On the Spectrum of Gauge/Gravity Duals with Reduced Supersymmetry

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Abstract

The topic of the present thesis is the study of some examples in gauge/string duality. We carefully study the orbifold gauge theory and orbifold string theory and show that the known integrability in AdS/CFT extends to the general supersymmetric orbifolds of $AdS^5 \times S_5$. There is an interesting interplay between the two descriptions of the orbifold gauge theory. Another interesting example is the Klebanov-Strassler (KS) background. We find the exhaustive list of the supergravity excitations in the $\mathcal{I}$-odd sector of the KS theory. These comprise the three $j = \frac{1}{2}$ massive supermultiplets each consisting of a (possibly pseudo) scalar, two fermions and a vector, and the two $j = 1$ supermultiplets whose bosonic content is a vector and a pseudovector. Surprisingly, the spectrum of the excitations which fit into the pure gauge sector strongly resembles the results obtained from the numeric studies in lattice gauge theory.

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Contents

Chapter 1

Introduction

The theory of strings made its appearance in physics as a result of an attempt to understand the strong interactions. However, the discovery of asymptotic freedom and emergence of QCD as well as the lack of any significant progress in string theory of strong interactions led to a shift of intrest of string theory. From the point of view of QCD, the modern theory of strong interactions, hadronic strings would be interpreted as color electric flux tubes between quarks. The main emphasis of string theory therefore shifted to the Planck scale in a hope of finding the unified theory of interactions.

Nevertheless, the interest in the web of dualities between gauge theory and string theory never ceased existing. A concrete version of gauge/string correspondence was proposed by 't Hooft in [1] (see also [2],[3],[4]) in the form of the $1/N$-expansion. The idea was to generalize the gauge group of QCD SU(3) to SU(N) and study the limit of large N. The central idea of $1/N$-expansion is that each Feynman graph of a non-Abelian gauge theory can be naturally drawn on some Riemann surface, and these surfaces are to represent some closed string worldsheets. Indeed, taking the large N limit while keeping the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ fixed results in a genus g graph acquiring the topological factor $N^{\chi} \equiv N^{2-2g}$. Then the $1/N$-expansion could be thought of as a perturbative expansion of some closed string theory with the string coupling via $g_S = 1/N$. However, there was never a worldsheet description of the string theory dual to large N QCD.

The gauge theory/closed string duality is expected to be a limit of a more general open/closed string correspondence, which should hold at the world sheet level. In particular, in terms of the Matrix Theory proposal [5] non-Abelian gauge degrees of freedom are just a part of a more general theory; and thus they naturally incorporate into the web of dualities.

Open string diagrams are equivalent to closed string world sheets with holes. The idea behind the open/closed string correspondence is that the holes can be replaced by closed string vertex operators, and absorbed into an adjustment of the sigma model that governs the motion of the closed string. From the perspective of the low energy effective field theory, this relation between open and closed strings gives rise to the famous duality between gauge theory and gravity, the central example of which is the celebrated AdS/CFT correspondence [6],[7],[8],[9]. The key physical insight that spurred this development was the discovery of D-branes [10], followed by understanding of the geometrical nature of the non-Abelian Chan-Paton factors in terms of stacks of coincident branes [11].

From the point of view of the perturbative open string theory D(irichlet)-branes are surfaces where open strings are allowed to end. Such a string with its end attached to a Dp-brane obeys Dirichlet boundary conditions in the $9-p$ directions transverse to the brane and Neumann boundary conditions in the p spatial directions along the brane. In this setup a stack of parallel D-branes naturally leads to the emergence of non-Abelian degrees of freedom: Chan-Paton factors label the branes a string is attached to. VEVs of the adjoint scalars in the low-energy effective gauge theory on the brane worldvolume are determined by the relative positions of the N D-branes. The full unbroken SU(N) gauge group is attained for the stack of N coincident branes. In this context the gauge degrees of freedom in the simplest form are the low energy excitations of open strings attached to a stack of N parallel D-branes.

On the other hand, D-branes can be viewed as solitons in the closed sting theory. Their appearance on the closed string side is not surprising since from the RR sector of the closed string theory there originate the p-form fields C_p; and there are no objects in perturbative string theory for these forms to couple to. D-branes serve as sources for these RR fields, and their interaction is constructed the same way as that of charged particles with the electromagnetic field. Namely, the coupling of a Dp-brane to a RR $p+1$-form is determined by the integral of the form over the brane's worldvolume:

$$S_{\text{int}} \sim \int_{V_{p+1}} C_{p+1} \,. \tag{1.1}$$

Since D-branes serve as the sources for the RR-fields as well, we expect the supergravity background to acquire some non-zero RR field strength. It turned out that the supergravity solutions describing the backreaction of D-branes had been known for a long time. In particular, of especial importance is the one describing a stack of N coincident D3-branes in a ten-dimensional flat space:

$$ds^2_{10} = h(r)^{-1/2}(-dx_0^2 + d\vec{x}^2) + h(r)^{1/2}(dr^2 + d\Omega_5^2) \,; \tag{1.2}$$

where the factor $h(r)$ is

$$h(r) = 1 + \frac{R^4}{r^4} \,; \qquad R^4 = 4\pi g_S \alpha'^2 N \,. \tag{1.3}$$

The induced RR flux is

$$C_4 = h^{-1}\, dx_0 \wedge \ldots \wedge dx^4 \,. \tag{1.4}$$

A major breakthrough in understanding the duality was made when Maldacena pointed out that the key region for understanding the dynamics of the dual gauge theory was the throat, where $r \ll L$. In this limit the metric (1.2) simplifies:

$$ds^2_{10} = \frac{R^2}{z^2}(-dx_0^2 + d\vec{x}^2 + dz^2) + R^2 d\Omega_5^2 \,; \tag{1.5}$$

where we have introduced $z = R^2/r$. The limitting metric (1.5) describes the direct product of a 5-sphere S^5 and an Anti-de Sitter space AdS_5, both having the same curvature. This background is believed to be an exact string theory solution at the quantum level. The AdS/CFT conjecture suggested that the $\mathcal{N}=4$ SYM at large N and fixed 't Hooft coupling λ was dual to the IIB String Theory on $AdS^5 \times S^5$ with radius

$$R^4 \quad = \quad 4\pi\alpha'^2\lambda\,. \tag{1.6}$$

A mere symmetry counting reveals the exact match between the isometries of the $AdS^5 \times S^5$ background (1.5) and the (super)symmetry group SU$(2,2|4)$ of $\mathcal{N} = 4$ SYM theory. The bosonic part of this group is SU$(2,2) \times$ SU(4). The first factor SU$(2,2)$ is the conformal group in four dimensions, and it corresponds to the isometry group of the AdS_5 space SO$(4,2)$. The SO(6) isometry group of the 5-sphere S^5 matches the SU(4) $\mathcal{R}$-symmetry group of $\mathcal{N}=4$ supersymmetry. This is obvious since the action of the $\mathcal{N}=4$ SYM theory in four dimensions is obtained from that of $\mathcal{N} = 1$ SYM in ten dimesions by performing dimensional reduction along the six dimensions. The SO(6) subgroup acting along these six dimensions becomes the $\mathcal{R}$-symmetry group of the resulting $\mathcal{N}=4$ SYM theory.

The identity of the conformal group in four dimensions and the isometry group of the AdS_5 is not an accidental coincidence as well. The reason is that the boundary of the AdS_{d+1} space is a conformal compactification of a Minkowski space M_d. Indeed, the Anti de Sitter space AdS_{d+1} can be embedded into a flat $d+2$-dimensional space $\mathbb{R}^{2,d}$ as a hyperboloid:

$$-X_0^2 + \sum_{i=1}^{d} X_i^2 - X_{d+1}^2 \quad = \quad -R^2\,. \tag{1.7}$$

In addition, there is to be imposed an equivalence $X^i \to -X^i$. This equation can be rewritten as

$$UV - \sum_{i,j=0}^{d-1} \eta_{ij} X^i X^j = R^2\,. \tag{1.8}$$

Taking the limit $U = \alpha u$, $V = \alpha V$, $X^i = \alpha x^i$, $\alpha \to +\infty$; it becomes

$$uv - \eta_{ij} x^i x^j \;=\; 0\,; \tag{1.9}$$

where u, v and x^i are determined only up to an overall scaling factor: $(u, v, x^i) \sim (su, sv, sx^i)$. In the patch where $v \neq 0$ one can use the scaling equivalence and set $v = 1$; then u is determined from the Minkowski space coordinates x^i. It is obvious that the SO$(d, 2)$ group naturally acts on the "quadric" (1.9), and this action is conformal. Therefore, equation (1.9) gives the conformal compactification of the Minkowski space M_d as a boundary of the Anti de Sitter space AdS_{d+1}.

At the level of algebra the identification of these groups is as follows. The algebra of the conformal group in a space of d dimensions ($d \geq 3$) consists of the following generators: translations P_μ, Lorentz transformations (spatial rotations and boosts) $M_{\mu\nu}$, dilatation D and special conformal transformations K_μ. The algebra can be written in the standard form of the SO$(d, 2)$ algebra with signature $(-, +, +, \cdots, +, -)$; generators of the latter being

$$J_{\mu\nu} \;=\; M_{\mu\nu}\,, \tag{1.10}$$

$$J_{\mu,d} \;=\; \frac{1}{2}\,(K_\mu - P_\mu)\;, \tag{1.11}$$

$$J_{\mu,d+1} \;=\; \frac{1}{2}\,(K_\mu + P_\mu)\;, \tag{1.12}$$

$$J_{d+1,d} \;=\; D\,. \tag{1.13}$$

In the works [7], [8] there was introduced the idea of holography which made further progress towards understanding the correspondence. It was suggested that the AdS background (1.5) was related to a non-critical string theory. Recall that in the conformal gauge the worldsheet action of the string theory contains the fields $X^\mu(\sigma)$ and the Liouville field φ; the former serving as the Weyl factor in the worldsheet metric:

$$g_{ij} \;=\; \mathrm{e}^{\varphi(\sigma)}\,\delta_{ij}\,. \tag{1.14}$$

The criticality condition implies the vanishing of the central charge c of the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n}\,. \tag{1.15}$$

In this case one can identify $L_n = -z^{n+1}\frac{\partial}{\partial z}$, and the Virasoro algebra reduces to the that of the two-dimensional conformal group. In this case by a gauge fixing one can get rid of the Weyl factor and field φ. In particular, the non-criticality condition fixes the number of the target space dimensions to be 26 for a bosonic string theory and 10 for superstrings. However, in a non-critical string theory the central charge c ("conformal anomaly") does not vanish, and the Liouville field φ cannot be gauged away. In fact, it becomes an effective extra coordinate in the target space. It was proposed in [7] that the AdS coordinate z used in eq. (1.5) is nothing but the Liouville field:

$$z = R\mathrm{e}^{-\varphi/R}\,. \tag{1.16}$$

Note that in these notations the regime $z \ll R$ is "far from the brane," and $z \gg R$ is "near the brane." The idea of holography is to identify the generating functional of the field gauge theory with the minimum of the supergravity action subject to some boundary conditions at $z = R$ and $z = 0$. The proposal was illustrated with an example of a massless scalar field.

Yet another idea was that of considering the semiclassical limit of the AdS/CFT correspondence [12]. It amounts to considering the dynamics of the classical strings in AdS. The gauge duals of such regime are the operators with a very high bare dimension. Conformal dimensions of such operators can be identified by virtue of the mentioned identity between the isometry group of AdS_5 and the conformal group of the four dimensional Minkowski space. In particular, in the global coordinates on AdS_5 the metric is

$$ds^2_{AdS_5} = R^2\left(-\cosh^2\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\Omega_3^2\right)\,; \tag{1.17}$$

and the conformal dimension of the dual operator is merely the energy of a spinning string.

However, there are some difficulties in studying the AdS/CFT conjecture. One of them is the fact that weak coupling on the gravity side (closed strings) corresponds to the strong coupling regime on the gauge theory side (open strings); and this prevents one from performing simple perturbative checks. Indeed, given that the gauge theory coupling λ is small; the curvature radius of $AdS^5 \times S^5$ R given by (1.6) becomes small as well, and there appears a need for quantum corrections on the string side. It was major breakthrough when it was realized that some integrable structures were present in the scalar subsector of $\mathcal{N}=4$ SYM [13], and this result was extended to the complete set of operators in [14],[15],[16]. At the same time there was investigated the integrability of the closed string motion in [12] and the subsequent works. This opened new opportunities for understanding the AdS/CFT duality beyond perturbation theory.

Another disappointing fact about the pure AdS/CFT is that $\mathcal{N}=4$ SYM theory is maximally supersymmetric and conformally invariant at both a classical and a quantum level, and this fact prevents it from being a good prototype of the models encountered in particle physics, such as QCD. That is why there is a need to study some examples with fewer supersymmetries. Indeed, there exists a way of reducing the number of supersymmetries. In order to do it one can consider a more general background of the form $AdS^5 \times X_5$; where X_5 is a compact Einstein manifold:

$$R_{ab} \;=\; \Lambda\, g_{ab}\,, \qquad \Lambda > 0\,. \tag{1.18}$$

A common way of achieving such supersymmetry reduction is known as “orbifolding” [17]; *i.e.*, taking the factor X^5 as a quotient of a 5-sphere w.r.t. a discrete group: $X_5 \simeq S^5/\Gamma$. Such theories were shown to be conformal in the planar (large N) limit [18]. It is also possible to construct the gauge theory duals to some spaces X^5 which are not locally equivalent to a sphere. Given that the AdS_5 part of the background remains unaltered, the dual gauge theory still has the full conformal group SO(4, 2); but the number of supersymmetries gets reduced compared to the pure AdS/CFT. A broad class of such backgrounds can be obtained by placing a large number N of D3 branes at the apex of a six-dimensional cone Y_6; then the near-horizon geometry is known to be that of $AdS_5 \times X^5$, where X^5 is the base of the cone Y_6.

A prominent example of such a background is the Klebanov-Witten (KW) solution [19] $AdS_5 \times T^{1,1}$. The five-dimensional space $T^{1,1}$ is the base of the conifold given by the following equation in the four complex variables:

$$\sum_{a=1}^{4} z_a^2 \;=\; 1\,. \tag{1.19}$$

The metric of $T^{1,1}$ can be described in terms of the five angular variables [20]:

$$ds^2_{T^{1,1}} \;=\; \frac{1}{9}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i\, d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i\, d\phi_i^2\right)\,. \tag{1.20}$$

After introducing the following basis one-forms,

$$\begin{aligned} g^1 &= \frac{e^1 - e^3}{\sqrt{2}}\,, \qquad g^2 = \frac{e^2 - e^4}{\sqrt{2}}\,,\\ g^3 &= \frac{e^1 + e^3}{\sqrt{2}}\,, \qquad g^4 = \frac{e^2 + e^4}{\sqrt{2}}\,,\\ &\qquad\qquad g^5 = e^5\,, \end{aligned} \tag{1.21}$$

where

$$\begin{aligned} &e^1 \equiv -\sin\theta_1 d\phi_1\,, \qquad e^2 \equiv d\theta_1\,,\\ &e^3 \equiv \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2\,,\\ &e^4 \equiv \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2\,,\\ &e^5 \equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\,; \end{aligned} \tag{1.22}$$

the metric on $T^{1,1}$ takes the form

$$ds^2_{T^{1,1}} \;=\; \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2\,. \tag{1.23}$$

The dual gauge theory is is an $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory coupled to the two chiral superfields A_i in the $(\mathbf{N}, \bar{\mathbf{N}})$ representation and the two chiral superfields B_i in the $(\bar{\mathbf{N}}, \mathbf{N})$ representation. Let us stress that in spite of the reduced number of supersymmetries ($\mathcal{N} = 1$), the KW gauge theory is conformal. Indeed, it is the S^5 factor that gets replaced by the $T^{1,1}$ on the gravity side; while the AdS^5 factor responsible for the conformal group $\mathrm{SO}(2, 4)$ remains unchanged.

During the subsequent studies there were constructed gravity duals for some gauge theories which were not conformal. These supergravity backgrounds no longer contain the AdS^5 factor. One such example is the Klebanov-Strassler solution, which established a duality between the cascading $SU(k(M + 1)) \times SU(kM)$ gauge theory and type IIB strings on the warped deformed conifold [21]. It generalizes the duality between the superconformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory with bi-fundamentals and string theory on $AdS_5 \times T^{1,1}$ [19]. Adding extra colors to one of the gauge groups breaks the conformal symmetry [22, 23, 24] and leads to the cascade behavior [21, 25, 26]. The gauge group $\mathrm{SU}(k(M + 1)) \times \mathrm{SU}(kM)$ shrinks to $\mathrm{SU}(M)$ at the bottom of the cascade and the KS theory reduces to the pure gauge $\mathcal{N} = 1$ SYM [21]. Unfortunately such a limit requires small $g_s M$, which makes the supergravity approximation invalid. Nevertheless this connection between the KS solution and the pure super-Yang-Mills theory strongly motivates the studies of the bi-fundamental free sector of the $\mathrm{SU}(k(M + 1)) \times \mathrm{SU}(kM)$ theory that survives at the bottom of the cascade.

1.1 Outline

The layout of the present thesis is as follows. In the second chapter we introduce the orbifold string theory. We review the integrability of the semi-classical closed string motion on the orbifolds. Then we introduce the orbifold gauge theory and discuss its different descriptions. In the third chapter we formulate the extension of the known one-loop integrability in AdS/CFT to the case of the general orbifolds and discuss a possible extension to the higher loop calculations. These results were published in [27] The main goal of the fourth chapter is the study of the $\mathcal{I}$-odd sector of the Klebanov-Strassler theory. It is based on the works [28], [29]. Together these constitute some interesting examples in the gauge/string duality. Appendices deal with some detailed calculations.

Chapter 2

Classical Aspects of Orbifold String Theory and Orbifold Gauge Theory

Besides exploring the integrability of the orbifold gauge theories, an important part of the present thesis is to test the correspondence between large N gauge theory and closed string theory. This chapter deals mainly with the classical aspects of orbifold string theory and orbifold gauge theory. The closed string dual to the orbifold gauge theories follows from the AdS/CFT dictionary. The stack of N D3-branes, located on the fixed point of the orbifold space $\mathbb{C}^3/\Gamma$, induce via their gravitational backreaction a near-horizon geometry that is given by

$$AdS_5 \times S^5/\Gamma. \tag{2.1}$$

The AdS/CFT correspondence states that the planar diagrams of the orbifold gauge theory span the worldsheet of closed strings propagating on this near-horizon geometry.

The orbifold group Γ can be an arbitrary finite subgroup of $SO(6)$, the isometry group of the sphere S^5. In general, the finite group does not commute with supersymmetry, and the resulting orbifold string theory is therefore non-supersymmetric.

It can be shown, however, that all such non-supersymmetric orbifolds of $AdS^5 \times S_5$ are unstable, due to the presence of localized tachyonic modes. For this reason we will restrict ourselves to supersymmetric orbifolds, for which Γ defines a finite subgroup of $SU(3)$. Let us parametrize S^5 as a sphere of radius R inside $\mathbb{C}^3$, with coordinates (Z_1, Z_2, Z_3):

$$\sum_I \overline{Z}_I Z_I = R^2\,. \tag{2.2}$$

$SU(3)$ naturally acts on $\mathbb{C}^3$ and on the S^5. In the special case that the finite group Γ fits inside an $SU(2)$ subgroup of $SU(3)$, the orbifold theory is $\mathcal{N} = 2$ supersymmetric.[1] (See Fig. 2.1 for an illustration.)

2.1 Closed Strings on Orbifolds: Semiclassical Treatment

In this section we will summarize the semiclassical treatment of closed strings moving on $AdS_5{\times}S^5/\Gamma$. We will mostly focus on string configurations in the twisted sectors, since the properties of untwisted states simply follow from the parent theory on $AdS_5{\times}S^5$. Twisted sector strings connect two different points on S^5 that are related

[1] Finite subgroups of $SU(2)$ have a well-known classification: they organize into an ADE series. The A-type subgroups are Abelian, while the D-type and exceptional type subgroups are non-Abelian. Under the McKay correspondence these correspond to the cyclic groups, the double covers of the dihedral groups, and the double covers of the rotational symmetry groups of the tetrahedron, cube/octahedron, and dodecahedron/octahedron, respectively. The finite subgroups of $SU(3)$ are less familiar, but have a similar classification. Finite subgroups of $SU(3)$ other than $SU(2)$ and direct products of Abelian phase groups fall into 2 series: analogues of dihedral subgroups, denoted by $\Delta(3n^2)$ with n a positive integer, and $\Delta(6n^2)$ with n a positive even integer, and analogues of exceptional subgroups, denoted by $\Sigma(60)$, $\Sigma(168)$, $\Sigma(360k)$, $\Sigma(36k)$, $\Sigma(72k)$, $\Sigma(216k)$ with $k = 1$, 3. The number in braces is the order of the group. As an example, the discrete $SU(3)$ subgroup $\Delta(3n^2)$ has $3n^2$ elements, generated by the three $\mathbb{Z}^3$ transformations (here $\omega = e^{\frac{2\pi i}{n}}$)

$$\begin{aligned} g_1 &: \quad (Z_1, Z_2, Z_3) \longrightarrow (\omega\, Z_1\,,\, \omega^2\, Z_2, Z_3)\,, \\ g_2 &: \quad (Z_1, Z_2, Z_3) \longrightarrow (\, Z_1, \omega\, Z_2\,, \omega^2\, Z_3)\,, \\ g_3 &: \quad (Z_1, Z_2, Z_3) \longrightarrow (\, Z_1,\; Z_2\;,\; Z_3\,)\,. \end{aligned} \tag{2.3}$$

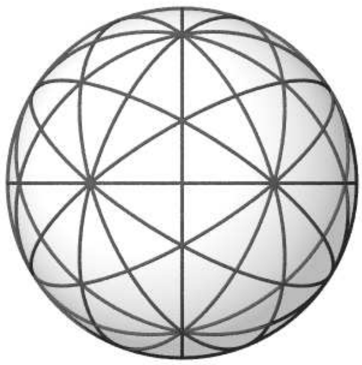



Figure 2.1: Finite group transformations acting on S^5 may have fixed points or act freely. In the former case the group action looks similar to the isometries that act on the sphere on the left. In the free case, the group element can be viewed as a combination of commuting Abelian isometries, analogous to the isometries that act on the torus on the right. For supersymmetric orbifolds, transformations with fixed points are contained inside an $SU(2)$ subgroup. (The above pattern on the sphere has icosahedral symmetry, which is one of the exceptional subgroups of $SU(2)$.)

via some element $g \in \Gamma$. Since the finite group still acts on the twist g by conjugation, twisted sectors are labeled by conjugacy classes in Γ.

To characterize the twisted string states, we note that the S^5 metric allows for three commuting Abelian isometries. In general, these are broken by the orbifold group. For a given twist element g, however, we can orient things such that g acts by a combination of the three isometries, and thus preserves all three of them. So to specify a given twisted sector, we are free to assume that the twist g acts via a diagonal matrix on the Z_I. If g is an element of order S inside Γ, $g^S = 1$, we can write

$$g \ : \ (Z_1,\ Z_2,\ Z_3) \ \to \ (\omega^{s_1} Z_1,\ \omega^{s_2} Z_2,\ \omega^{s_3} Z_3)\,, \tag{2.4}$$

$$\omega = e^{2\pi i/S}\,, \qquad\qquad \sum_I s_I = 0\,. \tag{2.5}$$

We see that, in this given twisted sector, the string is free to move along three circle directions, and one can define corresponding conserved angular momenta J_I, with $I = 1, 2, 3$. We further observe that in general, the group element g acts freely on

S^5. The corresponding twisted sector strings thus have a minimal length. However, when one of the three integers s_I, say s_1, vanishes — so when g in fact fits inside $SU(2)$ — the action of g on S^5 has an obvious fixed point at $(Z_1, Z_2, Z_3) = (R, 0, 0)$.

Next we summarize some relevant results on the classical motion of strings along the S^5 having in mind the future comparison with the gauge theory side. The more general calculations can be found in [30] and references therein; in particular, [31], [32] and [33]. We restrict ourselves to the strings moving in S^5 directions only and trivially embedded into AdS^5. This motion is governed by the sigma model action (restricted to the bosonic string coordinates)

$$S \sim \int d\tau d\sigma \left(\frac{1}{2} \partial_a \mathrm{t}\, \partial^a \mathrm{t} \, - \partial_a \overline{Z}_I \partial^a Z_I + \Lambda(\overline{Z}_I Z_I - R^2) \right) . \tag{2.6}$$

Here t denotes the AdS time coordinate, and Λ is a Lagrange multiplier field. This action is obtained as a reduction of the string worldsheet action in the conformal gauge; thus the equations of motion derived from this action must be supplemented by the corresponding Virasoro constraints:

$$\dot{\mathrm{t}}^2 = \dot{Z}_I \dot{\bar{Z}}_I + Z'_I \bar{Z}'_I \,, \tag{2.7}$$

$$0 = \dot{Z}_I \bar{Z}'_I \,. \tag{2.8}$$

We want to solve for the motion of the string in the twisted sector defined by the twist element (2.4). As it was emphasized in [34], the closedness requirement then allows for the fractional winding numbers,

$$Z_I(\sigma + 2\pi) = \mathrm{e}^{2\pi i \tilde{m}_I} Z_I(\sigma) \,, \qquad \tilde{m}_I = m_I + \frac{s_I}{S} \,. \tag{2.9}$$

The S^5 metric has three commuting isometries. In general, these are broken by the orbifold group. However, from the explicit form of the twist element given in (2.4), we see that *in this given twisted sector*, the string is free to move along three circle directions. It is therefore natural to choose the following *Ansatz,*

$$\mathrm{t} = \kappa \tau \,, \qquad\qquad Z_I = z_I(\sigma)\, e^{i\omega_I \tau} \,. \tag{2.10}$$

Inserting this Ansatz, the Lagrangian governing the dependence $z_I(\sigma)$ becomes

$$\mathcal{L} = \frac{1}{2}\sum_I(\bar{z}'_I z'_I - \omega_I^2 \bar{z}_I z_I) - \frac{1}{2}\Lambda(\sum_I \bar{z}_I z_I - R^2)\,. \tag{2.11}$$

The Virasoro constraints simplify to

$$\kappa^2 = \sum_I(\bar{z}'_I z'_I + \omega_I^2 \bar{z}_I z_I)\,, \tag{2.12}$$

$$0 = \sum_I \omega_I(\bar{z}'_I z_I - z'_I \bar{z}_I)\,. \tag{2.13}$$

Note that both Lagrangian and Virasoro constraints have a $U(1)^3$-invariance w.r.t. the multiplication by a phase,

$$z_I \to \mathrm{e}^{i\alpha_I} z_I\,, \qquad \bar{z}_I \to \mathrm{e}^{-i\alpha_I}\bar{z}_I\,. \tag{2.14}$$

This invariance leads to the three integrals of motion,

$$\ell_I = \frac{i}{2}\,(\bar{z}'_I z_I - z'_I \bar{z}_I)\,. \tag{2.15}$$

This allows us to eliminate the angular variables. Then denoting $r_I^2 = z_I \bar{z}_I$ and substituting this back into the action we get the following effective Lagrangian:

$$\mathcal{L} = \frac{1}{2}\sum_I\Big(r_I'^2 - w_I^2 r_I^2 + \frac{\ell_I^2}{r_I^2}\Big) - \frac{1}{2}\Lambda\Big(\sum_I^3 r_I^2 - 1\Big)\,. \tag{2.16}$$

This system is called the *Neumann-Rosochatius* (NR) integrable system (e.g., [35]); and its detailed analysis in the context of the closed string dynamics is given in [30]. Here we restrict ourselves to the simplest example, the circular strings. These solutions take the following forms:

$$z_I = a_I\,\mathrm{e}^{i\tilde{m}_I\sigma}\,, \qquad \Lambda = \mathrm{const.} \tag{2.17}$$

With this ansatz the integrals are $\ell_I = \tilde{m}_I a_I^2$; while the dynamical equations yield

$$w_I^2 = -\Lambda - \tilde{m}_I^2\,, \qquad \sum_{I=1}^3 a_I^2 = 1\,; \tag{2.18}$$

$$\kappa^2 = \sum_{I=1}^{3} a_I^2 (w_I^2 + \tilde{m}_I^2)\,, \qquad \sum_{I=1}^{3} a_I^2 w_I \tilde{m}_I = 0\,. \tag{2.19}$$

The space-time energy for the circular string configuration is

$$E = \sqrt{8\pi^2\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \dot{\mathrm{t}} \; = \sqrt{8\pi^2\lambda}\, \kappa\,; \tag{2.20}$$

while the spins are

$$J_I = \sqrt{8\pi^2\lambda}\, w_I \int_0^{2\pi} \frac{d\sigma}{2\pi}\, r_I^2(\sigma) = \sqrt{8\pi^2\lambda}\, w_I a_I^2\,. \tag{2.21}$$

One can define the total spin $L = \sum_{I=1}^{3} |J_I|$. The relations (2.18) and (2.19) can be rewritten in terms of the energy and the spins. Eqs. (2.19) read

$$\frac{E^2}{8\pi^2\lambda} = -\Lambda\,, \qquad \sum_{I=1}^{3} \tilde{m}_I J_I = 0\,; \tag{2.22}$$

while (2.18) becomes

$$\sum_{I=1}^{3} \frac{|J_I|}{\sqrt{-\Lambda - \tilde{m}_I^2}} = \sqrt{8\pi^2\lambda}\,. \tag{2.23}$$

We will consider the two spin solution with $J_3 = 0$. Then in the large L limit one can solve these equations and find the following expansion for the energy:

$$E = L + \frac{4\pi^2\lambda}{L}\, |\tilde{m}_1|\, |\tilde{m}_2|\,. \tag{2.24}$$

Given that $J_1, J_2 > 0$, one must have $\tilde{m}_1 \tilde{m}_2 < 0$. Recalling the definition of the fractional winding numbers $\tilde{m}_2$, $\tilde{m}_2$ one can write the string energy as

$$E = L + \frac{4\pi^2\lambda}{L} \left(m - \frac{s_1}{S}\right)\left(m' + \frac{s_2}{S}\right); \tag{2.25}$$

m and m' being some positive integers. After the formulation of the Bethe Ansatz Equations for the orbifold gauge theory in Section 3.3 we will see that this expression matches the one-loop anomalous dimensions for the corresponding $\mathfrak{su}_2$ subsector formed by the two scalars in the field theory.

2.2 Orbifold Gauge Theory

We now turn to the study of the class of quiver gauge theories obtained by taking an arbitrary (Abelian or non-Abelian) orbifold of $\mathcal{N}=4$ supersymmetric $U(N)$ gauge theory. Our motivation is to investigate to what extent the recently uncovered large N integrability of $\mathcal{N}=4$ SYM can be extended to this general class of orbifold gauge theories. In this section we will summarize some of their relevant properties. The relevant references are [36],[37],[38],[39],[18],[40].

It will be convenient to think of the quiver gauge theory as the low energy limit of the open string theory on a stack of N D3-branes located near an orbifold singularity. Before taking the orbifold quotient, the transverse space of the D3-branes is $\mathbb{R}^6 \simeq \mathbb{C}^3$. The low energy field theory on the D3-branes is $\mathcal{N}=4$ SYM, with its field content (in $\mathcal{N}=1$ superfield notation): a vector multiplet $\mathcal{A}$ and three chiral multiplets Φ^I, with $I=1,2,3$, that parametrize the three complex transverse positions of the D3-branes along $\mathbb{C}^3$.

