

W. Barreto · L. Castillo · E. Barrios

Bondian frames to couple matter with radiation

Received: 17 December 2009 / Accepted: 22 February 2010
© Springer Science+Business Media, LLC 2010

Abstract A study is presented for the non linear evolution of a self gravitating distribution of matter coupled to a massless scalar field. The characteristic formulation for numerical relativity is used to follow the evolution by a sequence of light cones open to the future. Bondian frames are used to endow physical meaning to the matter variables and to the massless scalar field. Asymptotic approaches to the origin and to infinity are achieved; at the boundary surface interior and exterior solutions are matched guaranteeing the Darmois–Lichnerowicz conditions. To show how the scheme works some numerical models are discussed. We exemplify evolving scalar waves on the following fixed backgrounds: (a) an atmosphere between the boundary surface of an incompressible mixed fluid and infinity; (b) a polytropic distribution matched to a Schwarzschild exterior; (c) a Schwarzschild–Schwarzschild spacetime. The conservation of energy, the Newman–Penrose constant preservation and other expected features are observed.

Keywords Characteristic formulation, Matter evolution, Einstein–Klein–Gordon system

1 Introduction

Numerical relativity has reached high sophistication levels to advance in the study of realistic solutions to the Einstein equations. [1]. Particularly, to simulate gravitational radiation from collapsing sources [2] and binary black hole mergers [3].

W. Barreto Facultad de Ciencias, Centro de Física Fundamental Universidad de Los Andes Mérida, Venezuela wbarreto@ula.ve · L. Castillo Departamento de Física, Escuela de Ciencias, Núcleo de Sucre Universidad de Oriente Cumaná, Venezuela · E. Barrios Postgrado en Física Fundamental, Facultad de Ciencias Universidad de Los Andes Mérida, Venezuela

Almost all the investigations have been done in the ADM 3+1 formulation [4], although the characteristic formulation offers a myriad of valuable advantages [5].

This work is motivated by the possibility of simulating gravitational radiation from axisymmetric matter sources, as part of a longer project [6; 7]. We met that purpose beginning with the most simplified and concomitant problem under spherical symmetry, that is, the self-gravitating massless scalar field. In the same vein we deal with the problem of matter coupled to radiation which eventually will lead us to get gravitational signals from bounded sources. As far as we can see, this problem stands alone as an important one in the field. Here we report a study in the aforementioned direction, limiting ourselves to spherical symmetry to follow the evolution of a massless scalar field interacting with a perfect fluid distribution of matter. This model problem offers a number of advantageous computational and geometrical features. It is well known that the scalar field mimics gravitational radiation and it has been used to study the global properties of the spacetime, black hole threshold and radiative signals [8; 9; 10; 11; 12].

The framework for computing a complete spacetime within the characteristic approach has been laid out by Tamburino and Winicour [13] and explicitly in [14], where is explored the production of gravitational waves for axially symmetric distribution of matter (for a complete review see [5] and references therein). Some years later, fundamental studies in 1D and 3D [8; 15; 16] were followed by a 3D code able to treat matter by Bishop et al. [17]. A relatively simpler system is a spherically symmetric distribution of a perfect fluid coupled with scalar radiation [18]. For other symmetries other than spherical, the mathematical problem has a very similar structure [2]. In these contexts we can take full advantage of the characteristic approach to treat matter and radiation. The first example of the use of characteristic numerical relativity for the study of dynamical neutron star spacetime, collapse and radiative signals was reported in [18]. Few investigations consider the scalar field interaction with fluid stellar distributions [19; 20].

We have recently discovered an unexpected unity in the treatment of matter in numerical relativity [21], using explicit Bondian observers [22]. These observers offer an Eulerian (noncomoving) description (global) with the spirit of Lagrangian observers (local and comoving). Following this line also we reported a disclosure of a central equation of state (CEoS), which is unique for all evolutions; it emerges as a conserved quantity from the field equations [23].

Authors commonly refer to the scalar field as a model of matter distribution that simplifies the treatment of the hydrodynamic issues. This approach has been useful to study non linear physics and asymptotic behaviors, especially for central regions [24]. Scalar field models have been extended to realistic situations such as gravitational radiation [15]. However, cases where the scalar field is coupled with radiation require a different approach due to both the confinement effect of matter and the dispersive nature of radiation. When a scalar field is coupled to matter it can be easily interpreted as an anisotropic fluid if we use Bondian frames explicitly.

We develop a numerical framework to deal with matter coupled to scalar radiation. We perform a detailed study of the central world line at $r = 0$, leading to a conformally flat spacetime in that region. We assume that the radial dependence of the geometrical and physical variables keep the same dependence as the static variables near the coordinate-origin.

We formulate in this work the characteristic evolution in terms of Bondian observers [21; 23]. The Eulerian formulations of numerical relativity [18; 25] actually use Bondian observers in the mathematical treatment of matter. We also match the interior solution with the exterior one in a clear and precise treatment of the boundary distribution of matter without the use of artificial atmosphere. Infinity is treated as usual in literature.

In what follows we write the field equations for Bondian observers when matter is coupled to scalar radiation, which makes the fluid manifestly anisotropic. Section 3 is devoted to the regularization and matching. In Sect. 4 we show how the scheme works by means of numerical test models. Finally we summarize with some remarks in Sect. 5.

