$\mathrm{Exp}(10^{76})$ Shades of Black:
Aspects of Black Hole Microstates

by

Orestis Vasilakis

A Dissertation Presented to the

FACULTY OF THE GRADUATE SCHOOL

UNIVERSITY OF SOUTHERN CALIFORNIA

In Partial Fulfillment of the

Requirements for the Degree

DOCTOR OF PHILOSOPHY

(PHYSICS)

August 2015

- ... and what did you say your objection with string theory is again?

- It's rather simple. These days theoretical physicists dream of a theory of everything that will describe nothing.

-So?

-Well, string theory is something.

Acknowledgments

Five years ago my research advisor Nicholas Warner told me that the relationship between the student and the advisor is like a small marriage. It comes to no surprise then that typing this thesis feels like some kind of divorce, necessary however for the emancipation of the student. But nothing else than an interesting and rich in knowledge and experiences marriage has it been. I am grateful to Nick for being next to me during the long process of my PhD and for shaping my line of thought as a researcher. His technical prowess and his ability to transform a physics problem into a mathematical one and then back, helped raise my technical skills and standards. Nick has been an tremendous source of knowledge about physics as well as other topics ranging from movies, to religion and politics. I can only hope that in the future our opinions will converge about the return of the Elgin marbles to their true home.

I am also thankful to Krzysztof Pilch who has been an excellent source regarding supergravity. He has been patient enough to introduce me to gauge supergravity by participating in my first paper in that area and thus providing me with useful working knowledge of the field. Whenever I had a question his door was always open. The above combined with his unique sense of humor significantly enriched my years at USC.

Nikolay Bobev is also someone to whom I owe a lot. He was one of the first people I contacted once I received my offer from USC. Since then he has been a valuable resource on many different levels from introducing me to new interesting topics in string theory to providing advice regarding aspects of the field and dealing with situations while a graduate student. His immense amounts of energy, his eagerness to help and his positive attitude will always be an inspiration.

I would also like to thank Iosif Bena and Andrea Puhm for being amazing collaborators and for enriching my toolbox with techniques about probes and D-branes. I am also grateful to Iosif for arranging my visits at IPhT-Saclay and for his hospitality while there.

Ben Niehoff played a significant role in some of the work presented in this thesis. Together we worked on one of my favorite papers which in addition with our discussions provided me with the inspiration for one more publication. Ben is someone I have been interacting with for the most part of my graduate studies and I hope that the special bond we have by sharing the same research advisor will remain strong in the years to come.

I am also thankful to the people that encouraged me to apply and come to USC while I was still planning this phase of my life. Fernando Quevedo suggested applying to USC by saying to me ”Nick Warner is a great guy and the university has an amazing football team”. Stephan Haas contacted me with the offer from USC and with his pleasant attitude and patience helped me adapt during my first year as a graduate student. Finally Konstantinos Sfetsos as a USC alumni provided useful advice that helped me finalize my decision about coming to USC.

I would like to thank the members of the high energy physics group Clifford Johnson, Krzysztof Pilch, Itzhak Bars, Dennis Nemeschansky, Nick Warner and Hubert Saleur for the great courses they offered and their willingness to answer my questions, thus enhancing my understanding of physics. Also thanks to Clifford Johnson, Francis Bonahon, Nick Warner, Dennis Nemeschansky, Moh El-Naggar, Itzhak Bars, Paolo Zanardi and Feodor Malikov for kindly participating in my qualifying exam and dissertation defense committees. I also want to thank Tameem Albash and Ibrahima Bah who as postdocs of the group who were eager to answer my physics questions.

I also want to thank my fellow graduate students Ben Niehoff, Scott MacDonald, Dmitry Rychkov, Isaiah Yoo, Min Woo Suh, Ignacio Araya Quezada, Albin James, Alexander Tyukov and Vasilis Stylianou for the fun discussions and for making the office a place full of interesting people. A special thanks to Vasilis who in addition to being a colleague has also become a close friend and his presence added the eastern Mediterranean flavor of the office thus reminding me of home.

My uncle Evan is someone to whom I also owe a lot, who long time ago starting from the island of Tinos eventually came to the US and was here to pick me up from the airport when I first arrived at Los Angeles. He had even bought me a cell phone! I will always be grateful to him and my aunt Catherine for their hospitality and for being there for me whenever I needed them.

Special thanks to my Greek friends here at USC. Harris and Daphney were the first Greeks I met during my first year and helped me a lot adapting in the US. Somehow I was always knocking their door whenever they were about to eat and thus we had many meals together! We also took our first road trip together and with them I have some of the most memorable experiences in the US. I am also grateful to Angeliki, Theodora, Christodoulos and Vasilis for the numerous time we had fun together but most of all for being there for me when I needed them. I admire Theodora and Vasilis for their patience and willingness to hear some of my inner thoughts whenever I needed to express them and thus help me preserve my sanity.

I am thankful to Leonardo and Erik who with their guitars and their attitude brought in contact with the alternative and sometimes underground art scene in Los Angeles far away from the Hollywood glamour. They provided with an artistic outlet which enriched my life in the US and has also been necessary for the satisfaction of certain aspects of my character.

