accounts for the intervalley scattering and can be easily calculated:

$$\mathcal{I}_1^{(l)} = -\left(\frac{2}{\hbar} v_F k_F^2 \zeta_F^2\right)\delta_{l,0}\,. \qquad \text{(A.5)}$$

In the equation above, we have used that

$$\frac{\partial f_0}{\partial \varepsilon} \approx -\frac{\delta(k-k_F)}{\hbar v_F}.$$

The terms with $l \neq 0$ vanish since the chirality relaxation mechanism that we have considered only accounts for the scattering between the zero-modes.

Let us now consider the contribution for the intravalley scattering coming from equation (A.4). For $l = 0$ the integral can be performed analytically, however, we are only interested in the range where the semiclassical picture is valid. If we restrict ourselves terms up to $\mathcal{O}(\zeta_F^2)$, we end up with:

$$\mathcal{I}_2^{(0)} = -\frac{2}{\hbar} v_F k_F^2 \left[\frac{1}{3} - \frac{7}{15}\zeta_F^2 + \mathcal{O}(\zeta_F^4)\right]. \qquad \text{(A.6)}$$

In order to calculate $\mathcal{I}_2^{(l)}$ for $l \neq 0$, it is convenient to define $x = \cos\theta + \chi\zeta_k$. Expanding the integrand up to $\mathcal{O}(\zeta_k^2)$, we find that:

$$\mathcal{I}_2^{(l)} = \int \mathrm{d}k\, k^2\, v_k^2 \,\frac{\partial f_0}{\partial \varepsilon}\, \exp\left[i\frac{\pi l}{2}\,(\zeta_k - 1/\zeta_k)\right] \qquad \text{(A.7)}$$
$$\times\left[-\chi\zeta_k\mathcal{Q}_1 + (1+\zeta_k^2)\mathcal{Q}_2 - \zeta_k\chi\mathcal{Q}_3 + \zeta_k^2\mathcal{Q}_4\right],$$

where

$$\mathcal{Q}_m \equiv \int_{-1+\chi\zeta_k}^{1+\chi\zeta_k} \mathrm{d}x\, x^m \, \exp\left(-i\frac{\pi l}{2\zeta_k}x^2\right). \tag{A.8}$$

Solving for odd values of m:

$$\mathcal{Q}_1 = \frac{2\zeta_k}{\pi l}\, e^{-i\frac{\pi l}{2\zeta_k}(\zeta_k^{-1}+\zeta_k)} \sin(\pi l \chi) = 0,$$
$$\mathcal{Q}_3 = \frac{4\chi i(-1)^l \zeta_k^2}{\pi l}\, e^{-i\frac{\pi l}{2}(\zeta_k^{-1}+\zeta_k)}.$$

However, $\zeta_k \mathcal{Q}_3 = \mathcal{O}(\zeta_k^3)$ and such term can be neglected. Let us now focus on m even. They can all be obtained through $\mathcal{Q}_0$ as follows:

$$\mathcal{Q}_m = \left(\frac{2i\zeta_k}{\pi}\right)^{m/2} \frac{d^{m/2}}{dl^{m/2}}\mathcal{Q}_0\,,$$

where l is set to be a non-zero integer at the end of the calculation. Clearly, $\mathcal{Q}_4 = \mathcal{O}(\zeta_k^3)$ and we only need to calculate $\mathcal{Q}_2$. The integral $\mathcal{Q}_0$ can be written as:

$$\mathcal{Q}_0 = \int_0^{1+\chi\zeta_k} \mathrm{d}x\, e^{-i\frac{\pi l}{2\zeta_k}x^2} + \int_0^{1-\chi\zeta_k} \mathrm{d}x\, e^{-i\frac{\pi l}{2\zeta_k}x^2}. \tag{A.9}$$

Let us focus on the right hand side of equation (A.9). Thus,

$$\int_0^{1\pm\chi\zeta_k} \mathrm{d}x\, e^{-i\frac{\pi l}{2\zeta_k}x^2} = \frac{1}{2}\sqrt{\frac{2\zeta_k}{il}} - \frac{1}{2}\sqrt{\frac{2\zeta_k}{\pi l}} F_0\left(\tfrac{\pi l(1\pm\chi\zeta_k)^2}{2\zeta_k}\right),$$

where we have defined:

$$F_m(t) = \int_t^{\infty} \mathrm{d}y\, y^{-m-1/2} e^{-iy}\,.$$

After integration by parts, one can show that:

$$F_m(t) = -i\frac{e^{-it}}{t^{m+1/2}} + i\left(m+\frac{1}{2}\right) F_{m+1}(t).$$