2 Field equations for Bondian frames

Bondi's metric in the spherical form reads [22]

$$ds^2 = e^{2\beta} \left(\frac{V}{r} du^2 + 2dudr \right) - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $\beta = \beta(u, r)$ and $V = V(u, r)$. In these coordinates the components of the energy-momentum tensor are distinguished by a bar. In spherical symmetry there exists a well defined notion of quasilocal energy, the Misner–Sharp mass function, $\tilde{m}(u, r)$ [26; 27; 28] introduced by means of

$$\tilde{m} = \frac{1}{2}(r - Ve^{-2\beta}), \quad (2)$$

which measures the energy content in the sphere of radius r and it reduces to the Arnowitt–Deser–Misner and Bondi masses in the appropriate limits.

Consider a stress-energy tensor for a perfect fluid and a massless scalar field

$$\bar{T}_{\mu\nu} = \bar{T}_{\mu\nu}^M + \bar{T}_{\mu\nu}^\Phi. \quad (3)$$

One can follow the Tamburino–Winicour formalism [13], in particular as applied in regular spacetimes, where the foliation of light cones emanates from a freely falling central observer [14; 15; 16; 17; 18]. But following Bondi, local Minkowski coordinates (t, x, y, z) are introduced by

$$dt = e^{2\beta} (1 - 2\tilde{m}/r)^{1/2} du + (1 - 2\tilde{m}/r)^{-1/2} dr, \quad (4a)$$

$$dx = (1 - 2\tilde{m}/r)^{-1/2} dr, \quad (4b)$$

$$dy = r d\theta, \quad (4c)$$

$$dz = r \sin \theta d\phi. \quad (4d)$$

Denoting the Minkowski components of the energy-momentum tensor by a caret we have

$$\bar{T}_{00} = \hat{T}_{00} e^{4\beta} (1 - 2\tilde{m}/r), \quad (5a)$$

$$\bar{T}_{01} = (\hat{T}_{00} + \hat{T}_{01}) e^{2\beta}, \quad (5b)$$

$$\bar{T}_{11} = (1 - 2\tilde{m}/r)^{-1} (\hat{T}_{00} + \hat{T}_{11} + 2\hat{T}_{01}), \quad (5c)$$

$$\bar{T}_2^2 = \bar{T}_3^3 = \hat{T}_3^3 = \hat{T}_2^2. \quad (5d)$$

Next one assumes that for an observer moving relative to these coordinates with velocity ω in the radial direction, the space contains an isotropic fluid with pressure p and energy density ρ .

For this Bondian observer, the covariant energy-momentum tensor of matter is:

$$\hat{T}_{\mu\nu}^M = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \quad (6)$$

Then a Lorentz transformation readily shows that

$$\bar{T}_{00} = e^{4\beta} (1 - 2\tilde{m}/r) \left(\frac{\rho + p\omega^2}{1 - \omega^2} \right), \quad (7a)$$

$$\bar{T}_{01} = e^{2\beta} \left(\frac{\rho - \omega p}{1 + \omega} \right), \quad (7b)$$

$$\bar{T}_{11} = (1 - 2\tilde{m}/r)^{-1} (\rho + p) \left(\frac{1 - \omega}{1 + \omega} \right), \quad (7c)$$

$$\bar{T}_2^2 = \bar{T}_3^3 = -p. \quad (7d)$$

The energy-momentum tensor for the massless scalar field minimally coupled with gravity

$$\bar{T}_{\mu\nu}^\Phi = \bar{\nabla}_\mu \Phi \bar{\nabla}_\nu \Phi - \frac{1}{2} g_{\mu\nu} \bar{\nabla}^\alpha \Phi \bar{\nabla}_\alpha \Phi, \quad (8)$$

can be read by an observer at rest in the frame of (1).

It can be shown that the Einstein–Klein–Gordon equations can be written as

$$\frac{\rho + \omega^2 p}{1 - \omega^2} + \rho^\Phi + \varepsilon^\Phi = \frac{1}{4\pi r} \left(-\frac{\tilde{m}_{,u} e^{-2\beta}}{r - 2\tilde{m}} + \frac{\tilde{m}_{,r}}{r} \right), \quad (9)$$

$$\frac{\rho - \omega p}{1 + \omega} + \rho^\Phi = \frac{\tilde{m}_{,r}}{4\pi r^2}, \quad (10)$$

$$\frac{1 - \omega}{1 + \omega}(\rho + p) + \rho^\Phi + p^\Phi = (1 - 2\tilde{m}/r) \frac{\beta_{,r}}{2\pi r}, \quad (11)$$

$$p + p_t^\Phi = -\frac{1}{4\pi} \beta_{,ur} e^{-2\beta} + \frac{1}{8\pi} (1 - 2\tilde{m}/r) (2\beta_{,rr} + 4\beta_{,r}^2 - \beta_{,r}/r) \\ + \frac{1}{8\pi r} [3\beta_{,r} (1 - 2\tilde{m}_{,r}) - \tilde{m}_{,rr}] \quad (12)$$

and

$$2(r\Phi)_{,ur} = r^{-1} [re^{2\beta} (r - 2\tilde{m}) \Phi_{,r}]_{,r}, \quad (13)$$