Let Γ be some finite group of order $|\Gamma|$, that acts on $\mathbb{C}^3$. The orbifold space is obtained by dividing out the action of the discrete group Γ. The transverse space to the D3-branes thus becomes $\mathbb{C}^3/\Gamma$. When viewed from the covering space, the stack of N D3-branes in the orbifold space give rise to the total of $|\Gamma|N$ image D3-branes. It is convenient to label the image D3-branes by a pair of Chan-Paton indices (i,h) with $i=1,\ldots,N$ and $h\in\Gamma$, so that the brane labeled by (i,h) is the image of the i-th brane inside the D3-stack under the group element $h\in\Gamma$. The group Γ thus acts on the Chan-Paton indices as

$$g \;:\; (i,h) \;\rightarrow\; (i,gh)\,. \tag{2.26}$$

When the N coincident D-branes all approach the orbifold fixed point, the image branes all coincide and the strings stretched between them have massless ground states. The vector multiplet $\mathcal{A}$ has a separate matrix entry for each open string stretching between two image branes, and thus defines an $|\Gamma|N\times|\Gamma|N$ matrix. Before imposing invariance under the orbifold group Γ, the full collection of image branes thus supports an $U(|\Gamma|N)$ gauge symmetry. The orbifold projection, however, selects only those fields that are invariant under the discrete group Γ. The discrete group acts on the vector multiplet $\mathcal{A}$ only via the Chan-Paton indices and on the chiral multiplets Φ^I via both the Chan-Paton and transverse indices.

This projection operator does not commute with the full $\mathcal{N}=4$ superconformal invariance, but in the special case that Γ forms a subgroup of $SU(3)$, the orbifold quotient preserves $\mathcal{N}=1$ superconformal invariance. More generally, one could consider orbifolds with Γ some subgroup of $SO(6)$. However, it has been shown that for non-supersymmetric orbifolds, the quantum theory has non-zero β-functions for certain double-trace operators and is therefore not conformally invariant. The renormalized Hamiltonian of non-supersymmetric orbifolds thus contains extra terms that do not descend from the $\mathcal{N}=4$ Hamiltonian [41]. For this reason we will restrict ourselves to the supersymmetric subclass.

Although the orbifold theories all have less supersymmetry, their action is assumed to be identical to that of the parent $\mathcal{N}=4$ theory, which in $\mathcal{N}=1$ superfield notation reads

$$\mathcal{L} = \int d^4\theta \ \mathrm{Tr}\left(\mathcal{W}^\alpha \mathcal{W}_\alpha + e^{\mathcal{A}}\Phi_I^\dagger e^{-\mathcal{A}}\Phi_I\right) \ + \ \int d^2\theta \, \epsilon^{IJK} \, \mathrm{Tr}\left(\Phi_I[\Phi_J, \Phi_K]\right) \ + \ c.c. \quad (2.27)$$

Here the trace Tr is over the adjoint representation of the full $U(|\Gamma|N)$ gauge group of the $\mathcal{N}=4$ theory. The fields $(\mathcal{A}, \Phi)$ of the orbifold theory, however, have to be Γ-invariant. This invariance condition can be solved as

$$\mathcal{A}_{h,hg} \;=\; \mathcal{A}(g)\,, \quad (2.28)$$

$$\Phi^I_{h,hg} \;=\; \mathfrak{R}(h)^I_{\ J}\phi^J(g)\,. \quad (2.29)$$

We see that after the projection, the Γ valued left and right Chan-Paton indices have collapsed to a single group valued index. The gauge and matter fields can thus be thought of as group algebra valued $N \times N$ matrices. We will refer to the above basis of orbifold projected fields as the *orbit basis* (as distinguished from the *quiver basis*, that will be introduced later).

Note that the orbifold projection does not commute with the full $U(|\Gamma|N)$, the gauge symmetry gets broken to a subgroup. This unbroken gauge group is identified as follows. Recall that the orbifold group acts on the Chan-Paton indices of the gauge field via the regular representation, and the latter decomposes into irreducible representations ρ_λ via

$$\gamma_{\text{reg}}(g) = \bigoplus_\lambda \rho_\lambda(g)^{\oplus N_\lambda}\,, \qquad\qquad N_\lambda = \dim \rho_\lambda\,. \tag{2.30}$$

In words, each irreducible representation ρ_λ occurs N_λ times in the decomposition of the regular representation. In explicit matrix notation, we have

$$\gamma_{\text{reg}} = \begin{pmatrix} \rho_1 \otimes \mathbb{1}_{N_1} & 0 & \dots & 0 \\ 0 & \rho_2 \otimes \mathbb{1}_{N_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_r \otimes \mathbb{1}_{N_r} \end{pmatrix}, \tag{2.31}$$

where each ρ_λ denotes an $N_\lambda \times N_\lambda$ matrix. By inspecting the explicit form (2.31) of γ_{reg}, it is not difficult to derive that the orbifold gauge group, defined as the subgroup of $U(|\Gamma|N)$ transformations that commutes with $\gamma(g)$ for all $g \in \Gamma$, takes the product form

$$\bigotimes_\lambda U(NN_\lambda)\,. \tag{2.32}$$

Here the product runs over all representations of Γ and each factor $U(NN_\lambda)$ is the subgroup that rearranges the NN_λ copies of the representation space V_λ of ρ_λ — it therefore obviously commutes with Γ. Using Schur's lemma, one proves that (2.32) indeed defines the maximal unbroken gauge group: physical operators need only be gauge invariant under this group.

2.2.1 Construction of Observables

The novel feature of the orbifold gauge theory is the emergence of the twisted states, which are new compared to the parent theory. The untwisted sector is the subclass of operators that come directly from the parent $\mathcal{N}=4$ theory. In the open string language, the untwisted operators can be thought of as arrays of concatenated open strings attached to several image D3-branes, as indicated on the left in the Fig 2.2. Such an operator is written as

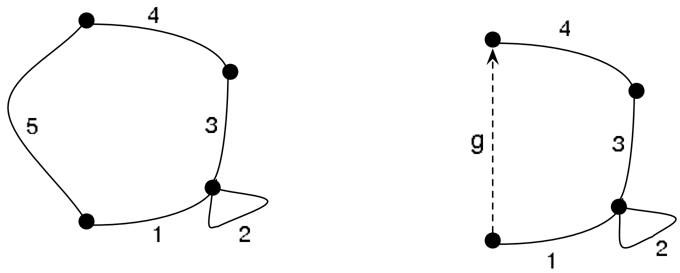


Figure 2.2: An untwisted state (left) and twisted state (right). Both are concatenated arrays of open strings (lines) stretched between D3-branes (dots). The end-point brane of the twisted state is the image under a finite group transformation g on its begin-point brane.

$$\mathcal{O} = \operatorname{Tr}\left(\mathcal{W}_{A_1}\mathcal{W}_{A_2}\ldots\mathcal{W}_{A_L}\right). \tag{2.33}$$

Here $\mathcal{W}_A$ stands for a (multiple) covariant derivative of one of the fields of the theory, in $\mathcal{N}=1$ notation:

$$\mathcal{W}_A \in \left\{\mathcal{D}^n\Phi_I,\, \mathcal{D}^m\mathcal{W}_\alpha\right\}; \tag{2.34}$$

and each operator $\mathcal{W}_A$ corresponds to a ground state of one of the open strings. The gauge invariant trace implies that the array is closed: it begins and ends on the same D3-brane, and thus represents a proper closed string state in the unorbifolded theory.

A twisted sector state, on the other hand, corresponds to a concatenated array of open strings that ends on a different D3-brane, related via a finite group transformation $g \in \Gamma$ to the D3-brane where it begins. This configuration looks like an open string on the covering space, but it represents a closed string on the orbifold space. Correspondingly, it is associated with an operator that is not gauge invariant under the full $U(|\Gamma|N)$ symmetry of the cover theory, but that *is* invariant under the gauge group (2.32) of the orbifold theory. In the gauge theory, the twisted states are represented as single trace expressions

$$\mathcal{O}(g) = \operatorname{Tr}\left(\gamma(g)\, \mathcal{W}_{A_1} \mathcal{W}_{A_1} \dots \mathcal{W}_{A_L}\right), \tag{2.35}$$

where we introduced a twist operator $\gamma(g)$, defined as follows. When $\gamma(g)$ acts from the left on a matrix-valued operator $\mathcal{W}_A$, it acts via the group action (2.26) — the regular representation $\gamma_{\text{reg}}(g)$ — on the left Chan-Paton index. When $\gamma(g)$ acts from the right, it acts via the complex conjugate group action $\overline{\gamma}_{\text{reg}}(g)$ on the right Chan-Paton index. The actions from the left and from the right are not identical; instead, the operators $\mathcal{W}_A$ of the orbifold theory satisfy a relation of the form

$$\gamma(g)\, \mathcal{W}_A = \mathfrak{R}(g^{-1})_A^{\ B}\, \mathcal{W}_B \gamma(g)\,, \tag{2.36}$$

where $\mathfrak{R}(g)_A^{\ B}$ denotes a matrix representation of the finite group Γ, acting on the $\mathbb{C}^3$ index of $\mathcal{W}_A$.[2]

As a consequence of the orbifold projection, some of the physical operators (2.35) vanish identically. Using equation (2.36) to commute $\gamma(g)$ past all the fields in the operator shows that a necessary condition for non-vanishing operators is that the total single trace operator must be invariant under the simultaneous action of $\mathfrak{R}(g)_A^{\ B}$ on all the spins. However, while necessary, this is not sufficient. More generally,

[2]Here $\mathfrak{R}(g) = 1$ in case $\mathcal{W}_A$ has no $\mathbb{C}^3$ index. Note further that inserting multiple twist operators in the trace does *not* introduce a new class of operators, since by using the exchange relation (2.36) and the property $\gamma(g_1)\gamma(g_2) = \gamma(g_1 g_2)$, one can always reduce any number of twist operators to a single overall twist. This is as one would have expected from the string interpretation.

physical operators are of the form

$$\mathcal{O}_{\mathcal{K}}(g) \;=\; \mathcal{K}^{A_1\,A_2\,\cdots\,A_L}\,\mathrm{Tr}\left(\gamma(g)\,\mathcal{W}_{A_1}\mathcal{W}_{A_1}\ldots\mathcal{W}_{A_L}\right), \tag{2.37}$$

where $\mathcal{K}$ must be an invariant tensor under the complete stabilizer subgroup S_g of g, defined as the subgroup within Γ of all elements that commute with g.[3] In the untwisted case, where g is the identity element in Γ, the stabilizer subgroup is the whole group Γ and indeed, as we saw before, untwisted operators are in one-to-one correspondence with Γ-invariant tensors. It is important to note that the basis (2.37) of operators is a complete basis, in the sense that any operator of the seemingly more general class given in (2.35), that is not of the form (2.37), vanishes identically. Detailed construction of twisted operators as well as the proof of their gauge invariance is given in Appendix A.1.1.

An extremely important fact is that the insertion of a single twist field $\gamma(g)$ suffices to produce all the operators within the twist class $[g]$ (cf. [42]). In this context the untwisted operators can naturally be viewed as those belonging to the sector with the conjugacy class of the unit element $[e]$. Then any single representative $g \in [g]$ can be diagonalized; and then one can apply the tools used for the Abelian orbifolds to the general non-Abelian case. If g is an element of order S in Γ, then its eigenvalues are some phase factors of the form $\exp(2\pi i s_k/S)$. This observation will be heavily exploited in Section 3.2 where we study the Bethe Ansatz Equations for the orbifold gauge theory.

2.3 Quiver Gauge Theory

In this section, we will make a comparison between the above group theoretic description of the physical operators with the quiver representation of the orbifold

[3] It can be the case that even for some S_g-invariant tensor $\mathcal{K}(g)$ the corresponding operator $\mathcal{O}_{\mathcal{K}}$ vanishes identically due to some symmetry reasons — for instance, this is the case in the $\mathbb{Z}_6$ quiver we consider in Section 3.3.

gauge theory. It is the quiver gauge theory representation that makes the physical field content of orbifold gauge theories most manifest. As discussed, the unbroken gauge group of the orbifold theory takes the product form

$$\bigotimes_{\lambda} U(NN_\lambda)\,, \tag{2.38}$$

where the product runs over all representations ρ_λ of the finite group Γ, and $N_\lambda = \dim V_\lambda$. Notice that, even in the case that $N=1$, i.e, for the world-brane theory of a single D3-brane near an orbifold singularity, this gauge group contains several, in general non-Abelian, factors. In the string theoretic construction, each gauge factor is associated to a stack on NN_λ so-called fractional D3-branes. There is one type of fractional brane for each representation ρ_λ of the finite group.

The vector multiplets $\mathcal{A}$ arise as the ground states of open strings attached to a given fractional brane. Let us denote by $\mathcal{A}_\lambda$ the vector multiplet of the fractional brane associated to ρ_λ. Hence $\mathcal{A}_\lambda$ is an $U(NN_\lambda)$ gauge multiplet. In terms of the *orbit basis* $\mathcal{A}(g)$ defined in (2.28), the *quiver basis* $\mathcal{A}_\lambda$ is obtained via the Fourier-like transformation (see Appendix A.1):

$$\mathcal{A}_\lambda = \sum_g \overline{\rho}_\lambda(g)\,\mathcal{A}(g) \tag{2.39}$$

Setting up the quiver terminology, we will refer to each gauge factor and its associated stack of fractional branes, as a *node* of the quiver diagram. There is one quiver node for each irreducible representation of Γ.

In a quiver diagram, the nodes are connected by oriented lines: these represent the chiral matter fields. In the string theory construction, the chiral matter fields Φ^I arise as the ground states of open strings that may have end-points on two different fractional branes. Correspondingly, they transform as bi-fundamental fields under the product gauge group (2.38). Algebraically the chiral matter fields $\Phi^{\lambda\overline{\mu}}$ correspond to the invariant tensors $(\mathbb{C}^3 \otimes V_\lambda \otimes \overline{V}_\mu)^\Gamma$. The number $n_{\lambda\bar{\mu}}$ of chiral matter fields between two given nodes λ and μ is determined by the multiplicity of

ρ_μ in the decomposition of the tensor product between the defining representation $\mathfrak{R}$ and ρ_λ:

$$\mathfrak{R} \otimes \rho_\lambda = \bigoplus_\mu n_{\lambda\bar{\mu}} \; \rho_\mu \, . \tag{2.40}$$

In the string construction, the number $n_{\lambda\bar{\mu}}$ is the intersection number between the two fractional branes. The fields $\Phi^{\lambda\bar{\mu}}$ thus transform in the $(NN_\lambda, \overline{NN_\mu})$ bi-fundamental representation of the gauge group (2.38).[4] This *quiver basis* $\Phi^{\lambda\bar{\mu}}$ is related to the *orbit basis* $\Phi^{I}(g)$ given in (2.29) via the linear transformation

$$\Phi^{\lambda\bar{\mu}} = \sum_{g,I} \mathcal{K}^{I}_{\lambda\bar{\mu}} \; \bar{\rho}_\mu(g) \, \Phi_{I}(g) \, , \tag{2.41}$$

where $\mathcal{K}_{\lambda\bar{\mu}}$ denotes one of the $n_{\lambda\bar{\mu}}$ basis elements that spans the space of invariant tensors in $\mathbb{C}^3 \otimes V_\lambda \otimes \overline{V}_\mu$.

In the quiver basis, it is now easy to specify all possible single trace operators of the orbifold gauge theory. For this, it is useful to introduce the notion of the *path algebra* of the quiver diagram. A path is a concatenated array of arrows that connect quiver nodes connected by oriented lines. The arrows are allowed to point back to the same node. We can multiply two paths if one ends at the same node as where the other begins. We can then connect them head to tail to produce a single longer path. In the quiver gauge theory, each arrow of the path represents a gauge or matter operator $\mathcal{W}_A$ of the general form (2.34), transforming in the corresponding representation of the quiver gauge group. Connecting two arrows amounts to taking their gauge invariant product at the corresponding quiver node. To write gauge invariant operators, we now simply choose arbitrary closed paths along the quiver, pick a corresponding array of operators, and in the end take the trace.

How does this description of gauge invariant single trace operators compare with that in terms of twisted sector states (2.37)? Let us pick some closed path $\mathcal{C}_\lambda$, that

[4] As a check, let us count the number of independent components of the chiral matter field Φ. For each arrow there are $N^2 N_\lambda N_\mu$ components, and each node therefore connects to $N^2 N_\lambda \sum_{\{\mu\}} \dim V_\mu$ independent components. Since $\mathfrak{R} \otimes V_\lambda = \oplus_{\{\mu\}} V_\mu$, dimension counting gives $\sum_{\{\mu\}} \dim V_\mu = 3 \dim V_\lambda$. Therefore, the total number of independent components of Φ is $3N^2 \sum_\lambda N_\lambda^2 = 3\,|G|\; N^2$. This is the expected result.

starts and ends at a given node λ but along the way visits the following sequence of quiver nodes

$$\mathcal{C}_\lambda \ : \ \lambda \leftarrow \mu \leftarrow \nu \leftarrow \ldots \leftarrow \sigma \leftarrow \lambda\,. \tag{2.42}$$

For each arrow along this path, we pick the corresponding field and multiply them together, and take the trace at the λ node

$$\mathcal{O}_{\mathcal{C}_\lambda} = \mathrm{Tr}_\lambda\big(\mathcal{W}_{\lambda\bar{\mu}}\mathcal{W}_{\mu\bar{\nu}}\cdots\mathcal{W}_{\sigma\bar{\lambda}}\big)\,. \tag{2.43}$$

This is a manifestly gauge invariant operator of the quiver gauge theory. The equivalence with the group algebraic description of the orbifold theory implies that this operator must be a linear combination of twisted state operators $\mathcal{O}_{\mathcal{K}}(g)$ defined in Eqn. (2.37). A straightforward calculation, described in Appendix A.1, indeed shows that

$$\mathcal{O}_{\mathcal{C}_\lambda} = \sum_g \mathcal{K}(g)^{A_1A_2\ldots A_L}\,\mathrm{Tr}\left(\gamma(g)\,\mathcal{W}_{A_1}\ldots\mathcal{W}_{A_L}\right), \tag{2.44}$$

where the S_g-invariant tensor $\mathcal{K}(g)$ is given by

$$\mathcal{K}(g)^{A_1A_2\ldots A_L} = \mathrm{Tr}_\lambda\big(\overline{\rho}_\lambda(g)\,\mathcal{K}^{A_1}_{\lambda\bar{\mu}}\mathcal{K}^{A_2}_{\mu\bar{\nu}}\cdots\mathcal{K}^{A_L}_{\sigma\bar{\lambda}}\big)\,. \tag{2.45}$$

The class of operators associated to closed loops on the quiver diagram span a complete basis of twisted sector operators, and vice versa.

Chapter 3

Bethe Ansatz Equations for General Orbifolds

This chapter is devoted to the study of some aspects of integrability of the orbifold gauge theories in the context of the AdS/CFT correspondence. Even though there are some works studying integrability in the context of these dualities for some special orbifolds or some special limits [43],[44],[45],[46],[34],[47],[48],[49],[50], they mainly deal with the Abelian orbifolds and the corresponding quiver gauge theories. We extend some of these studies to the generic orbifolds with an arbitrary non-Abelian orbifold group. Organization of the chapter is as follows. First we introduce the orbifold gauge theory which is the low-energy limit of the corresponding open string theory. There is a very simple description of the orbifold gauge theory, where the orbifoldization procedure requires an introduction of an extra field, the twist field in addition to the dynamic fields. Except for the appearance of the twisted sectors, the orbifolded theory very closely resembles the original one. However, some of the operators of the original theory are projected out by the orbifoldization procedure. There is no mixing between the different twisted sectors; and this superselection rule simplifies diagonalization of the matrix of anomalous dimensions. There exists an alternative description of the orbifold gauge theory, the one using the quiver diagram. We explain how to make a transition between these two descriptions. We formulate the Bethe Ansatz Equations (BAE) at the one-loop level for a general orbifold gauge theory. The key idea is the fact that one can diagonalize the twist field in a given twisted sector, after which the setup reduces to the Abelian case modulo some subtleties. It allows one to apply the techniques used for the Abelian orbifolds in the general case. Indeed, it turns out that the methods of studying the Abelian orbifold gauge theories can be extended to arbitrary non-Abelian setups with minor modifications.

3.1 Feynman Rules

As an example we go through the derivation of the Feynman rules for the scalar field ϕ^I; the other fields are treated in a similar way. We can parametrize the invariant configurations of the scalar field $\phi^I_{ig,jh}$ in terms of this group algebra valued object $\phi^I_{ij}(g)$, and this group algebra valued field ϕ is to be integrated over in the path integral. Using the parametrization (2.29) and the orthogonality of the defining representation $\mathfrak{R} : \Gamma \to \mathrm{SO}(6)$, we get the kinetic term for the scalar field in the form

$$\mathcal{L}_{\phi\phi} = \frac{1}{2}\sum_I \operatorname{Tr} \partial_\mu\phi^I\, \partial^\mu\phi^I = \frac{1}{2}|\Gamma| \sum_g \sum_{I,J} \mathfrak{R}(g)^I_{\ J}\, \partial_\mu \operatorname{Tr} \phi^I(g)\, \partial^\mu \phi^J(g^{-1})\,. \tag{3.1}$$

Then for the quadratic propagator the only modification compared to the original theory is "conservation of the group index" and renormalization:

$$\left\langle \phi^I_{ij,g}\, \phi^J_{kl,h} \right\rangle \;=\; |\Gamma|^{-1} \frac{\mathfrak{R}(g)^I_{\ J}}{p^2}\, \delta_{gh,e}\, \delta_{il}\, \delta_{jk}\,, \tag{3.2}$$

In terms of the original $\mathcal{N}=4$ fields (we omit the obvious Latin part of the Chan-Paton indices)

$$\left\langle \phi^I_{h,g}\, \phi^J_{f^{-1}g,\, f^{-1}h} \right\rangle \;=\; \frac{\mathfrak{R}(f)^I_{\ J}}{|\Gamma|\, p^2}\,. \tag{3.3}$$

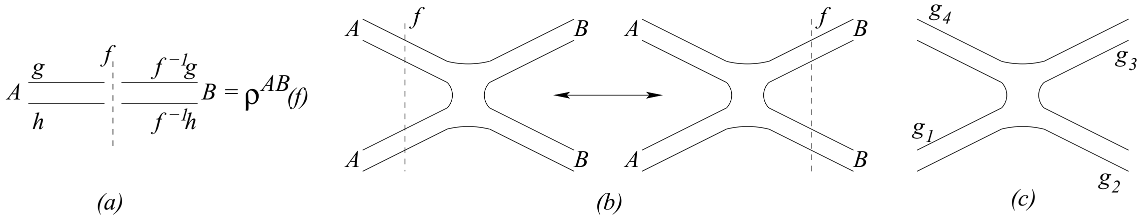


Figure 3.1: (a) When a line of the Feynman graph crosses the cut the Wick contraction $\langle \mathcal{W}_A \, \mathcal{W}_B \rangle$ is non-diagonal, and proportional to the matrix element $\mathfrak{R}^{AB}(f)$. (b) The twist lines can be deformed and moved through interaction vertices. (c) In the notation using the group valued fields untwisted vertices obey the conservation condition, similar to the conservation of momentum: for the vertex shown the product $g_1 g_2 g_3 g_4 = 1$.

Generally, for elementary fields (or their derivatives) $\mathcal{W}_A$ there takes place the following replacement in the propagator:[1]

$$\langle \mathcal{W}_A \, \mathcal{W}_B \rangle_{\mathcal{N}=4} = G(p) \, \delta^{AB} \quad \rightarrow \quad \langle \mathcal{W}_A \, \mathcal{W}_B \rangle = \frac{1}{|\Gamma|} G(p) \, \mathfrak{R}^{AB}(f) \, . \tag{3.4}$$

The factor of $1/|\Gamma|$ compensates for the overcounting of fields. The propagator is not simply diagonal on the group valued Chan-Paton indices (g, h), but there can be a twist by some group element f, that acts simultaneously on both the left and right index. The advantage of this (redundant) double line notation is that the interaction vertices coincide with those of the original theory, and the only modification is the introduction of these twists along the propagators.

Equivalently, we can think of the twist as the assignment of a group element f to each line of the dual graph to the Feynman diagram. We will call these lines on the dual graph 'cuts'. When a propagator crosses a cut, the propagator $\langle \phi^I \, \phi^J \rangle$ is non-diagonal: the conventional factor δ^{IJ} gets replaced by the matrix element $\mathfrak{R}^{IJ}(f^{-1})$ with f the twist along the cut. Vertices of the dual graph correspond to loops of the original Feynman graph. The product of the group elements that meet

[1] We ignore the ghost fields. Gauge fixing is easy to do via the Feynman gauge. Since the gauge field $\mathcal{A}$ can be treated as a group algebra valued, the gauge fixing and Faddeev-Popov ghosts can also be treated as group algebra valued.

at a dual vertex must multiply to the identity element in Γ. (Unless the amplitude involves the insertion of some twist operator at this dual vertex, see below.)

3.2 Integrability: Orbifolding the Bethe Ansatz

The field theoretic problem we are trying to solve on the gauge theory side is diagonalization of the matrix of anomalous dimensions in the large N (planar diagram) limit. It is convenient to represent the field theory operators as the spin chain states,

$$\mathcal{O}^{A_1 A_2 \dots A_L}[g] \equiv \operatorname{Tr}\left(\gamma(g)\mathcal{W}_{A_1}\mathcal{W}_{A_2}\dots\mathcal{W}_{A_L}\right) = |A_1 A_2 \dots A_L\rangle_g \,. \tag{3.5}$$

Using this terminology, the matrix of anomalous dimensions is represented as some spin chain Hamiltonian $\mathcal{H}$. Note that the basis (3.5) is overcomplete — some of the states are projected out. Another subtlety is that one can perform a cyclic permutation in the trace leading to a seemingly different spin chain representation. In the untwisted case this results in an extra requirement on the physical spin chain state (*i.e.*, one emerging from some gauge theory operator) — invariance w.r.t. the translation operator, the zero momentum condition. This particular choice of a representative makes the form of the Hamiltonian the most simple. When a nontrivial twist field $\gamma(g)$ is introduced, the zero momentum constraint gets modified.

Since all the terms in the action are untwisted, in the planar limit there should be no mixing between the sectors with different twists. A twisted sector is therefore a superselection sector: the twist $[g]$ is preserved under time-evolution defined by $\mathcal{H}$. This way we can restrict ourselves to the operators $\mathcal{O}[g]$ with a fixed class $[g]$. However, the representation of $\mathcal{H}$ as a spin chain Hamiltonian does depend on the twist sector. This dependence can be derived based on the form of the Feynman rules. The sum over the twist factors locally decouples from the remainder of the Feynman integral. In particular, the Γ-invariance of the interaction vertices of the original $\mathcal{N}=4$ Feynman diagram ensures that the cuts can be deformed and moved through the vertices, as it is indicated in Fig. 3.1. Following this procedure one can move the cuts, and translate them along the worldsheet spanned by the Feynman diagram. Evidently, we can merge cuts that are along homologous cycles on the worldsheet; the group element associated with the merged cut is the product of the original twists.[2] Proceeding this way, we can merge all the cuts and reduce the sum over the twist factors to a single set of twists associated to a generating set of non-contractible loops of the worldsheet spanned by the Feynman diagram.

Note that each operator insertion corresponds to a hole (puncture) on the graph surface, and a planar diagram that describes the leading order large N limit of amplitudes of some operators of the orbifold gauge theory (3.5), can be drawn on a cylinder (or a sphere with the two punctures). In the untwisted sector there is only one non-contractible loop wrapping the cylinder. Summation over this twist leads to projection onto the Γ-invariant states. Hence in this case, the amplitudes of the orbifold coincide with those of the $\mathcal{N}=4$ theory, as advocated. The miraculous integrability of the $\mathcal{N}=4$ theory therefore directly carries over to the untwisted sector of the orbifold gauge theory, provided it is supersymmetric.

The story with the twisted sectors is slightly more complicated. In terms of the dual graph each twist field can be represented as a tadpole ending in the corresponding puncture. A direct consequence is the fact that the standard form of the dual graph consists of one horizontal cut wrapping the cylinder and one vertical cut corresponding to the twist (Fig. 3.2). However, the extra cut can be moved away from the interaction region using the commutation relation (2.36). After this transformation the graph coincides with that of the $\mathcal{N}=4$ theory modulo renormalization $N \to |\Gamma|\, N$ and projection onto the S_g-invariant states in (3.5). Unfortunately, this equivalence extends only up to the $\ell < L$ loops. The reason for this restriction is that the ℓ-loop gauge theory Hamiltonian translates into a semi-local spin chain

[2] Summation over the different configurations leading to the same overall cut results in renormalization $N \to |\Gamma|\, N$ in the $1/N$ expansion; cf. [40].

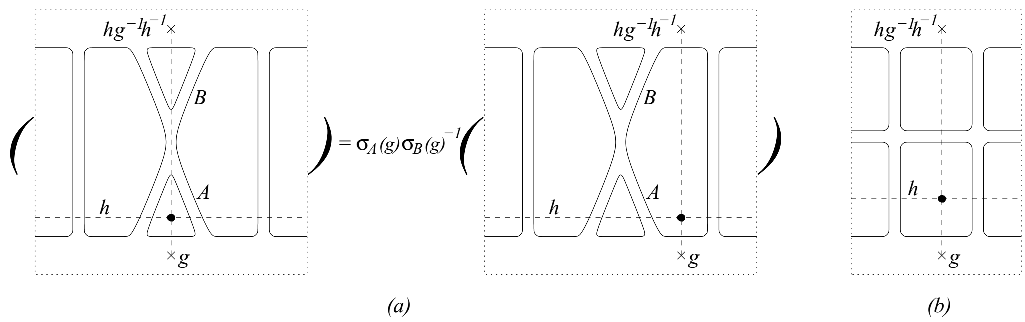


Figure 3.2: (a) A planar diagram on a cylinder. Introduction of the twist field causes appearance of an extra vertical cut in the dual graph (dotted lines). Should this cut be located in the interaction region it can be shifted away using the interchange relation (2.36). Representation matrix $\mathfrak{R}(g)$ being diagonalized, this shift results in a mere phase factor $\sigma_A(g)\sigma_B(g)^{-1}$. Note that the twist field gets conjugated, $g \to hgh^{-1}$, and this does not change its class. Then the summation over the cut h results in the projection onto the S_g-invariant subspace. (b) Diagrams with high number of loops can contain the wrapping interactions which do not reduce to the untwisted case. The diagram shown would be multiplied by an extra factor, the character $\operatorname{Tr}\mathfrak{R}(g)$ as a result of the horizontal loop wrapping the cylinder.

Hamiltonian that connects $\ell + 1$ adjacent spins. So when $\ell \geq L$, the Hamiltonian becomes fully delocalized, and includes the so-called wrapping terms, non-local interactions that wrap around the full length of the spin chain. When this is the case, the extra cut emerging from insertion of the twist field $\gamma(g)$ can no longer be shifted away from the interaction region, and some propagators inevitably cross it (Fig. 3.2b).

The conclusion is that locally, on any nearest neighbor set of spins, the interaction terms in $\mathcal{H}$ all act identically to the local interaction terms of the $\mathcal{N}=4$ Hamiltonian, as long as the local set of spins does not contain the twist operator $\gamma(g)$. If the twist generator is present in the interaction region, one could shift the twist operator to either side, until it is outside the interaction region. In this way we derive, for example, that the nearest neighbor interaction term, when acting on two spins separated by a twist $\gamma(g)$, gets modified via

$$\mathcal{H}_{[12]}\,\mathcal{W}_{A}\gamma(g)\,\mathcal{W}_{B} \;=\; \widetilde{\mathcal{H}}_{AB}^{CD}\,\mathcal{W}_{C}\gamma(g)\,\mathcal{W}_{D}, \tag{3.6}$$

where

$$\widetilde{\mathcal{H}}_{AB}^{CD} \;=\; \mathcal{H}_{AB'}^{CD'}\,\mathfrak{R}(g)_{B}^{B'}\,\overline{\mathfrak{R}}(g)_{D'}^{D}\,. \tag{3.7}$$

This relation (and analogous relations for the higher order terms) expresses the Γ-invariance of the local interaction terms of $\mathcal{H}$ — the twist field can be moved either to the left or to the right, which results in the same phase factor.