I also want to thank my friends from Greece, Nikos, Kostas, Eleni, Mikaela, Katerina, Simos and Yiannis whom I have known for more than ten years and who for the most part now live in different parts of the world but always find time to meet with me whenever we are in Greece. The long and numerous phone conversations we have had over the last few years definitely helped me feel less far away from home and value my relationship with them even more.

Finally and most of all I want to express my gratitude to my parents since without their constant encouragement and support all these years I wouldn't be were I am today.

While working towards my thesis I received support from various sources. I would like to acknowledge the support from IPhT-Saclay during my stay there. Also my research has been supported by the Gerondelis Fellowship, the USC Graduate School Travel Award, the USC College Doctorate Fellowship, the USC Myronis Fellowship, the USC Russell Fellowship and the DOE grants DE-FG03-84ER-40168 and DE-SC0011687.

Abstract

In this thesis we examine smooth supergravity solutions known as "microstate geometries". These solutions have neither a horizon, nor a singularity, yet they have the same asymptotic structure and conserved charges as black holes. Specifically we study supersymmetric and extremal non-supersymmetric solutions. The goal of this program is to construct enough microstates to account for the correct scaling behavior of the black hole entropy with respect to the charges within the supergravity approximation. For supersymmetric systems that are $\frac{1}{8}$-BPS, microstate geometries account so far only for $Q^{5/4}$ of the total entropy $S \sim Q^{3/2}$, while for non-supersymmetric systems the known microstate geometries are sporadic.

For the supersymmetric case we construct solutions with three and four charges. Five-dimensional systems with three and four charges are $\frac{1}{8}$-BPS. Thus they admit macroscopic horizons making the supergravity approximation valid. For the three-charge case we present some steps towards the construction of the superstratum, a microstate geometry depending on arbitrary functions of two variables, which is expected to provide the necessary entropy for this class of solutions. Specifically we construct multiple concentric solutions with three electric and two dipole magnetic charges which depend on arbitrary functions of two variables and examine their properties. These solutions have no KKM charge and thus are singular.

For the four-charge case we construct microstate geometries by extending results available in the literature for three charges. We find smooth solutions in terms of bubbled geometries with ambipolar Gibbons-Hawking base space and by constructing the relevant supertubes.

In the non-supersymmetric case we work with a three-charge system of extremal black holes known as almost-BPS, which provides a controlled way of breaking sypersymmetry. By using supertubes we construct the first systematic example of a family of almost-BPS microstate geometries and examine the moduli space of solutions. Furthermore by using brane probe analysis we show that, despite the breaking of supersymmetry, almost-BPS solutions receive no quantum corrections and thus must be subject to some kind of non-renormalization theorem.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In addition to attracting the eye, light has always captured the interest of human mind and perception. Necessarily then, objects so massive from which "not even light can escape" concentrate even more interest. That was actually the very first inception of black holes by John Michell already from 1783. Quite astonishingly, by considering an object with an escape velocity the speed of light, at the surface of a planet of mass "M" and by using the non-relativistic laws for kinetic and gravitational potential energy, we find that the radius of the planet is equal to the Schwarzschild radius, that is the radius of the horizon of a relativistic black hole of mass "M"! These early thoughts, although provocative, didn't gain much ground as the idea of a massless particle like light interacting gravitationally didn't make much sense in the context of classical mechanics. General Relativity on the other hand provided the appropriate theoretical framework for the study of black holes. It was within this theoretical framework that the first black hole solution was discovered by Karl Schwarzschild [1] in the trenches of the first world war.

Since then, the systematic study of black holes has produced many remarkable results which even led to contradictions and paradoxes. These combined with the fact that black holes are objects where the gravitational force is extremely strong, point to the direction that the study of black holes takes us out of the regime of validity of general relativity and a more fundamental theory of gravity is needed. String theory, as a theory of quantum gravity, should provide answers to these puzzles.

In general relativity black holes are realized as objects with infinite mass density. The whole mass of the object is concentrated at a region of spacetime, which forms a curvature singularity. This singularity is surrounded by an event horizon, which is broadly defined as the spacetime boundary beyond which an outside observer can not be affected by events within the horizon. In more folklore terms it is the area beyond which "not even light can escape". The spacetime between the horizon and the singularity is empty and an infalling observer would not notice anything special while crossing the horizon. In four dimensions a black hole is specified by its asymptotic charges, that is its mass "M", charge "Q" and angular momentum "J". Actually there are uniqueness theorems [2] (also known as no hair theorems) which state that in four dimensional asymptotically flat spacetimes, black holes are uniquely specified by their asymptotic charges M, J, Q and that for specific values of these charges there is a single solution in general relativity that describes them. Later it was realized [3], [4], [5, 6] that black holes have thermodynamic properties and their asymptotic charges should be realized as macroscopic variables of a thermodynamic system. Admitting a microscopic origin of black holes thermodynamic properties means that black holes should have a set of microstates that give rise to a specific macrostate, which can not be realized anymore by a single solution. The latter is obviously in contradiction with uniqueness.

Realizing how enormous the set of microstates of a black hole is makes the contradiction even more striking. A solar mass black hole has entropy $10^{60} erg \cdot K^{-1}$ which means it consists of about $e^{10^{76}}$ states!