where the comma denotes partial differentiation respect to the indicated coordinate, and the scalar energy flux ε^Φ , the scalar energy density ρ^Φ , the scalar radial pressure p^Φ , the scalar tangential pressure p_t^Φ , respectively are defined by

$$\varepsilon^\Phi = e^{-2\beta} [e^{-2\beta} (1 - 2\tilde{m}/r)^{-1} \Phi_{,u}^2 - \Phi_{,u} \Phi_{,r}], \quad (14)$$

$$\rho^\Phi = p^\Phi = (1 - 2\tilde{m}/r) \Phi_{,r}^2 / 2, \quad (15)$$

$$p_t^\Phi = \Phi_{,u} \Phi_{,r} e^{-2\beta} - p^\Phi. \quad (16)$$

From this point of view the scalar field can be interpreted as a radiating and anisotropic fluid [29] whose energy-momentum tensor can be written as

$$T_{\mu\nu}^\Phi = (\rho^\Phi + p_t^\Phi) u_\mu u_\nu + \varepsilon^\Phi l_\mu l_\nu - p_t^\Phi g_{\mu\nu} + (p^\Phi - p_t^\Phi) \chi_\mu \chi_\nu, \quad (17)$$

with $u^\mu u_\mu = 1$, $l^\mu l_\mu = 0$, $\chi^\mu \chi_\mu = -1$, if we identify the four velocity for an observer at rest in the frame of (1), the null and the space-like vectors as

$$u^\mu = (1 - 2\tilde{m}/r)^{-1/2} e^{-2\beta} \delta_0^\mu, \quad (18)$$

$$l_\mu = (1 - 2\tilde{m}/r)^{1/2} e^{2\beta} \delta_\mu^0, \quad (19)$$

$$\chi_\mu = (1 - 2\tilde{m}/r)^{-1/2} \delta_\mu^1. \quad (20)$$

Note that Bondian observers can be purely Lagrangian when we deal only with radiation [21].

The conservation equation $T_{1;\mu}^\mu = 0$, or Eqs. (10)–(12), lead us to the generalized Tolman–Oppenheimer–Volkoff (TOV) equation for non static radiative situations

$$\tilde{p}_{,r} - e^{-2\beta} \left(\frac{\tilde{\rho} + \tilde{p}}{1 - 2\tilde{m}/r} \right)_{,u} + \left(\frac{\tilde{\rho} + \tilde{p}}{1 - 2\tilde{m}/r} \right) [4\pi r (\tilde{p} + p_r^\Phi) + \tilde{m}/r^2] = \frac{2}{r} (p - \tilde{p}), \quad (21)$$

where

$$\tilde{\rho} = \frac{\rho - \omega p}{1 + \omega}, \quad (22)$$

$$\tilde{p} = \frac{p - \omega \rho}{1 + \omega}, \quad (23)$$

are the named effective variables [30],

From the field equation (9) is straightforward that

$$\frac{d\tilde{m}}{du} = -4\pi r^2 \left[(p - p^\Phi) \frac{dr}{du} + (1 - 2\tilde{m}/r) e^{2\beta} \varepsilon^\Phi \right], \quad (24)$$

where

$$\frac{dr}{du} = e^{2\beta} (1 - 2\tilde{m}/r) \frac{\omega}{1 - \omega}, \quad (25)$$

is the matter velocity. Integrating the wave equation (13) we obtain

$$2r\Phi_{,u} = e^{2\beta} (r - 2\tilde{m}) \Phi_{,r} + \int_0^r e^{2\beta} (1 - 2\tilde{m}/r) \Phi_{,r} dr, \quad (26)$$

which combined with (24) lead us to

$$\begin{aligned} \frac{d\tilde{m}}{du} = & -4\pi e^{2\beta} N^2 - 4\pi r^2 \frac{dr}{du} p \\ & + 2\pi r^2 (1 - 2\tilde{m}/r) \left(\frac{1 + \omega}{1 - \omega} \right) e^{2\beta} \rho^\Phi, \end{aligned} \quad (27)$$

where

$$N = \frac{1}{2} e^{-2\beta} \int_0^r e^{2\beta} (1 - 2\tilde{m}/r) \Phi_{,r} dr.$$

In absence of matter, $\rho = p = 0$, for the exterior region the field equations reduce to

$$R_{\mu\nu} = -8\pi \Phi_{,u} \Phi_{,v}, \quad (28)$$

or explicitly to the hypersurface equations

$$\beta_{,r} = 2\pi r \Phi_{,r}^2, \quad (29)$$

$$\tilde{m}_{,r} = 2\pi r (r - 2\tilde{m}) \Phi_{,r}^2, \quad (30)$$

and to the wave equation (13) [8].