It is important that in each given twisted sector $[g]$ one can diagonalize the twist field $\gamma(g)$ and apply the methods that are used for the Abelian orbifolds to the general case.

3.2.1 Bethe Equations: A Brief Introduction

We will start with the simplest example, the periodic Heisenberg $\mathfrak{su}_2$ spin chain of length L. Each of the L spins has a two-dimensional space of states $\mathbb{C}^2$ with the basis vectors $|\!\downarrow\rangle$ and $|\!\uparrow\rangle$ corresponding to the spin being oriented downward or upward. On the field theory side this picture corresponds to the $\mathfrak{su}_2$ subsector consisting of the two scalar fields Z and W. Our goal is to diagonalize the Hamiltonian

$$\mathcal{H} \;=\; \sum_{i=1}^{L}\Big(1-\mathbf{P}_{i,i+1}\Big)\,, \tag{3.8}$$

where $\mathbf{P}_{i,i+1}$ is the interchange operator acting between the i-th and the $i+1$-th sites. We choose a vacuum state $|\!\downarrow\downarrow\ldots\downarrow\rangle$ with all spins pointing down. ($\operatorname{Tr} Z^L$ operator in field theory.) The next step is to consider states with one excitation,

$$|n\rangle \;=\; |\!\downarrow\downarrow\ldots\downarrow\uparrow_n\downarrow\ldots\downarrow\rangle \tag{3.9}$$

with the spin up being at the n-th position. One can try to find a plane wave solution in the form

$$|k\rangle \;=\; \sum_{n=1}^{L}\mathrm{e}^{ikn}\,|n\rangle\;. \tag{3.10}$$

Acting with the Hamiltonian, we get for the eigenvalue $\epsilon(k) = 1 - \cos k$. One still has to identify $|0\rangle \equiv |L\rangle$, and this leads to the periodicity condition

$$e^{ikL} = 1 . \tag{3.11}$$

Physically such a solution corresponds to a standing wave. The next step is to consider a solution with several waves. A remarkable feature of this system is its integrability. It manifests itself in the fact that the scattering reduces to the two-particle scattering, and the two-particle scattering is a mere exchange of quantum numbers. The state with the l interacting waves writes as

$$|k_1, k_2, \ldots k_l\rangle = \sum_{1 \leq n_1 < \ldots < n_l \leq L} a_{n_1, n_2, \ldots n_l}(k_1, k_2, \ldots k_l)\, |n_1, n_2, \ldots n_l\rangle . \tag{3.12}$$

The corresponding coefficients

$$a_{n_1, n_2, \ldots n_l}(k_1, k_2, \ldots k_l) = \sum_{\sigma \in \mathcal{S}_l} S(\sigma, k)\, \exp i[k_{\sigma(1)} n_1 + \cdots + k_{\sigma(l)} n_l] . \tag{3.13}$$

Here $\mathcal{S}_l$ is the group of permutations, and the phase factor $S(\sigma, k)$ obeys the group property

$$S(\sigma_1 \sigma_2, k) = S(\sigma_2, k)\, S(\sigma_1, \sigma_2 k) . \tag{3.14}$$

For the interchange of the two neighboring excitations $\sigma_{i,i+1}$ the phase factor

$$S(\sigma_{i,i+1}, k) = S(k_i, k_{i+1}) = -\frac{e^{i(k_i + k_{i+1})} + 1 - 2e^{ik_{i+1}}}{e^{i(k_i + k_{i+1})} + 1 - 2e^{ik_i}} \tag{3.15}$$

reduces to the two-particle scattering phase. Then the periodicity condition reads

$$e^{ik_1 L} \prod_{j \neq 1} S(k_1, k_j) = 1 . \tag{3.16}$$

The set of equations (3.16) is known as the Bethe ansatz equations (BAE).

These equations get modified for the orbifold gauge theory. After the diagonalization the action of the twist field $\gamma(g)$ on the fields Z and W can be brought to the form

$$g : \begin{pmatrix} Z \\ W \end{pmatrix} \to \begin{pmatrix} \omega^{s_Z} & 0 \\ 0 & \omega^{s_W} \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} , \tag{3.17}$$

where $\omega = \mathrm{e}^{2\pi i/S}$; S being the order of the element g, $g^S = 1$. As it was argued, interaction terms are unaffected by the orbifoldization procedure except for the interaction between the first and the L-th site. As it was emphasized in [51], one can use the same bulk solution (3.12), though the periodicity condition as well as the zero momentum constraint will both acquire an extra phase factor. The simplest way to find these phases is to consider the plane wave solution.

Bethe Ansatz Equations can be generalized to the chains with an arbitrary underlying symmetry (super)algebra [52],[53],[54],[55],[56]. It is convenient to use the *rapidities* λ to describe the excitations. There exist the r types of excitations, corresponding to the r simple roots. Since there can be multiple excitations of the same type it is convenient to number the corresponding spectral parameters as $\lambda_{j,k}$; where $j = 1, 2, \ldots r$ and $k = 1, 2, \ldots K_j$, K_j being the number of excitations of type j. The set of the BAE becomes

$$\mathrm{e}^{iP_{j,k}L} = \prod_{(j',k')\neq(j,k)} S_{jj'}(\lambda_{j,k}, \lambda_{j',k'})\,; \tag{3.18}$$

where the scattering matrix and momenta are given by

$$S_{jj'} = \frac{\lambda_{j,k} - \lambda_{j',k'} + \frac{i}{2}a_{j,j'}}{\lambda_{j,k} - \lambda_{j',k'} - \frac{i}{2}a_{j,j'}}\,, \qquad \mathrm{e}^{iP_{j,k}} = \frac{\lambda_{j,k} + \frac{i}{2}V_j}{\lambda_{j,k} - \frac{i}{2}V_j}\,. \tag{3.19}$$

(Here V_j are the Dynkin labels of the representation via which the algebra acts on each site — twice the spin in the $\mathfrak{su}_2$ case; and $a_{jj'}$ are the elements of the Cartan matrix.) The total energy of the corresponding eigenstate is

$$\epsilon = \sum_{j=1}^{r}\sum_{k=1}^{K_j} \epsilon_j(\lambda_{j,k})\,, \qquad \epsilon_j(\lambda_{j,k}) = \frac{V_j}{\lambda_{j,k}^2 + \frac{1}{4}V_j^2}\,. \tag{3.20}$$

The algebra behind the $\mathcal{N}=4$ supersymmetry is the $\mathfrak{su}_{2,2|4}$ superalgebra. Thus generic operators of the field theory get identified with some states of the $\mathfrak{su}_{2,2|4}$-symmetric spin chain. The whole $\mathcal{N} = 4$ theory was proved to be integrable in [14],[15],[16]. The energy eigenvalues $E_{\mathcal{O}}$ of $\mathcal{H}$ are related to the anomalous dimensions $\Delta_{\mathcal{O}}$ of local single trace operators $\mathcal{O}$ by

$$\Delta_{\mathcal{O}} = \lambda\, E_{\mathcal{O}}, \tag{3.21}$$

with $\lambda = \frac{g_{\rm YM}^2 N}{8\pi^2}$ the 't Hooft coupling.

Note that the spin chains with the different representations of $\mathfrak{su}_{2,2|4}$ correspond to different subclasses of operators in field theory. The two bosonic subalgebrae $\mathfrak{su}_{2,2}$ and $\mathfrak{su}_4$ of $\mathfrak{su}_{2,2|4}$ are nothing but the algebra of the conformal group in four dimensions and the $\mathcal{R}$-symmetry algebra. Unlike the bosonic semisimple Lie algebrae, the Dynkin diagram of a superalgebra is not unique. For the $\mathfrak{su}_{2,2|4}$ there exist the two distinguished choices of the root system, the so-called "Beauty" and the "Beast"; and they are discussed in [57]. Though the "Beast" is the most obvious system with one fermionic root, the "Beauty" root system proves useful in the context of $\mathcal{N}=4$ supersymmetry.

3.2.2 General Orbifolds

As it was argued, there is no mixing between the different twisted sectors. Furthermore, in each given twisted sector $[g]$ one can construct all the states inserting one twist field $\gamma(g)$, g being any fixed representative of the conjugacy class $[g]$. In conjunction with the fact that one can diagonalize the action of any given element $g \in \Gamma$ — the problem reduces to the Abelian case modulo some subtleties. In particular, the S_g-invariance does not completely incorporate into Bethe equations; and it is to be imposed by hand — that is why some of the Bethe eigenstates may be projected out.

Therefore, one can apply techniques similar to those used in [49] for the study of Abelian orbifolds. Then each given element $g \in \Gamma \subset \mathrm{SU}_4$ can be brought to the diagonal form so that in SU(4) it becomes

$$\mathfrak{R}(g) = \begin{pmatrix} e^{-2\pi i t_1/S} & & & 0 \\ & e^{2\pi i(t_1-t_2)/S} & & \\ & & e^{2\pi i(t_2-t_3)/S} & \\ 0 & & & e^{2\pi i t_3/S} \end{pmatrix}. \tag{3.22}$$

Here S is the order of the element g, *i.e.*, $g^S = 1$. For supersymmetric orbifolds that we consider the group Γ embeds into SU(3), and this imposes the extra restrictions on the weights t_i. Even though we need only the two independent parameters in order to describe embedding $\Gamma \subset$ SU(3), it may be convenient to keep all the three parameters t_1, t_2 and t_3 in the calculations. In particular, it may account for different embeddings SU(3) $\subset$ SU(4) or different choices of the vacuum state.

The Bethe equations for the complete $\mathfrak{su}_{2,2|4}$ algebra acquire some extra phases:

$$\left(\frac{\lambda_{j,k} + \frac{i}{2}V_j}{\lambda_{j,k} - \frac{i}{2}V_j}\right)^L = \mathfrak{R}_j(g) \prod_{(j'.k')\neq(j,k)} \frac{\lambda_{j,k} - \lambda_{j',k'} + \frac{i}{2}a_{j,j'}}{\lambda_{j,k} - \lambda_{j',k'} - \frac{i}{2}a_{j,j'}}. \tag{3.23}$$

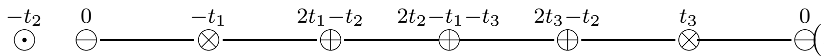
Similarly, the momentum constraint reads

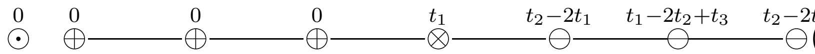
$$\mathfrak{R}_0(g) \prod_{j=1}^{7} \prod_{k=0}^{K_j} \frac{\lambda_{j,k} + \frac{i}{2}V_j}{\lambda_{j,k} - \frac{i}{2}V_j} = 1\,. \tag{3.24}$$

The phase factors

$$\mathfrak{R}_j(g) = e^{2\pi i q_j/S}\,, \tag{3.25}$$

where the integers q_j depend on the choice of the root system:

"Beauty" :  (3.26)

"Beast" :  (3.27)

(The leftmost "root" corresponds to the phase $\mathfrak{R}_0(g) = e^{2\pi i q_0/S}$.) Let us stress that this structure is the direct generalization of that in the $\mathfrak{su}_2$ subsector: the bulk ansatz remains unaltered, while the boundary conditions get modified. Recall that in the $\mathfrak{su}_2$ case there is a single root $\gamma_1 = \alpha_{12}$, and the weight $q_1 = s_W - s_Z \equiv s_2 - s_1$. Analogously, for an excitation associated with some simple root $\gamma = \alpha_{ij}$ the corresponding weight $q_\gamma = s_j - s_i$ is the difference of the two corresponding charges. The number q_0 is determined by the choice of the Bethe vacuum.

There is an elegant way to summarize all the Bethe equation and momentum constraint together. In order to do this one introduces the two new types of excitations to the existing seven types ($j = 1, \ldots, \operatorname{rk}\mathfrak{su}_{2,2|4} = 7$). The quasi-excitation of type $j = 0$ corresponds to the insertion of a new spin chain site. In order to have a length L chain one is to insert exactly the $K_0 = L$ excitations of type 0. The quasi-excitation of type $j = -1$ corresponds to the twist field. The scattering phases of the excitations are[3]

$$S_{j,j'} = \frac{\lambda_{j,k} - \lambda_{j',k'} + \frac{i}{2}a_{j,j'}}{\lambda_{j,k} - \lambda_{j',k'} - \frac{i}{2}a_{j,j'}}, \tag{3.28}$$

$$S_{j,0} = \frac{\lambda_{j,k} - \frac{i}{2}V_j}{\lambda_{j,k} + \frac{i}{2}V_j}, \quad S_{j,-1} = \mathfrak{R}_j(g); \tag{3.29}$$

$$S_{0,0} = 1, \quad S_{0,-1} = \mathfrak{R}_0(g). \tag{3.30}$$

Type 0 excitation do not have an associated spectral parameter, while type -1 excitations can have different twist classes $[g]$. Excitations of both type 0 and -1 do not contribute to the total energy.

With these notations we can therefore summarize all the Bethe equations as

$$\prod_{j'=-1}^{J} \prod_{\substack{k'=1 \\ (j',k')\neq(j,k)}}^{K_{j'}} S_{j,j'}(\lambda_{j,k}, \lambda_{j',k'}) = 1. \tag{3.31}$$

The equations for $j = 1, \ldots, 7$ give the BAE (3.23), equation for $j = 0$ gives the

[3]Note that the scattering phase $S_{-1,-1}$ is not needed as we restrict ourselves to one excitation of type -1. Even though one may introduce several such excitations it would cause some unnecessary technical difficulties. As it was shown, insertion of a single twist field suffices to produce all the orbifold states.

momentum constraint (3.24),[4] and equation for $j = -1$ gives the "zero charge condition"

$$\mathfrak{R}_0(g)^L \prod_{j'=1}^{7} \mathfrak{R}_{j'}(g)^{K_{j'}} . \tag{3.32}$$

It implies the g-invariance of the corresponding state in the field theory. Again, let us stress that for a generic orbifold this condition is not sufficient, and there should be imposed a more restrictive invariance condition w.r.t. the full stabilizer S_g. As a result, some of the Bethe eigenstates may be projected out in field theory.

3.3 The Two Example Quivers

Here we study application of the Bethe equations to the two example quivers, ones with both Abelian and non-Abelian orbifold group. For these simple examples one can easily determine the anomalous dimensions of operators in the twisted sectors. Then these operators can be recast into the quiver notation. Generally operators corresponding to the closed paths in the quiver are *not* the eigenvectors of the matrix of anomalous dimensions. In other words, an operator corresponding to a closed loop in the quiver is typically a mix of operators with different conformal dimensions; neither does it belong to a given twisted sector.

3.3.1 Abelian $\mathbb{Z}_6$ Quiver

Here we consider a simple example, $\mathbb{Z}_6$ quiver (see Fig. 3.3). We restrict ourselves to the $\mathfrak{su}_2$ subsector formed by the two scalars, Z with charge $s_Z = 1$ and W with charge $s_W = -2$. We will study the twisted sector with the twist γ^n, $n = 0, \ldots, 5$; γ being the generating element of $\mathbb{Z}_6$. Let us choose the length of the spin chain $L = 3$; then the vacuum can be chosen as $\mathrm{Tr}\left[\gamma ZZZ\right]$ — note that it will be projected out.

[4]Although there are L quasi-excitations of type 0, there is only one corresponding Bethe equation, because all of these quasi-excitations are equivalent, and they have no spectral parameter which might distinguish them.

There also exist the excited states with one or three W's, while the states with the two excitations will also be projected out. Our goal will be to describe these Bethe

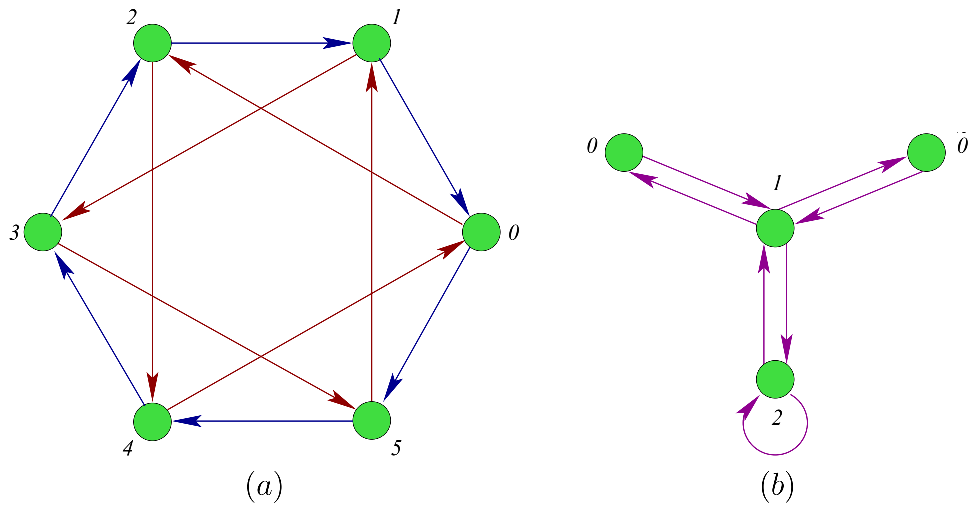


Figure 3.3: (a) The $\mathbb{Z}_6$ quiver. There are the six nodes corresponding to the six representations of $\mathbb{Z}_6$. We show only the scalar lines corresponding to the fields Z (blue lines) transforming in $\mathfrak{R}_Z \simeq \rho_1$ and W (red lines) transforming in $\mathfrak{R}_W \simeq \rho_4 \simeq \rho_{-2}$. (b) The D_5 quiver with the two-dimensional defining representation $\mathfrak{R} \simeq \rho_1$. Note that for these two quivers we show only the lines corresponding to the $\mathfrak{su}_2$ subsector; *i.e.*, the two scalar fields.

vectors in terms of the quiver notation. By $\mathcal{O}_{ijki} \equiv \operatorname{Tr} \Phi^i_j \Phi^j_k \Phi^k_i$ we will denote the quiver gauge theory operator corresponding to the closed cycle between the three nodes $k \to j \to i$ in the quiver.

Note that the state with the three excitations is unique in each given twisted sector, and it corresponds to the field theory operator $\operatorname{Tr}\left[\gamma^n WWW\right]$. Commuting the twist field γ^n with one of the fields W we find that

$$\operatorname{Tr}\left[\gamma^n WWW\right] = \mathrm{e}^{2\pi i n s_W/6} \operatorname{Tr}\left[\gamma^n WWW\right]; \tag{3.33}$$

i.e., this state is projected out in all sectors except for $n = 0$ and $n = 3$. The reason for this is the extra symmetry: it is sufficient to commute the twist field with only one of the three W fields. Note that the total charge of the three fields W is zero, and normally one would expect $\operatorname{Tr}\left[\gamma WWW\right]$ to be a non-trivial operator.

Using the formula (2.44) we find that

$$\mathcal{O}_{0420} = \mathrm{Tr}\left[WWW\right] + \mathrm{Tr}\left[\gamma^3 WWW\right], \qquad (3.34)$$

$$\mathcal{O}_{1531} = \mathrm{Tr}\left[WWW\right] - \mathrm{Tr}\left[\gamma^3 WWW\right]; \qquad (3.35)$$

or

$$\mathrm{Tr}\left[WWW\right] = \frac{1}{2}\mathcal{O}_{0420} + \frac{1}{2}\mathcal{O}_{1531}, \qquad (3.36)$$

$$\mathrm{Tr}\left[\gamma^3 WWW\right] = \frac{1}{2}\mathcal{O}_{0420} - \frac{1}{2}\mathcal{O}_{1531}. \qquad (3.37)$$

Graphically the operators $\mathcal{O}_{042}$ and $\mathcal{O}_{153}$ correspond to the two closed triangles formed by the red lines. Applying the Hamiltonian we find the anomalous dimensions

$$\Delta_{\mathrm{Tr}\left[\gamma^3 WWW\right]} = \Delta_{\mathrm{Tr}\left[\gamma^3 WWW\right]} = 0. \qquad (3.38)$$

The states with one excitation have the form $\mathrm{Tr}\left[\gamma^n ZZW\right]$, and there is one such state in each given twisted sector. These operators correspond to the triangles with the two blue (field Z) and one red (field W) line. There are six such triangles and there are six different operators with $n = 1, \ldots, 5$ — these numbers coincide as we expect. The transition formula between these two descriptions is

$$\mathcal{O}_{l,\,l+1,\,l+2,\,l} = \sum_{n=0}^{5} \mathrm{e}^{-2\pi i \frac{ln}{6}}\, \mathrm{Tr}\left[\gamma^n ZZW\right]; \qquad (3.39)$$

performing the Fourier transform yields

$$\mathrm{Tr}\left[\gamma^n ZZW\right] = \frac{1}{6}\sum_{l=0}^{5} \mathrm{e}^{2\pi i \frac{ln}{6}} \mathcal{O}_{l,\,l+1,\,l+2,\,l}. \qquad (3.40)$$

These operators diagonalize the matrix of anomalous dimensions. Direct application of the Hamiltonian shows that the corresponding eigenvalues are

$$\Delta_{\mathrm{Tr}\left[\gamma^n ZZW\right]} = 4\lambda \sin^2 \frac{\pi n}{6}. \qquad (3.41)$$

This simple example illustrates the interrelation of the two descriptions in the orbifold gauge theory. First, the quiver description gives a very clear understanding of what the physical fields and gauge invariant operators are, while in the "orbit" description using the twist fields some of the operators may be projected out. On the other hand, the description using the twist fields proves to be more robust for studying the field theory dynamics (the matrix of anomalous dimensions). In order to illustrate this let us write the part of the action responsible for the non-trivial part of the interaction Hamiltonian, $\mathrm{Tr}\left[ZWZ^\dagger W^\dagger + WZW^\dagger Z^\dagger\right]$. In terms of the quiver notation

$$\begin{aligned}\mathrm{Tr}\left[ZWZ^\dagger W^\dagger + WZW^\dagger Z^\dagger\right] &= \sum_l \left[\mathcal{O}_{l,l+1,l-1,l-2,l} + \mathcal{O}_{l,l-2,l-1,l+1,l}\right] = \\ &= \sum_l \mathrm{Tr}\left[Z^l_{l+1}\, W^{l+1}_{l-1}\, Z^{l-2\,\dagger}_{l-1}\, W^{l-2\,\dagger}_{l} + W^l_{l-2}\, Z^{l-2}_{l-1}\, W^{l+1\,\dagger}_{l-1}\, Z^{l\,\dagger}_{l+1}\right]. \qquad (3.42)\end{aligned}$$

Here Z^k_l denotes the field corresponding to the quiver arrow going from node l to node k. Note that the conjugation changes the direction of the corresponding arrow; *e.g.*, $Z^{1\,\dagger}_2$ is an arrow going from node 1 to node 2. Indeed, as we see, studying the matrix of anomalous dimensions using the quiver notation would have been more complicated.

3.3.2 Non-Abelian D_5 Quiver

Next we consider a simple orbifold with a non-Abelian discrete group D_5 (the facts about the dihedral group D_S as well as its representation ring are given in Appendix A.2.) The corresponding quiver is shown in Fig. 3.3. Again, we study the $\mathfrak{su}_2$ sector, and the scalar field Φ^I transforms in the two-dimensional representation $\mathfrak{R} \simeq \rho_1$. From the quiver representation it is clear that there are the four different operators of length $L = 2$; namely, those are

$$\mathcal{O}_{010}\,, \quad \mathcal{O}_{\tilde{0}1\tilde{0}}\,, \quad \mathcal{O}_{121}\,, \quad \mathcal{O}_{222}\,. \qquad (3.43)$$

On the other hand, there are the four different twist classes, $\{[e],\ [r],\ [r^2],\ [\sigma]\}$. Applying the definitions of the operators (A.28), we see that in each twist class there is exactly one non-trivial operator; thus there are the total of four operators of length two:

$$\mathcal{O}_e = \operatorname{Tr}\big[ZW\big]\,,\ \ \mathcal{O}_r = \operatorname{Tr}\big[\gamma(r)ZW\big]\,,\ \ \mathcal{O}_{r^2} = \operatorname{Tr}\big[\gamma(r^2)ZW\big]\,,\ \ \mathcal{O}_\sigma = \operatorname{Tr}\big[\gamma(\sigma)ZZ\big]\,. \tag{3.44}$$

Here Z and W denote the first and second components of the field Φ^I. Note that the product of the two fields ZZ has transforms non-trivially under the action of r; nevertheless, in the sector with twist $[\sigma]$ the operator $\mathcal{O}_\sigma = \operatorname{Tr}\big[\gamma(\sigma)ZZ\big]$ is non-trivial as $r \notin S_\sigma$. The absence of mixing between the different twist classes ensures that the set of operators $\{\mathcal{O}_e,\, \mathcal{O}_r,\, \mathcal{O}_{r^2},\, \mathcal{O}_\sigma\}$ diagonalize the matrix of anomalous dimensions. Acting with the Hamiltonian we find the corresponding anomalous dimensions as

$$\begin{aligned} \Delta_{\mathcal{O}_e} &= 0\,, & \Delta_{\mathcal{O}_r} &= 4\lambda\sin^2\frac{\pi}{5} = \frac{5-\sqrt{5}}{2}\,\lambda\,, \\ \Delta_{\mathcal{O}_{r^2}} = 4\lambda\sin^2\frac{2\pi}{5} &= \frac{5+\sqrt{5}}{2}\,\lambda\,, & \Delta_{\mathcal{O}_\sigma} &= 0\,. \end{aligned} \tag{3.45}$$

The same eigenvalues can be obtained solving the Bethe equations. In this formalism the three operators $\mathcal{O}_e$, $\mathcal{O}_r$ and $\mathcal{O}_{r^2}$ are the states with one excitation. Diagonalizing the twist field as

$$\gamma(g) = \begin{pmatrix} \mathrm{e}^{i\alpha} & 0 \\ 0 & \mathrm{e}^{-i\alpha} \end{pmatrix}\,, \qquad g = e, r, r^2\,; \tag{3.46}$$

we find that the Bethe equation and the momentum constraint reduce to

$$\frac{\lambda + \frac{i}{2}}{\lambda - \frac{i}{2}} = \mathrm{e}^{i\alpha}\,, \qquad \epsilon = \frac{1}{\lambda^2 + \frac{1}{4}}\,. \tag{3.47}$$

This gives

$$\lambda = \frac{1}{2}\cot\frac{\alpha}{2}\,, \qquad \epsilon = 4\sinh^2\frac{\alpha}{2}\,. \tag{3.48}$$

For the twist element $g = e,\ r,\ r^2$ we have $\alpha = 0,\ 2\pi/5,\ 4\pi/5$ correspondingly. This reproduces the correct result. The twist field $\gamma(g)$ is non-diagonal. After the diagonalization of $\gamma(\sigma)$ operator $\mathcal{O}_\sigma$ maps to the vacuum state, and that is why $\Delta_{\mathcal{O}_\sigma} = 0$.

The next step is to find the dictionary between the two notations. In order to do this one can start with the quiver notation and rewrite the corresponding operators using the transition rules (A.21) and (A.22) from Appendix A.1. The two operators $\mathcal{O}_{010}$ and $\mathcal{O}_{\tilde{0}1\tilde{0}}$ correspond to the closed paths $\rho_0 \leftarrow \rho_1 \leftarrow \rho_0$ and $\rho_{\tilde{0}} \leftarrow \rho_1 \leftarrow \rho_{\tilde{0}}$. Since the representations ρ_0 and $\rho_{\tilde{0}}$ are one-dimensional, the corresponding invariant tensors

$$\mathcal{K}_{AB}{}_1^1 = \mathcal{K}_{AB} \tag{3.49}$$

(the indices A, B belong to the defining representation $\mathfrak{R} \simeq \rho_1$.) The non-zero components are

$$\mathcal{K}_{12} = \mathcal{K}_{21} = \frac{1}{\sqrt{2}} \tag{3.50}$$

(note that the normalization respects the unitarity condition.) Then

$$\mathcal{K}_{AB}(g) = \mathcal{K}_{AB}\,\overline{\rho_\lambda}(g)\,, \qquad \lambda = 0, \tilde{0}\,. \tag{3.51}$$

This gives

$$\begin{aligned} \mathcal{O}_{010} &= \sqrt{2}\,\mathrm{Tr}\left[ZW + (1+\omega)\gamma(r)ZW + (1+\omega^2)\gamma(r^2)ZW + 5\gamma(\sigma)ZZ\right] \\ &= \sqrt{2}\Big[\mathcal{O}_e + (1+\omega)\mathcal{O}_r + (1+\omega^2)\mathcal{O}_{r^2} + 5\mathcal{O}_\sigma\Big]\,; \qquad (3.52) \\ \mathcal{O}_{\tilde{0}1\tilde{0}} &= \sqrt{2}\,\mathrm{Tr}\left[ZW + (1+\omega)\gamma(r)ZW + (1+\omega^2)\gamma(r^2)ZW - 5\gamma(\sigma)ZZ\right] \\ &= \sqrt{2}\Big[\mathcal{O}_e + (1+\omega)\mathcal{O}_r + (1+\omega^2)\mathcal{O}_{r^2} - 5\mathcal{O}_\sigma\Big]\,. \qquad (3.53) \end{aligned}$$

(We have used the permutation relation (A.31).)

Next, $\mathcal{O}_{121}$ corresponds to the product of the two tensors,

$$\mathcal{K}_{AB}{}_l^k = \sum_{p\in\rho_2} \mathcal{K}_{Al}^p\,\mathcal{K}_{Bp}^k\,, \qquad k, l \in \rho_1\,. \tag{3.54}$$

The non-trivial coefficients corresponding to the decomposition $\mathfrak{R} \otimes \rho_1 \to \rho_2$ are $\mathcal{K}^1_{11} = \mathcal{K}^2_{22} = 1$, while those corresponding to the decomposition $\mathfrak{R} \otimes \rho_2 \to \rho_1$ are $\mathcal{K}^1_{21} = \mathcal{K}^2_{12} = 1$. This gives the corresponding invariant tensor in (3.54):

$$\mathcal{K}_{12}{}^1_1 = \mathcal{K}_{21}{}^2_2 = 1\,. \tag{3.55}$$

Therefore, one identifies

$$\mathcal{O}_{121} \;=\; 2\big[\mathcal{O}_e + (\omega^2+\omega^4)\mathcal{O}_r + (\omega^3+\omega^4)\mathcal{O}_{r^2}\big]\,. \tag{3.56}$$

Similarly, for the operator $\mathcal{O}_{222}$ we need to find the decomposition $\mathfrak{R}\otimes\, \rho_2 \to \rho_2$. The non-trivial coefficients are $\mathcal{K}^2_{11} = \mathcal{K}^1_{22} = 1$. Consequently,

$$\mathcal{K}_{12}{}^1_1 = \mathcal{K}_{21}{}^2_2 = 1 \tag{3.57}$$

and

$$\mathcal{O}_{222} \;=\; 2\big[\mathcal{O}_e + 2\omega^3\mathcal{O}_r + (1+\omega)\mathcal{O}_{r^2}\big]\,. \tag{3.58}$$

These formulae give the transition between the two bases in the operator space.

3.4 Discussion

Integrability of the $AdS^5 \times S_5$ duality extends to the orbifold theories with minor modifications. In particular, in a given twisted sector $[g]$ the BAE reduce to those in the Abelian theory; though some of the states may still be projected out. As a general rule, which states survive the projection is determined by the invariant tensors of the stabilizer subgroup S_g; although there can be present extra symmetries projecting out some of the conceivably non-trivial states. Exactly as in the Abelian case, orbifoldization amounts to appearance of the fractional mode numbers, on both the closed string and the Bethe equations side [34]. Given the well established full one-loop agreement between the classical energies and anomalous dimensions as functions of the mode numbers in [58],[59],[60], the correspondence holds for the arbitrary orbifold at one loop. On the other hand, in the quiver gauge theory notation problems with some states being projected out do not appear, but the matrix of anomalous dimensions becomes more complicated. We demonstrate the equivalence of the two approaches using some example orbifolds. The higher loop techniques described in [49], [61] are likely to apply to the non-Abelian case as well. This would open a possibility of applying the existing powerful integrability techniques to the quiver gauge theories with reduced supersymmetry. One of the things to be verified is that the duality relations between the roots of different types are not violated (*e.g.*, by some states being projected out).