From the thermodynamic properties of black holes the Bekenstein-Hawking entropy law states that the entropy of a black hole is proportional to the area of its horizon. Entropy in general is an extensive quantity and thus proportional to the volume of the system. The fact that for black holes the entropy is proportional to the area instead of the volume implies that the degrees of freedom of a black hole are encoded in one dimension less than that of the actual spacetime.

Another mystery that arises from black holes physics is the information paradox. By considering quantum fields at the vicinity of the horizon, Hawking showed [5] that black holes evaporate by emitting thermal radiation and thus promoted the thermodynamic laws of black holes from a mathematical analogy to actual physical laws requiring physical interpretation. The fact that Hawking radiation is purely thermal leads to the loss of unitarity (a cornerstone of quantum mechanics), as a pure state that is thrown inside a black hole will eventually be emitted as a mixed state while the black hole evaporates. In other words the information about anything that falls inside the black hole is irreversibly lost. This occurs because the black hole horizon is simply empty space and thus there is a unique vacuum state at the horizon (Unruh vacuum). As a result the two particles of the particle-antiparticle pairs nucleated at the horizon are maximally entangled and carry no information about the black hole.

String theory has taken great strides in providing intuition and shedding light in some of the puzzles of black holes. As we mentioned earlier the Bekenstein-Hawking entropy law suggests that the degrees of freedom of a black hole are encoded in the surface of the horizon, which has one dimension less than the actual spacetime. The AdS/CFT correspondence [7], [8], [9] provided a precise connection between a theory of gravity and a field theory at the boundary with no gravity. This implies that since the theory at the boundary is unitary then information has to be preserved.

In [10] Strominger and Vafa made an important step towards understanding the statistical nature of the black hole entropy. They studied a class of supersymmetric black holes in string theory which consist of D-brane bound states. While at strong gravitational coupling the D-branes collapse to a singularity and form a black hole, at zero gravitational coupling they form a system of free objects which can be combined in a specific number of ways. It was then found that the counting of the possible combinations of the D-branes at zero gravitational coupling, matches the number of microstates obtained from the Bekenstein-Hawking entropy law at strong gravitational coupling. The above result was made possible because supersymmetry guarantees that the states persist as one varies the coupling. The result was rederived and generalized to black holes with mildly broken supersymmetry by Callan and Maldacena [11]. Although these calculations reveal the origin of the microscopic structure of black holes, they do not provide us with a picture of those microstates at strong gravitational coupling.

The fuzzball proposal of Mathur [12] offers a suggestion of how these microstates might actually look. According to the fuzzball proposal, the traditional macroscopic picture of a black hole should be replaced with that of a fuzzball, a horizonless superposition of strings and branes extending all the way up to the area of spacetime where there would be a horizon. It basically says that black holes should be thought as string stars. This picture is quite revolutionary compared to the classical idea of black holes and comes in contrast with other considerations which support that string theory resolves the singularity with string scale corrections to the singularity behind the horizon. The advantage of the fuzzball proposal is that in addition to resolving the singularity it allows information to be preserved since we don't have a horizon from which information can not escape. Instead of empty space in the area of a horizon we have structure made of string states which allow for unitary scattering. The radiation emanating from a fuzzball doesn't pop out of the vacuum, but has been in contact with the object. Then the radiation is not exactly thermal anymore since by being correlated with the object carries information to the outside.

Fuzzballs are a drastic departure from the traditional picture of black holes but there seems to be an accumulation of evidence in favor of them. In [13] Mathur proved that one needs $\mathcal{O}(1)$ corrections at the level of the horizon in order to resolve the information paradox. This is based on Page's argument [14] according to which when we have a macroscopically large burning system in a typical state, the von Neumann entropy of the radiation emitted from the system starts increasing from zero, reaches a maximum when half of the system has evaporated and then decreases until it becomes zero again when all the information has been released. Based on that Mathur shows that when black holes evaporate through the Hawking process or with small corrections to it, the von Neumann entropy of the radiation increases monotonically over time and never becomes zero again. Thus preservation of information and unitarity require $\mathcal{O}(1)$ corrections to the Unruh vacuum, which means that black holes should have non-trivial structure at the horizon. Also in [15], [16] the authors provided further support to the idea that there must be some horizon-scale structure by finding that an infalling observer won't be able to freely cross the horizon, but just before he will encounter a firewall that will crisp him to death.

In this thesis we are going to examine the fate of the black hole microstates at strong gravitational coupling. In particular we are going to look for microstates with a geometric description within the regime of validity of supergravity (i.e. we take the limit where the string length is arbitrarily small). These microstate geometries are a semiclassical approximation and offer a coarse grained description of the full set of microstates of a black hole. The goal of this program is to construct enough microstate geometries to account for the correct scaling of the black hole entropy with respect to its charges. The microstate geometries we will describe correspond to smooth horizonless configurations with the same asymptotic structures as a black hole and thus are in agreement with the physical description of black holes as fuzzballs. In that sense these geometries represent different shades, with huge levels of detailed structure, of black holes. Specifically we are going to describe aspects of supersymmetric and extremal non-supersymmetric microstate geometries, offer specific examples and examine their physical properties as well as the moduli space of solutions.