Since we are restricting ourselves to terms up to $\mathcal{O}(\zeta_k^2)$, we obtain:

$$\mathcal{Q}_0 = \sqrt{\frac{2\zeta_k}{il}} + \frac{2\zeta_k(-1)^l}{\pi l}\frac{e^{-i\frac{\pi l}{2}(\zeta_k^{-1}+\zeta_k)}}{1-\zeta_k^2}\,,$$
$$\mathcal{Q}_2 = -\frac{e^{\frac{i\pi}{4}}}{\pi}\sqrt{\frac{2\zeta_k^3}{l^3}} + \frac{2\zeta_k(-1)^l}{\pi l}e^{-i\frac{\pi l}{2}(\zeta_k^{-1}+\zeta_k)}\left(i+\frac{2\zeta_k}{\pi l}\right).$$

Plugging all determined values for $\mathcal{Q}_m$ into (A.7):

$$\mathcal{I}_2^{(l)} = -\frac{2k_F^2 v_F\zeta_F}{\hbar\pi l}e^{i\pi l}\left(i+\frac{2\zeta_F}{\pi l}\right) - \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{\hbar\pi l^{3/2}}\int \mathrm{d}\varepsilon\, k^2 v_k \zeta_k^{-3/2}\frac{\partial f_0}{\partial\varepsilon}e^{i\frac{\pi l}{2}(\zeta_k^{-1}+\zeta_k)}\,. \tag{A.10}$$

Here we have used that we can invert the dispersion relation and write $k(\varepsilon)$. The energy integral is performed at the vicinity of the Fermi surface. Since we assume that $T/\varepsilon_F \ll 1$, all the integrand besides the oscillating exponential is consider to vary slowly in the temperature range. In addition to that, we must expand the exponent near the Fermi energy. Keeping only linear deviations in the exponent, we are left with:

$$\mathcal{I}_2^{(l)} \approx -\frac{2k_F^2 v_F\zeta_F}{\hbar\pi l}e^{i\pi l}\left(i+\frac{2\zeta_F}{\pi l}\right) - \frac{e^{i\frac{\pi}{4}(1+2l\zeta_F^{-1})}}{2\hbar\pi}k_F^2 v_F\left(\frac{2\zeta_F}{l}\right)^{3/2}\int_{-\infty}^{\infty}\mathrm{d}t\frac{e^{(1+i\lambda l/\pi)t}}{(e^t+1)^2}. \tag{A.11}$$

In the equation (A.11), we have defined $t=(\varepsilon-\mu)/T$ and $\lambda = \pi^2 T/(\hbar v_F k_F\zeta_F)$. The integral can be solved using the residue theorem, and its value is given by:

$$\int_{-\infty}^{\infty}\frac{e^{(1+i\lambda l/\pi)t}}{(e^t+1)^2}\mathrm{d}t = \frac{\lambda l}{\sinh\lambda l}.$$

Therefore, the conductivity can be expressed as:

$$\sigma_{zz}^{(\chi)} = \sigma_{zz}^{(0)} + 2\sum_{l=1}^{\infty}\sigma_{zz}^{(l)}\cos\left(\frac{\pi l}{2\zeta_F}+\frac{\pi}{4}\right), \tag{A.12}$$

where

$$\sigma_{zz}^{(0)} = \frac{n_\chi e^2 v_F}{\hbar k_F} \left[\frac{3\zeta_F^2}{i\omega + \tau_v^{-1}} + \frac{1}{i\omega + \tau^{-1}} \left(1 - \frac{12}{5}\zeta_F^2 \right) \right], \tag{A.13}$$

and

$$\sigma_{zz}^{(l)} = \frac{n_\chi e^2 v_F}{\hbar k_F} \frac{1}{i\omega + 1/\tau} \frac{3}{2\pi} \frac{\lambda l}{\sinh \lambda l} \left(\frac{2\zeta_F}{l} \right)^{3/2}. \tag{A.14}$$

Appendix B

Dingle Factor

In the treatment of quantum oscillations, the Dingle factor in equation (2.47) comes from the smearing of LLs due to impurity scattering. In the previous section, we assumed that the density of states have sharp peaks at each Landau level. However, this is not true in a more realistic scenario. The presence of impurities breaks the energy degeneracy of the Landau levels and as a net result they get smeared by the presence of impurities.