Chapter 4

$\mathcal{I}$-odd Sector of the KS Theory

Klebanov-Strassler solution provides a rich yet solvable example of gauge/string correspondence. For earlier work leading up to this duality, see [19, 22, 23, 24], and for reviews [62, 63]. This background demonstrates in a geometrical language such features of the $SU(M)$ supersymmetric gluodynamics as color confinement and the breaking of the $\mathbb{Z}_{2M}$ chiral R-symmetry down to $\mathbb{Z}_2$ via gluino condensation [21]. In fact, it has been argued [21] that by reducing the continuous parameter $g_s M$ one can interpolate between the cascading theory solvable in the supergravity limit and $\mathcal{N} = 1$ supersymmetric $SU(M)$ gauge theory.

An important aspect of the low-energy dynamics is that the baryonic $U(1)_B$ symmetry is broken spontaneously by the condensates of baryonic operators $\mathcal{A}$ and $\mathcal{B}$, whose explicit expressions in the infrared $SU(2M) \times SU(M)$ gauge theory are

$$\begin{aligned} \mathcal{A} &\sim \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}}(A_1)_1^{\alpha_1}(A_1)_2^{\alpha_2}\ldots(A_1)_M^{\alpha_M}(A_2)_1^{\alpha_{M+1}}(A_2)_2^{\alpha_{M+2}}\ldots(A_1)_M^{\alpha_{2M}} \ , \\ \mathcal{B} &\sim \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}}(B_1)_1^{\alpha_1}(B_1)_2^{\alpha_2}\ldots(B_1)_M^{\alpha_M}(B_2)_1^{\alpha_{M+1}}(B_2)_2^{\alpha_{M+2}}\ldots(B_1)_M^{\alpha_{2M}} \ . \end{aligned} \tag{4.1}$$

This phenomenon, anticipated in the cascading gauge theory in [21, 64], was later demonstrated on the supergravity side where the fluctuations corresponding to the pseudoscalar Goldstone boson and its scalar superpartner [65], as well as the fermionic superpartner [66], were identified. Furthermore, finite deformations along the scalar direction give rise to a continuous family of supergravity solutions [67, 68, 69] dual to the baryonic branch, $\mathcal{AB} = \text{const}$, of the gauge theory moduli space.

The main goal of the present chapter is to find an exhaustive list of bound states in the KS theory which are singlet w.r.t. the global SU(2) $\times$ SU(2) symmetry and odd w.r.t. the $\mathbb{Z}_2$ $\mathcal{I}$-symmetry. In particular, this would give a deeper understanding of the GHK scalar fluctuations (U, a) [65] and their radial excitations as well as their supermultiplet structure. On the other, we would like to shed new light on the normal modes of the warped deformed conifold throat embedded into a string compactification, which has played a role in models of moduli stabilization [70] and D-brane inflation [71, 72].

The problem of finding the spectra of bound states at large g_sM can be mapped to finding normalizable fluctuations around the supergravity background. This problem is complicated by the presence of 3-form and 5-form fluxes, but some results on the spectra are already available in the literature [73, 74, 65, 75, 76, 66, 77]. A particularly impressive effort was made by Berg, Haack and Mück (BHM) who used a generalized PT ansatz [78] to derive and numerically solve a system of seven coupled scalar equations [75, 76]. Each of the resulting glueballs is even under the charge conjugation $\mathcal{I}$-symmetry preserved by the KS solution, and therefore has $J^{PC} = 0^{++}$. In the present thesis we study some other families of glueballs, those which are odd under the $\mathcal{I}$-symmetry. We find several scalar and vector excitations which organize into the four-dimensional supersymmetry multiplets. Recall that a massive multiplet of the supersymmetry algebra in four dimensions consists of the members with spin $(j\,, j+\frac{1}{2}\,, j-\frac{1}{2}\,, j)$ (e.g., [79]). We find three multiplets with $j = \frac{1}{2}$ and two multiplets with $j = 1$. The $j = \frac{1}{2}$ multiplets generalize the zero momentum case studied in [65].

The chapter is structured as follows. In the next two sections we write down the exhaustive list of the $\mathcal{I}$-odd SU(2) $\times$ SU(2)-singlet excitations over the KS background and discuss their behavior in the conformal KW limit. In Section 4.4 we study a generalization of the ansatz for the NSNS 2-form and metric perturbations that allows us to study radial excitations of the GHK scalar mode. We derive a system of coupled radial equations with the mass of the excitation as a spectral parameter. In Section 4.5 we show that a similar ansatz for the RR 2-form perturbation decouples from the metric giving rise to a single decoupled equation for pseudoscalar glueballs. In Sections 4.6 and 4.7 we write the equations of motion for the most general vector ansatz and disentangle them. The resulting glueballs give the vector superpartners of the scalars found in Section 4.4 and 4.5, thus completing the bosonic content of the three $j = \frac{1}{2}$ multiplets. In addition to these there are found the two $j = 1$ multiplets, bosonic content of each of them being the two vectors. In Section 4.8 we find the mass spectra of the members of the multiplets we have found. This is done using either the shooting method or its generalization, the determinant method. Section 4.9 discusses some subtleties related to the non-uniqueness of the equations of motion. In Section 4.10 we construct the operators of the dual gauge theory. Section 4.11 discusses the effects of compactification and possible cosmological applications. We give a perturbative treatment of the coupled equations for small mass that allows us to study the scalar mass in models where the length of the throat is finite.

4.1 Geometry of the KS Solution

The Klebanov-Strassler supergravity solution, which corresponds to a certain vacuum of the $\mathrm{SU}(k(M+1)) \times \mathrm{SU}(kM)$ gauge theory [21], provides an interesting and rich example of the gauge/string duality.

The ten dimensional metric for the KS solution is

$$ds^2_{10} = h(\tau)^{-1/2}(-dt^2 + dx^2 + dy^2 + dz^2) + h(\tau)^{1/2} ds^2_6 \ , \tag{4.2}$$

where

$$\begin{aligned} ds^2_6 = \frac{\epsilon^{4/3} K}{2} \bigg[& \frac{1}{3K^3}(d\tau^2 + (g_5)^2) \\ & + \cosh^2\left(\frac{\tau}{2}\right)((g^3)^2 + (g^4)^2) + \sinh^2\left(\frac{\tau}{2}\right)((g^1)^2 + (g^2)^2) \bigg] \end{aligned} \tag{4.3}$$

is the usual warped deformed conifold metric. The volume form is

$$\mathrm{vol} = \frac{\epsilon^4}{96} h^{1/2} \sinh^2 \tau dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 \,. \tag{4.4}$$

Here the auxiliary function $K(\tau)$ is

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3} \sinh \tau} \,, \tag{4.5}$$

and the warp factor h is

$$h(\tau) = (g_s M \alpha')^2 2^{2/3} \varepsilon^{-8/3} I(\tau) \,; \tag{4.6}$$

where

$$I(\tau) \equiv \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3} \,. \tag{4.7}$$

The NSNS two-form field and corresponding field strength are

$$B_2 = \frac{g_s M \alpha'}{2} [f(\tau) g^1 \wedge g^2 + k(\tau) g^3 \wedge g^4] \,, \tag{4.8}$$

$$\begin{aligned} H_3 = dB_2 \quad &= \quad \frac{g_s M \alpha'}{2} \Big[d\tau \wedge (f' g^1 \wedge g^2 + k' g^3 \wedge g^4) \\ &\quad + \frac{1}{2} (k - f) g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \Big] \;; \end{aligned} \tag{4.9}$$

while the RR three-form field strength is

$$\begin{aligned} F_3 \quad &= \quad \frac{M\alpha'}{2} \Big[g^5 \wedge g^3 \wedge g^4 + d[F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4)] \Big] \qquad (4.10) \\ &= \quad \frac{M\alpha'}{2} \Big[g^5 \wedge g^3 \wedge g^4 (1 - F) + g^5 \wedge g^1 \wedge g^2 F + F' d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \Big] \,. \end{aligned}$$

The auxiliary functions in these forms are

$$\begin{aligned} F(\tau) \quad &= \quad \frac{\sinh \tau - \tau}{2 \sinh \tau} \,, \\ f(\tau) \quad &= \quad \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau - 1) \,, \\ k(\tau) \quad &= \quad \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau + 1) \,. \end{aligned} \tag{4.11}$$

The five-form field strength is given by

$$F_5 \;=\; (1+*)B_2 \wedge F_3 \;\equiv\; (1+*)\,\ell(\tau)\omega_2 \wedge \omega_3\,. \tag{4.12}$$

The forms ω_2 and ω_3 are

$$\omega_2 \;=\; \frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4)\,, \tag{4.13}$$

$$\omega_3 \;=\; \omega_2 \wedge g^5\,. \tag{4.14}$$

The auxiliary function ℓ can be expressed as

$$\ell(\tau) \;=\; 2f + 4FF' \;\equiv\; 2f(1-F) + 2kF\,. \tag{4.15}$$

Note that in the KT (large τ) limit [24] one has

$$H_3 \;=\; \frac{g_S M\alpha'}{2} d\tau \wedge \omega_2\,, \tag{4.16}$$

$$F_3 \;=\; \frac{M\alpha'}{2}\omega_3\,. \tag{4.17}$$

The KS solution is invariant under the $\mathbb{Z}_2$ symmetry $\mathcal{I}$, which acts by exchanging the two two-spheres of the deformed conifold accompanied by the inversion of sign of the 3-form flux H_3 and F_3. On the field theory side this symmetry exchanges and conjugates the bi-fundamental fields A and B. Thus the KS solution corresponds to one particular $\mathcal{I}$-invariant vacuum $|A|^2 = |B|^2$. The latter spontaneously breaks $U(1)_{\rm Baryon}$ symmetry $A \rightarrow Ae^{ia}$, $B \rightarrow Be^{-ia}$. The corresponding massless Goldstone pseudoscalar a combines with the scalar $U \sim |A|^2 - |B|^2$ into a $\mathcal{I}$-odd scalar supermultiplet [65]. While a corresponds to the longitudinal part of the $U(1)_{\rm Baryon}$ current $J_\mu = \partial_\mu a$, the fluctuation of U changes the expectation values of the baryon operators $\mathcal{A}$, $\mathcal{B}$ and moves the theory along the baryonic branch of the moduli space [65, 67, 68, 69]. Out of the SU(2) $\times$ SU(2)-invariant forms on $T^{1,1}$, g^5 and dg^5 are $\mathcal{I}$-even; while $g^1 \wedge g^2$, $g^3 \wedge g^4$ and $g^1 \wedge g^3 + g^2 \wedge g^4$ are $\mathcal{I}$-odd.

4.2 General Ansatz for $\mathcal{I}$-odd Excitations

We consider the $\mathcal{I}$-odd supergravity excitations over the KS background which are singlets w.r.t. the action of the global SU(2) × SU(2) symmetry group. The $\mathcal{I}$-symmetry of the KS solution acts on the conifold geometry by interchanging the two spheres (θ_1, ϕ_1) and (θ_2, ϕ_2), simultaneously changing the sign of F_3 and H_3. Hence we are looking for the perturbations of B_2 and C_2 invariant under the exchange of the two-spheres and the perturbations of metric and C_4 which are odd under the interchange $(\theta_1, \phi_1) \leftrightarrow (\theta_2, \phi_2)$.

The list of SU(2) × SU(2)-invariant forms on the conifold includes the one-form $\mathrm{d}\tau$ along the radius and invariant forms on the "base" of the deformed conifold $T^{1,1}$. There is a unique invariant one-form g^5, which is $\mathcal{I}$-even. It satisfies

$$\star \mathrm{d} \star \mathrm{d}g^5 = 8g^5. \tag{4.18}$$

(Below $\star$ will denote the Hodge operation on $T^{1,1}$, while $*$ and $*_4$ will refer to the same operation in the ten-dimensional or four-dimensional spaces respectively.)

There are the three $\mathcal{I}$-odd SU(2) × SU(2)-invariant two-forms: $g^1 \wedge g^2$, $g^3 \wedge g^4$ and $g^1 \wedge g^3 + g^2 \wedge g^4$. In addition there are the two $\mathcal{I}$-even two-forms $\mathrm{d}g_5 = -(g^1 \wedge g^4 + g^3 \wedge g^2)$ and $\mathrm{d}\tau \wedge g^5$, which are not independent. Indeed, any fluctuation including $\mathrm{d}g_5$ can be transformed into the fluctuation with g_5 or $\mathrm{d}\tau \wedge g^5$ with the help of a suitable gauge transformation.

The invariant two-forms mentioned above can be combined into two eigenvectors of the Laplace-Beltrami operator on $T^{1,1}$ as follows:

$$\omega_2 = g^1 \wedge g^2 + g^3 \wedge g^4 \,, \tag{4.19}$$

$$Y_2 = (g^1 \wedge g^2 - g^3 \wedge g^4) + i(g^1 \wedge g^3 + g^2 \wedge g^4) \,. \tag{4.20}$$

They satisfy

$$\mathrm{d} \star \omega_2 = 0 \,, \qquad \mathrm{d}\omega_2 = 0 \,, \tag{4.21}$$

$$\mathrm{d} \star Y_2 = 0 \,, \qquad \star \mathrm{d}Y_2 = 3i\, Y_2 \,. \tag{4.22}$$

There are also three and four-forms on $T^{1,1}$ invariant under $\mathrm{SU}(2)\times\mathrm{SU}(2)$, but they all can be obtained from the forms above using the exterior differentiation and the Hodge transformation. The only $\mathcal{I}$-odd $\mathrm{SU}(2)\times\mathrm{SU}(2)$-invariant metric fluctuation is $g^1\cdot g^2+g^3\cdot g^4$.

The Hodge duality in Minkowski space allows one to relate the p- and $(4-p)$-forms to each other. That is why the general ansatz can be written in terms of zero, one and two-forms in Minkowski space. It is also known that any form has a Hodge decomposition into the sum of an exact, co-exact and harmonic parts. The field theory in the $\mathbb{Z}_2$-symmetric vacuum dual to the KS background does not have any spontaneously broken symmetries besides $U(1)_{\mathrm{Baryon}}$. Therefore we do not expect any $\mathrm{SU}(2)\times\mathrm{SU}(2)$ singlet massless particles in addition to those associated with the baryonic branch of the moduli space. The latter were studied in [65, 66]. That is why we are looking only for massive excitations; i.e., all four-dimensional forms P_k in our ansatz satisfy

$$\Box_4 P_k = m^2 P_k \tag{4.23}$$

with some non-zero m^2. It means that the harmonic part is absent from the decomposition (which is not generally the case for the four-dimensional massless modes). Therefore, any two-form P_2 can be written using the two vectors (one-forms) $\mathbf{M}$ and $\mathbf{N}$:[1]

$$P_2 \;=\; \mathrm{d}_4\mathbf{M} + *_4\mathrm{d}_4\mathbf{N}\,. \tag{4.24}$$

Similarly, any vector $\mathbf{N}$ can be represented as a sum of an exact and a co-closed parts:

$$\mathbf{N} \;=\; \mathrm{d}_4\chi + \tilde{\mathbf{N}}\,, \tag{4.25}$$

where

$$\mathrm{d}_4 *_4 \tilde{\mathbf{N}} \;=\; 0\,. \tag{4.26}$$

[1] We use the boldface notation for the spin 1 excitations (vectors) in four dimensions.

This consideration shows that all the $\mathcal{I}$-odd excitations over the KS background reduce to some ansatz involving vectors and scalars. At this point we do not make a distinction between the particles with different behavior with respect to parity; i.e., vectors and axial vectors, scalars and axial scalars. For the sake of simplicity we call all states of spin 1 "vectors" and all states of spin 0 "scalars". The quantum numbers of the physical states, including parity, will be given in Figure 4.1 in Section 4.8.

The most general scalar ansatz consists of the two decoupled systems of excitations:

$$\begin{aligned} \delta B_2 &= \chi(x,\tau)\,\mathrm{d}g^5 + \partial_\mu \sigma(x,\tau)\,\mathrm{d}x^\mu \wedge g^5\,, \\ \delta G_{13} &= \delta G_{24} = U(x,\tau)\,; \end{aligned} \tag{4.27}$$

and

$$\delta C_2 = \tilde{\chi}(x,\tau)\,\mathrm{d}g^5 + \partial_\mu \tilde{\sigma}(x,\tau)\,\mathrm{d}x^\mu \wedge g^5\,. \tag{4.28}$$

As it was explained, the terms proportional to $\mathrm{d}\tau \wedge g^5$ are absent since they can be transformed into the form of (4.27) and (4.28) with the help of a gauge transformation. One could seemingly add the $\mathcal{I}$-odd scalar excitations of F_5,

$$\delta F_5 = (1+*)\big[\mathrm{d}\tau \wedge (\mathrm{d}_4 a \wedge g^1 \wedge g^2 + \mathrm{d}_4 b \wedge g^3 \wedge g^4) \wedge g^5\big]\,; \tag{4.29}$$

or

$$\delta F_5 = (1+*)\big[\mathrm{d}_4 c \wedge \mathrm{d}\tau (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5\big]\,. \tag{4.30}$$

However, equations of motion would require the functions a, b and c to vanish identically.[2]

The most general SU(2) $\times$ SU(2)-singlet $\mathcal{I}$-odd vector excitation of the 3-forms is as follows:

$$\delta F_3\,, \delta H_3 = \mathbf{C}^{(1)} \wedge \mathrm{d}\tau + \mathbf{C}^{(2)} \wedge g^5 + *_4 \mathrm{d}_4 \mathbf{C}^{(3)}\,. \tag{4.31}$$

[2]This is not the case for the massless particles [65].

For the 5-form the most general vector perturbation is

$$\begin{aligned}
\delta F_5 &= (1+*)\ \big[\mathbf{F}^{(1)}\wedge \mathrm{d}\tau\wedge g^5\wedge g^1\wedge g^2+\mathbf{F}^{(2)}\wedge \mathrm{d}\tau\wedge g^5\wedge g^3\wedge g^4\,+\\
&+\ \mathbf{F}^{(3)}\wedge \mathrm{d}\tau\wedge g^5\wedge (g^1\wedge g^3+g^2\wedge g^4)+(\mathrm{d}_4\mathbf{F}^{(4)}+*_4\mathrm{d}_4\mathbf{F}^{(5)})\wedge g^5\wedge g^1\wedge g^2\,+\\
&+\ (\mathrm{d}_4\mathbf{F}^{(6)}+*_4\mathrm{d}_4\mathbf{F}^{(7)})\wedge g^5\wedge g^3\wedge g^4\,+\\
&+\ (\mathrm{d}_4\mathbf{F}^{(8)}+*_4\mathrm{d}_4\mathbf{F}^{(9)})\wedge g^5\wedge (g^1\wedge g^3+g^2\wedge g^4)\big]\ . \qquad (4.32)
\end{aligned}$$

It turns out that not all fifteen (3+3+9) real vectors introduced here are independent. The ansatz has only seven independent vector degrees of freedom. We illustrate this in the next section by considering the conformal KW case.

4.3 Supermultiplet Structure in the Conformal Case

We start our analysis with the scalar U of [65] dual to the operator $\mathrm{Tr}\,(|A|^2-|B|^2)$ of dimension 2 [68]. In the conformal case this operator is responsible for the resolution of the conifold. The corresponding state belongs to the Betti multiplet [80]. The latter also contains a 5d-massless gauge vector of dimension 3 dual to the baryonic current. Its presence on the gravity side is guaranteed by the nontrivial harmonic three-form $w_3 = \star\, w_2$ on $T^{1,1}$. The Betti multiplet is a “massless” Vector Multiplet I according to the classification of the superconformal multiplets given in [81, 82]. It is a short version of the Vector Multiplet I, which contains just two bosonic states of dimensions 2 and 3.

In the table 4.1 we match the components of the five-dimensional superconformal multiplets of [81] to the four-dimensional fluctuations considered in the previous section. The identification of U as ϕ from the table 4.1 is straightforward. The Betti vector ϕ_μ,

$$\delta C_4 = \phi_\mu \wedge \omega_3\,, \qquad (4.33)$$

Table 4.1: Shortened Gravitino Multiplets II, IV and Vector Multiplet I [81, 82]. Field notations are inherited from [83].

Shortened Gravitino Multiplets II, IV				
Field	reps	Δ	$\mathcal{R}$	Mode
a_μ	$(1/2, 1/2)$	5	0	$\mathbf{C}^{(2)}$, $(\chi,\,\tilde{\chi})$
$b^{\pm}_{\mu\nu}$	$(1,0)$, $(0,1)$	5	∓ 2	$\mathbf{F}^{(1)} - \mathbf{F}^{(2)}$, $\mathbf{F}^{(3)}$
$a_{\mu\nu}$	$(1,0)$, $(0,1)$	6	0	$\mathbf{C}^{(3)}$

Vector Multiplet I				
Field	reps	Δ	$\mathcal{R}$	Mode
ϕ_μ	$(1/2, 1/2)$	3	0	$\mathbf{F}^{(1)} + \mathbf{F}^{(2)}$
ϕ	$(0,0)$	2	0	U

is contained in (4.32). The combination $\mathbf{F}^{(1)} + \mathbf{F}^{(2)}$ is identified with the derivative of ϕ_μ with respect to τ and the remaining functions $\mathbf{F}^{(3)}, .., \mathbf{F}^{(9)}$ are dependent on $\mathbf{F}^{(1)} + \mathbf{F}^{(2)}$.

The scalars $\chi, \tilde{\chi}$ have dimension $5 = 2 + \sqrt{1 + \mathbf{8}}$ as it follows from (4.18). The same result follows from the large τ behavior of their equations of motion. Consequently $\chi, \tilde{\chi}$ are the longitudinal modes of the five-dimensional vectors a_μ from the Gravitino Multiplets of type II and IV. It is convenient to consider these multiplets together combining the modes into the complex combinations like

$$\delta B_2 + i\,\delta C_2 = a_\mu \wedge g^5 \ . \tag{4.34}$$

Similarly the complex vector $\mathbf{C}^{(2)}$ from (4.31) corresponds to the vector part of a_μ. It has dimension 5 in the KW case as well. The complex vectors $\mathbf{C}^{(1)}, \mathbf{C}^{(3)}$ correspond to the antisymmetric tensor $a_{\mu\nu}$ and have dimension 6. Only one of them is independent on-shell.

Although the fluctuations of the RR four-form C_4 are real they can be parametrized with help of complex $b_{\mu\nu}$,

$$\delta C_4 = b_{\mu\nu} \wedge Y_2 + c.c. \; . \tag{4.35}$$

By comparing (4.35) to (4.32) we identify the real components of $b_{\mu\nu}$ with $\mathbf{F}^{(1)} - \mathbf{F}^{(2)}$ and $\mathbf{F}^{(3)}$. All other vectors $\mathbf{F}^{(4)}, .., \mathbf{F}^{(9)}$ are not independent on-shell. These fluctuations have dimension $5 = 2 + |\mathbf{3}|$ due to (4.22) and also belong to the Gravitino Multiplets II and IV.

In the SU(2) $\times$ SU(2) invariant sector only shortened version of the Gravitino Multiplets II and IV appear. Thus we do not expect any other massive bosonic states in the $\mathcal{I}$-odd sector. This agrees with our study of the ansatz (4.31), (4.32) in the following section.

There are two ways one can look at the system given by the vector ansatz (4.31), (4.32) and the scalar ansatz (4.27), (4.28). First, one can classify the states according to the complex representation of the superconformal symmetry. Second, one can look for the states of definite parity. The second approach is more straightforward. In particular, as it is demonstrated in [28] the definite parity R-R and NS-NS sectors decouple from each other, although they are a mixture of states from the superconformal Gravitino Multiplets. Therefore instead of dealing with the Gravitino Multiplets II and IV independently we will refer to the combination of the Gravitino Multiplets II and IV just as to the “Gravitino Multiplets” and specify the parity where appropriate. That is why we combined the Gravitino Multiplets II and IV together in the table 4.1.

More precisely, the study of the equations of motion for the full KS case reveals the following. The scalar U from the Vector Multiplet I mixes with the scalar χ from NS-NS sector of the Gravitino Multiplets, while the pseudoscalar $\tilde{\chi}$ from the R-R sector decouples. At the same time the calculation done there in the large τ approximation shows that the Betti pseudovector mixes with the pseudovector part of a_μ from the R-R sector, though both decouple from the vector part of a_μ from the NS-NS sector. This suggests that the vector excitations from the Gravitino Multiplets and the Vector Multiplet I split into the following two non-interacting systems. One includes the spin 1 states of positive parity from a_μ, $a_{\mu\nu}$ (NS-NS sector) and one of the $b_{\mu\nu}$ modes. Another consists of the spin 1 states of negative parity and includes the vectors from a_μ, $a_{\mu\nu}$ (R-R sector) and another $b_{\mu\nu}$ mode together with the Betti pseudovector.

4.4 Radial Excitations of the GHK Scalar

The ansatz that produced a normalizable scalar mode independent of the four-dimensional coordinates x^μ was [65]

$$\delta B_2 = \chi(\tau)\, dg^5 \ , \qquad \delta G_{13} = \delta G_{24} = \psi(\tau) \ . \tag{4.36}$$

We find a generalization of this ansatz that will allow us to study the radial excitations of this massless scalar; i.e., the series of modes that exist at non-vanishing $k_\mu^2 = -m_4^2$. Thus, we must include the dependence of all fields on x^μ. Such an ansatz that decouples from other fields at linear order is

$$\begin{aligned} \delta F_3 &= 0\,, \\ \delta F_5 &= 0\,, \\ \delta B_2 &= \chi(x,\tau)\, dg^5 + \partial_\mu \sigma(x,\tau)\, dx^\mu \wedge g^5\,, \\ \delta H_3 \equiv d\delta B_2 &= \chi'\, d\tau \wedge dg^5 + \partial_\mu(\chi - \sigma)\, dx^\mu \wedge dg^5 + \partial_\mu \sigma'\, d\tau \wedge dx^\mu \wedge g^5\,, \\ \delta G_{13} = \delta G_{24} &= \psi(x,\tau)\,. \end{aligned} \tag{4.37}$$

The ansatz for δB_2 originates from the longitudinal component of a 5-d vector:

$$\delta B_2 = (A_\tau d\tau + A_\mu dx^\mu) \wedge g^5 \ . \tag{4.38}$$

Requiring the 4-d field strength to vanish, $F_{\mu\nu} = 0$, restricts A_μ to be of the form ∂_μ acting on a function. Then, choosing

$$A_\tau = -\chi' \ , \qquad A_\mu = \partial_\mu(\sigma - \chi) \ , \tag{4.39}$$

we recover the ansatz (4.37) up to a gauge transformation. Yet another gauge equivalent way of writing (4.37) is

$$\delta B_2 = (\chi - \sigma)\, dg^5 \ - \sigma'\, d\tau \wedge g^5 \ . \tag{4.40}$$

The new feature of our ansatz compared to the generalized PT ansatz used in [75, 76] is the presence of the second function in δB_2 which multiplies $d\tau \wedge g^5$. Terms of this type, which are allowed by the 4-d Lorentz symmetry, turn out to be crucial for studying the modes that are odd under the $\mathcal{I}$-symmetry. While the functions χ and ψ are contained in the general PT ansatz, they were forced to vanish by the constraints imposed on the modes studied in [75, 76], which as a result were even under the $\mathcal{I}$-symmetry. It turns out that the closure of our ansatz for an odd mode requires the addition of the term involving σ, which is not contained in the PT ansatz.

In order to find the dynamic equations for the functions ψ, χ and σ in (4.37) we study the linearized supergravity equations.

It turns out that all the Bianchi identities as well as the self-duality equation for F_5 and the dynamic equations for $*F_3$ are automatically satisfied with the ansatz (4.37). The only non-trivial equation is that for $*H_3$. When written in terms of the form components it reduces to the two equations (see Appendix B.1):

$$2(g_s M\alpha')\,\frac{K(\tau)^2}{\epsilon^{4/3}\,\sqrt{h(\tau)}}\,\psi + \chi' \ = \ \frac{3}{16}\,\epsilon^{4/3}\,h(\tau)\,K(\tau)^4\,\sinh^2\tau\,\Box_4\sigma' \ , \tag{4.41}$$

$$\partial_\mu(\chi - \sigma) + \frac{9}{8}\,K(\tau)^2\,\partial_\tau\Big\{K^4\,\sinh^2\tau\,\partial_\mu\sigma'\Big\} \ = \ 0\,. \tag{4.42}$$

The first order perturbation of the Ricci curvature tensor is given by

$$\delta R_{ij} = \frac{1}{2}\left(-\delta G_a{}^a{}_{;ij} - \delta G_{ij;a}{}^a + \delta G_{ai;j}{}^a + \delta G_{aj;i}{}^a\right) \ , \tag{4.43}$$

where covariant derivatives and contractions of indices are performed using the unperturbed metric. The first term in this expression vanishes for our case because the metric perturbation is traceless. The remaining three terms combine to give the only non-zero perturbations $\delta R_{13} = \delta R_{24}$:

$$\begin{aligned}
\delta R_{13} &= -\frac{3}{\epsilon^{4/3}} K^3 \sinh(\tau) z \left[\frac{K''}{K} + \frac{1}{2}\frac{h''}{h} + \frac{z''}{z} + \frac{(K')^2}{K^2} - \frac{1}{2}\frac{(h')^2}{h^2} + \frac{K'}{K}\frac{h'}{h} + \right. \\
&\left. + 2\frac{K'}{K}\frac{z'}{z} + \coth\tau \left(\frac{h'}{h} + 4\frac{K'}{K} + 2\frac{z'}{z} \right) + 2 - \frac{1}{\sinh(\tau)^2} - \frac{4}{9}\frac{1}{\sinh(\tau)^2 K^6} \right] - \\
&- \frac{1}{2} h(\tau) K \sinh(\tau)\, \Box_4 z \\
&= -\frac{3}{\epsilon^{4/3}} K^3 \sinh\tau z \left[\frac{1}{2} \frac{\left((K\sinh(\tau))^2 (\ln h)' \right)'}{(K\sinh(\tau))^2} + \frac{\left((K\sinh(\tau))^2 z' \right)'}{(K\sinh(\tau))^2 z} \right. \\
&\left. - \frac{2}{\sinh(\tau)^2} - \frac{8}{9}\frac{1}{K^6 \sinh(\tau)^2} + \frac{4}{3}\frac{\cosh(\tau)}{K^3 \sinh(\tau)^2} \right] - \frac{1}{2} h(\tau) K \sinh\tau\, \Box_4 z\,; \qquad (4.44)
\end{aligned}$$

where $z(x,\tau)$ is defined by

$$\psi(x,\tau) = h^{1/2} K \sinh(\tau)\, z(x,\tau) = 2^{-1/3} [\sinh(2\tau) - 2\tau]^{1/3} h^{1/2} z(x,\tau)\,. \qquad (4.45)$$

The source terms T_{ij} on the right hand side of the Einstein equation $R_{ij} = T_{ij}$ are due to the deformations of the metric and B_2 form. The only nontrivial deformations of sources are those with indices 13 or 24, and $\delta T_{13} = \delta T_{24}$. As it is explained in Appendix B.1, these deformations can be written as

$$\delta T_{13} = \Big[A_1(\tau) + A_2(\tau) \Big]\, \psi(x,\tau) + B(\tau)\, \chi'(x,\tau) \qquad (4.46)$$

with some auxiliary functions A_1, A_2 and B. Then eliminating χ' with the help of (4.41) yields

$$\begin{aligned}
\delta T_{13} &= \frac{3}{2^{2/3}} \frac{(g_s M \alpha')^4}{\epsilon^{20/3} h^2} \frac{(\tau\coth\tau - 1)^2 [\sinh 2\tau - 2\tau]^{5/3}}{\sinh^6 \tau} z(\tau) + \\
&+ \frac{3}{8 \cdot 2^{1/3}} \frac{(g_s M \alpha')^2}{\epsilon^4 h} \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{\sinh^6(\tau)} \Big[\cosh(4\tau) + 8(1+\tau^2)\cosh(2\tau) - \\
&- 24\tau \sinh(2\tau) + 16\tau^2 - 9 \Big]\, z(x,\tau) - \frac{9}{16} \frac{g_s M \alpha'}{\epsilon^{4/3}} \frac{\sinh 2\tau - 2\tau}{\sinh\tau} K^5\, \Box_4 \sigma'(x,\tau) \\
&= -\frac{3}{\epsilon^{4/3}} K^3 \sinh\tau \left[-\frac{1}{2}\frac{(h')^2}{h^2} + \frac{1}{2}\frac{h''}{h} + \frac{K'}{K}\frac{h'}{h} + \coth\tau \frac{h'}{h} \right] z - \\
&- \frac{9}{16} \frac{g_s M \alpha'}{\epsilon^{4/3}} \frac{\sinh 2\tau - 2\tau}{\sinh\tau} K^5\, \Box_4 \sigma'\,. \qquad (4.47)
\end{aligned}$$

As it was mentioned, the perturbations $\delta T_{13} = \delta T_{24}$ are the only non-zero components of δT_{ij}. Equating (4.44) and (4.47) we obtain the final form of the linearized Einstein equations.