Table 1 Coefficients for the physical variables expansion as a power of r near the center

$\mathcal{O}\pi\rho$	πp	ω
$0 \frac{3}{4}m_3 - \frac{1}{2}\pi\Phi_1^2$	$\beta_2 - \frac{3}{4}m_3 - \frac{1}{2}\pi\Phi_1^2$	0
$1 \omega_1(\beta_2 - \pi\Phi_1^2) + \pi\Phi_1\Phi_2\omega_1(\beta_2 - \pi\Phi_1^2) - 2\pi\Phi_1\Phi_2\omega_1$		

3 Regularization and matching

Some previous investigations consider regularization near $r = 0$ [18; 25; 31; 32; 33]. The conditions for the scalar field, as a matter model, do not necessarily apply to distributions of matter. Depending on gauge conditions each procedure to get regular spacetimes may be cumbersome and tricky, even in vacuum. Initially regular spacetimes can eventually develop singularities [24; 34]. We show a simple way to construct regular spacetimes, near the coordinate-origin, when the inner spacetime corresponds to a spherical distribution of baryonic matter coupled to a massless scalar field. To construct regular and general enough spacetimes, which eventually recover equilibria, collapse, form singularities and horizons, we do an asymptotic study close to the special regions: $r = 0$ and $r \rightarrow \infty$. The treatment is basically the same for these two zones, that is, power expansions of r and r^{-1} , respectively. For $r = R(u)$, the boundary surface, the Darmois–Lichnerowicz [35; 36; 37] conditions are guaranteed to match the interior and exterior solutions on a moving boundary. Integrating from $r = 0$ no additional conditions are required at the surface to describe its evolution.

3.1 Close to the origin

Consider the following asymptotic expansions for the metric functions near $r = 0$, which represents a regularly and conformally flat spacetime [23]:

$$\tilde{m} = m_3(u)r^3 + \mathcal{O}(r^5), \quad (31)$$

$$\beta = \beta_0(u) + \beta_2(u)r^2 + \mathcal{O}(r^4), \quad (32)$$

and for the scalar field

$$\Phi = \Phi_0(u) + \Phi_1(u)r + \Phi_2(u)r^2 + \mathcal{O}(r^3). \quad (33)$$

Using the field equations we get the physical variables expansion as a function of r as showed in Table 1.

From the evolution equation (13) we get

$$e^{-2\beta_0} \frac{d\Phi_0}{du} = \Phi_1, \quad (34)$$

and

$$e^{-2\beta_0} \frac{d\Phi_1}{du} = \frac{3}{2}\Phi_2. \quad (35)$$

From the field equation (12)

$$e^{-2\beta_0} \frac{d\beta_2}{du} = [\pi\Phi_1\Phi_2 - 2\omega_1(\beta_2 - \pi\Phi_1^2)], \quad (36)$$

and from (9)

$$\frac{dm_3}{du} = 2 \frac{d\beta_2}{du}. \quad (37)$$

This last equation, together with expansions showed in Table 1, is readily integrated to give

$$\rho_c + 3p_c = \text{constant}, \quad (38)$$

where $\rho_c = \rho_0 + \rho_0^\Phi$ and $p_c = p_0 + p_0^\Phi$.

The lapse of the coordinate time du is related to the corresponding lapse of time $d\tau$ measured by a central observer as

$$d\tau = e^{2\beta_0} du. \quad (39)$$

We have preference for the central time τ to describe the studied system. The reason is mainly numerical: proceeding with the radial integration from $r = 0$, where the world line is therefore geodesic, in consistence with the conformally flat result as a consequence of regularity. Therefore, the replacements $\beta \rightarrow \beta - \beta_0$ and $u \rightarrow \tau$ left invariant the field equations (9)–(13). The same situation will be analog for the matching surface, which behaves asymptotically as Vaidya or Schwarzschild, and for the asymptotically flat infinity.

3.2 Matching at the surface

Boundary conditions at the surface $r = R(u)$ are needed for β and \tilde{m} in order to perform radial integrations. We match the interior solution with the exterior at $r = R$ by means of the Darmois–Lichnerowicz conditions. These conditions are equivalent to the continuity of the functions β and \tilde{m} across the boundary, and to the continuity of the spin coefficient [37]

$$\gamma = (1 - 2\tilde{m}/r)\beta_{,r} - \frac{\tilde{m}_{,r}}{2r} - \beta_{,u}e^{-2\beta}. \quad (40)$$

Considering the expansion of β around the surface, we have

$$\beta_{,u}^\pm = \frac{dB}{du} - \frac{dR}{du}\beta_{,r}^\pm, \quad (41)$$

where the superscript \pm indicates the evaluation of the function at $r = R + 0$ or $r = R - 0$, and $B = \beta(u, r = R)$. Therefore, $\gamma^+ = \gamma^-$ leads us to

$$\omega_R = 1 - 2RF \frac{\delta\beta_{,r}}{\delta\tilde{m}_{,r}}, \quad (42)$$

where $F = 1 - 2M/R$, $M = \tilde{m}_R$, the subscript R indicates that the quantity is evaluated at the surface, and $\delta\Psi = \Psi^+ - \Psi^-$ represents the jump of the indicated function across the boundary. Observe that $\delta\dot{R} = 0$, where over dot indicates derivative respect to time. Thus, we get

$$\tilde{p}_R + \omega_R \tilde{\rho}_R = (1 + \omega_R) \delta\rho^\Phi, \quad (43)$$

which leads us directly to

$$p_R = \left(\frac{1 + \omega_R}{1 - \omega_R} \right) \delta\rho^\Phi. \quad (44)$$

If the scalar field gradient is continuous across the boundary $r = R$ we have a pressureless surface, which can be expressed as $\tilde{p}_R = -\omega_R \tilde{\rho}_R$ or equivalently

$$\omega_R = \frac{\tilde{m}_{,r}^- - 2RF\beta_{,r}^- + 4\pi R^2 \rho_R^\Phi}{\tilde{m}_{,r}^- - 4\pi R^2 \rho_R^\Phi}. \quad (45)$$