The assumption that $k_F \ell_B^2/(v_F \tau) \ll 1$, allows us to disregard corrections to the plane-wave scattering due to the magnetic field[1]. Within this approximation, the density of states is still isotropic, however, it gets a contribution coming from the smearing of energy levels, namely:

$$\begin{aligned} \nu(\xi) &= \int_{\mathcal{BZ}} \mathrm{d}^3k \, \Im\left[G^R(\xi, \boldsymbol{k})\right], \\ \nu(\xi) &= \frac{1}{\pi}\int_{\mathcal{BZ}} \mathrm{d}^3k \, \frac{\Gamma(\xi, k)}{[\xi - \varepsilon(k)]^2 + \Gamma^2(\xi, k)}. \end{aligned} \tag{B.1}$$

In the limit when $\Gamma \to 0$, we recover the well-know result

$$\nu(\xi) = \int_{\mathcal{BZ}} \mathrm{d}^3k \; \delta(\xi - \varepsilon(k)) = \left.\frac{4\pi k^2}{\hbar v_k}\right|_{\varepsilon=\xi}.$$

In fact, the smearing of the energy levels can be introduced by following replacement:

$$\delta(\xi - \varepsilon(k)) \to \frac{1}{\pi}\frac{\Gamma(\xi, k)}{[\xi - \varepsilon(k)]^2 + \Gamma^2(\xi, k)}.$$

[1]Otherwise, we must consider the whole matrix elements of the impurity potential in the presence of magnetic field.

We can thus rewrite equation (A.10) in a more convenient way:

$$\mathcal{I}_2^{(l)} = -\frac{2k_F^2 v_F \zeta_F}{\hbar \pi l}(-1)^l \left(i + \frac{2\zeta_F}{\pi l}\right) - \frac{\sqrt{2} e^{i\frac{\pi}{4}}}{\pi l^{3/2}} \tag{B.2}$$

$$\times \int \mathrm{d}\xi \int\limits_{\frac{1}{\sqrt{2}\ell_B}}^{\infty} \mathrm{d}k\, \delta(\xi - \varepsilon(k))\, k^2 v_k^2\, \zeta_k^{3/2} \frac{\partial f_0}{\partial \xi} e^{i\frac{\pi l}{2}(\zeta_k^{-1}+\zeta_k)}\,.$$

The choice of the lower limit of integration is for later convenience. As previously mentioned, the smearing can be taken into account by replacing the delta function in the integral above by a Lorentzian distribution. Therefore, let us focus on:

$$\mathcal{S}_l(\xi) = \frac{1}{\pi} \int\limits_{\frac{1}{\sqrt{2}\ell_B}}^{\infty} \mathrm{d}k\, \frac{\Gamma(\xi,k)\, e^{i\frac{\pi l}{2}(\zeta_k^{-1}+\zeta_k)}}{[\xi - \varepsilon(k)]^2 + \Gamma^2(\xi,k)} k^2 v_k^2 \zeta_k^{3/2}. \tag{B.3}$$

The cutoff $1/(\sqrt{2}\ell_B)$ guarantees the integral convergence. One can solve equation (B.3) using the steepest descent approximation. For that, let us analytically continue the integrand and define $z \equiv \zeta_k^{-1/2}$. Hence,

$$\mathcal{S}_l(\xi) = \frac{1}{\sqrt{8}\pi \ell_B^3} \int\limits_1^{\infty} \mathrm{d}z\, H\left(\xi, \tfrac{z}{\sqrt{2}\ell_B}\right) \frac{e^{i\frac{\pi l}{2}(z^2+1/z^2)}}{z},$$

where

$$H(\xi,k) \equiv \frac{\Gamma(\xi,k)\, v_k^2}{[\xi - \varepsilon(k)]^2 + \Gamma^2(\xi,k)}.$$

Expanding the exponent near $z = 1$, we obtain:

$$\mathcal{S}_l(\xi) \approx \frac{(-1)^l}{\sqrt{8}\pi \ell_B^3} H\left(\xi, \tfrac{1}{\sqrt{2}\ell_B}\right) \int_{\mathcal{C}} \mathrm{d}z\, e^{i2\pi l(z-1)^2} \tag{B.4}$$

$$+ \frac{i}{\sqrt{2}\ell_B^3} \mathrm{Res}_{\varepsilon \to \xi + i\Gamma} \left[H\left(\xi, \tfrac{z}{\sqrt{2}\ell_B}\right) \frac{e^{i\frac{\pi l}{2}(z^2+1/z^2)}}{z} \right].$$