Chapters 2 and 3 contain material available in the literature. Chapters 4, 5, 6, 7 contain work I did in collaboration with Iosif Bena, Ben Niehoff, Andrea Puhm and Nick Warner [17–21].

In chapter 2 we introduce some key concepts to understand the contents of this dissertation. We provide a short review of black holes as well as their thermodynamics and stress the similarities of the extremal Reissner-Nordström black hole with the black holes examined in this thesis. We also discuss some aspects of string theory and supergravity and the origin of black hole entropy in string theory.

In chapter 3 we present material available in the literature [22] and set up the formalism for what will be presented in the rest of the thesis. We discuss the brane content of three-charge solutions from the point of view of eleven-dimensional supergravity and write down the BPS equations. Then we display smooth horizonless solutions in a Gibbons-Hawking base and examine their regularity. In the end we move on to supertubes, which are the archetype microstate solutions for two-charge black holes.

In chapter 4 we provide the microstate description in terms of topological bubbles and supertubes of BPS geometries with four electric and four dipole magnetic charges [18]. These geometries have been explicitly constructed and uplifted in eleven-dimensional supergravity in [23]. These solutions were found to belong to a CFT with central charge different compared to the one of the three-charge model. The fourth charge results in an additional flux through the topological cycles that resolve the brane singularities. The analog of these solutions in the IIB frame yield a generalized regular supertube with three electric charges and one dipole charge. Direct comparison is also made with the with the smooth solutions of chapter 3.

In chapter 5 we set up the formalism for "almost-BPS" solutions, a specific class of five-dimensional three-charge solutions with broken supersymmetry, and present the material of [17, 21]. The supersymmetry is broken in a very controlled manner using holonomy and this enables a close comparison with a scaling, BPS microstate geometry. For a flat base metric we obtain a new relation between BPS and almost-BPS systems. Furthermore we use a multi-species supertube solution to construct an example of a scaling microstate geometry for non-BPS black rings in five dimensions. Requiring that there are no closed time-like curves near the supertubes places additional restrictions on the moduli space of physical, non-BPS solutions when compared to their BPS analogs. For large holonomy the scaling non-BPS solution always has closed time-like curves while for smaller holonomy there is a "gap" in the non-BPS moduli space relative to the BPS counterpart. Certain details of this chapter including the asymptotic charges of the microstate geometry and how the solution is related to the corresponding non-BPS black ring and relegated to appendix B.

In chapter 6 we continue our investigation of almost-BPS geometries [21]. A key feature of BPS multi-center solutions is that the equations controlling the positions of these centers are not renormalized as one goes from weak to strong coupling. In particular, this means that brane probes can capture the same information as the fully back-reacted supergravity solution. We investigate this non-renormalization property for non-supersymmetric, extremal almost-BPS solutions at intermediate coupling when one of the centers is considered as a probe in the background created by the other centers. We find that despite the lack of supersymmetry, the probe action reproduces exactly the equations underlying the fully back-reacted solution, which indicates that these equations also do not receive quantum corrections. In the course of our investigation we uncover the relation between the charge parameters of almost-BPS supergravity solutions and their quantized charges, which solves an old puzzle about the quantization of the charges of almost-BPS solutions.

In chapter 7, based on [19, 20], we obtain new BPS solutions of six-dimensional, $\mathcal{N} = 1$ supergravity coupled to a tensor multiplet. These solutions are sourced by multiple "superthreads" carrying D1-D5-P charges and two magnetic dipole charges. These new solutions are sourced by multiple threads with independent and arbitrary shapes and include new shape-shape interaction terms. Because the individual superthreads can be given independent profiles, the new solutions can be smeared together into continuous "supersheets," described by arbitrary functions of two variables. The supersheet solutions have singularities like those of the three-charge, two dipole-charge generalized supertube in five dimensions and we show how such five-dimensional solutions emerge from a very simple choice of profiles. We lay down the formalism to construct multiple supersheets with arbitrary and independent profiles. For a more general choice of profiles we construct multi-supersheet solutions that after smearing still are genuinely six-dimensional. We also derive the conditions under which different supersheets can touch, or even intersect through each other. The solutions are by construction free of Dirac strings in contrast to the five-dimensional construction where one has to separately solve integrability conditions. The new solutions obtained here also represent an important step in finding superstrata, which are expected to play a role in the description of black-hole microstates, due to their ability to store a large amount of entropy in their two-dimensional profile.

Finally conclude in chapter 8 by summarizing the main results and discussing the some of the directions the investigation of black hole microstate geometries has to follow in the future.

Chapter 2

Black holes in general relativity and string theory

The solutions we will be describing in this thesis will be microstate geometries of extremal black holes in supergravity. The purpose of this chapter is to briefly present some of the notions of black holes, string theory and supergravity that we will be referring to.