When $\rho_R = 0$, we solve as usual the indetermination taking the limit

$$\omega_R = -\lim_{r \rightarrow R} \frac{\tilde{p}}{\tilde{\rho}} = -\left. \frac{\tilde{p}_{,r}}{\tilde{\rho}_{,r}} \right|_R, \quad (46)$$

where

$$\begin{aligned} \tilde{p}_{,r} &= \frac{2}{r}(\tilde{\rho} + \rho^\Phi) - \frac{\tilde{m}_{,rr}}{4\pi r^2} + \frac{\beta_{,rr}}{2\pi r^2}(r - 2\tilde{m}) \\ &+ \frac{\beta_{,r}}{2\pi r^2}(4\tilde{m} - 2\tilde{m}_{,r} - 1) - \rho_{,r}^\Phi \end{aligned} \quad (47)$$

and

$$\tilde{\rho}_{,r} = \frac{\tilde{m}_{,rr}}{4\pi r^2} - \frac{2}{r}(\tilde{\rho} + \rho^\Phi) - \rho_{,r}^\Phi. \quad (48)$$

Two comments are in order here. First, (46) can be obtained from the continuity of the spin coefficient gradient $\gamma_r^+ = \gamma_r^-$. Second, exactly the same result is obtained from the field equations, that is, Eq. (46) proceed from (21) evaluated at the surface [30].

Once satisfied the matching conditions across the boundary $r = R$ we need surface equations to follow the radius evolution and the exchange of energy on it. Evaluating (27) at the surface and defining

$$\Omega = (1 - \omega_R)^{-1}$$

and

$$\mathcal{R}^\Phi = 4\pi R^2 \rho_R^\Phi,$$

we get

$$e^{-2B} \frac{dM}{du} = -4\pi N_R^2 + \frac{1}{2} \mathcal{R}^\Phi (2\Omega - 1)F, \quad (49)$$

which clearly establishes the transfer of energy at the boundary of the matter distribution. Additionally, evaluating (25) at the surface we get

$$e^{-2B} \frac{dR}{du} = F(\Omega - 1), \quad (50)$$

which conforms together with (45) and (49) the system of equations at the surface for $\rho_R \neq 0$. These equations determine completely the evolution at the surface distribution.

3.3 Close to infinity

This section is standard in literature but we include here a résumé for the sake of completeness. Assuming that the scalar field has an asymptotic expansion [8]

$$\Phi = \frac{Q_1(u)}{r} + \frac{Q_2}{r^2} + \mathcal{O}(r^{-3}), \quad (51)$$

the metric functions read

$$\beta = H(u) - \frac{\pi Q_1^2}{r^2} + \mathcal{O}(r^{-3}), \quad (52)$$

$$\tilde{m} = \mathcal{M}(u) - \frac{2\pi Q_1^2}{r} + \frac{2\pi Q_1(\mathcal{M} Q_1 - 2Q_2)}{r^2} + \mathcal{O}(r^{-3}). \quad (53)$$

The coefficient Q_1 is the scalar monopole moment and Q_2 the Newman–Penrose constant. The asymptotic Bondi mass \mathcal{M} also can be expressed globally as

$$\mathcal{M}(u) = 4\pi \int_0^R r^2 \tilde{\rho} dr + 4\pi \int_0^\infty r^2 \rho^\Phi dr = \tilde{m}|_{\mathcal{I}^+}, \quad (54)$$

and the scalar news function [8]

$$\mathcal{N}(u) = \frac{1}{2} e^{-2H} \int_0^\infty e^{2\beta} (1 - 2\tilde{m}/r) \Phi_{,r} dr. \quad (55)$$

The Bondi mass loss equation is

$$e^{-2H} \frac{d\mathcal{M}}{du} = -4\pi \mathcal{N}^2. \quad (56)$$

With these definitions, the total radiated energy can be established

$$\Delta \equiv \mathcal{M}(u) - \mathcal{M}(u_0) = -J(u), \quad (57)$$

where

$$J = 4\pi \int_{u_0}^u \mathcal{N}^2 e^{2H} du. \quad (58)$$

It is easy to check from (54) that the most general Killing propagator to get the energy conservation in the spherical context is

$$\xi^\mu = e^{-2\beta} \delta_0^\mu. \quad (59)$$

As a matter of fact, we get (59) when the Linkage [13], a finite representation of the Bondi–Metzner–Sachs asymptotic group,

$$\mathcal{C} = \int T_\mu^\nu \xi^\mu d\Sigma_\nu, \quad (60)$$

is compared with (54).

Up to now the system has been described with enough generality to proceed with the hydrodynamical solver developing, which was not planted as a goal in this paper. However to show how the scheme works some numerical models are discussed.

4 Models

4.1 Ghost scalar fluid

This model exemplifies the junction conditions, the scalar field energetics and the preservation of the Newman–Penrose constant.