Combining the equations for the field strengths and the Einstein equations we obtain the following system:

$$(g_sM\alpha')\,\frac{\sinh 2\tau - 2\tau}{\epsilon^{4/3}\,\sinh^2\tau}\,z + \chi' = \frac{3}{16}\,\epsilon^{4/3}\,h(\tau)\,K(\tau)^4\,\sinh^2\tau\,\Box_4\sigma' \,, \tag{4.48}$$

$$\partial_\mu(\chi - \sigma) = -\frac{9}{8}\,K(\tau)^2\,\partial_\tau\Big\{K^4\,\sinh^2\tau\,\partial_\mu\sigma'\Big\} \,, \tag{4.49}$$

$$\frac{\left((K\sinh\tau)^2\,z'\right)'}{(K\sinh\tau)^2} + \frac{\epsilon^{4/3}h}{6\,K^2}\Box_4 z = \left(\frac{2}{\sinh^2\tau} + \frac{8}{9}\frac{1}{K^6\sinh^2\tau} - \frac{4}{3}\frac{\cosh\tau}{K^3\sinh^2\tau}\right) z + \frac{3}{16}\,(g_sM\alpha')\,\frac{\sinh 2\tau - 2\tau}{\sinh^2\tau}\,K^2\,\Box_4\sigma' \,. \tag{4.50}$$

Note that χ can be eliminated between (4.48) and (4.49). Further, a change of variables

$$\tilde{z} = zK\sinh(\tau) \,, \tag{4.51}$$

$$\tilde{w} = \frac{\epsilon^{4/3}}{g_sM\alpha'}K^5\sinh(\tau)^2\sigma' \,, \tag{4.52}$$

leads to a more symmetric pair of equations:

$$\tilde{z}'' - \frac{2}{\sinh^2\tau}\tilde{z} + \frac{\epsilon^{4/3}h}{6K^2}\Box_4\tilde{z} = \frac{3(g_sM\alpha')^2}{16\epsilon^{4/3}}\frac{\sinh 2\tau - 2\tau}{K^2\sinh^3\tau}\Box_4\tilde{w} \,, \tag{4.53}$$

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\tilde{w} + \frac{\epsilon^{4/3}h}{6K^2}\Box_4\tilde{w} = \frac{8}{9}\frac{\sinh 2\tau - 2\tau}{K^2\sinh^3\tau}\tilde{z} \,. \tag{4.54}$$

Introducing the dimensionless mass-squared $\tilde{m}^2$ according to (here we explicitly restore the parameters g_s, M and α')

$$\tilde{m}^2 = m_4^2\,\frac{2^{2/3}(g_sM\alpha')^2}{6\,\epsilon^{4/3}} \,, \tag{4.55}$$

we can rewrite the equations for $\tilde{z}$ and $\tilde{w}$ as

$$\tilde{z}'' - \frac{2}{\sinh^2\tau}\tilde{z} + \tilde{m}^2\frac{I(\tau)}{K^2(\tau)}\tilde{z} = \tilde{m}^2\frac{9}{4\cdot 2^{2/3}}\,K(\tau)\,\tilde{w} \,, \tag{4.56}$$

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\tilde{w} + \tilde{m}^2\frac{I(\tau)}{K^2(\tau)}\tilde{w} = \frac{16}{9}\,K(\tau)\,\tilde{z} \,. \tag{4.57}$$

This is a system of coupled equations which defines the mass spectrum of certain scalar glueballs with positive 4-d parity. The natural charge conjugation symmetry of the KS background is the $\mathcal{I}$-symmetry, under which these modes are odd. Therefore, we assign $J^{PC} = 0^{+-}$ to this family of glueballs.[3]

In the massless case these equations lead to the GHK solution [65]. If we assume $\Box_4 = -k_\mu^2 = m_4^2 = 0$, then there are two solutions [65], $\tilde{z}_1 = \coth\tau$ and $\tilde{z}_2 = \tau\coth\tau - 1$. The solution for $\tilde{z}$ which is non-singular at the origin is $\tilde{z} = \tau\coth\tau - 1$. Substituting it into the second equation, we find

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\tilde{w} = \frac{16}{9}\,K(\tau)\,(\tau\coth\tau - 1) \equiv -\frac{2^{2/3}\,8}{9}\,I'(\tau)\sinh\tau\,. \tag{4.58}$$

The two solutions of the homogeneous equation are $\tilde{w}_1 = 1/\sinh\tau$ and $\tilde{w}_2 = (\sinh 2\tau - 2\tau)/\sinh\tau$; both of them are singular either at zero or at infinity. This means that the regular solution of the inhomogeneous equation is uniquely fixed. With the Wronskian $W(\tilde{w}_1, \tilde{w}_2) = \tilde{w}_1\tilde{w}_2' - \tilde{w}_1'\tilde{w}_2 = 4$, we can find a general solution

$$\begin{aligned}\tilde{w}(\tau) = -\frac{2^{2/3}\,8}{9}\Big\{\tilde{w}_1(\tau)\big[C_1 - \int^\tau dx\,\frac{\tilde{w}_2(x)}{W(x)}\,I'(x)\sinh x\big] + \\ +\tilde{w}_2(\tau)\big[C_2 + \int^\tau dx\,\frac{\tilde{w}_1(x)}{W(x)}\,I'(x)\sinh x\big]\Big\}\,.\end{aligned} \tag{4.59}$$

Integrating by parts and choosing the particular homogeneous solution to make $\tilde{w}$ well behaved at both zero and infinity we get

$$\tilde{w}(\tau) = -\frac{2^{2/3}\,8}{9}\,\frac{1}{\sinh\tau}\int_0^\tau dx\,I(x)\sinh^2 x\,. \tag{4.60}$$

Let us also note that the non-zero $\tilde{w}$ in the zero momentum case $k_\mu = 0$ is not in contradiction with the GHK solution. This is because $\tilde{w}$ enters (4.37) only through $\partial_\mu\sigma$ which is zero as long as the momentum vanishes.

[3]For comparison, the glueballs found in [75, 76] are 0^{++}. The glueballs whose spectrum comes from the minimal scalar equation [73] resulting from the analysis of graviton fluctuations are 2^{++}. The axial vector $U(1)_R$ fluctuations [77] give rise to 1^{++} glueballs whose masses are also determined by the minimal scalar equation.

4.5 Pseudoscalar Modes from the RR Sector

The type of ansatz used in section 4.4 works even more simply for the RR 2-form field:

$$\begin{aligned} \delta H_3 &= 0\,, \\ \delta F_5 &= 0\,, \\ \delta C_2 &= \chi(x,\tau)\,dg^5 + \partial_\mu\sigma(x,\tau)\,dx^\mu\wedge g^5\,, \\ \delta F_3 \equiv d\delta C_2 &= \chi'\,d\tau\wedge dg^5 + \partial_\mu(\chi-\sigma)\,dx^\mu\wedge dg^5 + \partial_\mu\sigma'\,d\tau\wedge dx^\mu\wedge g^5\,. \end{aligned} \tag{4.61}$$

This ansatz is odd under the $\mathcal{I}$-symmetry. It is similar to, but somewhat simpler than the GHK pseudoscalar ansatz [65] which involved mixing with δF_5. Since $\delta F_3\wedge H_3 = 0$, now it is consistent to set $\delta F_5 = 0$. We also have $F_5\wedge\delta F_3 = 0$, so it is consistent to take $\delta H_3 = 0$. Finally, one needs to study mixing with metric fluctuations. At a first glance it seems that δG_{12} and δG_{34} might need to be turned on, but a more detailed analysis shows that their sources vanish:

$$\delta T_{12} = F_{13\tau}\delta F_2{}^{3\tau} + \delta F_{14\tau}F_2{}^{4\tau} = \frac{M\alpha'}{2}\,G^{33}\,G^{55}\big[F'\chi' - F'\chi'\big] = 0\,, \tag{4.62}$$

$$\delta T_{34} = F_{31\tau}\delta F_4{}^{1\tau} + \delta F_{32\tau}F_4{}^{2\tau} = 0\,. \tag{4.63}$$

Thus, the perturbation (4.61) decouples from all other modes, and the only non-trivial linearized equation of motion is

$$d*\delta F_3 = 0\,. \tag{4.64}$$

The calculation we need to perform is the same as in Section 4.4, except we now set $\psi = 0$ and find

$$\chi' = \frac{3}{16}\,\epsilon^{4/3}\,h(\tau)\,K(\tau)^4\,\sinh^2\tau\,\Box_4\sigma'\,, \tag{4.65}$$

$$0 = \partial_\mu(\chi-\sigma) + \frac{9}{8}\,K(\tau)^2\,\partial_\tau\Big\{K^4\,\sinh^2\tau\,\partial_\mu\sigma'\Big\}\,. \tag{4.66}$$

Eliminating χ and changing variables,

$$\tilde{w} = \frac{\epsilon^{4/3}}{g_s M\alpha'}K^5\sinh(\tau)^2\sigma'\,, \tag{4.67}$$

we find

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\tilde{w} + \frac{\epsilon^{4/3}h}{6K^2}\Box_4\tilde{w} = 0\,. \tag{4.68}$$

Again, after introducing the dimensionless mass as in (4.55), we get a non-minimal scalar equation

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\tilde{w} + \tilde{m}^2\frac{I(\tau)}{K(\tau)^2}\,\tilde{w} = 0\,. \tag{4.69}$$

Since the 4-d parity operation includes sign reversal of RR fields, we identify the family of glueballs coming from this eigenvalue problem as *pseudoscalars* whose J^{PC} quantum numbers are 0^{--}.

If we set $\tilde{m} = 0$ the solution regular at small τ is $(\sinh 2\tau - 2\tau)/\sinh\tau$. Since this blows up at large τ we conclude that this equation does not contain a massless glueball. A simple numerical analysis using the shooting method would allow one to find the mass spectrum.

4.6 Triplet of Vectors from the Gravitino Multiplets

This section analyzes the vector fluctuations from the Gravitino Multiplets, more precisely a combination of the Gravitino Multiplets II and IV with negative parity. The system of the linearized equations in this subsector reduces to three coupled equations, which can be disentangled.

We start with writing down a general ansatz for the spin 1 excitations in the "NS-NS sector" of the Gravitino Multiplets and show that they decouple from the other vectors. The deformations of the three and five-forms are:

$$\delta B_2 = *_4 \mathrm{d}_4\mathbf{H} + \mathbf{A}\wedge g^5\,, \tag{4.70}$$

$$\delta C_2 = \mathbf{E}\wedge \mathrm{d}\tau\,; \tag{4.71}$$

$$\begin{aligned}\delta F_5 &= (1+*)\left[\mathrm{d}_4\mathbf{K}\wedge \mathrm{d}\tau\wedge g^1\wedge g^2 + \mathrm{d}_4\mathbf{L}\wedge \mathrm{d}\tau\wedge g^3\wedge g^4\right.\\ &\left.+\mathrm{d}_4\mathbf{M}\wedge(g^1\wedge g^3+g^2\wedge g^4)\wedge g^5+\mathbf{N}\wedge\mathrm{d}\tau\wedge(g^1\wedge g^3+g^2\wedge g^4)\wedge g^5\right].\end{aligned} \tag{4.72}$$

As it was discussed in Section 4.3, vector $\mathbf{A}$ corresponds to a_μ, vectors $\mathbf{E}$ and $\mathbf{H}$ to $a_{\mu\nu}$ and $\mathbf{K}$, $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$ to $b_{\mu\nu}$ of the conformal case. The equations of motion are analyzed in detail in the Appendix B.2, and they show that $\mathbf{E}$, $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$ depend on the $\mathbf{A}$, $\mathbf{H}$, $\mathbf{N}$ algebraically. The latter describe the physical degrees of freedom. After redefining $\mathbf{N}$ and $\mathbf{A}$,

$$\frac{G^{55}}{\sqrt{h}}\mathbf{N} = \Box_4\tilde{\mathbf{N}}\,, \tag{4.73}$$

$$K^2\sinh\tau\,\mathbf{A} = \tilde{\mathbf{A}}\,; \tag{4.74}$$

the resulting equations take the form:

$$\tilde{\mathbf{N}}'' - \left(\frac{\cosh^2\tau+1}{\sinh^2\tau}+\frac{4\cdot 2^{1/3}(F')^2}{IK^2\sinh^2\tau}\right)\tilde{\mathbf{N}} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{N}} + \\ + F'\mathbf{H}' - \frac{2^{1/3}F'\ell}{IK^2\sinh^2\tau}\mathbf{H} + \frac{F'}{K^2\sinh\tau}\tilde{\mathbf{A}} = 0\,, \tag{4.75}$$

$$\tilde{\mathbf{A}}'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\tilde{\mathbf{A}} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{A}} + \tilde{m}^2\frac{4\cdot 2^{1/3}F'}{K^2\sinh\tau}\tilde{\mathbf{N}} = 0\,, \tag{4.76}$$

$$\mathbf{H}'' + \left(2\frac{\left(K\sinh\tau\right)'}{K\sinh\tau}+\frac{I'}{I}\right)\mathbf{H}' - \left(\frac{2^{1/3}\ell'}{IK^2\sinh^2\tau}+\frac{2^{2/3}\ell^2}{I^2K^4\sinh^4\tau}\right)\mathbf{H} + \tilde{m}^2\frac{I}{K^2}\mathbf{H} - \\ - \frac{4\cdot 2^{1/3}}{IK^2\sinh^2\tau}\left(F'\tilde{\mathbf{N}}\right)' - \frac{4\cdot 2^{2/3}F'\ell}{I^2K^4\sinh^4\tau}\tilde{\mathbf{N}} = 0\,. \tag{4.77}$$

One can diagonalize this system. In particular, we expect to identify the massive vector superpartner of the scalar (4.28). Although the equations (4.75)-(4.77) look bulky it is quite easy to split them into the three independent equations. First we notice that the constraint $\tilde{\mathbf{N}} = 0$ implies

$$\mathbf{H} = \frac{(\sinh\tau\tilde{\mathbf{A}})'}{\tilde{m}^2 I\sinh^2\tau}\,, \tag{4.78}$$

and it reduces the system (4.75)-(4.77) to one equation

$$\tilde{\mathbf{A}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{\mathbf{A}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{A}} = 0\ . \tag{4.79}$$

This equation coincides with the one for the scalar $\tilde{\chi}$ (4.68). Hence the vector mode above and the scalar (4.68) form a massive vector $j = 1/2$ multiplet.[4] It is interesting to notice that $\tilde{\mathbf{N}} = 0$ does not imply $\delta F_5 = 0$ as it would in the conformal case. Rather $F_5 = (1 + *)\, \mathrm{d}_4\mathbf{H} \wedge H_3$ with $\mathbf{H}$ being related to $\tilde{\mathbf{A}}$ by (4.78).

To find the two remaining modes we impose a constraint

$$\tilde{\mathbf{H}} = -\frac{K(\sinh\tau\tilde{\mathbf{A}})'}{\tilde{m}^2\sqrt{I}\sinh\tau}\ , \tag{4.80}$$

where $\tilde{\mathbf{H}} = \sqrt{I}K\sinh\tau\,\mathbf{H}$. This constraint guarantees that the two remaining modes are orthogonal to the vector mode from above. The disentanglement procedure is described in detail in Appendix B.2. Eliminating $\tilde{\mathbf{H}}$ from the above equations one obtains

$$\tilde{\mathbf{A}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{\mathbf{A}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{A}} - \frac{2\tilde{m}^2 I'}{K^3\sinh\tau}\,\tilde{\mathbf{N}} = 0\,, \tag{4.81}$$

$$\tilde{\mathbf{N}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{\mathbf{N}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{N}} - \frac{2^{-1/3} I'}{K^3\sinh\tau}\,\tilde{\mathbf{A}} = 0\,. \tag{4.82}$$

After a trivial rescaling and change of variables $\mathbf{X}_\pm = \tilde{\mathbf{A}} \pm 2^{2/3}\tilde{m}\tilde{\mathbf{N}}$ these two equations decouple,

$$\mathbf{X}_\pm'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\mathbf{X}_\pm + \tilde{m}^2\,\frac{I}{K^2}\,\mathbf{X}_\pm \mp \frac{2^{5/3}\tilde{m}F'}{K^2\sinh\tau}\,\mathbf{X}_\pm = 0\,. \tag{4.83}$$

These particles are members of the two $j = 1$ gravitino multiplets. We are going to identify their superpartners as a part of the analysis performed in the next section.

4.7 Betti Vector and Axial Vector Triplet

In this section we consider the vector excitations in the parity even "R-R sector" of the combination of the Gravitino Multiplets and the axial Betti vector from Vector

[4] We use spin j to characterize the massive supermultiplets $(j - 1/2) \oplus j \oplus j \oplus (j + 1/2)$.

Multiplet I. We expect this system of four vectors to contain the superpartners of the scalar excitations (4.27) and the two vectors $\mathbf{X}^{\pm}$ from Section 4.6.

We consider the following deformations of the 3-form potentials:

$$\delta B_2 = \mathbf{J} \wedge \mathrm{d}\tau \,, \tag{4.84}$$

$$\delta C_2 = \mathbf{C} \wedge g^5 + *_4 \mathrm{d}_4 \mathbf{D} \,; \tag{4.85}$$

and the 5-form:

$$\delta F_5 = (1+*)\big[\mathbf{F} \wedge \mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^5 + \mathbf{G} \wedge \mathrm{d}\tau \wedge g^3 \wedge g^4 \wedge g^5 + \mathrm{d}_4\mathbf{P} \wedge g^1 \wedge g^2 \wedge g^5 + \mathrm{d}_4\mathbf{Q} \wedge g^3 \wedge g^4 \wedge g^5 + d_4\mathbf{R} \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\big] \,. \tag{4.86}$$

Clearly in the conformal limit $\mathbf{C}$ corresponds to a_μ, while $\mathbf{J}$ and $\mathbf{D}$ correspond to $a_{\mu\nu}$. The fluctuations of F_5 correspond to both $b_{\mu\nu}$ and ϕ_μ.

Detailed analysis of the supergravity equations for this system is given in Appendix B.3. Choosing $\mathbf{C}, \mathbf{D}, \mathbf{F}$,and $\mathbf{G}$ as independent variables we end up with the following system of the four coupled equations:

$$\mathbf{B}_+'' - \frac{2}{\sinh^2\tau}\mathbf{B}_+ + \tilde{m}^2 \frac{I}{K^2}\mathbf{B}_+ + K^3 \sinh\tau(\mathbf{D}' - \mathbf{J}) - K\tilde{\mathbf{C}} = 0 \,, \tag{4.87}$$

$$\mathbf{B}_-'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\mathbf{B}_- + \tilde{m}^2 \frac{I}{K^2}\mathbf{B}_- + 2^{-1/3}\frac{I'}{K}(\mathbf{D}' - \mathbf{J}) + \frac{2^{-1/3}I'}{K^3\sinh\tau}\tilde{\mathbf{C}} = 0 \,, \tag{4.88}$$

$$\tilde{\mathbf{C}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\tilde{\mathbf{C}} + \tilde{m}^2 \frac{I}{K^2}\tilde{\mathbf{C}} - 2^{1/3}\tilde{m}^2 K \mathbf{B}_+ + \tilde{m}^2 \frac{I'}{K^3\sinh\tau}\mathbf{B}_- = 0, \tag{4.89}$$

$$\mathbf{D}'' + \Big(\log(IK^2\sinh^2\tau)\Big)'\mathbf{D}' + \tilde{m}^2\frac{I}{K^2}\mathbf{D} + \frac{(I'K^2\sinh^2\tau)'}{IK^2\sinh^2\tau}\mathbf{D} + + \frac{I'}{I}\mathbf{J} - \frac{1}{IK^2\sinh^2\tau}\left(2^{1/3}K^3\sinh\tau\,\mathbf{B}_+ + \frac{I'}{K}\mathbf{B}_-\right)' = 0 \,; \tag{4.90}$$

where

$$\mathbf{J} = -\frac{I'}{I}\mathbf{D} + \frac{2^{1/3}K}{I\sinh\tau}\mathbf{B}_+ + \frac{I'}{IK^3\sinh^2\tau}\mathbf{B}_- \,. \tag{4.91}$$

Here we introduced the new variables as follows:

$$\frac{G^{55}}{\sqrt{h}}\coth^2\frac{\tau}{2}\,\mathbf{F} \;=\; \coth\frac{\tau}{2}\,\Box_4\tilde{\mathbf{F}}\,, \qquad (4.92)$$

$$\frac{G^{55}}{\sqrt{h}}\tanh^2\frac{\tau}{2}\,\mathbf{G} \;=\; \tanh\frac{\tau}{2}\,\Box_4\tilde{\mathbf{G}}\,, \qquad (4.93)$$

$$\mathbf{B}_\pm \;=\; \tilde{\mathbf{F}} \pm \tilde{\mathbf{G}}\ , \qquad (4.94)$$

$$\tilde{\mathbf{C}} \;=\; K^2\sinh\tau\,\mathbf{C}\ . \qquad (4.95)$$

The system of the equations (4.87)-(4.90) can be further reduced. The hint is to consider a conformal limit when the Betti vector decouples from the Gravitino Multiplet states. The former is associated with $\mathbf{F}=\mathbf{G}$ while the perturbation $b_{\mu\nu}$ from the Gravitino Multiplet corresponds to $\mathbf{F}=-\mathbf{G}$. We put

$$\mathbf{B}_- \;=\; 0 \qquad (4.96)$$

in order to "turn of" the excitation of $b_{\mu\nu}$ in the system (4.87)-(4.90). This implies for $\mathbf{D}$:

$$\mathbf{D} = \frac{(\sinh\tau\,\tilde{\mathbf{C}})'}{\tilde{m}^2 I\sinh^2\tau}\ . \qquad (4.97)$$

The remaining equations form a self-consistent subsystem of two equations:

$$\mathbf{B}_+'' - \frac{2}{\sinh^2\tau}\mathbf{B}_+ + \tilde{m}^2\frac{I}{K^2}\mathbf{B}_+ \;=\; 2K\tilde{\mathbf{C}}\,, \qquad (4.98)$$

$$\tilde{\mathbf{C}}'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{C}} \;=\; 2^{1/3}\tilde{m}^2K\mathbf{B}_+\,. \qquad (4.99)$$

After a trivial rescaling of variables it reproduces the scalar equations (4.56) and (4.57). Thus these modes represent the mixing of the Betti vector with the vector part of a_μ. They are the vector superpartners of the scalar excitations z and w of Section 4.4.

To extract the remaining degrees of freedom we "turn off" the Betti vector by choosing

$$\mathbf{B}_+ \;=\; 0\ . \qquad (4.100)$$

Using this equation one can eliminate $\mathbf{D}$ from the remaining equations as follows:

$$\mathbf{D} = -\frac{(\sinh\tau\,\tilde{\mathbf{C}})'}{\tilde{m}^2 I \sinh^2\tau} \,. \tag{4.101}$$

The remaining self-consistent subsystem of the two equations for $\mathbf{B}_-$ and $\tilde{\mathbf{C}}$ is

$$\mathbf{B}_-'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\mathbf{B}_- + \tilde{m}^2\frac{I}{K^2}\mathbf{B}_- = -\frac{2^{2/3}I'}{K^3\sinh\tau}\tilde{\mathbf{C}} \,, \tag{4.102}$$

$$\tilde{\mathbf{C}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{C}} = -\tilde{m}^2\frac{I'}{K^3\sinh\tau}\mathbf{B}_- \,. \tag{4.103}$$

After a trivial rescaling and change of variables $\mathbf{Y}_\pm = 2^{-1/3}\tilde{m}\mathbf{B}_- \mp \tilde{\mathbf{C}}$ the equations become

$$\mathbf{Y}_\pm'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\mathbf{Y}_\pm + \tilde{m}^2\frac{I}{K^2}\mathbf{Y}_\pm \mp \frac{2^{5/3}\tilde{m}F'}{K^2\sinh\tau}\mathbf{Y}_\pm = 0 \,. \tag{4.104}$$

These equations exactly coincide with the system (4.83), which suggests that we have found the members of the same supermultiplets. Namely, we have the two $j = 1$ supermultiplets each containing a vector $\mathbf{X}$, an axial vector $\mathbf{Y}$ and two fermions of spin 1/2 and 3/2.

4.8 Numerical Analysis: Finding the Spectra

As a result of our analysis we have identified all the $\mathcal{I}$-odd excitations of the KS solution invariant w.r.t. the global SU(2) $\times$ SU(2)-symmetry and organized them into the supermultiplets.

First, there is a system of the radial excitations of the GHK scalar (4.56), (4.57) and their vector superpartners (4.98), (4.99). These glueballs are the members of the $j = \frac{1}{2}$ supermultiplet. There is another $j = \frac{1}{2}$ multiplet consisting of a scalar (4.68) and a vector (4.79). In addition to these there are the two $j = 1$ multiplets $\mathbf{X}_\pm$ (4.83) and $\mathbf{Y}_\pm$ (4.104).

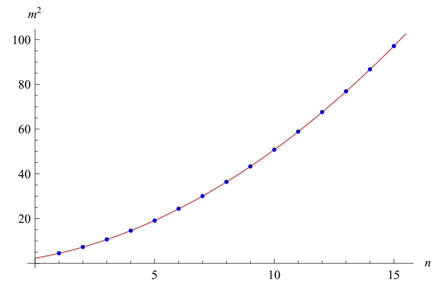
To determine the spectrum of glueballs in the field theory, we need to solve the eigenvalue problem for $\tilde{m}^2$ in the infinite throat limit. None of the mentioned

Table 4.2: Non-zero eigenvalues with $\tilde{m}^2 < 100$. There are the two distinct spectra. Both spectra can be fitted by quadratic polynomials in the eigenvalue number n (the red line in the plots).

Spectrum I

n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$
1	4.53	5	19.1	9	43.3	13	76.9
2	7.30	6	24.4	10	50.8	14	86.7
3	10.7	7	30.1	11	58.9	15	97.1
4	14.6	8	36.4	12	67.6		

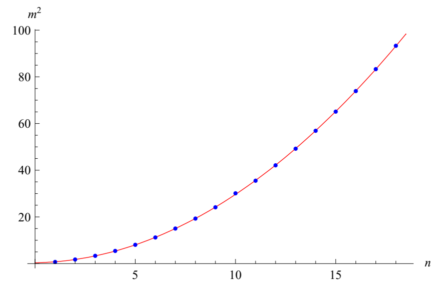
Quadratic Fit



Spectrum II

n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$
1	0.129	6	8.06	11	30.1	16	65.1
2	0.703	7	11.2	12	35.5	17	73.9
3	1.76	8	15.0	13	42.1	18	83.3
4	3.33	9	19.3	14	49.2	19	93.3
5	5.43	10	24.1	15	56.9		

Quadratic Fit



equations seems amenable to analytical solution and we have to employ a numerical approach to find the spectrum of normalizable solutions. In most of the cases the spectrum can be found using the shooting method. However, the system of the two coupled equations (4.56) and (4.57) requires the use of the determinant method, which generalizes the standard shooting technique to a system of several equations (see, e.g.,[76]). The detailed description of the determinant method as well as the subtleties specific to the system (4.56), (4.57) is given in Appendix B.4.

The result is that the spectrum of (4.56), (4.57) consists of the two distinct series, each one with a quadratic growth of $\tilde{m}_n^2$ for large n. These series are interpreted as the radial excitation spectra of the two different particles. The lowest eigenvalues ($\tilde{m}^2 < 100$) for these spectra are shown in Table 4.2. The quadratic fit for spectrum

I is

$$\tilde{m}^2_{In} = 2.31 + 1.91\,n + 0.294\,n^2\,. \tag{4.105}$$

For spectrum II (we drop the lowest eigenvalue when doing the fit)

$$\tilde{m}^2_{IIn} = 0.36 + 0.14\,n + 0.279\,n^2\,. \tag{4.106}$$

It is interesting to compare these results with those found for the 0^{++} modes by Berg, Haack and Mück (BHM) [76]. The conventions of [76] correspond to a particular choice of the KS parameters (see Appendix B.5), and the relation between the masses is

$$m^2_{BHM} = (3/2)^{2/3} I(0)\,\tilde{m}^2 \approx 0.9409\,\tilde{m}^2\,. \tag{4.107}$$

Using this relation one can convert the mass eigenvalues to the BHM normalization. We note that the lightest glueball we find, the first entry from spectrum II in Table 4.2, has $m^2_{BHM} \approx 0.121$. For comparison, the lightest 0^{++} eigenvalue found in [76] has $m^2_{BHM} \approx 0.185$. The fact that the 0^{+-} sector has the lightest glueballs may be qualitatively understood as follows. Roughly speaking, glueball masses increase with the dimensions of the operators that create them. The lowest dimension operator from the 0^{++} sector is the gluino bilinear $\mathrm{Tr}\lambda\lambda$ of dimension three, but the 0^{+-} sector contains an operator of dimension two, namely $\mathrm{Tr}(\bar{A}A - \bar{B}B)$.

Converting the asymptotics of the two spectra to BHM units, we find

$$m^2_{I\ BHM} \approx 2.17 + 1.79\,n + 0.277\,n^2\,, \tag{4.108}$$

$$m^2_{II\ BHM} \approx 0.34 + 0.13\,n + 0.262\,n^2\,. \tag{4.109}$$

The coefficients of the quadratic terms are close to those found in [76]. The quadratic dependence on n, which is characteristic of Kaluza-Klein theory, is a special feature of strongly coupled gauge theories that have weakly curved gravity duals (see [84] for a discussion). Note that m_4^2 is obtained from $\tilde{m}^2$ through multiplying by a factor $\sim T_s/(g_s M)$, where T_s is the confining string tension. Thus, for $n \ll \sqrt{g_s M}$ these modes are much lighter than the string tension scale, and therefore much lighter than all glueballs with spin greater than two. Such anomalously light bound states appear to be special to gauge theories that stay very strongly coupled in the UV, such as the cascading gauge theory; they do not appear in asymptotically free gauge theories. Therefore, the anomalously light glueballs could perhaps be used as a 'special signature' of gauge theories with gravity duals if they are realized in nature.