2.1 Black Holes

2.1.1 Schwarzschild black hole

The Schwarzschild solution [1, 24] is the simplest spherically symmetric solution of Einstein equations in vacuum after flat spacetime. The metric describing the spacetime geometry refers to a point source of mass M and is

$$ds^2 = -\left(1 - \frac{2G_4 M}{r}\right) dt^2 + \left(1 - \frac{2G_4 M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \qquad (2.1)$$

The above metric looks like it has singularities at $r = 0$ and at $r = 2G_4M$. The locus $r = r_H \equiv 2G_4M$ is merely a coordinate singularity since by going to Kruskal-Szekeres coordinates one finds the metric to be completely regular at $r = r_H$. The surface $r = r_H$ is null and satisfies $g^{rr} = 0$. Thus it is the horizon of the black hole. By calculating some of the curvature invariants of the metric we get

$$R = 0 , \quad R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{12 G_4^2 M^2}{r^6} , \qquad (2.2)$$

which blow up only at $r = 0$. Thus $r = 0$ is a true singularity of the solution and at this point the curvature of the spacetime becomes infinite. The latter means that near the black hole singularity we out of the regime of validity of general relativity.

2.1.2 Reissner-Nordström black hole

The Reissner-Nordström (RN) solution [25, 26], [24] is the simplest example of a black hole with electromagnetic charges. Thus in terms of macroscopic parameters, in addition to the mass M, the solution carries an electric charge q and magnetic charge p. RN has an extremal limit and thus it serves as an important case study for the extremal solutions we will be studying later on. This black hole is a solution of the Einstein-Maxwell equations and has the following metric

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{2.3}$$

where

$$\Delta = 1 - \frac{2G_4 M}{r} + \frac{G_4 \left(q^2 + p^2\right)}{r^2} . \tag{2.4}$$

It also has gauge field strength

$$F = -\frac{q}{r^2} dt \wedge dr + p \sin\theta d\theta \wedge d\phi . \tag{2.5}$$

By setting $q = p = 0$ we see that we get the Schwarzschild solution (2.1) described above. To find the horizon once again we look for the points at which the metric looks singular and $g^{rr} = \Delta = 0$. We find two such points given by

$$r_\pm = G_4 M \pm \sqrt{G_4^2 M^2 - G_4 \left(q^2 + p^2\right)}. \tag{2.6}$$

As in the Schwarzschild case the loci $r = r_\pm$ are simply coordinate singularities and by examining curvature invariants we see that $r = 0$ is a curvature singularity . Then, based on the values of the macroscopic parameters M, q, p, there are three distinct cases we can consider.

- $G_4^2 M^2 < G_4 \left(q^2 + p^2\right)$:

  Then the equation $\Delta = 0$ has no real solutions and $\Delta > 0$ always. Consequently there are no horizons and the only singularity of the metric is at $r = 0$ and is called a "naked singularity". Naked singularities are considered unphysical and to avoid them the *cosmic censorship conjecture* has been introduced. According to this conjecture naked singularities can not be the product of the gravitational collapse.

- $G_4^2 M^2 > G_4 \left(q^2 + p^2\right)$:

  This case corresponds to a physical solution and is the non-extremal Reissner-Nordström black hole. The equation $\Delta = 0$ has two real solutions at $r = r_\pm$, which correspond to null surfaces and thus are event horizons of the black hole.

- $G_4^2 M^2 = G_4 \left(q^2 + p^2\right)$:

  This is the extremal Reissner-Nordström black hole. For a given mass it has the maximum amount of charge allowed by the cosmic censorship conjecture. Apart from the curvature singularity at $r = 0$ the metric has only one coordinate singularity at $r = r_+ = r_-$ which is the event horizon.

2.1.3 Extremal Reissner-Nordström

At this point let us study the extremal RN solution in greater detail. Because of the extremality condition the electromagnetic force balances that of gravity and the non-linearities of the gravitational equations cancel. Thus one can easily generalize to multiple black hole solutions. For simplicity let us set $G_4 = 1$ [1]. The metric is

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{2.7}$$

[1] G_4 can easily be restored by making the substitution $M \to G_4 M$.

To further analyze the properties of (2.7) it is useful to rewrite the metric in isotropic coordinates by making the change $\rho = r - M$. Then the metric becomes

$$ds^2 = -H^{-2}dt^2 + H^2\left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2\right) , \tag{2.8}$$

where

$$H = 1 + \frac{M}{\rho} . \tag{2.9}$$

Also by setting the magnetic charge $p = 0$ we have $M^2 = q^2$ and

$$A = -H^{-1}dt . \tag{2.10}$$

In this coordinate system the last factor in the metric is simply flat $\mathbb{R}^3$ and the horizon is at $\rho = 0$. In these coordinates the spacetime is not geodesically complete as for the part of the spacetime behind the horizon one would need to consider $\rho < 0$. Taking the near horizon limit $\rho \to 0$, and changing coordinates to $\rho = M^2/z$ the metric becomes

$$ds^2 = M^2\left(\frac{-dt^2 + dz^2}{z^2}\right) + M^2 d\Omega_2^2 . \tag{2.11}$$

The latter spacetime is $AdS_2 \times S^2$ with electromagnetic flux F on it. This is also known as the Robinson-Bertotti solution [27, 28]. Both AdS_2 and S^2 have radius M and S^2 is the shape of the black hole's event horizon. A spacetime of the type $AdS_p \times S^q$ is a general characteristic of the near horizon region of extremal black holes even in higher dimensions.