Consider a mixture of two components fluid of pressure $p + p^\Phi \rightarrow p$ and energy density $\rho + \rho^\Phi \rightarrow \rho$ which is incompressible and remains static if a layout outer scalar field guarantee the Darmois–Lichnerowicz conditions. Clearly from (45) these conditions reduce to

$$\tilde{m}_{,r}^- - 2RF\beta_{,r}^- + 4\pi R^2 \rho_R^\Phi = 0, \quad (61)$$

which is satisfied if

$$F\beta_{,r}^+ = 2\pi R\rho_R. \quad (62)$$

Under this scenario $\omega = 0$ everywhere, and derivatives respect to the timelike coordinate dropping to zero. The set of Eqs. (9)–(12) simplify considerably to get the well known interior solution [30]

$$\tilde{m} = M(r/R)^3, \quad (63)$$

$$e^{2\beta} = \frac{1}{2} \left\{ 3 \left(\frac{F}{\zeta} \right)^{1/2} - 1 \right\}, \quad (64)$$

$$p = \rho \left\{ \frac{F^{1/2} - \zeta^{1/2}}{\zeta^{1/2} - 3F^{1/2}} \right\}, \quad (65)$$

where

$$\zeta = [1 - (1 - F)(r/R)^2] \quad (66)$$

Table 2 Exterior/initial data set for the massless scalar field

Φ	Family	Parameters
$\lambda/(R+r)$	(a)	λ, R
$\lambda(r_a-r)^4(r_b-r)^4/[(r_b-r_a)/2]^8$	(b)	λ, r_a, r_b
$\lambda \exp(r-r_0)^2/\sigma^2$	(c)	λ, σ, r_0

Fig. 1 Mass as a function of the normalized amplitude $\chi = \pi\lambda^2/(6R^2 + \pi\lambda^2)$, for the exterior datum (a). If the datum is (b) the amplitude is $\lambda \rightarrow 13.46\lambda$; if the datum is (c) $\lambda \rightarrow \pi\lambda$. For any choice of the initial datum and any choice of parameters r_a, r_b, σ and r_0 —which leave partially immersed the scalar field in the fluid distribution—the overlapping is true finding the appropriate rescaling in the amplitude

and

$$\rho = \frac{3(1-F)}{8\pi R^2} = \frac{3M}{4\pi R^3}. \quad (67)$$

Observe that $\beta_R = 0$, $F\beta_{,r} = 4\pi R\rho_R = 3M/R^2$ and $p_R = 0$. In this way we have constructed a “fixed” background of a incompressible perfect fluid coupled to an exterior scalar field which remains “frozen” in some hydrodynamic characteristic time scale.

Three different families of exterior/initial scalar fields are showed in Table 2. All them are specified in such a way that smooth metric functions across the matching surface $r = R$ are assured. For each set and any R , it is obtained M as a function of the scalar field amplitude. Taking as a reference the data set (a), for instance, it can be fitted the two others to it simply doing the appropriate rescaling in amplitude, as is showed in Fig. 1. The limit mass for the background is $R/2$; this feature seems to be general. The limit mass to keep the incompressible static fluid is the well known Buchdhal mass limit $4R/9$. Therefore, if a black hole form it occurs by means of scalar radiation accretion.

The exterior spacetime is determined integrating the field equations (29) and (30). To evolve the scalar wave on the specified background we have the set of ordinary differential equations at $r = 0$ for the scalar field (34) and (35). This system requires only initial conditions for Φ_0, Φ_1 and Φ_2 , which are fitted from the initial condition for the scalar field. Besides M and R are given as parameters, to define the matter distribution.

The wave equation can be integrated following the null parallelogram method [38]. Note that, as is required by the used method, writing down (13)

$$2g_{,ur} - [e^{2\beta}(1 - 2\tilde{m}/r)g_{,r}]_{,r} = -[e^{2\beta}(1 - 2\tilde{m}/r)]_{,r}g/r, \quad (68)$$

where $g = r\Phi$, we have to take care considering the RHS. To avoid the numerical derivative, that term is replaced by $-\Lambda g$, where

$$\Lambda = \frac{e^{2\beta}}{r^2} \{4\pi r^2(p - \rho) + 2\tilde{m}/r\}. \quad (69)$$

A subtle issue of this model is the Killing propagator to get the energy conservation as showed in Fig. 2. Because the spacetime is fixed as a background, the right propagator is

Fig. 2 Energy conservation (multiplied by 10^5) as a function of the Bondi time for the exterior/initial datum (a). The parameters and conditions of integration are: $M = 1.6921$, $R = 5$, $\lambda = 10$ and 10^3 radial points. The evolution corresponds to a relativistic case with $F = 0.3232$. The *descending curve* corresponds to Δ with \mathcal{M} calculated by Eq. (54). The *ascending curve* corresponds to the energy radiated to infinity given by Eq. (58) with \mathcal{N} calculated by means of Eq. (55). Thus, in accordance with Eq. (57), the *horizontal curve* represents the global conservation of energy

Fig. 3 Decay of the scalar field (multiplied by 10^3) preserving the Newman–Penrose constant, that is, $g_{,x}|_{\mathcal{I}^+} = \text{constant}$, any time of the evolution. The compactified coordinate x is related with r by means of $r = 15Rx/[8(1-x^4)]$. The initial datum, the parameters and conditions are the same as for Fig. 2

$$\xi^\mu = e^{-2H} \delta_0^\mu. \quad (70)$$

Figure 3 displays the evolution of the initial data (a); it is evident that the Newman–Penrose constant is conserved.