One may be puzzled why the spectrum in Table 4.2 does not include the GHK massless mode. This is because in solving the coupled equations (4.56), (4.57) we required that both wave-functions $\tilde{z}$ and $\tilde{w}$ vanish as $\tau \to \infty$. This excludes the GHK zero mode which grows as $\tilde{z} \sim \tau$. On the other hand, this growth is a lot slower than the exponential growth found for generic solutions. The meaning of the GHK mode as the baryonic branch modulus seems to be well established since even the solutions at finite distance along this modulus are available [67, 68]. Thus, the GHK scalar zero-mode should be normalizable with a proper definition of norm. In fact, the GHK pseudoscalar and its fermionic superpartner are normalizable [65, 66]; therefore, the supersymmetry of the problem implies that the GHK scalar is normalizable as well and is part of the spectrum.

One can use the shooting method in order to find the spectrum of the scalar (4.68) and its superpartner (4.79). The lowest eigenvalues ($\tilde{m}^2 < 100$) are listed in Table 4.3. The quadratic fit is

$$\tilde{m}^2_{IIIn} = 0.994 + 1.16\, n + 0.288\, n^2\,; \qquad (4.110)$$

and in the BHM normalization it is given by

$$m^2_{III\ BHM} = 0.935 + 1.09\, n + 0.271\, n^2\,. \qquad (4.111)$$

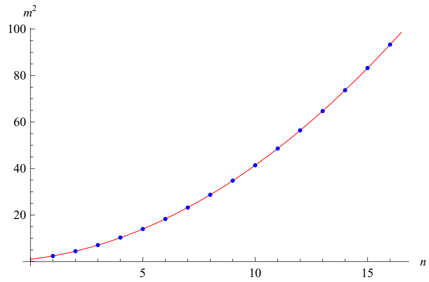
The shooting method for $\mathbf{X}_\pm$ (4.83) and their superpartners gives the two spectra, listed in the table 4.4. These spectra can be fitted by the following quadratic

Table 4.3: Non-zero eigenvalues with $\tilde{m}^2 < 100$ in the RR sector. This spectrum can also be fitted by a quadratic polynomial (red line).

Spectrum III

n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$
1	2.41	5	14.0	9	34.8	13	64.7
2	4.47	6	18.3	10	41.4	14	73.7
3	7.08	7	23.2	11	48.6	15	83.2
4	10.3	8	28.7	12	56.4	16	93.3

Quadratic Fit



polynomials:

$$\tilde{m}_-^2 = 0.633 + 1.02\,n + 0.287\,n^2\,, \tag{4.112}$$

$$\tilde{m}_+^2 = 1.44 + 1.31\,n + 0.288\,n^2\,. \tag{4.113}$$

In the units used by Berg et.al. [75] the lowest states have masses

$$m^2_{BHM\,-} = 1.78\,, \tag{4.114}$$

$$m^2_{BHM\,+} = 2.83\,. \tag{4.115}$$

In Figure 4.1 we collected the information about the spectrum of the $\mathcal{I}$-odd sector. It contains two massless scalars [65], the lightest massive scalars from massive vector multiplets [28], and lightest vectors from the seven vector towers discovered in this work. We have also added to the figure two $\mathcal{I}$-even bosonic states from the lightest graviton multiplet, a tensor 2^{++} state [85] and a vector 1^{++} dual to the $U(1)_{\mathcal{R}}$ current [86]. These states share the spectrum of the "minimal" scalar and hence the lowest mass of their spectrum is a natural reference point. More $\mathcal{I}$-even scalar glueballs were found in the works [75, 76].

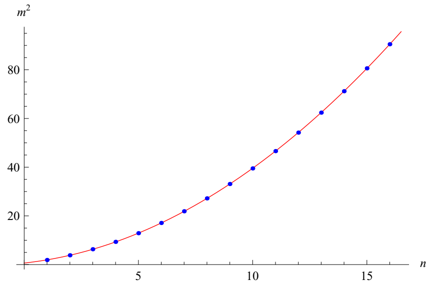
The quantum numbers of the $\mathcal{I}$-odd scalars from Figure 4.1 were identified in [28]. The massless states are a scalar and a pseudoscalar; 0^{+-} and 0^{--}. The corresponding

Table 4.4: Lowest values of $\tilde{m}^2$ and quadratic fit for the $j = 1$ multiplets described by (4.83), (4.104).

Spectrum IV ($\mathbf{X}_-$)

n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$
1	1.89	5	12.9	9	33.1	13	62.4
2	3.83	6	17.1	10	39.5	14	71.2
3	6.31	7	21.9	11	46.6	15	80.6
4	9.34	8	27.2	12	54.2	16	90.5

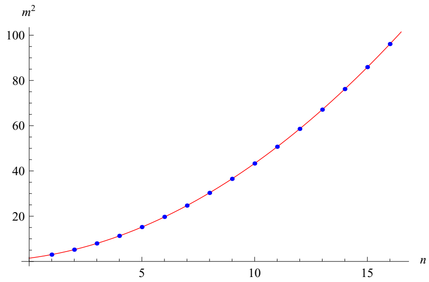
Quadratic Fit



Spectrum V ($\mathbf{X}_+$)

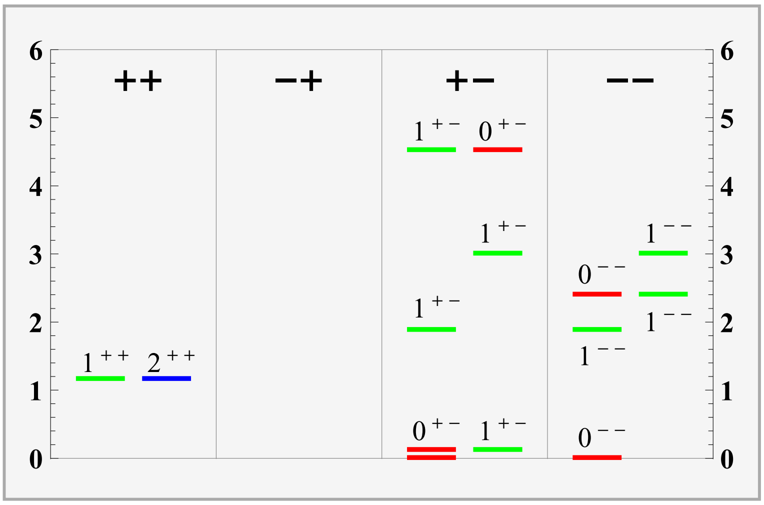
n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$
1	3.01	5	15.2	9	36.5	13	67.1
2	5.20	6	19.7	10	43.3	14	76.2
3	7.96	7	24.7	11	50.7	15	85.9
4	11.3	8	30.3	12	58.6	16	96.1

Quadratic Fit



tower of massive states is described by a vector multiplet, which contains a scalar 0^{+-} and a pseudovector 1^{+-}. The latter mixes with another massive vector multiplet from the ansatz (4.84), (4.85) and (4.86). Hence both of them should have the same quantum numbers from above 1^{+-}. The vector state from the vector multiplet described by (4.70), (4.71) and (4.72) have opposite parity transformations and therefore describes the 1^{--} vector state. One can draw the same conclusion by looking at the supermultiplet structure: this vector lies in the same supermultiplet with the pseudoscalar 0^{--}. The quantum numbers of the remaining four vectors are straightforward. The ones described by (4.84)-(4.86) are pseudovectors 1^{+-} and the other two from (4.70)-(4.72) are vectors 1^{--}.

Two mixing vector multiplets consisting of the 0^{+-} scalar and the 1^{+-} vector correspond to the operators of different dimensions. Therefore their spectra are significantly different. To identify the spectra we associate the lighter modes with



J^{PC}	$\tilde{m}^2$
1^{+-}, 0^{+-}	4.53
1^{+-}, 1^{--}	3.01
0^{--}, 1^{--}	2.41
1^{+-}, 1^{--}	1.89
1^{++}, 2^{++}	1.12
0^{+-}, 1^{+-}	0.129
0^{+-}, 0^{--}	0

Figure 4.1: Values of $\tilde{m}^2$ and J^{PC} quantum numbers of the states from the $SU(2)\times SU(2)$ invariant $\mathcal{I}$-odd sector. Each infinite tower is represented by it's lightest massive mode. Also in the figure: the massless scalar multiplet and the lightest states of the $\mathcal{I}$-even Graviton multiplet $1^{++}, 2^{++}$.

the operators of lower dimensions. Thus following [28], we identify the lightest massive multiplet in the figure 4.1 to correspond to the $U(1)_{\text{Baryon}}$ current (Betti) multiplet, which contains a scalar and a vector of dimensions 2 and 3 respectively.

As seen from the figure 4.1, the states from the Betti multiplet are much lighter than the other glueballs from the $\mathcal{I}$-odd sector and the known states from the $\mathcal{I}$-even sector. It would be interesting to compare the mass of the lightest state from the Betti Multiplet with the mass of the lightest glueball created by the chiral operator $\text{Tr}(AB)$. Despite a charge under the $SU(2)\times SU(2)$ symmetry, the latter has the lowest dimension in the KS theory; $\Delta = 3/2$. Therefore the corresponding state is a natural candidate to be the lightest massive mode in the KS spectrum.

4.9 Scaling Dimensions and SQM

The KS solution explicitly breaks both conformal and $U(1)_{\mathcal{R}}$ symmetries. Therefore the fluctuations with different scaling dimensions and $\mathcal{R}$-charges can mix with each other. Indeed we saw earlier in section 4.7 that the uncharged Betti vector mixes with the perturbation of the R-R four-form which carries $U(1)_{\mathcal{R}}$-charge ± 2. Similarly the scalar of dimension 2 mixes with the scalar of dimension 5 in (4.56)-(4.57).

The mixing between different multiplets of different dimensions can confuse the dimension analysis. Namely one cannot derive the dimension of the mode by merely analyzing the corresponding equations of motion in the large τ limit as it is usually done in the conformal case. A proper choice of basis fluctuations may be required to identify the corresponding multiplet structure and the dimensions. To illustrate this point we consider an example of the decoupled vector multiplet.

In Section 4.5 the scalar particle $\tilde{\chi}$ described by (4.69) was found to be degenerate with the vector fluctuation $\tilde{\mathbf{A}}$ that satisfies the same equation (4.79). Clearly both states must belong to the same $j = 1/2$ multiplet. As they satisfy the same equation the naive large τ analysis implies that they have the same dimension $\Delta = 5$. This must be wrong as the bosonic states from the $j = 1/2$ multiplet have the dimensions Δ_1, Δ_0 that differ by $\Delta_1 - \Delta_0 = 1$.

To resolve the puzzle we notice that the vector $\tilde{\mathbf{A}}$ mixes with other degrees of freedom, namely $\mathbf{H}$ and $\tilde{\mathbf{N}}$. In section 3 we chose $\tilde{\mathbf{A}}$ to be an independent variable, but we can choose $\mathbf{H}$ to be an independent variable instead ($\tilde{\mathbf{N}}$ cannot be chosen as an independent variable as it vanishes in this case). After eliminating $\tilde{\mathbf{A}}$ and redefining $\tilde{\mathbf{H}} = \sqrt{I} K \sinh\tau\, \mathbf{H}$ the system (4.75)-(4.77) reduces to the equation

$$\tilde{\mathbf{H}}'' + \left(\frac{1}{2}\frac{I''}{I} - \frac{(K\sinh\tau)''}{K\sinh\tau} + \frac{I'}{I}\frac{(K\sinh\tau)'}{K\sinh\tau} - \frac{3}{4}\frac{I'^2}{I^2}\right)\tilde{\mathbf{H}} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{H}} = 0 \;. \quad (4.116)$$

At the large τ limit this equation behave as

$$\tilde{\mathbf{H}}'' - \frac{16}{9}\tilde{\mathbf{H}} \simeq 0 \;, \quad (4.117)$$

which indicates that $\mathbf{H}$ has dimension $\Delta = 6$, in accordance with the $j = 1/2$ multiplet structure. This is exactly what we expected since $\mathbf{H}$ corresponds to the fluctuation $a_{\mu\nu}$ from table 1. The later indeed has dimension six.

Let us note that one cannot favor (4.116) over (4.79) without knowledge of the supermultiplet structure. In fact both equations (4.79) and (4.116) possess the same spectrum as they can be related to each other by the Supersymmetric Quantum Mechanics (SQM) transformation. More precisely this means that there are two first order differential operators Q_+ and Q_-, such that $Q_+Q_-\psi = m^2\psi$ gives the equation (4.79), while $Q_-Q_+\psi = m^2\psi$ leads to (4.116). The SQM transformation $\psi \to Q\psi$ which turns the solution of one equation into the solution of another changes the dimension of the corresponding mode. For the multiplets with half-integer j the bosonic states should have different dimensions $|\Delta_{+-} - \Delta_{-+}| = 1$ and the SQM transformation is a five-dimensional truncation of the ten-dimensional supersymmetry transformation. Among explicit examples there are the $j = 1/2$ multiplet considered in this paper and the graviton multiplet studied in [86]. The latter contains two bosonic states of dimension 3 and 4, and the corresponding equations are also related by a SQM transformation.

Our logic also suggests that in addition to the equations (4.98)-(4.99) there should be a SQM-related system of equations governing the dynamics of the vectors $\mathbf{B}_+, \tilde{\mathbf{C}}$ with the same spectrum and with the large τ behavior that corresponds to the correct dimensions 3 and 6. It would be interesting to find this system explicitly by choosing $\mathbf{D}$ as an independent variable instead of $\mathbf{C}$.

The bosonic states from the multiplets with integer j have the same dimensions and hence should be described by the same equation. Thus each $j = 1$ multiplet containing vector $\mathbf{X}$ and axial vector $\mathbf{Y}$ is described by a single equation governing both particles.

These ideas can be nicely illustrated on the example of the scalar of Section 4.5. The effective potential in (4.69) is singular at $\tau = 0$ which does not allow us to use the conventional semiclassical (WKB) approximation. Yet we can cast the equation (4.69) in the form $Q_1Q_2\tilde{w} = m^2\tilde{w}$, where Q_i are first-order differential operators and then consider an equation $Q_2Q_1\tilde{\tilde{w}} = m^2\tilde{\tilde{w}}$, which must give rise to the same spectrum up to a zero mode. Namely, in our case this means that for A such that

$$A^2 + A' = \frac{\cosh^2\tau + 1}{\sinh^2\tau} \,, \tag{4.118}$$

equation (4.69) shares the spectrum with an equation

$$\tilde{\tilde{w}}'' - (B^2 + B')\tilde{\tilde{w}} + \tilde{m}^2 \frac{I(\tau)}{K(\tau)^2}\,\tilde{\tilde{w}} = 0\,, \tag{4.119}$$

$$B = -A - \frac{1}{2}\frac{d}{d\tau}\log\frac{I(\tau)}{K(\tau)^2} \,. \tag{4.120}$$

A general solution of (4.118) reads

$$A = -\coth\tau + \frac{2\sinh^2\tau}{\cosh\tau\sinh\tau - \tau + \mathbf{C}} \,. \tag{4.121}$$

For (4.119) to be non-singular at the origin $\mathbf{C}$ has to be non-zero. For a finite $\mathbf{C}$ the potential is regular everywhere but not monotonic and (4.119) admits a zero mode. A most convenient choice is to take infinite $\mathbf{C}$, which reduces A to $A = -\coth\tau$. In this case the WKB approximation is applicable in it simplest form (see [73] for similar considerations) and yields the same result as the shooting method up to the third digit.

4.10 Operators of the Dual Gauge Theory

In section 4.3 we explained how the four-dimensional massive multiplets discussed above are embedded in the structure of the superconformal multiplets of the KW theory [81]. Namely they exhaustively match the spectrum of the shortened SU(2) × SU(2) singlet multiplets of Vector type I and Gravitino types II and IV. Let us remind the reader of the operators that correspond to those superconformal multiplets.

The Betti multiplet, which is the "massless" type I Vector Multiplet (here quotes indicate that massless refer to the five-dimensional mass), corresponds to the operator

$$\mathcal{U} = \mathrm{Tr}\, Ae^V \bar{A}e^{-V} - \mathrm{Tr}\, Be^V \bar{B}e^{-V}. \tag{4.122}$$

The lowest component of this operator $\mathrm{Tr}\,\left(A\bar{A} - B\bar{B}\right)$ is dual to the scalar U [68] and has dimension $\Delta = 2$.

The complex type IV Gravitino multiplet corresponds to the operator

$$\bar{L}^{2k}_{\dot{\alpha}} = \mathrm{Tr}\, e^V \bar{W}_{\dot{\alpha}} e^{-V} W^2 (AB)^k \ , \tag{4.123}$$

where k labels representations of the R-symmetry group. The lowest (spin 1/2) component of this operator has dimension $\Delta = 3/2\,k + 9/2$. The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant sector corresponds to $k = 0$. In this case the dependence on the bi-fundamental fields A and B vanishes

$$\mathcal{O} = \mathrm{Tr}\, e^V \bar{W}_{\dot{\alpha}} e^{-V} W^2 \ . \tag{4.124}$$

This is very interesting as this operator belongs to the pure gauge $\mathcal{N} = 1$ SYM sector of the dual field theory. For $k = 0$ the Gravitino multiplets of types II and IV are similar to each other. In particular, the type II multiplet corresponds to the complex conjugate of the operator L^{20}_{α} (4.124).

The five-dimensional superconformal multiplets split into the irreducible representations of the superalgebra in four dimensions. We saw that the Gravitino II and Gravitino IV multiplets split into four towers of massive supermultiplets, from which the lightest ones are presented in Figure 4.1. Down the throat they mix with the Betti multiplet and with each other. This means that the dual operators mix with each other at low energies. It would be interesting to understand how this mixing affects the masses of the corresponding glueballs from the field theory point of view.

4.11 Effects of Compactification

Now we will embed the KS throat into a flux compactification, along the lines of [87], and estimate the mass of the Higgs scalar. Generally, glueballs are dual to the normalizable modes localized near the bottom of the throat, and one does not expect them to be strongly affected by the bulk of the Calabi-Yau. This is indeed the case for all the massive radial excitations found in sections 4.4 and 4.5. We will see, however, that the case of the GHK scalar is more subtle and exhibits some UV sensitivity.

To model a compactification, we will introduce a UV cut-off on the radial coordinate, $\tau_{\rm max}$. We also need to include a deformation of the KS solution introduced by bulk effects. On the field theory side this corresponds to perturbing the Lagrangian of the cascading gauge theory by some irrelevant operators. Here we are not interested in classifying all of them but rather model the compactification effects in the simplest way by considering one perturbation which simulates the main features of the compactified solution. We consider a shift of the warp factor $\delta h = \text{const}$ which corresponds to the dimension 8 operator on the field theory side [88, 89]. This also has a simple geometrical meaning: the warp factor of the compactified solution is a finite constant in the bulk of the Calabi-Yau and therefore should not drop below a certain value along the throat.

Let us introduce a small parameter δ which shifts the rescaled warp factor, $I(\tau) \to I(\tau) + \delta$, and consider the system (4.56)-(4.57) in perturbation theory near $\tilde{m}^2 = 0$:

$$\tilde{z} = \tilde{z}_0 + \tilde{m}^2 \tilde{z}_1 \ , \tag{4.125}$$

$$\tilde{w} = \tilde{w}_0 + \tilde{m}^2 \tilde{w}_1 \ , \tag{4.126}$$

$$\tilde{z}_0 = \tau \coth \tau - 1 \ , \tag{4.127}$$

$$\tilde{w}_0(\tau) = -\frac{2^{2/3}\, 8}{9} \frac{1}{\sinh \tau} \int_0^\tau dx\, I(x) \sinh^2 x \, . \tag{4.128}$$

At the leading order in $\tilde{m}^2$

$$\begin{aligned}\tilde{z}_1 &= (\tau\coth\tau - 1)\int_0^\tau dx\, u(x)\coth x - \coth\tau\int_0^\tau dx\, u(x)\,(x\coth x - 1)\,, & (4.129)\\ \tilde{w}_1 &= -\frac{1}{4\sinh\tau}\int_0^\tau dx\, v(x)\,\frac{\sinh 2x - 2x}{\sinh x} - \frac{\sinh 2\tau - 2\tau}{4\sinh\tau}\int_\tau^\infty dx\, v(x)\,\frac{1}{\sinh x}\,; & (4.130)\end{aligned}$$

where

$$u(\tau) = -\frac{I(\tau)}{K^2(\tau)}\,\tilde{z}_0 + \frac{9}{4\cdot 2^{2/3}}\,K(\tau)\,\tilde{w}_0 - \frac{\delta}{K^2}\,\tilde{z}_0\,, \quad (4.131)$$

$$v(\tau) = -\frac{I(\tau)}{K^2(\tau)}\,\tilde{w}_0 + \frac{16}{9}\,K(\tau)\,\tilde{z}_1 - \frac{\delta}{K^2}\,\tilde{w}_0\,. \quad (4.132)$$

Keeping in mind that for large τ, $u \simeq -2^{-2/3}\delta\tau\, \mathrm{e}^{2\tau/3}$, one finds the following asymptotic behavior:

$$\tilde{z}_1(\tau) \simeq -2^{-2/3}\delta\int_0^\tau dx\,(\tau - x)x\mathrm{e}^{2x/3} \simeq -\frac{9\,\delta}{4\,2^{2/3}}\,\tau\mathrm{e}^{2\tau/3}\,. \quad (4.133)$$

This yields $v \simeq -2^{2/3}\delta\tau\,\mathrm{e}^{\tau/3}$ and

$$\begin{aligned}\tilde{w}_1 &= -\frac{1}{4\sinh\tau}\int_0^\tau dx\, v_0(x)\,\frac{\sinh 2x - 2x}{\sinh x} - \frac{\sinh 2\tau - 2\tau}{4\sinh\tau}\int_\tau^\infty dx\, v_0(x)\,\frac{1}{\sinh x}\\ &\simeq \frac{9\cdot 2^{2/3}\delta}{8}\,\tau\mathrm{e}^{\tau/3}\,. & (4.134)\end{aligned}$$

Finally, up the first order in the mass squared and δ:

$$\tilde{z} \simeq \tau\left[1 - \frac{9\,\delta\tilde{m}^2}{4\cdot 2^{2/3}}\,\mathrm{e}^{2\tau/3}\right]\,, \quad (4.135)$$

$$\tilde{w} \simeq -2^{4/3}\tau\mathrm{e}^{-\tau/3}\left[1 - \frac{9\,\delta\tilde{m}^2}{8\cdot 2^{2/3}}\,\mathrm{e}^{2\tau/3}\right]\,. \quad (4.136)$$

This suggests that for generic boundary conditions the cut-off value is

$$\tau_{\max} \simeq -\log\delta^{3/2}\tilde{m}^3\,. \quad (4.137)$$

This prediction can be tested numerically. In order to do this one can specify some

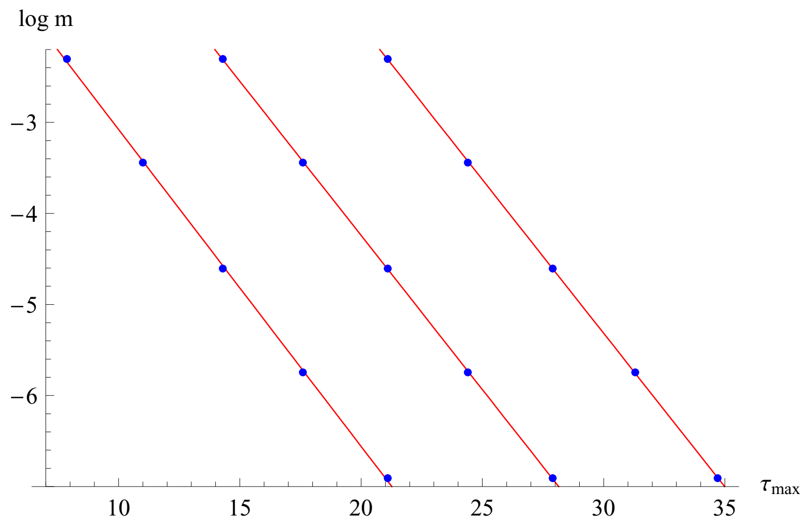


Figure 4.2: The dependence of $\log \tilde{m}$ on $\tau_{\rm max}$ is linear with the slope equal to -1/3. The three lines shown correspond to $\delta = 1$, $\delta = 0.01$ and $\delta = 0.0001$.

small $\tilde{m}$ and plot the determinant

$$\det \begin{pmatrix} \tilde{z}_1(\tau) & \tilde{z}_2(\tau) \\ \tilde{w}_1(\tau) & \tilde{w}_2(\tau) \end{pmatrix}, \tag{4.138}$$

of the two linearly independent solutions regular at $\tau = 0$ as a function of τ. The first zero marks the point $\tau_{\rm max}$ such that there is a regular solution with $z(\tau_{\rm max}) = w(\tau_{\rm max}) = 0$. Hence $\tau_{\rm max}$ is the corresponding cut-off value. As Fig. 4.2 shows, the relation (4.137) holds for $\tau_{\rm max}$ large enough so that

$$\tilde{m}^2 \sim \delta^{-1} e^{-2\tau_{\rm max}/3} \ll 1 \tag{4.139}$$

is small.

Let us consider a simple model of compactification where the throat is embedded into an asymptotically conical space that terminates at some large cut-off value $\tau_{\rm max}$. To calculate the mass from (4.139) we need to know δ as well as $\tau_{\rm max}$. The former is the asymptotic value of the (rescaled) warp factor. The point where the field theory warp factor approaches δ marks the UV cutoff of the field theory

$$I(\tau_{UV}) \sim \tau_{UV} e^{-4\tau_{UV}/3} \simeq \delta \ . \tag{4.140}$$

Using this in (4.139) we find $\tilde{m}^2 \sim \mathrm{e}^{(4\tau_{UV}-2\tau_{\max})/3}$. This shows that the Higgs mass becomes parametrically small only for $\tau_{\max} \gg 2\tau_{UV}$. This is not satisfied in general; the geometry requires only that $\tau_{\max} > \tau_{UV}$ because τ_{UV} is the length of the throat embedded into a CY space. With the ratio between the UV and IR scales of the field theory around $4 \cdot 10^3$ [71] we estimate that $\tau_{UV} \simeq 25$ [68]. The cut-off $\tau_{\max}$ can be related to the warped volume of the Calabi-Yau which, in a singular conifold approximation, is

$$V_6^w = \mathrm{Vol}(T^{1,1}) \int_0^{r_{\max}} dr h(r) \sqrt{\frac{\det\, g_6}{\det\, g_{T^{1,1}}}} \ , \tag{4.141}$$

where $r \sim \epsilon^{2/3} e^{\tau/3}$. The integral from zero to r_{UV} is the warped volume of the throat, and from r_{UV} to $r_{\max}$ is the bulk volume. Assuming that the latter dominates,

$$V_6^w \simeq \frac{16\pi^3}{27} \epsilon^{4/3} (g_s M \alpha')^2 \left[r_{\max}^6 - r_{UV}^6 \right] r_{UV}^{-4} \ . \tag{4.142}$$

Requiring $\tau_{\max} \gg 50$ leads to an enormous V_6^w, far larger than, for example, $V_6^w \simeq 5^6 \alpha'^3$ in [71].

Thus, while for $\tau_{\max} \gg 2\tau_{UV}$ the Higgs scalar becomes parametrically lighter than the other normal modes, in compactifications with realistic parameters it may actually be heavier. This is due to the special feature of its wave function $\tilde{z}$ which grows linearly with τ in the throat. The only conclusion we can draw from our simplified model of compactification is that this mode is rather UV sensitive, so to determine its mass we need to know the details of the compactification.

4.12 Discussion

We have found all the $\mathcal{I}$-odd $\mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant bosonic supergravity excitations over the KS solution. At the massless level there are two spin 0 zero states:

a Goldstone pseudoscalar that corresponds to the spontaneously broken $U(1)_{\rm Baryon}$ and a scalar related to the expectation value of the baryon operators. Together with fermions these states form a $j = \frac{1}{2}$ scalar supermultiplet. At the massive level the supersymmetry representation changes so that the pseudoscalar is eaten by the Betti pseudovector giving rise to a tower of $j = \frac{1}{2}$ vector supermultiplets. In the conformal case the $j = \frac{1}{2}$ multiplets are embedded into the "massless" Vector Multiplet of type I [81].

There are two more towers of massive spin 0 modes (scalar and pseudoscalar) and six more massive spin 1 towers (3 vector and 3 axial vector). In the conformal case they belong to a combination of the shortened Gravitino Multiplets II and IV.

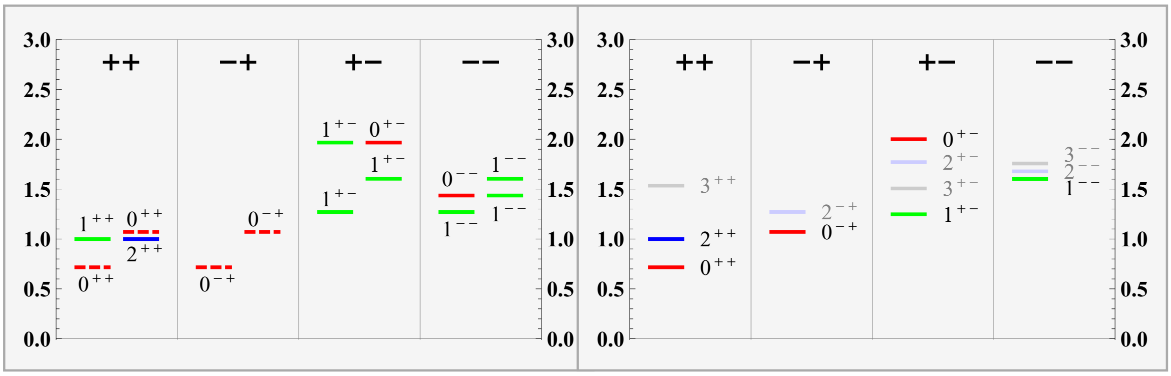
The two massive scalar excitations mix with each other while the massive pseudoscalar excitation decouples. Similarly the seven massive (pseudo)vectors split into two non-interacting subsystems of three vectors and four axial vectors. The system of three vectors contains the superpartner of the only massive pseudoscalar and two vectors $\mathbf{X}_{\pm}$. The system of four axial vectors contains two superpartners of the two coupled massive scalars and the two axial vectors $\mathbf{Y}_{\pm}$. The states $\mathbf{X}_{+}, \mathbf{Y}_{+}$ and $\mathbf{X}_{-}, \mathbf{Y}_{-}$ are degenerate in pairs and form two $j = 1$ "gravitino" multiplets that consist of a vector, an axial vector and the spin 1/2 and 3/2 fermions.

We identify the the spin 0 massive modes from the $\mathcal{I}$-odd along with their vector superpartners together with the remaining $\mathcal{I}$-odd vector states and compute numerically the spectra of these multiplets. The results for the lightest states together with their J^{PC} quantum numbers are presented in the Figure 4.1.

An interesting task for the future would be to generalize our analysis to the $\mathcal{I}$-even sector and identify all $\mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant bosonic modes of the KS theory. Some $\mathcal{I}$-even states are already known. Among them are the vector and the spin two states from the Graviton multiplet (the lightest modes are shown in figure 4.1). In fact these states are likely to be the only bosonic non-scalar states in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant $\mathcal{I}$-even sector. Indeed there are no spin 1 $\mathcal{I}$-even excitations of B_2 and C_2 and the only possible spin 1 fluctuations of the metric were considered in [75] and [85]. Some of the scalar states, namely a system of seven 0^{++} excitations were studied by M. Berg et al. in [75, 76]. They calculated the spectra of the particles but did not identify the corresponding operators. Besides an obvious task to find the corresponding pseudoscalar superpartners it would be interesting to match the resulting supermultiplets to the superconformal multiplets of [81].