Now let us note that in (2.8), (2.10) both the metric and the gauge potential are expressed in terms of the function H. H is the function of the electric potential of a point particle in $\mathbb{R}^3$ and is also harmonic, that is it satisfies the Laplace equation on $\mathbb{R}^3$

$$\nabla^2 H = 0\,. \tag{2.12}$$

This equation is linear and thus we can immediately generalize to a solution containing multiple sources located at points $\vec{y}_j$ of $\mathbb{R}^3$

$$H = 1 + \sum_{j=1}^{N} \frac{q_j}{|\vec{y} - \vec{y}_j|}\,. \tag{2.13}$$

This is known as the Majumdar-Papapetrou solution [29, 30] which describes a system of N extremal RN black holes. The balancing between gravity and the electromagnetic force due to the extremality condition $M_j^2 = q_J^2$, is necessary in establishing the above result as it reduces the equations of the system to a linear one (2.12) which in turn allows us to superimpose solutions.

It can be shown that the extremal RN black hole and its multi-centered generalizations are supersymmetric solutions of $\mathcal{N} = 2$ supergravity in four dimensions [31]. Actually the fields of the Einstein-Maxwell system (the metric $g_{\mu\nu}$ and the gauge field A) are the bosonic matter content of the gravity multiplet of this supergravity. A characteristic of the states of supersymmetric theories is the BPS bound, $M \geq |Z|$, where Z is the central charge of the supersymmetry algebra. Supersymmetric solutions saturate the BPS bound and preserve half the supersymmetries. Thus they are also called BPS or sometimes $\frac{1}{2}$-BPS. The extremal RN black hole is indeed BPS with central charge $Z = q + ip$ and saturates the BPS bound

$M = \sqrt{q^2 + p^2}$, which is identified with the extremality condition.

2.1.4 Black hole thermodynamics

By studying various physical processes one can derive laws about the behavior of black holes. Some of these laws look like the four laws of thermodynamics and suggest there is an analogy between black holes and thermodynamic systems[3, 4, 32] [2]. In the classical context of general relativity, that these laws are being derived, they can be nothing more than a mathematical analogy since black holes have zero temperature and uniqueness theorems dictate that black holes are described by a single state rather than a set of microstates. However by a studying quantum fields at the vicinity of the horizon Hawking showed that black holes emit thermal radiation [5, 6]. As a result the analogy has a physical meaning and black holes are thermodynamical systems. At this point let us write down the four laws of black hole thermodynamics and comment on their implications in black hole physics.

- **Zeroth law:**

  The surface gravity [3], κ, on the event horizon of a stationary black hole is constant. The relevant thermodynamical law states that the temperature of a physical system in thermal equilibrium is constant. This suggests that the surface gravity on the horizon is proportional to the temperature of the black hole $\kappa \sim T$.

- **First law:**

  For a process that results in an infinitesimal change of the parameters of the black hole

[2] A review can be found in [33].

[3] Surface gravity is the acceleration, as exerted at infinity, required to keep an object on the horizon of a black hole

when it is perturbed we have

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ \,, \tag{2.14}$$

where M, Q, J are the mass, charge and angular momentum of the black hole, A is the area of the event horizon, Ω_H is the angular velocity of the horizon and Φ_H is the electrostatic potential at the horizon. This resembles the first law of thermodynamics according to which

$$dE = TdS + \text{Work terms} \,. \tag{2.15}$$

The mass is the energy of the black hole and since from the zeroth law we have that $\kappa \sim T$ then from then first law we deduce $S \sim A$.

- **Second law:** There is no physical process under which the area of the black hole decreases, thus $\delta A \geq 0$. This is reminiscent of the second law of thermodynamics which states that in all physically allowed processes $\delta S \geq 0$. This law argues in favor of the entropy of the black hole being proportional to the area of its event horizon. Their relation is given by the Bekenstein-Hawking formula

$$S = \frac{A}{4G} \tag{2.16}$$

Consequently from (2.14) we have

$$T = \frac{\kappa}{2\pi} \,. \tag{2.17}$$

and $E = M$.

When a thermodynamic system falls into a black hole its entropy is invisible to an external observer who will thus realize the entropy being reduced, in violation of the second law of thermodynamics. To rectify that one has to take into account the black hole entropy as well. The latter is expressed with the generalized second law of thermodynamics (GSL). GSL states that there are no physical processes such that the sum of ordinary entropy S_o outside black holes and the total black hole entropy S_{BH} decreases.

$$\delta S_o + \delta S_{BH} \geq 0\,. \tag{2.18}$$

- **Third law:**

  Reaching the limit $\kappa = 0$ requires and infinite number of physical processes. After replacing κ with T the last sentence is the unattainablility statement of the third law of thermodynamics. For thermodynamical systems the third law comes also in the Planck-Nernst form which states that $S \to 0$ as $T \to 0$. This fails for extremal black holes as they have $\kappa = 0$ with finite horizon area and thus entropy. However apart from black holes, there are other quantum systems that violate the Planck-Nernst form of the third law [34, 35].

For the Schwarzschild black hole then we have

$$T_{Sch} = \frac{1}{4\pi r_H}\,, \quad S_{Sch} = \frac{\pi r_H^2}{G_4}\,, \tag{2.19}$$

with $r_H = 2G_4M$.