The contour $\mathcal{C}$ is defined by $\Re[(z-1)^2] = 0$ together with $\Im[(z-1)^2] \geq 0$ and $|z| \geq 1$. Let us assume for simplicity that $\Gamma(\xi,k) = \Gamma(\xi)$. Using that only

$\xi \sim \mu$ contributes to $\mathcal{I}_2^{(l)}$, we find that:

$$\mathcal{S}_l(\xi) \approx \frac{(-1)^l v_0^2}{8\pi \ell_B^3 \sqrt{2l}} \frac{\Gamma}{\xi^2} + \frac{k^2 v_k \zeta_k^{3/2}}{\hbar} \exp\left(\frac{i\pi l}{2\zeta_k}\right), \tag{B.5}$$

where

$$v_0 \equiv v_k\big|_{k=\frac{1}{\sqrt{2}\ell_B}},$$

and k is taken to be $k(\xi + i\Gamma)$. Plugging it into $\mathcal{I}_2^{(l)}$, we end up with:

$$\begin{aligned}\mathcal{I}_2^{(l)} = &-\frac{2k_F^2 v_F \zeta_F}{\hbar \pi l}(-1)^l \left(i + \frac{2\zeta_F}{\pi l}\right) + \frac{k_F^3 v_0^2 \Gamma \zeta_F^{3/2} e^{i\frac{\pi}{4}}}{\pi^2 l^2 \varepsilon_F^2} \\ &- \frac{e^{i\frac{\pi l}{2\zeta_F} + i\frac{\pi}{4} - \lambda_D l}}{2\hbar\pi} k_F^2 v_F \left(\frac{2\zeta_F}{l}\right)^{3/2} \int\limits_{-\infty}^{\infty} \mathrm{d}t \, \frac{e^{(1+i\lambda l/\pi)t}}{(e^t+1)^2}.\end{aligned}$$

Here, we have defined $\lambda_D = \pi\Gamma/(\hbar k_F v_F \zeta_F)$ and neglected terms of $\mathcal{O}(\Gamma e^{-\lambda_D l})$. However, from (2.30),

$$\Gamma/\varepsilon_F \sim \Gamma/(\hbar v_F k_F) \ll \zeta_F$$

and consequently $\zeta_F^{3/2}\Gamma/\varepsilon \ll \zeta_F^2$. Therefore, the second term in $\mathcal{I}_2^{(l)}$ can also be neglected within our approximation.

The only modification in the conductivity expression coming from the smearing of LLs occurs in equation (B.6), which must be replaced by:

$$\sigma_{zz}^{(l)} = \frac{n_\chi e^2 v_F}{\hbar k_F} \frac{e^{-\lambda_D l}}{i\omega + 1/\tau} \frac{3}{2\pi} \frac{\lambda l}{\sinh \lambda l} \left(\frac{2\zeta_F}{l}\right)^{3/2}. \tag{B.6}$$

Appendix C

Sympletic Form and Poisson Structure

We can think of the space of functions as an infinite dimensional vector space which can be endowed with a metric structure. Let us define the inner product between two functions f and g as

$$\langle f, g\rangle \equiv \int_{\mathbb{R}^3} f(x) g(x)\, \mathrm{d}^3 x.$$

This inner product defines the so-called L^2 metric. Clearly, the convergence of this integral imposes some conditions on $f(x)$ and $g(x)$, although let us not bother about that for now. Under this, we can understand functions the same way we understand vectors. Thus, let us rewrite the action in a generic form:

$$S = \int_{t_i}^{t_f} \Big(\langle \pi_I, \dot{\phi}^I\rangle - H\Big)\, dt\,. \tag{C.1}$$

Variation of the action gives us:

$$\begin{aligned}
\delta S = \int_{t_i}^{t_f} &\left(\langle \pi_I, \partial_t \delta\phi^I\rangle - \left\langle \frac{\delta H}{\delta \phi^I}, \delta\phi^I \right\rangle\right) \mathrm{d}t\, + \\
&+ \int_{t_i}^{t_f} \left\langle \delta\pi_I, \dot{\phi}^I - \frac{\delta H}{\delta \pi_I}\right\rangle \mathrm{d}t\,, \\
\delta S = \langle \pi_I, \delta\phi^I\rangle\big|_{t_i}^{t_f} &+ \int_{t_i}^{t_f} \left\langle \delta\pi_I, \dot{\phi}^I - \frac{\delta H}{\delta \pi_I}\right\rangle \mathrm{d}t\, + \\
&- \int_{t_i}^{t_f} \left\langle \dot{\pi}_I + \frac{\delta H}{\delta \phi^I}, \delta\phi^I\right\rangle \mathrm{d}t\,.
\end{aligned}$$