Comparing our results with those for a pure gauge non-supersymmetric theory may give a sensible prediction for the masses of some of the lightest $\mathcal{I}$-even scalars. As we observed above, some of the fluctuations considered in this paper are dual to the operators that do not contain the bi-fundamental fields A and B. In particular, the graviton multiplet, which contains 1^{++} and 2^{++} states, is dual to the "super-current" operator $V_{\alpha\dot{\alpha}} = \mathrm{Tr}\, W_\alpha e^V \bar{W}_{\dot{\alpha}} e^{-V}$ [90]. Also the states of the Gravitino Multiplets correspond to the components of the superfield $\mathcal{O} = \mathrm{Tr}\, e^V \bar{W}_{\dot{\alpha}} e^{-V} W^2$ in the conformal case. In the KS theory however, the latter mix with the states from the Betti multiplet, dual to A and B dependent operators. Below we plot the lightest states from the pure gauge sector of the KS theory (Figure 4.3.a) and compare them with those of the pure SU(3) theory (Figure 4.3.b). In Figure 4.3.a we employ a qualitative approach, ignoring the mixing between the states from the pure gauge sector (i.e. A and B independent) and from the KK sector (with A or B).

In Figure 4.3.a we present only those states from Figure 4.1 that belong to the pure gauge sector of the KS theory. The masses of the states are normalized to the mass of the 2^{++} state. We have also plotted two light $\mathcal{I}$-even scalar multiplets, which we expect to see in the spectrum. These two multiplets should correspond to a mixture of the following pure $\mathcal{N} = 1$ SYM operators: the gluino bilinear $\lambda\lambda$ of dimension 3 and the dimension 4 operators $\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$ and $\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$. These multiplets have not been identified yet and we mark their position with dashed lines. Their masses in figure 2.a are conjectured based on the comparison with the pure glue SU(3) theory. It is also possible that some of the two 0^{++} particles in



(a) (b)

Figure 4.3: (a) Pure gauge sector of the KS theory. Conjectured positions of 0^{++} and 0^{-+} states are marked by dashed lines. (b) Spectrum of non-supersymmetric pure glue SU(3) theory [91]. Both spectra are normalized to the mass of 2^{++} state.

question is a part of the seven scalar system of [75, 76].

In Figure 4.3.b we plot the lattice results of Morningstar and Peardon [91] for spectrum of the pure glue SU(3) theory, which we also normalize to the mass of the 2^{++} state. We shade the irrelevant high spin states, which cannot be described in the supergravity approximation. Although the two theories are very different, the relative masses of the states are surprisingly similar. Indeed each state from the pure glue SU(3) theory has a counterpart with the same quantum numbers and a similar mass (measured in the units of 2^{++} mass) in the pure gauge sector of the KS theory. Besides the counterparts of the pure glue SU(3) theory states, Figure 4.3.a also contains their superpartners and even one "extra" vector multiplet (a 0^{--} scalar and a 1^{--} vector). In general the additional states are attributed to the fermionic degrees of freedom which are absent from the pure glue SU(3) theory. Let us emphasize that the reason for the similarity between Figure 4.3.a and 4.3.b is not immediately clear and could be coincidental. To examine this issue in more detail is an interesting problem for the future.

Appendix A

Orbifolds: a Detailed Construction

A.1 Quiver vs Orbit Description

We show that the two descriptions of the quiver gauge theory field content are equivalent and develop explicit transition formulae between them. First we introduce the basis in the field space of the parent $\mathcal{N}=4$ theory. As we saw in Section 2.2, Chan-Paton indices of the fields transform in the regular representation of the orbifold group Γ. Namely, the fields belong to

$$V^{\perp} \otimes V_{\text{reg}}^{\oplus N} \otimes \bar{V}_{\text{reg}}^{\oplus N} \simeq V^{\perp} \otimes V_{\text{reg}} \otimes \bar{V}_{\text{reg}} \otimes \mathbb{C}^{N} \otimes \mathbb{C}^{*N} \tag{A.1}$$

$V^{\perp}$ being the representation corresponding to the transverse indices. We are going to use the two orthonormal bases in the group algebra $V_{\text{reg}} \simeq \mathbb{C}[\Gamma]$,

$$\{\mathrm{e}_g = g\} \tag{A.2}$$

and

$$\{\mathrm{E}^{\lambda}_{mn} = \frac{1}{\sqrt{S_\lambda}} \sum_g \overline{\rho^{\lambda}_{nm}}(g)\, g\}\,, \quad S_\lambda = \frac{|\Gamma|}{\dim V_\lambda}\,. \tag{A.3}$$

The group acts on them according to

$$h\,:\,\mathrm{e}_g \to \mathrm{e}_{hg}\,, \tag{A.4}$$

$$h\,:\,\mathrm{E}^{\lambda}_{mn} \to \sum_k \overline{\rho^{\lambda}_{nk}}(h^{-1})\,\mathrm{E}^{\lambda}_{mk} = \sum_k \mathrm{E}^{\lambda}_{mk}\,\rho^{\lambda}_{kn}(h)\,. \tag{A.5}$$

The relation between these bases is

$$\mathrm{e}_g = \sum_\lambda \frac{1}{\sqrt{S_\lambda}} \sum_{mn} \rho^\lambda_{nm}(g)\, \mathrm{E}^\lambda_{mn}\,, \tag{A.6}$$

$$\mathrm{E}^\lambda_{mn} = \frac{1}{\sqrt{S_\lambda}} \sum_g \overline{\rho^\lambda_{nm}}(g)\, \mathrm{e}_g\,. \tag{A.7}$$

The dual bases are introduced according to $\mathrm{e}^*_g(\mathrm{e}_h) = \delta_{gh}$ and $\mathrm{E}^{\lambda *}_{kl}(\mathrm{E}^\mu_{mn}) = \delta^{\lambda\mu}\,\delta_{km}\,\delta_{ln}$. Next we construct the two bases in the field space. These basis vectors are to label the invariant configurations in the field space, thus we need to find the invariant configurations

$$\left(V^\perp \otimes V^{\oplus N}_{\mathrm{reg}} \otimes \bar{V}^{\oplus N}_{\mathrm{reg}}\right)^\Gamma \simeq \left(V^\perp \otimes V_{\mathrm{reg}} \otimes \bar{V}_{\mathrm{reg}}\right)^\Gamma \otimes \mathbb{C}^N \otimes \mathbb{C}^{*N}\,. \tag{A.8}$$

In what follows we are going to drop the trivial $\mathbb{C}^N \otimes \mathbb{C}^{*N}$ factor

Let us start with the gauge field. The "orbit" basis

$$\mathbf{t}_g = \sum_h \mathrm{e}_h \otimes \mathrm{e}^*_{hg} \tag{A.9}$$

has a natural interpretation in terms of invariant combinations of strings stretching between image branes. Similarly, the product $\mathbf{t}_g \circ \mathbf{t}_h = \mathbf{t}_{gh}$ has a natural interpretation in terms of gluing the ends of open strings. Note that the hermitian conjugate $\mathbf{t}^\dagger_g = \mathbf{t}_{g^{-1}}$. In order to build the "quiver" basis we note that E^λ_{mn} do *not* transform in the first index (recall that each representation $\mathfrak{R}_\lambda$ enters $\mathfrak{R}_{\mathrm{reg}}$ with multiplicity equal to $N_\lambda = \dim V_\lambda$ — and this is the first index of E^λ_{mn} that numbers these copies). Therefore, the combination

$$\mathbf{T}^\lambda_{mn} = \sum_k \mathrm{E}^\lambda_{mk} \otimes \mathrm{E}^{\lambda *}_{nk} \tag{A.10}$$

is Γ-invariant. The multiplication rule is $\mathbf{T}^\lambda_{mn} \circ \mathbf{T}^\mu_{kl} = \delta^{\lambda\mu}\,\delta_{kn}\,\mathbf{T}^\lambda_{ml}$. Hermitian conjugate $\mathbf{T}^{\lambda\dagger}_{mn} = \mathbf{T}^\lambda_{nm}$. Here we recognize the matrix algebra $\bigoplus_\lambda \mathfrak{gl}(v_\lambda) \simeq \mathbb{C}[\Gamma]$. Thus, in these two calculations we get the same answer; *i.e.*, the algebrae of $\mathbf{t}_g$ and $\mathbf{T}^\lambda_{mn}$

are both isomorphic to the group algebra. A straightforward calculation shows that the two bases are related by a discrete Fourier transform,

$$\mathbf{t}_g = \sum_{\lambda} \sum_{km} \overline{\rho^{\lambda}_{mk}}(g)\, \mathbf{T}^{\lambda}_{mk} \,. \tag{A.11}$$

Now we can do a similar calculation for the scalar and spinor fields which have transverse indices with non-trivial transformation rules. In this case we have to find the invariant subspace $\left(V^{\perp} \otimes V_{\text{reg}} \otimes \bar{V}_{\text{reg}}\right)^{\Gamma}$. Denote the basis of the transverse representation $V^{\perp}$ as $\{\mathrm{f}_A \equiv \mathrm{e}_{\alpha,A}\}$. Then the "orbit" basis has the form

$$\mathbf{t}_{A,g} = \sum_h (h \triangleright \mathrm{f}_A) \otimes \mathrm{e}_h \otimes \mathrm{e}^*_{hg} \,; \tag{A.12}$$

where in terms of components the action is $h \triangleright \mathrm{f}_A = \sum_B \rho^{\alpha}_{BA}(h)\, \mathrm{f}_B$. Since the representation $\mathfrak{R}_{\alpha}$ is real, hermitian conjugation acts according to $\mathbf{t}^{\dagger}_{A,g} = \sum_B \rho_{BA}(g^{-1})\mathbf{t}_{B,g^{-1}}$. To find the "quiver" basis we will need to find the invariant tensors in the product

$$\left(V^{\perp} \otimes V_{\text{reg}} \otimes \bar{V}_{\text{reg}}\right)^{\Gamma} \simeq \bigoplus_{\lambda,\mu} \left(V_{\alpha} \otimes V_{\lambda} \otimes V^*_{\mu}\right)^{\Gamma} \otimes \mathbb{C}^{N_{\lambda}} \otimes \mathbb{C}^{*N_{\mu}} \,.$$

To do it we decompose the product of representations $\mathfrak{R}_{\alpha}$ and $\mathfrak{R}_{\lambda}$ into a direct sum of irreducible representations. In particular, in terms of basis vectors

$$\mathrm{e}_{\alpha A} \otimes \mathrm{e}_{\lambda l} = \sum_{\mu m} \mathcal{K}^{\mu m}_{\alpha A,\lambda l}\, \mathrm{e}_{\mu m} \quad \text{and} \quad \mathrm{e}_{\mu m} = \sum_{A,l} \overline{\mathcal{K}^{\mu m}_{\alpha A,\lambda l}}\, \mathrm{e}_{\alpha A} \otimes \mathrm{e}_{\lambda l} \,.$$

Therefore, the invariant configuration is

$$\sum_m \mathrm{e}_{\mu m} \otimes \mathrm{e}^*_{\mu m} = \sum_{A,l,m} \overline{\mathcal{K}^{\mu m}_{\alpha A,\lambda l}}\, \mathrm{e}_{\alpha A} \otimes \mathrm{e}_{\lambda l} \otimes \mathrm{e}^*_{\mu m} \,.$$

(The field components of the invariant configuration are given by the invariant tensor, $\Phi^{Al}_m \sim \overline{\mathcal{K}^{\mu m}_{\alpha A,\lambda l}}$.) This gives

$$\mathbf{T}^{\lambda\mu}_{lm} = \sum_{A,i,j} \overline{\mathcal{K}^{\mu j}_{\alpha I,\lambda i}}\, \mathrm{f}_A \otimes \mathrm{E}^{\lambda}_{li} \otimes \mathrm{E}^{\mu *}_{mj} \,. \tag{A.13}$$

Note that here the indices l and m are the indices of the gauge groups at the corresponding nodes $\mathfrak{R}_{\lambda}$ and $\mathfrak{R}_{\mu}$. These indices appear owing to the fact that each representation $\mathfrak{R}_\lambda$ enters the decomposition of the regular representation $\mathfrak{R}_{\rm reg}$ with multiplicity N_λ. A calculation similar to that for the gauge field gives

$$\mathbf{t}_{A,g} = \sum_{\lambda,l} \sum_{\mu,km} \frac{\dim V_\lambda}{\dim V_\mu} \mathcal{K}^{\mu k}_{A,\lambda l} \, \overline{\rho^\mu_{km}}(g) \, \mathbf{T}^{\lambda\mu}_{lm} \, . \tag{A.14}$$

The presence of the factor $\mathcal{K}^{\mu\cdot}_{A,\lambda\cdot}$ restricts the sum over (λ, μ) only to those pairs which are connected by a line in the quiver. Using (A.14), we can find the relation between the field components in the two notations,

$$\phi^{\lambda\mu}_{lm} = \sum_g \sum_k \frac{N_\lambda}{N_\mu} \mathcal{K}^{\mu k}_{A,\lambda l} \, \overline{\rho^\mu_{km}}(g) \, \phi^A_g \, . \tag{A.15}$$

The matrix product of the quiver gauge theory gives rise to the (modified) convolution in terms of the group algebra. Particularly, the product of the two fields ϕ and ψ with the transverse indices transforming in the representations $\mathfrak{R}_\alpha$ and $\mathfrak{R}_\beta$ is

$$\sum_m \phi^{\lambda\mu}_{lm} \, \psi^{\mu\nu}_{mn} = \frac{N_\lambda}{N_\nu} \sum_{g,h} \mathcal{K}^{\nu r}_{\alpha A, \beta B, \lambda l} \, \overline{\rho^\nu_{nr}}(h^{-1} g^{-1}) \; \rho^\beta_{BC}(g) \, \phi^A_g \, \psi^C_h \, . \tag{A.16}$$

Here $\mathcal{K}^{\nu r}_{\alpha A, \beta B, \lambda l} = \sum_p \mathcal{K}^{\mu p}_{\alpha A, \lambda l} \, \mathcal{K}^{\nu r}_{\beta B, \mu p}$ is (one of) the invariant tensors corresponding to the decomposition $\mathfrak{R}_\alpha \otimes \mathfrak{R}_\beta \otimes \mathfrak{R}_\lambda \to \mathfrak{R}_\nu$. Note that (A.16) has the same structure as (A.15), the product $\phi \circ \psi$ having the defining representation $\mathfrak{R}_\alpha \otimes \mathfrak{R}_\beta$. The convolution rule is

$$(\phi \circ \psi)^{AB}_g = \sum_h \phi^A_h \; \rho_{BC}(h) \, \psi^C_{h^{-1}g} \, . \tag{A.17}$$

Both formulae (A.16) and (A.17) are also valid for the gauge fields which have no transverse indices (trivial representation). In this case some matrix elements and decomposition tensors become degenerate. Let us stress that the multiplication rule (A.17) naturally corresponds to the standard matrix multiplication in the parent SU($|\Gamma| \, N$) theory. This means that

$$\sum_f \phi^A_{hf} \, \psi^B_{fg} = \; \rho^\alpha_{AA'}(h) \; \rho^\beta_{BB'}(h) \, (\phi \circ \psi)^{A'B'}_{h^{-1}g} \, . \tag{A.18}$$

This way of multiplication is induced from the original theory, and that is why it respects the gauge transformations. Another nice feature of the formula (A.16) is that when there exist several arrows going between different nodes the choice of a given arrow affects only the choice of the invariant tensors and does not affect the convolution product (A.17). It means that all the operators corresponding to the different paths (not necessarily closed) in the quiver formed by L consequent scalar lines $\lambda_1 \to \lambda_2 \to \cdots \to \lambda_{L+1}$ are contained in the product $\phi^{A_1} \dots \phi^{A_L}$.

We can summarize these results as follows. An operator formed in the quiver notation as the product $\phi^{\lambda\nu_1}\,\phi^{\nu_1\nu_2} \dots \phi^{\nu_{L-1}\mu}$ can be recast as

$$\Big(\phi \circ \cdots \circ \phi\Big)^{\lambda\mu}_{lm} \;=\; \sum_g \sum_k \frac{N_\lambda}{N_\mu}\, \mathcal{K}_{A_1 \dots A_L}{}^{\mu k}_{\lambda l}\; \overline{\rho^{\mu}_{km}}(g)\, \Big(\phi \circ \cdots \circ \phi\Big)^{A_1 \dots A_L}_{g} \; ; \tag{A.19}$$

where the invariant tensor $\mathcal{K}$ is the one corresponding to the decomposition $\mathfrak{R}_\lambda \otimes \mathfrak{R}_{\nu_1} \otimes \cdots \otimes \mathfrak{R}_{\nu_{L-1}} \to \mathfrak{R}_\mu$. The product of fields in the r.h.s. is calculated according to (A.17). In its turn it is related to the product of the fields of the original $\mathcal{N} = 4$ theory as

$$\Big(\phi^{A_1}, \dots \phi^{A_L}\Big)_{h,hg} \;=\; \sum_{B_1, \dots, B_L} \rho_{A_1 B_1}(h) \dots\; \rho_{A_L B_L}(h)\, \Big(\phi \circ \cdots \circ \phi\Big)^{B_1 \dots B_L}_{g} \tag{A.20}$$

These formulae will be of crucial importance for constructing the gauge invariant observables.

A.1.1 Construction of Observables

In order to construct gauge invariant observables it is convenient to use the quiver notation. Taking a closed loop in the quiver and using (A.19) one can write the corresponding operator as

$$\mathrm{Tr}_{\lambda_1} \phi^{\lambda_1\lambda_2}\, \phi^{\lambda_2\lambda_3} \dots \phi^{\lambda_L\lambda_1} \;=\; \sum_g \sum_{k,l} \mathcal{K}_{A_1,\dots A_L}{}^{\lambda_1 k}_{\lambda_1 l}\; \overline{\rho^{\lambda}_{kl}}(g) \Big(\phi \circ \dots \circ \phi\Big)^{A_1,\dots A_L}_{g} \; ; \tag{A.21}$$

where the invariant tensor

$$\mathcal{K}_{A_1,\dots A_L}{}^{\lambda_1 k}_{\lambda_1 l} \;=\; \sum_{l_2,\dots,l_L} \mathcal{K}^{\lambda_2 l_2}_{A_1\,\lambda_1 l} \mathcal{K}^{\lambda_3 l_3}_{A_2\,\lambda_2 l_2} \dots \mathcal{K}^{\lambda_L l_L}_{A_{L-1}\,\lambda_{L-1} l_{L-1}}\, \mathcal{K}^{\lambda_1 k}_{A_L\,\lambda_L l_L} \tag{A.22}$$

corresponds to the closed path $\lambda_1 \to \lambda_L \to \cdots \to \lambda_2 \to \lambda_1$. Note that the l.h.s. is explicitly symmetric w.r.t. the cyclic permutations of the fields under the trace. There also exists a different way to construct gauge invariant operators. Namely, let us start with the ansatz

$$\mathcal{O}[\mathcal{K}] \;=\; \sum_g \sum_{A_1,\dots A_L} \mathcal{K}_{A_1\dots A_L}(g) \left(\phi \circ \ldots \circ \phi\right)_g^{A_1,\dots A_k} . \tag{A.23}$$

Generally such an expression represents a sum of operators corresponding to some paths in the quiver, not necessarily closed. That is why the gauge invariance condition has to be imposed separately, and it yields

$$\mathcal{K}_{B_1\dots B_L}(h^{-1}gh) \;=\; \sum_{A_1\dots A_L} \mathcal{K}_{A_1\dots A_L}(g)\, \rho_{A_1 B_1}(h) \ldots \rho_{A_L B_L}(h) . \tag{A.24}$$

A straightforward consequence of this result is that $\mathcal{K}[g]$ has to be an invariant tensor w.r.t. the stabilizer subgroup S_g. Note that in (A.21) we had

$$\mathcal{K}_{A_1\dots A_L}(g) \;=\; \sum_{k,l} \mathcal{K}_{A_1,\dots A_L}{}_{\lambda l}^{\lambda k}\, \overline{\rho^{\lambda}_{kl}}(g) , \tag{A.25}$$

and it obviously satisfies (A.24). On the other side, tensor $\mathcal{K}(g)$ can be expanded in Fourier series as a function on the group,

$$\mathcal{K}_{A_1\dots A_L}(g) \;=\; \sum_{\lambda} \sum_{k,l} \tilde{\mathcal{K}}^{(\lambda)}_{A_1,\dots A_L}{}_{\lambda l}^{\lambda k}\, \overline{\rho^{\lambda}_{kl}}(g) ; \tag{A.26}$$

and then the condition (A.24) translates into the requirement that the coefficients $\tilde{\mathcal{K}}^{(\lambda)}_{A_1,\dots A_L}{}_{\lambda l}^{\lambda k}$ are invariant tensors. These considerations provide a dictionary between the two notations in the quiver gauge theory.

It is very important that the gauge invariance condition (A.24) relates the values of the tensor $\mathcal{K}(g)$ within the same conjugacy class, and there is no relation between the values of $\mathcal{K}$ on the different conjugacy classes. That is why one can build a gauge invariant operator with $\mathcal{K}(g) \neq 0$ only on a given conjugacy class $[g]$. Such operators are said to belong to the *twisted sector* with the twist $[g]$ (determined only up to a conjugation). One can choose a reference element g in the conjugacy class $[hgh^{-1}]$ and set $\mathcal{K}_{A_1 \dots A_L}(g) = \mathcal{K}_{A_1 \dots A_L}$, $\mathcal{K}_{A_1 \dots A_L}$ being some S_g-invariant tensor. Then (A.24) determines the values of $\mathcal{K}(h^{-1}gh)$ on all the elements of the conjugacy class. The corresponding operator is

$$\mathcal{O}[\mathcal{K}] \;=\; \sum_{g,h} \sum_{A_1 \dots A_L} \sum_{B_1 \dots B_L} \mathcal{K}_{A_1 \dots A_L}\, \rho_{A_1 B_1}(h) \dots \rho_{A_L B_L}(h) \big(\phi \circ \cdots \phi\big)_{h^{-1}gh}^{B_1 \dots B_L} \;; \tag{A.27}$$

and it rewrites in terms of the fields of the parent $\mathcal{N} = 4$ theory as

$$\mathcal{O}[\mathcal{K}] \;=\; \sum_{g,h} \sum_{A_1 \dots A_L} \mathcal{K}_{A_1 \dots A_L} \,\mathrm{Tr}\left[\gamma(g) \phi^{A_1} \dots \phi^{A_L}\right]. \tag{A.28}$$

The twist field $\gamma(g)$ acts on the dynamical fields as follows,

$$\big(\phi^A \gamma(g)\big)_{h_1,h_2} \;=\; \phi^A_{h_1, g h_2}\,, \tag{A.29}$$

$$\big(\gamma(g)\, \phi^A\big)_{h_1,h_2} \;=\; \phi^A_{g^{-1} h_1, h_2}\,. \tag{A.30}$$

Invariance condition imposed by the orbifold projection on the fields implies the interchange relation

$$\big(\gamma(g)\, \phi^A\big) = \rho_{AB}(g^{-1})\, \big(\phi^B \gamma(g)\big)\,. \tag{A.31}$$

A.2 Representation Ring of the Dihedral Group

The dihedral group D_S is generated by the two elements, r and σ, with the additional relations

$$r^S = \sigma^2 = 1\,, \qquad\qquad r\sigma = \sigma r^{-1}\,. \tag{A.32}$$

The order of the group $|D_S| = 2S$. We will restrict ourselves to the odd $S = 2n+1$. Then there are the $n+2$ conjugacy classes, $\mathcal{O}_1 = \{e\}$, $\mathcal{O}_2 = \{r, r^{2n}\}, \dots,$ $\mathcal{O}_{n+1} = \{r^n, r^{n+1}\}$, $\mathcal{O}_{n+2} = \{\sigma, \sigma r, \dots, \sigma r^{2n}\}$. Thus there exist the $n+2$ irreducible representations. Among them there are the n two-dimensional representations ρ_m:

$$\rho_m(r) = \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix}, \qquad \rho_m(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad m = 1, 2, \ldots n\,. \quad \text{(A.33)}$$

Here $\omega = \mathrm{e}^{2\pi i/S}$ There are also the two one-dimensional representations ρ_0 and $\rho_{\tilde{0}}$:

$$\rho_0(r) = 1\,, \quad \rho_0(\sigma) = 1\,; \qquad \rho_{\tilde{0}}(r) = 1\,, \quad \rho_{\tilde{0}}(\sigma) = -1\,. \quad \text{(A.34)}$$

The table of characters as well as the representation ring and the stabilizer subgroups of each element of the group D_{2n+1} are shown in Table A.1.

Table A.1: Table of characters and representation ring (multiplication table) of the dihedral group $D_{S=2n+1}$. Stabilizer subgroups S_g for a representative of each conjugacy class of the group D_5.

	$[e]$	$[r^m]$	$[\sigma]$
χ_0	1	1	1
$\chi_{\tilde{0}}$	1	1	-1
χ_l	2	$2\cos\left(2\pi\frac{lm}{S}\right)$	0

$\otimes$	ρ_0	$\rho_{\tilde{0}}$	ρ_k
ρ_0	ρ_0	$\rho_{\tilde{0}}$	ρ_k
$\rho_{\tilde{0}}$	$\rho_{\tilde{0}}$	ρ_0	ρ_k
ρ_l	ρ_l	ρ_l	$\begin{cases} \rho_{k+l} \oplus \rho_{k-l}\,, & k \neq l \\ \rho_{2l} \oplus \rho_0 \oplus \rho_{\tilde{0}}\,, & k = l \end{cases}$

g	e	r	r^2	$[\sigma]$
S_g	D_5	$\{e, r, \ldots, r^4\} \simeq \mathbb{Z}_5$	$\{e, r, \ldots, r^4\} \simeq \mathbb{Z}_5$	$\{e, \sigma\}$

Appendix B

Specific Calculations for the KS Background

Here we succinctly list the supergravity equations of motion required to study the perturbations of the forms H_3, F_3, F_5 and the metric tensor. Since the dilaton and RR scalar do not enter at linear order, we set them to zero.

Bianchi identities:

$$\begin{aligned} dF_3 &= 0\,, \\ dH_3 &= 0\,, \\ dF_5 &= H_3 \wedge F_3\,. \end{aligned} \tag{B.1}$$

Dynamic equations:

$$\begin{aligned} d\star H_3 &= -g_s^2 F_5 \wedge F_3\,, \\ d\star F_3 &= F_5 \wedge H_3\,, \\ F_5 &= \star F_5\,. \end{aligned} \tag{B.2}$$

Einstein equation:

$$R_{ij} = T_{ij} = \frac{g_s^2}{96} F_{iabcd} F_j{}^{abcd} + \frac{1}{4} H_{iab} H_j{}^{ab} - \frac{1}{48} G_{ij} H_{abc} H^{abc} + \frac{g_s^2}{4} F_{iab} F_j{}^{ab} - \frac{g_s^2}{48} G_{ij} F_{abc} F^{abc}\,. \tag{B.3}$$

Let us make a small digression about our conventions. The 1-forms (vectors) are shown in boldface. We work with the $(-+++)$ Minkowski signature. The four dimensional operations such as the Hodge star $*_4$ and Laplacian $\Box_4$ are performed w.r.t. the standard Minkowski metric (without the warp factor). As it was explained, the four dimensional one-forms are all divergence free:

$$\mathrm{d}_4 *_4 \mathbf{F} \;=\; 0\,. \tag{B.4}$$

The eigenvalue of the 4-Laplacian $\Box_4$ is m_4^2; however, for compactness we shall express all our formulae in terms of the dimensionless combination $\tilde{m}^2$:

$$m_4^2 \;=\; \frac{3\,\epsilon^{4/3}}{2\cdot 2^{2/3}}\,\tilde{m}^2\,. \tag{B.5}$$

Below we present some technical details related to the derivation of the equations of motion for different subsectors of our ansätze.

B.1 Scalar Glueballs

Here we present the details of the calculation for the scalar system (4.37). The r.h.s. of the equation for $*H_3$ vanishes identically. The variation

$$\delta * H_3 = *\delta H_3 + \delta_G * H_3 \tag{B.6}$$

consists of two parts: $*\delta H_3$ accounting for the deformation of the form H_3 itself, and $\delta_G * H_3$ arising from the deformation of the Hodge star. Explicit calculation shows that

$$\begin{aligned}
*\delta H_3 \;&=\; -\sqrt{-G}\,G^{11}\,G^{33}\,G^{55}\,\chi'\,d^4x\wedge dg^5\wedge g^5\\
&\quad -\sqrt{-G}\,G^{11}\,G^{33}\,|G^{\mu\mu}|\,\partial_\mu(\chi-\sigma)\;*_4\,dx^\mu\wedge d\tau\wedge dg^5\wedge g^5\\
&\quad +\tfrac{1}{2}\sqrt{-G}\,(G^{55})^2\,|G^{\mu\mu}|\,\partial_\mu\sigma'\;*_4\,dx^\mu\wedge dg^5\wedge dg^5\,,\\
\delta_G * H_3 \;&=\; -\tfrac{g_sM\alpha'}{2}\,\sqrt{-G}\,G^{11}\,G^{33}\,G^{55}\big[f'G^{11}+k'G^{33}\big]\psi\,d^4x\wedge dg^5\wedge g^5\,.
\end{aligned} \tag{B.7}$$

Here the four-dimensional Hodge star $*_4$ is taken w.r.t. the standard Minkowski metric. Differentiating this expression for $\delta * H_3$ and equating to zero the coefficients multiplying linearly independent forms gives the following three equations:

$$2\, G^{11}\, G^{33} \left[\frac{g_s M\alpha'}{2}\left[f' G^{11} + k' G^{33}\right]\psi + \chi'\right] = G^{55}\, h^{\frac{1}{2}}\, \Box_4 \sigma' \,, \tag{B.8}$$

$$\partial_\tau \left\{\sqrt{-G}\, G^{11}\, G^{33}\, G^{55} \left[\frac{g_s M\alpha'}{2}\left[f' G^{11} + k' G^{33}\right]\psi + \chi'\right]\right\} + \\ + \sqrt{-G}\, G^{11}\, G^{33}\, h^{\frac{1}{2}}\, \Box_4(\chi - \sigma) = 0\,, \tag{B.9}$$

$$2\sqrt{-G}\, G^{11}\, G^{33}\, h^{\frac{1}{2}}\, \partial_\mu(\chi - \sigma) + \partial_\tau \left\{\sqrt{-G}\, (G^{55})^2\, h^{\frac{1}{2}}\, \partial_\mu \sigma'\right\} = 0\,, \tag{B.10}$$

where we have substituted for the warp factor $|G^{\mu\mu}| = h^{\frac{1}{2}}$ (no summation over μ is implied). Not all of these equations are independent. Indeed, using (B.8) equation (B.9) simplifies to

$$\partial_\tau \left\{\sqrt{-G}\, (G^{55})^2\, h^{\frac{1}{2}}\, \Box_4 \sigma'\right\} + 2\sqrt{-G}\, G^{11}\, G^{33}\, h^{\frac{1}{2}}\, \Box_4(\chi - \sigma) = 0\,. \tag{B.11}$$

This is exactly what we obtain by acting on (B.10) with ∂^μ and contracting indices. Thus only (B.8) and (B.10) are independent. When written in terms of the auxiliary functions K and h, these two equations reproduce (4.41) and (4.42).