For the Reissner-Nordström we have

$$T_{RN} = \frac{r_+ - r_-}{4\pi r_+^2}\,, S_{RN} = \frac{\pi r_+^2}{G_4}\,. \tag{2.20}$$

At the extremal limit we have $r_+ = r_-$ and thus we get zero temperature $T_{RN} = 0$ but finite entropy $S_{RN} = \pi G_4 M^2$ in violation of the Planck-Nernst statement of the third law.

The laws of black hole thermodynamics suggest that black holes are truly thermodynamic systems whose macroscopic parameters should have a statistical origin. To explain black hole entropy there should be $N = e^{S_{BH}}$ microstates with the same asymptotic charges. This description comes in contrast with uniqueness theorems of general relativity stating that in four-dimensional asymptotically flat spacetime there is only one solution for a given set of asymptotic charges. Also general relativity does not provide an explanation as to why the entropy of a black hole is proportional to the area of its event horizon. Consequently the study of black holes takes us out of the regime of validity of general relativity. To truly understand their microscopic nature and evade uniqueness theorems we need a quantum theory of gravity and more than four spacetime dimensions. String theory is such a theory and so far has provided substantial evidence about the origin of black hole entropy.

2.2 Black holes in string theory

In general relativity black holes occur when matter collapses under its own gravity and there is no other phase of matter to withhold the collapse. The matter that forms the black hole consists of the well known collection of fundamental particles. The black holes we will be examining in this thesis are bosonic solutions of string and M-theory. For this reason let us briefly review the bosonic matter content of these theories.

2.2.1 Matter content of string & M-theory

Instead of particles in string theory and M-theory the fundamental objects are one-dimensional strings and other higher dimensional objects called branes [36–38]. In the same way that zero-dimensional particles couple to an electromagnetic field given by a vector gauge potential, these higher dimensional objects of string theory couple to electromagnetic fields given by higher rank tensors. Specific kinds of black holes that carry electromagnetic charges and angular momentum thus occur from particular arrangements of strings and branes.

M-theory lives in eleven spacetime dimensions and is still unknown in detail. Its low energy limit is $\mathcal{N} = 1$ eleven-dimensional supergravity, which is unique. The bosonic matter content of the low energy theory consists the metric $g_{\mu\nu}$ and a three-form gauge potential A_3, which in total accounts for 128 bosonic degrees of freedom. The presence of A_3 suggests that M-theory has two fundamental objects, M2 and M5 branes which respectively couple electrically and magnetically to the field strength.

Superstring theory lives in ten dimensions. We will be looking for solutions of IIA and IIB supergravity theories which are the low energy limit of IIA and IIB string theory. The bosonic excitations of the closed superstring in these two theories comes in two sectors NS-NS and R-R. The massless part of those excitations forms the matter content of the supergravity theories and gives 128 bosonic degrees of freedom. For the NS-NS sector we have the metric $g_{\mu\nu}$ the dilaton ϕ and the two-form potential B_2. The R-R sector contains several p-form gauge potentials C_p with $p = 0, ..., 4$. The NS-NS sector is the same for both theories while in the R-R sector the IIA theory has odd p and the IIB theory has even p. The gauge potentials are again sourced by extended objects. The potential B_2 in the NS-NS sector couples electrically to a F1 strings and magnetically to a NS5 brane. In the R-R sector the IIA theory contains D0, D2, D4, D6 and D8 branes and the IIB theory has D1, D3, D5 and D7 branes.

The above theories are related by dimensional reduction and dualities. By reducing M-theory on a circle we get IIA superstring theory in ten dimensions. Also one can go from IIA to IIB superstring theory and back by performing a string theory duality known as T-duality. Thus although in various chapters of this thesis we will discussing solutions at the IIA, IIB or M-theory frame, we will always be referring to the same system and the choice of frame is mostly based on convenience.

2.2.2 A black hole in string theory and its entropy

Different combinations of strings and branes correspond to different solutions of string theory. At strong gravitational coupling these objects collapse and form black holes with the same charges as the original brane configuration. At zero gravitational coupling these configurations as simply a stack of branes and by counting the degeneracy of their quantum states Strominger and Vafa were able to count their entropy and show that it matches the Bekenstein-Hawking entropy of the relevant black hole [10]. A similar analysis was performed by Callan and Maldacena [11] which we will review here.

We use the IIB frame compactified on $T^4 \times S^1$ to describe a three charge black hole in five dimensions. To that end we consider D5 branes wrapped on $T^4 \times S^1$ and D1 branes on S^1. There are also strings stretching between the D1 and D5 branes which are given momentum

P along one direction in S^1. Each D-brane is a BPS solution of string theory and thus cuts the supersymmetries in half. It can also be shown that momentum along one direction also cuts the supersymmetries by half [4]. Thus a solution with charges D1, D5 and P is $\frac{1}{8}$-BPS.