The last two terms give the equations of motion in the Hamiltonian form and the first term introduces the sympletic form. When the momentum variables depend only the fields ϕ^I and not on their time derivative, we can express the sympletic form as an anti-symmetric bilinear map in space of functions.

$$\Omega = \frac{1}{2}\int\int \Omega_{IJ}(x,x')\delta\phi^I(x)\wedge\delta\phi^J(x')\,\mathrm{d}^3x\,\mathrm{d}^3x', \tag{C.2}$$

$$\Omega_{IJ}(x,x') = \left(\frac{\delta\pi_I(x)}{\delta\phi^J(x')} - \frac{\delta\pi_J(x')}{\delta\phi^I(x)}\right). \tag{C.3}$$

In our case, the sympletic form becomes diagonal in position space and can be written as:

$$\begin{aligned}\Omega = &-\int_{\mathbb{R}^3}\left[\alpha\,\delta\rho\wedge\delta\beta + \frac{C}{6}\,\partial_i\alpha\,B^i\,\delta\theta\wedge\delta\beta\right]d^3x + \\ &-\int_{\mathbb{R}^3}\left[\delta\rho\wedge\delta\theta + \frac{C}{6}\,\partial_i\beta\,B^i\,\delta\alpha\wedge\delta\theta\right]d^3x + \\ &-\int_{\mathbb{R}^3}\left(\rho + \frac{C}{6}\,(A_i - \partial_i\theta)\,B^i\right)\delta\alpha\wedge\delta\beta\,d^3x\,. \end{aligned}\tag{C.4}$$

Poisson brackets are obtained by inverting the sympletic form (C.4), what gives us:

$$\{\alpha(x),\theta(x')\} = -\frac{\alpha}{\rho_-}\,\delta^3(x-x')\,, \tag{C.5}$$

$$\{\alpha(x),\beta(x')\} = \frac{1}{\rho_-}\,\delta^3(x-x')\,, \tag{C.6}$$

$$\{\rho(x),\theta(x')\} = -\frac{\pi_\beta}{\alpha\rho_-}\,\delta^3(x-x')\,, \tag{C.7}$$

$$\{\rho(x),\beta(x')\} = \frac{C\,\partial_i\beta\,B^i}{6\rho_-}\,\delta^3(x-x')\,, \tag{C.8}$$

$$\{\rho(x),\alpha(x')\} = \frac{C\,\partial_i\alpha\,B^i}{6\rho_-}\,\delta^3(x-x')\,, \tag{C.9}$$

where,

$$\rho_- = \rho - \frac{C}{6}(\xi_i - A_i)B^i\,. \tag{C.10}$$

For the time time dependent Hamiltonian, we need to specify Poisson bracket between (ξ_i,ρ,θ). The hydrodynamical Poisson brackets then become:

$$\{\rho(x), \rho(x')\} = 0\,, \tag{C.11}$$

$$\{\xi_i(x), \theta(x')\} = \frac{\partial_i\theta - \xi_i}{\rho_-}\,\delta^3(x - x')\,, \tag{C.12}$$

$$\{\xi_i(x), \xi_j(x')\} = \frac{\partial_i\xi_j - \partial_j\xi_i}{\rho_-}\,\delta^3(x - x')\,, \tag{C.13}$$

$$\{\rho(x), \theta(x')\} = \frac{\rho + \frac{C}{6}(A_i - \partial_i\theta)\,B^i}{\rho_-}\,\delta^3(x - x')\,, \tag{C.14}$$

$$\{\xi_i(x), \rho(x')\} = \left[\frac{C}{6}B^k\frac{(\partial_i\xi_k - \partial_k\xi_i)}{\rho_-} - \partial_i\right]\delta^3(x - x')\,. \tag{C.15}$$

They simplify if we write the Poisson structure in terms of the free charge density $\rho_+ = \rho + \frac{C}{6}\xi_i B^i$.