The source terms on the right hand side of the Einstein equation $R_{ij} = T_{ij}$ (B.3) are due to the deformations of the metric and B_2 form. The only nontrivial deformations are those with indices 13 or 24, and they are equal: $\delta T_{13} = \delta T_{24}$. Say, for the 13 component δT_{13} we have the following contributions:

$$\begin{aligned} \frac{1}{4}\,\delta_B(H_{1ab}\, H_3{}^{ab}) &= \frac{1}{4}\left[H_{1ab}\,\delta H_3{}^{ab} + \delta H_{1ab}\, H_3{}^{ab}\right] \\ &= \frac{1}{2}\left[G^{11}\, H_{12\tau}\,\delta H_{32\tau} + G^{33}\,\delta H_{14\tau}\, H_{34\tau}\right] G^{55} \\ &= -\frac{1}{4}\,(g_s M\alpha')\, G^{55}\left[G^{11}\, f' + G^{33}\, k'\right]\chi'\,, \end{aligned} \tag{B.12}$$

$$\frac{g_s^2}{96}\,\delta_G(F_{1abcd}F_3{}^{abcd}) = \frac{g_s^2}{4}\,(G^{11})^2\,(G^{33})^2\, G^{55}\,(F_{12345})^2\,\psi\,, \tag{B.13}$$

$$\begin{aligned} \frac{1}{4}\,\delta_G(H_{1ab}H_3{}^{ab}) &= \frac{1}{2}\left[H_{135}\, H_{315}\,\delta G^{13}\, G^{55} + H_{12\tau}\, H_{34\tau}\,\delta G^{24}\, G^{\tau\tau}\right] \\ &= \frac{1}{2}\left[(H_{135})^2 - H_{12\tau}\, H_{34\tau}\right] G^{11}\, G^{33}\, G^{55}\,\psi \\ &= \frac{1}{8}\,(g_s M\alpha')^2\left[\frac{1}{4}(k-f)^2 - f'k'\right] G^{11}\, G^{33}\, G^{55}\,\psi\,, \end{aligned} \tag{B.14}$$

$$\begin{aligned}
\frac{g_s^2}{4}\,\delta_G(F_{1ab}F_3{}^{ab}) &= \frac{g_s^2}{2}\left[F_{125}\,F_{345}\,\delta G^{24}\,G^{55} + F_{13\tau}\,F_{31\tau}\,\delta G^{31}\,G^{\tau\tau}\right] \\
&= \frac{g_s^2}{2}\left[(F_{13\tau})^2 - F_{125}\,F_{345}\right]G^{11}\,G^{33}\,G^{55}\,\psi \\
&= \frac{1}{8}\,(g_sM\alpha')^2\left[F'^2 - F(1-F)\right]G^{11}\,G^{33}\,G^{55}\,\psi\,, \qquad \text{(B.15)} \\
-\frac{1}{48}\,\delta_G\left[G_{13}(H_{abc}H^{abc} + g_s^2F_{abc}F^{abc})\right] &= -\frac{1}{8}\,(H^2 + g_s^2F^2)\,\psi \\
&= -\frac{1}{32}\,(g_sM\alpha')^2\,G^{55}\,\big[(G^{11})^2\,f'^2 + (G^{33})^2\,k'^2 \\
&\quad +\frac{1}{2}\,G^{11}\,G^{33}\,(k-f)^2 + (G^{11})^2\,F^2 + (G^{33})^2\,(1-F)^2 \\
&\quad +2\,G^{11}\,G^{33}\,F'^2\big]\psi\,. \qquad \text{(B.16)}
\end{aligned}$$

Denoting

$$\delta T_{13} = \Big[A_1(\tau) + A_2(\tau)\Big]\,\psi(x,\tau) + B(\tau)\,\chi'(x,\tau)\,, \qquad \text{(B.17)}$$

where A_1 stands for the contribution from F_5, we get

$$\begin{aligned}
A_1(\tau) &= \frac{3(g_sM\alpha')^4}{2^{1/3}\epsilon^{20/3}h^{5/2}}\,\frac{(\tau\coth\tau - 1)^2[\sinh(2\tau) - 2\tau]^{4/3}}{\sinh^6(\tau)}\,, \qquad \text{(B.18)} \\
A_2(\tau) &= -\frac{3\,(g_sM\alpha')^2}{8\epsilon^4h^{3/2}\sinh^6\tau}\Big[3\,\cosh 4\tau - 8\tau\,\sinh 2\tau \qquad \text{(B.19)} \\
&\quad -8\tau^2\,\cosh 2\tau - 8\,\cosh 2\tau + 16\tau^2 + 5\Big]\,, \\
B(\tau) &= -3\,(g_sM\alpha')\,\frac{(\sinh 2\tau - 2\tau)\,K(\tau)}{\epsilon^{8/3}\,h(\tau)\,\sinh^3\tau}\,. \qquad \text{(B.20)}
\end{aligned}$$

B.2 3-Vector System

With the ansatz (4.70), (4.71) and (4.72), Bianchi identity for F_5 at the linear order in perturbation leads to four independent equations when written in components. Those are

$$\begin{aligned}
\frac{1}{2}\,\mathbf{K} - \frac{1}{2}\,\mathbf{L} + \mathbf{M}' + \mathbf{N} &= -F'(\mathbf{A} + \mathbf{E})\,, \qquad \text{(B.21)} \\
h\,\sqrt{-G}G^{55}\big((G^{11})^2\mathbf{K} + (G^{33})^2\mathbf{L}\big) &= \mathbf{H}\,, \qquad \text{(B.22)} \\
h\,\sqrt{-G}(G^{33})^2G^{55}\Box_4\mathbf{L} - h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2\mathbf{N} &= F\Box_4\mathbf{H}\,, \qquad \text{(B.23)} \\
\Big[h\sqrt{-G}(G^{33})^2G^{55}\mathbf{L}\Big]' - h\sqrt{-G}G^{11}G^{33}G^{55}\mathbf{M} &= F\mathbf{H}'\,. \qquad \text{(B.24)}
\end{aligned}$$

Equations of motion for F_3 give the two equations:

$$-2h\sqrt{-G}G^{55}\Box_4\mathbf{E} \;=\; 2(k-f)h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2\mathbf{N} + \ell\,\Box_4\mathbf{H}\,, \qquad \text{(B.25)}$$

$$\Big[2h\sqrt{-G}G^{55}\mathbf{E}\Big]' \;=\; -2h\sqrt{-G}G^{55}\big(f'(G^{11})^2\mathbf{K} + k'(G^{33})^2\mathbf{L}\big) - $$
$$-2(k-f)h\sqrt{-G}G^{11}G^{33}G^{55}\mathbf{M} - \ell\,\mathbf{H}'\,. \qquad \text{(B.26)}$$

Another pair of equations appear from H_3 equation of motion:

$$\Big[h^{1/2}\sqrt{-G}(G^{55})^2\mathbf{A}'\Big]' - 2h^{1/2}\sqrt{-G}G^{11}G^{33}\mathbf{A} + h\sqrt{-G}G^{55}\Box_4\mathbf{A} =$$
$$= -2F'h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2\mathbf{N}\,, \qquad \text{(B.27)}$$

$$\Big[2h\sqrt{-G}G^{55}\mathbf{H}'\Big]' + 2h^{3/2}\sqrt{-G}\Box_4\mathbf{H} \;=\; 2(1-F)\mathbf{K} + 2F\mathbf{L} + 4F'\mathbf{M} - \ell\,\mathbf{E}\,. \qquad \text{(B.28)}$$

No other supergravity equations contribute. In fact, some equations in the system (B.21)-(B.28) are algebraic and can be solved for the functions **E**, **K**, **L**, **M** in terms of the functions **N** and **H**. After doing so and redefining **N** according to (4.73), one can notice that equation (B.26) becomes an identity. Thus, there are only three independent second order differential equations for three unknown functions $\tilde{\mathbf{N}}$, **H** and **A**. Introducing $\tilde{\mathbf{A}} = K^2\sinh\tau\,\mathbf{A}$, those reduce to the system (4.75), (4.76), (4.77).

As mentioned in the section 4.6, to separate the eigenmodes one can first impose $\tilde{\mathbf{N}} = 0$. Then the remaining equations for **H** and $\tilde{\mathbf{A}}$ are equivalent. After setting $\tilde{\mathbf{N}} = 0$, the equation (4.75) becomes the first order equation (4.78). Using it, one can eliminate the first and second derivatives of **H** from (4.77) and express **H** in terms of $\tilde{\mathbf{A}}$ and its derivative. This reduces the system to just one equation (4.79). Let us stress that in this case the ansatz for δF_5 simplifies,

$$\delta F_5 \;=\; (1+*)\,\mathrm{d}_4\mathbf{H}\wedge H_3\,; \qquad \text{(B.29)}$$

which gives a natural generalization of the KT limit ansatz in [28] to the complete KS background (recall that in the KT limit $H_3 \sim \mathrm{d}\tau \wedge \omega_2$).

To extract the remaining two modes the equations (4.75)-(4.77) can be written in the following form (we have done the trivial rescaling $\tilde{\mathbf{H}} \to 2^{7/6}\tilde{\mathbf{H}}$, $\tilde{\mathbf{A}} \to 2^{7/6}\tilde{m}\tilde{\mathbf{A}}$):

$$\tilde{\mathbf{A}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{\mathbf{A}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{A}} - \frac{2^{-1/6}\tilde{m}I'}{K^3\sinh\tau}\,\tilde{\mathbf{N}} = 0, \quad \text{(B.30)}$$

$$\tilde{\mathbf{H}}'' + \left(\frac{1}{2}\,\frac{I''}{I} - \frac{(K\sinh\tau)''}{K\sinh\tau} + \frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau} - \frac{3}{4}\,\frac{I'^2}{I^2}\right)\tilde{\mathbf{H}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{H}} + \frac{2^{-1/6}\sqrt{I}}{K\sinh\tau}\left(\frac{I'\tilde{\mathbf{N}}}{IK}\right)' = 0, \quad \text{(B.31)}$$

$$\tilde{\mathbf{N}}'' - \left(\frac{\cosh^2\tau + 1}{\sinh^2\tau} + \frac{2^{-1/3}I'^2}{IK^4\sinh^2\tau}\right)\tilde{\mathbf{N}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{N}} - \frac{2^{-1/6}I'}{IK}\left(\frac{\sqrt{I}\tilde{\mathbf{H}}}{K\sinh\tau}\right)' - \frac{2^{-1/6}\tilde{m}I'}{K^3\sinh\tau}\,\tilde{\mathbf{A}} = 0. \quad \text{(B.32)}$$

It follows from above that the three vectors $\tilde{\mathbf{A}}$, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{N}}$ are collinear. Therefore it suffices to consider the three scalar equations for the three variables A, H, N. The problem reduces to finding the spectrum of the Hamiltonian $\mathcal{H}$,

$$-\mathcal{H}\begin{pmatrix} A \\ H \\ N \end{pmatrix} = \begin{pmatrix} A'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,A - \frac{2^{-1/6}\tilde{m}I'}{K^3\sinh\tau}\,N \\ H'' + \left(\frac{1}{2}\,\frac{I''}{I} - \frac{(K\sinh\tau)''}{K\sinh\tau} + \frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau} - \frac{3}{4}\,\frac{I'^2}{I^2}\right)H + \frac{2^{-1/6}\sqrt{I}}{K\sinh\tau}\left(\frac{I'N}{IK}\right)' \\ N'' - \left(\frac{\cosh^2\tau+1}{\sinh^2\tau} + \frac{2^{-1/3}I'^2}{IK^4\sinh^2\tau}\right)N - \frac{2^{-1/6}I'}{IK}\left(\frac{\sqrt{I}H}{K\sinh\tau}\right)' - \frac{2^{-1/6}\tilde{m}I'}{K^3\sinh\tau}\,A \end{pmatrix}.$$

Let us stress that this Hamiltonian is Hermitian w.r.t. the inner product

$$\langle 1|2\rangle = \int_0^\infty \mathrm{d}\tau\,\frac{I}{K^2}\big(A_1A_2 + H_1H_2 + N_1N_2\big)\,, \quad \text{(B.33)}$$

and the mass eigenvalues are found from the equation

$$\mathcal{H}\begin{pmatrix} A \\ H \\ N \end{pmatrix} = \tilde{m}^2\,\frac{I}{K^2}\begin{pmatrix} A \\ H \\ N \end{pmatrix}. \quad \text{(B.34)}$$

As a consequence, different eigenvectors are orthogonal with the weight I/K^2.

We have found the decoupled mode which corresponds to setting $N \equiv 0$. This corresponds to the subspace of the form (see equation (B.32)):

$$(A\,,H\,,N) \;=\; \left(-\frac{K^2\sinh\tau}{\tilde{m}I}\left(\frac{\sqrt{I}H}{K\sinh\tau}\right)'\,,H\,,0\right)\,. \tag{B.35}$$

It is natural to suggest that the two remaining modes $(\hat{A}, \hat{H}, \hat{N})$ belong to the orthogonal complement of this subspace. Namely,

$$\int \mathrm{d}\tau\,\frac{I}{K^2}\left(-\hat{A}\frac{K^2\sinh\tau}{\tilde{m}I}\left(\frac{\sqrt{I}H}{K\sinh\tau}\right)' + \hat{H}\,H\right) = \\ \int \mathrm{d}\tau\left(\frac{\sqrt{I}}{K\sinh\tau}\left(\frac{\hat{A}\sinh\tau}{\tilde{m}}\right)' + \frac{I}{K^2}\,\hat{H}\right)H = 0. \tag{B.36}$$

The latter is satisfied by

$$\tilde{m}\hat{H} = -\frac{K}{\sqrt{I}\sinh\tau}\left(\hat{A}\sinh\tau\right)'\,, \tag{B.37}$$

or

$$\hat{A}' = -\tilde{m}\,\frac{\sqrt{I}}{K}\,\hat{H} - \coth\tau\,\hat{A}. \tag{B.38}$$

Using this expression one can eliminate all the derivatives of A from (B.30) and obtain another first order relation,

$$\hat{H}' = -\left(\log\frac{\sqrt{I}}{K\sinh\tau}\right)'\hat{H} + \tilde{m}\,\frac{\sqrt{I}}{K}\,\hat{A} - \frac{2^{-1/6}I'}{\sqrt{I}K^2\sinh\tau}\,\hat{N}. \tag{B.39}$$

Differentiating (B.39) and eliminating $\hat{A}$ and $\hat{A}'$ using (B.38) and (B.39) one recovers the equation (B.31) for $\hat{H}$. Thus the equation (B.31) can be omitted from the system, and $\hat{H}$ can be expressed via $\hat{A}$ using (B.38). After the elimination of $\hat{H}$ the system of the two equations (B.30) and (B.32) for $\hat{A}$ and $\hat{N}$ reproduces the system (4.81), (4.82). As it is shown in the main text, these two equations decouple giving rise to the two modes $\mathbf{X}_{\pm}$.

B.3 4-Vector System

Similarly to the previous example the excitations (4.84), (4.85) and (4.86) lead to the following linearized equations. The Bianchi identity gives five equations

$$-\frac{1}{2}\,h\sqrt{-G}G^{55}\Big((G^{11})^2\mathbf{P}-(G^{33})^2\mathbf{Q}\Big)+\Big(h\sqrt{-G}G^{11}G^{33}G^{55}\mathbf{R}\Big)' \;=\; F'\mathbf{D}'\,, \tag{B.40}$$

$$-\Big(h^{1/2}\sqrt{-G}(G^{33})^2(G^{55})^2\mathbf{G}\Big)'+h\sqrt{-G}(G^{33})^2G^{55}\Box_4\mathbf{Q} \;=\; f'\Box_4\mathbf{D}\,, \tag{B.41}$$

$$-\Big(h^{1/2}\sqrt{-G}(G^{11})^2(G^{55})^2\mathbf{F}\Big)'+h\sqrt{-G}(G^{11})^2G^{55}\Box_4\mathbf{P} \;=\; k'\Box_4\mathbf{D}\,, \tag{B.42}$$

$$\mathbf{F}+\mathbf{P}'-\mathbf{R} \;=\; F\mathbf{J}+f'\mathbf{C}\,, \tag{B.43}$$

$$\mathbf{G}+\mathbf{Q}'+\mathbf{R} \;=\; (1-F)\mathbf{J}+k'\mathbf{C}\,. \tag{B.44}$$

A pair of equations come from the F_3 equation of motion:

$$\Big[h^{1/2}\sqrt{-G}(G^{55})^2\mathbf{C}'\Big]'-2h^{1/2}\sqrt{-G}G^{11}G^{33}\mathbf{C}+ \\ +h\sqrt{-G}G^{55}\Box_4\mathbf{C}=h^{1/2}\sqrt{-G}(G^{55})^2\big(f'(G^{11})^2\mathbf{F}+k'(G^{33})^2\mathbf{G}\big), \tag{B.45}$$

$$\Big[2h\sqrt{-G}G^{55}\mathbf{D}'\Big]'+2h^{3/2}\sqrt{-G}\Box_4\mathbf{D} \;=\; 2k'\mathbf{P}+2f'\mathbf{Q}+4F'\mathbf{R}+\ell\mathbf{J}\,; \tag{B.46}$$

and a pair of equations from the equation of motion for H_3:

$$2h\sqrt{-G}G^{55}\Box_4\mathbf{J} \;=\; 2h^{1/2}\sqrt{-G}(G^{55})^2\big(F(G^{11})^2\mathbf{F}+(1-F)(G^{33})^2\mathbf{G}\big)+\ell\Box_4\mathbf{D}, \tag{B.47}$$

$$\Big[2h\sqrt{-G}G^{55}\mathbf{J}\Big]' \;=\; 2h\sqrt{-G}G^{55}\big(F(G^{11})^2\mathbf{P}+(1-F)(G^{33})^2\mathbf{Q}\big)+ \\ +4F'h\sqrt{-G}G^{11}G^{33}G^{55}\mathbf{R}+\ell\mathbf{D}'. \tag{B.48}$$

As in the case of the previous ansatz, one of the equations is not independent and it is easy to demonstrate that any of the equations (B.40)-(B.42) or (B.47)-(B.48) can be eliminated. Thus, we obtain a system of eight equations for eight unknown forms. To write it in a more convenient form we introduce $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ as in (4.92) and (4.93).

We solve the algebraic equations for ansatz functions $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{J}$, which we express in terms of the functions $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$. The remaining four coupled second order differential equations are most conveniently written in terms of the functions I, K, $\sinh\tau$ and their derivatives. This way we obtain a system

$$\tilde{\mathbf{F}}'' - \left[\frac{2}{\sinh^2\tau} + \frac{1}{2}\right]\tilde{\mathbf{F}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{F}} + \frac{1}{2}\,\tilde{\mathbf{G}} + \left(\frac{1}{2}\,K^3\sinh\tau + 2^{-4/3}\frac{I'}{K}\right)(\mathbf{D}' - \mathbf{J}) = = \frac{1}{2}\,K\tilde{\mathbf{C}} - \frac{2^{-4/3}I'}{K^3\sinh\tau}\,\tilde{\mathbf{C}}\,, \quad \text{(B.49)}$$

$$\tilde{\mathbf{G}}'' - \left[\frac{2}{\sinh^2\tau} + \frac{1}{2}\right]\tilde{\mathbf{G}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{G}} + \frac{1}{2}\,\tilde{\mathbf{F}} + \left(\frac{1}{2}\,K^3\sinh\tau - 2^{-4/3}\frac{I'}{K}\right)(\mathbf{D}' - \mathbf{J}) = = \frac{1}{2}\,K\tilde{\mathbf{C}} + \frac{2^{-4/3}I'}{K^3\sinh\tau}\,\tilde{\mathbf{C}}\,. \quad \text{(B.50)}$$

$$\tilde{\mathbf{C}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{C}} \;=\; 2^{1/3}\,\tilde{m}^2K(\tilde{\mathbf{F}} + \tilde{\mathbf{G}}) - \tilde{m}^2\,\frac{I'}{K^3\sinh\tau}\,(\tilde{\mathbf{F}} - \tilde{\mathbf{G}})\,, \quad \text{(B.51)}$$

$$\mathbf{D}'' + \Big(\log(IK^2\sinh^2\tau)\Big)'\mathbf{D}' + \tilde{m}^2\frac{I}{K^2}\,\mathbf{D} + \frac{(I'K^2\sinh^2\tau)'}{IK^2\sinh^2\tau}\,\mathbf{D} = = -\frac{I'}{I}\,\mathbf{J} + \frac{1}{IK^2\sinh^2\tau}\,\left(2^{1/3}K^3\sinh\tau(\tilde{\mathbf{F}} + \tilde{\mathbf{G}}) + \frac{I'}{K}(\tilde{\mathbf{F}} - \tilde{\mathbf{G}})\right)'\;; \quad \text{(B.52)}$$

where $\tilde{\mathbf{C}} = K^2\sinh\tau\,\mathbf{C}$, and $\tilde{m}$ is defined in (B.5). $\mathbf{J}$ is expressed in terms of given functions as follows:

$$\mathbf{J} \;=\; -\frac{I'}{I}\,\mathbf{D} + \frac{2^{1/3}K}{I\sinh\tau}\,(\tilde{\mathbf{F}} + \tilde{\mathbf{G}}) + \frac{I'}{IK^3\sinh^2\tau}(\tilde{\mathbf{F}} - \tilde{\mathbf{G}})\,. \quad \text{(B.53)}$$

The form of the equations in (B.49)-(B.52) suggests that we introduce $\mathbf{B}_\pm = \tilde{\mathbf{F}} \pm \tilde{\mathbf{G}}$, so that the equations take the form (4.87), (4.88), (4.89), (4.90) and (4.91).

The system of the equations (4.87)-(4.90) can be further reduced. We show that it can be split into the two decoupled pairs of equations by imposing the two different constraints, $\mathbf{B}_\pm = 0$; each of them leading to a consistent reduction.

First, we set

$$\mathbf{B}_- \;=\; 0\,; \quad \text{(B.54)}$$

then (4.88) implies

$$\mathbf{D}' - \mathbf{J} = -\frac{1}{K^2 \sinh\tau}\tilde{\mathbf{C}}. \tag{B.55}$$

Differentiating this equation, using (4.91) and plugging it into the equation (4.90), one gets, after eliminating $\mathbf{D}'$ via (B.55), a simple relation

$$\tilde{\mathbf{C}}' = \tilde{m}^2 I \sinh\tau\,\mathbf{D} - \coth\tau\,\tilde{\mathbf{C}}. \tag{B.56}$$

Note that differentiating (B.56) and then eliminating the derivatives of $\tilde{\mathbf{C}}$ from (4.89) we recover (B.55) (and therefore (4.90) as well). Thus, the constraint (B.54) singles out a consistent subsystem of the two equations:

$$\mathbf{B}_+'' - \frac{2}{\sinh^2\tau}\mathbf{B}_+ + \tilde{m}^2\frac{I}{K^2}\mathbf{B}_+ \;=\; 2K\tilde{\mathbf{C}}\,, \tag{B.57}$$

$$\tilde{\mathbf{C}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{C}} \;=\; 2^{1/3}\tilde{m}^2 K\mathbf{B}_+\,. \tag{B.58}$$

After a trivial rescaling of variables it reproduces the scalar equations (4.56) and (4.57).

To find the complementary pair of equations, one can instead set

$$\mathbf{B}_+ \;=\; 0\,. \tag{B.59}$$

Equation (4.87) implies a first order constraint

$$\mathbf{D}' \;=\; -\frac{I'}{I}\mathbf{D} + \frac{I'}{IK^3\sinh^2\tau}\mathbf{B}_- + \frac{1}{K^2\sinh\tau}\tilde{\mathbf{C}}\,. \tag{B.60}$$

Using this equation one can eliminate the derivatives of $\mathbf{D}$ from (4.90) and get the relation

$$\tilde{\mathbf{C}}' = -\tilde{m}^2 I \sinh\tau\,\mathbf{D} - \coth\tau\,\tilde{\mathbf{C}}. \tag{B.61}$$

Note that after eliminating the $\tilde{\mathbf{C}}$ derivatives from (4.89) using this equation we recover (B.60) (and thus (4.87) and (4.90)). There remains a consistent subsystem of the two equations for $\mathbf{B}_-$ and $\tilde{\mathbf{C}}$, (4.102) and (4.103). As it is shown in the main text, they can be further decoupled, yielding the two equations identical to (4.83).

B.4 Numerics: Determinant Method

A standard method of finding the spectrum of a single second-order differential equation is the shooting technique. For a system of several coupled linear equations the shooting method has to be generalized [76]. Here we will focus on the subtleties specific to the system of equations (4.56) and (4.57). The idea of the calculation (called the determinant method [76]) is to set the initial conditions at infinity corresponding to the two solutions regular at infinity, $\begin{pmatrix} \tilde{z}_1(\tau) \\ \tilde{w}_1(\tau) \end{pmatrix}$ and $\begin{pmatrix} \tilde{z}_2(\tau) \\ \tilde{w}_2(\tau) \end{pmatrix}$, and extend them numerically to small τ. Then the matrix

$$\begin{pmatrix} \tilde{z}_1(0) & \tilde{z}_2(0) \\ \tilde{w}_1(0) & \tilde{w}_2(0) \end{pmatrix} \tag{B.62}$$

becomes degenerate at the critical points (eigenvalues) in the spectral parameter space.

Let us find the asymptotic behavior of regular and singular solutions near both zero and infinity. At small τ equations (4.56) and (4.57) decouple,

$$\tilde{z}'' - \frac{2}{\tau^2}\,\tilde{z} = 0\,, \tag{B.63}$$

$$\tilde{w}'' - \frac{2}{\tau^2}\,\tilde{w} = 0\,. \tag{B.64}$$

There are the two regular solutions with $\tilde{z},\ \tilde{w} \sim \tau^2$ and the two singular solutions with $\tilde{z},\ \tilde{w} \sim 1/\tau$. For large τ we have

$$\tilde{z}'' = \tilde{m}^2 \frac{9}{4\cdot 2^{1/3}}\,\mathrm{e}^{-\tau/3}\tilde{w}\,, \tag{B.65}$$

$$\tilde{w}'' - \tilde{w} = \frac{16\cdot 2^{1/3}}{9}\,\mathrm{e}^{-\tau/3}\tilde{z}\,. \tag{B.66}$$

The asymptotic behavior of the two regular solutions is

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{w}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2^{4/3}\,\mathrm{e}^{-\tau/3} \end{pmatrix}, \qquad \begin{pmatrix} \tilde{z}_2 \\ \tilde{w}_2 \end{pmatrix} = \begin{pmatrix} \frac{81}{64\cdot 2^{1/3}}\,\tilde{m}^2\mathrm{e}^{-4\tau/3} \\ \mathrm{e}^{-\tau} \end{pmatrix}; \tag{B.67}$$

and the singular solutions are

$$\begin{pmatrix} \tilde{z}_3 \\ \tilde{w}_3 \end{pmatrix} = \begin{pmatrix} \tau \\ -2^{4/3}\left(\tau - \frac{3}{4}\right)\mathrm{e}^{-\tau/3} \end{pmatrix}, \qquad \begin{pmatrix} \tilde{z}_4 \\ \tilde{w}_4 \end{pmatrix} = \begin{pmatrix} \frac{81}{16\cdot 2^{1/3}}\,\tilde{m}^2\mathrm{e}^{2\tau/3} \\ \mathrm{e}^{\tau} \end{pmatrix}. \tag{B.68}$$

A particular subtlety of this setup is that at large τ the two singular solutions don't diverge equally fast: one of them grows exponentially while the other is only linear in τ. This makes it difficult to start shooting from zero: imposing the regularity condition at infinity would require vanishing of both linear and exponential terms. To cancel the linear term in the presence of the exponential one is difficult to do numerically. That is why for this particular system it is convenient to start shooting from large τ, since both singular solutions at zero share the same behavior ($\sim 1/\tau$).

B.5 BHM Normalization

Here we show how to find the conversion factor between the dimensionless mass squared $\tilde{m}^2$ and the mass in the normalization of Berg, Haack and Mück [76]. We note that the BHM conventions correspond to the KS solution with an extra relation between ϵ and M. The authors of [76] use the notations of the general PT ansatz (as given in Eq. (3.8) of [75]):

$$ds^2 = \mathrm{e}^{2p-x}ds_5^2 + (\mathrm{e}^{x+g} + a^2\mathrm{e}^{x-g})(e_1^2 + e_2^2) + \mathrm{e}^{x-g}[e_3^2 + e_4^2 - 2a(e_1e_3 + e_2e_4)] + \mathrm{e}^{-6p-x}e_5^2, \tag{B.69}$$

$$ds_5^2 = dr^2 + \mathrm{e}^{2A(r)}\eta_{ij}dx^i dx^j\,. \tag{B.70}$$

After setting[1]

$$a = \tanh y = \frac{1}{\cosh\tau}\,, \quad \mathrm{e}^{-g} = \cosh y = \coth\tau\,; \tag{B.71}$$

[1]The Papadopoulos-Tseytlin [78] variables are $(x, p, y, \Phi, b, h_1, h_2)$.

it reduces to the KS form

$$\begin{aligned} ds^2 &= \mathrm{e}^{2A+2p-x}\eta_{ij}dx^i dx^j + \frac{\mathrm{e}^x}{\sinh\tau}\Big[\coth\tau(e_1^2+e_2^2+e_3^2+e_4^2) \\ &\quad + \frac{2}{\sinh\tau}(e_1e_3+e_2e_4) + \mathrm{e}^{-6p-2x}(d\tau^2+e_5^2)\Big]\,. \end{aligned} \tag{B.72}$$

The radial KS coordinate τ is introduced according to

$$\partial_\tau = \mathrm{e}^{-4p}\partial_r\,. \tag{B.73}$$

Note that in the KS notation the conifold metric (4.3) can be rewritten as

$$ds_6^2 = \frac{\epsilon^{4/3}K(\tau)}{2}\left[\frac{1}{3K^3}(d\tau^2+(e_5)^2) + \frac{1}{2}\cosh\tau(e_1^2+\mathrm{e}_2^2+e_3^2+e_4^2) + e_1e_3+e_2e_4\right]. \tag{B.74}$$

In terms of τ, the PT variables necessary to describe the metric for the KS background solution take the form

$$\Phi = \Phi_0\,, \tag{B.75}$$

$$\mathrm{e}^y = \tanh(\tau/2)\,, \tag{B.76}$$

$$\frac{2}{3}\mathrm{e}^{6p+2x} = \coth\tau - \frac{\tau}{\sinh^2\tau}\,, \tag{B.77}$$

$$\mathrm{e}^{2x/3-4p} = 6^{-2/3}M^2\mathrm{e}^{\Phi_0}I(\tau)\sinh^{4/3}\tau\,. \tag{B.78}$$

In the BHM normalization

$$\mathrm{e}^{-2A-8p} = \left(\mathrm{e}^{-6p-2x}\sinh\tau\right)^{2/3}\frac{I(\tau)}{I_0}\,, \qquad I_0 \equiv I(0)\,. \tag{B.79}$$

These equations give for the coefficients

$$\mathrm{e}^{6p+2x} = \frac{3}{2}K^3\sinh\tau\,, \tag{B.80}$$

$$\mathrm{e}^x = 2^{-2/3}\mathrm{e}^{\Phi_0/2}MK(\tau)\sinh\tau\sqrt{I(\tau)}\,, \tag{B.81}$$

$$\mathrm{e}^{2A+2p-x} = \sqrt{\mathrm{e}^{2x/3-4p}}\sinh^{-2\tau/3}\frac{I_0}{I} = 6^{-1/3}\mathrm{e}^{\Phi_0/2}M\frac{I_0}{\sqrt{I}}\,. \tag{B.82}$$

Comparing these coefficients with those of the KS solution we find[2]

$$\frac{\epsilon^{4/3}}{M^2} = 3^{-1/3}\mathrm{e}^{\Phi_0/2}I_0\,, \tag{B.83}$$

$$\mathrm{e}^{\Phi_0/2} = \frac{1}{2}\,. \tag{B.84}$$

[2] We set $g_s = \alpha' = 1$ according to [76].

This yields $\epsilon^{4/3}/M^2 = 3^{-1/3} I_0/2$. Then using (4.55) we get for the four-dimensional mass in the BHM normalization

$$m_{BHM}^2 = m_4^2 = \tilde{m}^2 \, \frac{6}{2^{2/3}} \, \frac{\epsilon^{4/3}}{M^2} = (3/2)^{2/3} I_0 \, \tilde{m}^2 \,. \tag{B.85}$$

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