4.2 Scattering off a polytrope

In the case of a static polytrope

$$p = K\rho^\Gamma, \quad (71)$$

where K is the polytropic constant and Γ is the adiabatic exponent, related with the adiabatic index n by $\Gamma = 1 + 1/n$, the pressure and the energy density vanish at $r = R$. Again, we integrate numerically the system (10), (11) and (21) for any choice of K and Γ , typically 100 and 2 respectively. However, it is interesting to note that (21) written as

$$\frac{p_{,r}}{\rho + p} + \left[\frac{1}{2} \ln(1 - 2\tilde{m}/r) + 2\beta \right]_{,r} = 0, \quad (72)$$

can be integrated to get

$$e^{2\beta} = \frac{F^{1/2}(1 - 2\tilde{m}/r)^{-1/2}}{(1 + p/\rho)^{n+1}}. \quad (73)$$

This result let us to conclude that no limit appears for the total mass distribution except the black hole mass limit $R/2$. It clearly connects the surface with the center of the distribution by means of $e^{2\beta_0}(1 + p_0/\rho_0)^{n+1} = F^{1/2}$.

In this model, a polytrope with a vacuum Schwarzschild exterior as a background, the scalar field is scattered off and radiated to infinity. Figure 4 shows the energy conservation in this case. It should be stressed here that the scalar field gradient of the initial datum has to be zero at $r = R$ and the scalar field itself partially immersed in the distribution.

Fig. 4 Energy conservation (multiplied by 10^3) as a function of the Bondi time for the initial datum (c). In this case the matter fluid corresponds to a polytrope with $\rho_0 = 4 \times 10^{-3}$, $K = 100$ and $\Gamma = 2$. The parameters and conditions of integration are: $M = 0.774$, $R = 11.512$, $\lambda = 10^{-2}$, $\sigma = R$, $r_0 = R$ and 10^3 radial points; the evolution corresponds to a $F = 0.865$. The *descending curve* corresponds to Δ with \mathcal{M} calculated by Eq. (54). The *ascending curve* corresponds to the energy radiated to infinity given by Eq. (58) with \mathcal{N} calculated by means of Eq. (55). Thus, in accordance with Eq. (57), the *horizontal curve* represents the global conservation of energy

Fig. 5 Quasinormal mode and tail decaying as a function of the Bondi time for the initial datum (b). In this case the matter distribution corresponds to an incompressible fluid, that is, a Schwarzschild interior; the exterior is a Schwarzschild vacuum. The parameters and conditions of integration are: $M = 1$, $R = 2.857$, $\lambda = 1$, $r_a = 0$, $r_b = 10$, and 10^3 radial points; the evolution corresponds to a $F = 0.3$. The late time behavior of the signal decays as an inverse power law. In the window of time [200, 245] the power of the tail decay is approximately -2.1 , which corresponds to a 5% deviation from the expected value of -2 [39; 10]

Fig. 6 Energy conservation as a function of the Bondi time for the initial datum (b). In this case the matter distribution corresponds to an incompressible fluid, that is, a Schwarzschild interior; the exterior is a Schwarzschild vacuum. The parameters and conditions of integration are: $M = 1$, $R = 2.857$, $\lambda = 1$, $r_a = 0$, $r_b = 10$, and 10^3 radial points; the evolution corresponds to a $F = 0.3$. The *descending curve* corresponds to Δ with \mathcal{M} calculated by Eq. (54). The *ascending curve* corresponds to the energy radiated to infinity given by Eq. (58) with \mathcal{N} calculated by means of Eq. (55). Thus, in accordance with Eq. (57), the *horizontal curve* represents the global conservation of energy

4.3 Quasinormal mode and late time tail decay

For a Schwarzschild–Schwarzschild background an expected feature is showed when an initial compact support scalar field (b) is evolved. In this case the distribution of matter corresponds to an incompressible fluid, that is, a Schwarzschild interior spacetime; the exterior is a Schwarzschild vacuum. Figure 5 display the quasinormal mode ringing and the final tail decay. The energy conservation is displayed in Fig. 6.

5 Concluding remarks

We have used corner stones in the characteristic formulation of general and numerical relativity to present in this paper a framework which couples matter with radiation [13; 22; 40]. The old point of view of Bondian observers are mistakenly considered as Eulerian in numerical relativity for the mathematical treatment of matter [21].

Although we only explored toy models, we believe they deserve future attention at least as initial conditions. The partially immersed scalar field in the fluid is hidden. This mixture constitutes a fixed and well behaved background to evolve a cloned scalar field. The face value of model A is its usefulness as a playground to study asymptotic and matching regions. It is striking how the matching leads us to a connection between the mass distribution and the amplitude of the initial scalar field, and how the limit mass of $R/2$ does not depend apparently upon the

initial data. The energy conservation and the Newman–Penrose constant tests give confidence on the possible physical consequences of this simple model. The other two models represent simple tests which are in agreement with expectations.

We stress that this paper does not develop a general code, it just reports how Bondian observers are plausible in Numerical Relativity. Numerics is simplified to show how these observers are consistent with previous results. The underlying problem, gravitational radiation coupled to matter, is in fact difficult. We show that even in the spherical case we have to consider special regions, center and boundary surface carefully as well. We are currently developing a general code based on Sect. 3, which considers regularization and matching regions in a clearer manner.