$$\{\rho_+(x), \theta(x')\} = \delta^3(x - x')\,, \tag{C.16}$$

$$\{\xi_i(x), \rho_+(x')\} = -\partial_i\,\delta^3(x - x')\,, \tag{C.17}$$

$$\{\xi_i(x), \theta(x')\} = \frac{\partial_i\theta - \xi_i}{\rho_-}\,\delta^3(x - x')\,, \tag{C.18}$$

$$\{\rho_+(x), \rho_+(x')\} = -\frac{C}{3}B^i\partial_i\,\delta^3(x - x')\,, \tag{C.19}$$

$$\{\xi_i(x), \xi_j(x')\} = \frac{\partial_i\xi_j - \partial_j\xi_i}{\rho_-}\,\delta^3(x - x')\,. \tag{C.20}$$

Appendix D

Hamiltonian Equations for Hydrodynamic with Anomalies

The Hamiltonian (3.31) in terms of ρ_+ and $\widetilde{\xi}_i$ reads:

$$H = \int_{\mathbb{R}^3} \left[\left(\rho_+ - \frac{C}{6}\,\widetilde{\xi}_i B^i \right) \sqrt{\mu^2 + \widetilde{\xi}_k \widetilde{\xi}^k} - P(\mu) \right] d^3x + \\ + \int_{\mathbb{R}^3} \Phi \left[\rho_+ + \frac{C}{6} \left(\epsilon^{ijk} \widetilde{\xi}_i\, \partial_j \widetilde{\xi}_k + 2(\widetilde{\xi}_i + A_i) B^i \right) \right] d^3x \,. \tag{D.1}$$

The last term depends only on the external fields and it gives no contribution to Hamiltonian equations.

The time evolution of ρ_+ is given by:

$$\dot{\rho}_+ = \{H, \rho_+\}\,, \\ \dot{\rho}_+ = -\partial_i \left(\frac{C}{3} B^i \frac{\delta H}{\delta \rho_+} + \frac{\delta H}{\delta \widetilde{\xi}_i} \right),$$

and for $\widetilde{\xi}_i$:

$$\dot{\widetilde{\xi}}_i = \{H, \widetilde{\xi}_i\}\,, \\ \dot{\widetilde{\xi}}_i = -\partial_i \left(\frac{\delta H}{\delta \rho_+} \right) + \frac{\delta H}{\delta \widetilde{\xi}_j} \frac{1}{\rho_-} \left(\partial_i \widetilde{\xi}_j - \partial_j \widetilde{\xi}_i + \epsilon_{ijk} B^k \right).$$

The variation of Hamiltonian (D.1) provides us:

$$\frac{\delta H}{\delta \widetilde{\xi}_i} = J^i + \frac{C}{6}(2\Phi - \gamma\mu)B^i + \frac{C}{6}\epsilon^{ijk}\left(2\Phi\partial_j\widetilde{\xi}_k + \widetilde{\xi}_j E_k\right),$$
$$\frac{\delta H}{\delta \rho_+} = \Phi + \gamma\mu\,,$$

where J^i is defined in (**??**) and,

$$E_k = -\partial_k\Phi,$$
$$\gamma\mu = \sqrt{\mu^2 + \widetilde{\xi}_i\widetilde{\xi}^i}.$$

The free charge conservation is given by:

$$\dot{\rho}_+ + \partial_i\left[J^i + \frac{C}{6}\left(\gamma\mu B^i - \epsilon^{ijk}\widetilde{\xi}_j E_k\right)\right] = \frac{2}{3}CE_iB^i\,, \tag{D.2}$$

where we reproduce (3.11). The relativistic Euler equation in the presence of anomalies becomes:

$$\begin{aligned}\dot{\widetilde{\xi}}_i =& - v^k\partial_k\widetilde{\xi}_i - \frac{1}{\rho}\partial_i P + E_i + \epsilon_{ijk}v^jB^k + \frac{C}{6}\frac{\mu\gamma}{\rho_-}\epsilon_{ijk}\times \\ &\times \epsilon^{jlm}\,\partial_l\widetilde{\xi}_m\,B^k + \frac{C}{6\rho_-}\left(\widetilde{\xi}_lB^l\,v^j - \epsilon^{jlm}\,\widetilde{\xi}_lE_m\right)\times \\ &\times\left(\partial_i\widetilde{\xi}_j - \partial_j\widetilde{\xi}_i + \epsilon_{ijk}B^k\right).\end{aligned} \tag{D.3}$$