The most simple axisymmetric case from the Bondian point of view is out of scope of the present investigation. There is a unique geometrical way to define Bondian observers in the absence of spherical symmetry: (i) choose a tetrad to get the local Minkowskian frame; (ii) make a double Lorentz boost to go to the frame comoving with the fluid. In this sense could be interesting to revise results from [2] to consider gravitational radiation considering explicitly Bondian frames and regularization on special regions such as the axis of symmetry and boundary surface.

An advanced study of incompressible-like fluids and polytropic matter indicates the relevance of central equation of state at the center of the star. This motivated us to prepare a hydrodynamic solver which will be reported elsewhere for an adiabatic situation. A more general study of matter coupled to radiation within this framework is in progress.

Acknowledgments We are grateful to Luis Rosales and Carlos Peralta for their comments. Finishing this work WB was on sabbatical leave from ULA. We thank to the referees for their valuable comments.

References

1. L. Baiotti I. Hawke P.J. Montero F. Löffler L. Rezzolla N. Stergioulas J.A. Font E. Seidel (2005) *Phys. Rev. D* **71** 024035
2. F. Siebel J.A. Font E. Müller P. Papadopoulos (2003) *Phys. Rev. D* **67** 124018
3. C. Reisswig N.T. Bishop D. Pollney B. Szilagyi (2009) *Phys. Rev. Lett.* **103** 221101
4. R. Arnowitt S. Deser C. Misner (1962) The dynamics of general relativity L. Witten eds Gravitation: an introduction to current research Wiley New York
5. J. Winicour (2009) *Living Rev. Rel.* **12** 3
6. N.T. Bishop R. Gómez L. Lehner M. Maharaj J. Winicour (1999) *Phys. Rev. D* **60** 024005
7. N.T. Bishop R. Gómez S. Husa L. Lehner J. Winicour (2003) *Phys. Rev. D* **68** 084015
8. R. Gómez J. Winicour (1992) *J. Math. Phys.* **33** 1445
9. R.L. Marsa M.W. Choptuik (1996) *Phys. Rev. D* **54** 4929

10. Gundlach, C., Price, H., Pullin, J.: *Phys. Rev. D* **49**, 883, 890 (1994)
11. P. Papadopoulos P. Laguna (1997) *Phys. Rev. D* **55** 2038
12. Price, R.H.: *Phys. Rev. D* **5**, 2419, 2429 (1972)
13. L. Tamburino J. Winicour (1966) *Phys. Rev.* **150** 1039
14. R. Isaacson J. Welling J. Winicour (1983) *J. Math. Phys.* **24** 1824
15. R. Gómez P. Papadopoulos J. Winicour (1994) *J. Math. Phys.* **35** 4184
16. N.T. Bishop R. Gómez L. Lehner J. Winicour (1996) *Phys. Rev. D* **54** 6153
17. N.T. Bishop R. Gómez L. Lehner M. Maharaj J. Winicour (1997) *Phys. Rev. D* **56** 6298
18. F. Siebel J.A. Font P. Papadopoulos (2001) *Phys. Rev. D* **65** 024021
19. S.H. Hawley M.W. Choptuik (2000) *Phys. Rev. D* **62** 104024
20. V. Pavlidou K. Tassis T.W. Baumgarte S.L. Shapiro (2000) *Phys. Rev. D* **62** 084020
21. W. Barreto (2009) *Phys. Rev. D* **79** 107502
22. H. Bondi (1964) *Proc. R. Soc. Lond. A* **281** 39
23. W. Barreto L. Castillo E. Barrios (2009) *Phys. Rev. D* **80** 084007
24. D. Garfinkle (1995) *Phys. Rev. D* **51** 5558
25. D. Neilsen M.W. Choptuik (2000) *Class. Quantum Grav.* **17** 733
26. C.W. Misner D.H. Sharp (1964) *Phys. Rev.* **136** 571
27. S.A. Hayward (1994) *Phys. Rev. D* **49** 831
28. S.A. Hayward (1996) *Phys. Rev. D* **53** 1938
29. L. Herrera N.O. Santos (1997) *Phys. Rep.* **286** 53
30. L. Herrera J. Jiménez G.J. Ruggeri (1980) *Phys. Rev. D* **22** 2305
31. K. Lake (2003) *Phys. Rev. D* **67** 104015
32. A. Arbona C. Bona (1999) *Comput. Phys. Commun.* **118** 229
33. M. Ruiz M. Alcubierre D. Núñez (2008) *Gen. Relativ. Gravit.* **40** 159
34. M.W. Choptuik (1993) *Phys. Rev. Lett.* **70** 9
35. Darmois, G.: Memorial des Sciences Mathematiques. Gauthier-Villars, Paris, Fasc 25 (1927)
36. A. Lichnerowicz (1955) *Theories Relativistes de la Gravitation et de l'Electromagnetisme* Masson Paris
37. L. Herrera J. Jiménez (1983) *Phys. Rev. D* **28** 2987
38. R. Gómez R. Isaacson J. Winicour (1992) *J. Comput. Phys.* **98** 11
39. R. Gómez J. Winicour B.G. Schmidt (1994) *Phys. Rev. D* **49** 2828
40. H. Bondi M.G.J. van der Burg A.W.K. Metzner (1962) *Proc. R. Soc. A* **269** 21