After compactifying down to five dimensions the solution is simply the generalization of the extremal RN black hole (2.7) in five dimensions with $U(1)^3$ gauge symmetry. The metric is

$$ds^2 = -\left(Z_1 Z_2 Z_3\right)^{-2/3} dt^2 + \left(Z_1 Z_2 Z_3\right)^{1/3} \left(d\rho^2 + \rho^2 d\Omega_3^2\right) \,, \tag{2.21}$$

and the gauge potentials

$$A^I = Z_I^{-1} dt \,, I = 1, 2, 3 \,, \tag{2.22}$$

where

$$Z_I = 1 + \frac{Q_I}{\rho^2} \,, \tag{2.23}$$

are harmonic functions representing pointlike electric sources on $\mathbb{R}^4$. The solution is supersymmetric and saturates the BPS bound

$$M = Q_1 + Q_2 + Q_3 \,. \tag{2.24}$$

The near horizon region is $AdS_3 \times S_3$ and the horizon has topology S^3. By taking the near horizon limit $\rho << 1$ and setting ρ, t to be constant we find the horizon area to be

$$A = 2\pi^2 \sqrt{Q_1 Q_2 Q_3} \,, \tag{2.25}$$

[4] Momentum along opposite directions of S^1 would result in a non-extremal solution.

which after using the conventions of appendix A and the relation between the supergravity charges Q_I and the brane charges N_I gives us the entropy

$$S_{BH} = \frac{A}{4G_5} = 2\pi\sqrt{N_1 N_2 N_3}\,. \tag{2.26}$$

Since the solution is supersymmetric the existence of the states of the configuration (like their degeneracy) should be protected from quantum correction while varying the coupling. Thus the states are not renormalized. Consequently we should be able to reproduce (2.26) at zero gravitational coupling by examining the degeneracy of the ground states of the conformal field theory (CFT) living on the branes. The supersymmetric massless modes of the D1D5 CFT are living on the worldvolume of the strings stretching between D1 and D5 branes ($(1,5)$ and $(5,1)$ strings) along the common direction S^1. Thus the field theory lives on the cylinder (t, S^1). For each momentum mode the D1D5 CFT has $4N_1N_2$ bosons and $4N_1N_2$ fermions. Thus the central charge is

$$c = \frac{3}{2} 4N_1N_2 = 6N_1N_2\,, \tag{2.27}$$

and the degeneracy of states of level N_3 with large momentum P ($N_3 >> 1$) in the CFT is given by the Cardy formula [39]

$$d(N_3) \sim e^{2\pi\sqrt{\frac{cN_3}{6}}}\,. \tag{2.28}$$

After combining (2.27) with (2.28) and taking the logarithm of the latter we find the entropy of the D1D5P system to be

$$S_{CFT} = 2\pi\sqrt{N_1N_2N_3} \tag{2.29}$$

which exactly matches the Bekenstein-Hawking entropy (2.26).

This is one of the most remarkable results of string theory and reveals the statistical origin of the black hole entropy. However the result is established at zero gravitational coupling and does not provide information about the microscopic structure of black holes when gravity is present. Our purpose in the rest of this thesis is to move towards this direction and advance our understanding by providing relevant solutions.

Chapter 3

BPS microstates in five dimensions

In this chapter we will review results already available in the literature, which will form the backbone for the material presented in the rest of the thesis. Specifically we are going to set up the formalism for the study of three charge solutions in five-dimensional supergravity. We will also describe two and three charge microstate geometries with the same asymptotic charges as a black hole. A review for BPS solutions and microstate geometries in five dimensions can be found in [22] and the contents of this section are mostly taken from there. Microstate geometries are smooth, horizonless solutions that have the same asymptotic structure as black holes. We choose to present the material in inverse chronological order and present first the three-charge solutions. After having set the three-charge formalism one can easily go to the two-charge one by setting one of the charges to zero.

These microstate geometries provide examples of non-uniqueness in string theory and thus are important in understanding the mechanisms by which uniqueness can be violated. They resolve the black hole singularity through the mechanism of geometric transition, where the originally singular brane sources are replaced by charges dissolved in fluxes supported by non-trivial topology. The topological nature of the solutions is crucial in evading certain 'No-go' theorems which completely exclude non-singular soliton solutions that are regular in four spacetime dimensions [40]. We will see that, at least classically, there are infinite-dimensional families of such geometries and thus it is important to understand their implications to black hole physics.

These results indicate that microstate geometries reflect to the structure of black hole microstates at strong gravitational coupling and that the traditional black hole picture is an artifact of symmetry. They also in agreement with the fuzzball proposal and the idea that one needs $\mathcal{O}(1)$ corrections to the black hole geometry to resolved the information paradox. Under that light microstate geometries, although classical, imply that quantum gravity effects can extend all the way up to the black hole horizon instead of a Planck scale region around the singularity. For such an interpretation to be correct one would need to construct enough microstates to account for the black hole entropy. In general these microstates are not expected to have a geometric interpretation and thus microstate geometries offer a coarse-grained semiclassical description of the black hole phase space. The hope is that they adequately sample the phase space and provide at least the correct scaling behavior of the entropy with respect to the black hole charges.

For two-charge black holes we need $S \sim Q$ and the above goal has actually been achieved [12], [41], [42]. Supertubes , which we present at section 3.4, are smooth supergravity solutions and it has been shown that by quantizing all possible oscillation modes one can reconstruct the black hole entropy. However the issue with two-charge systems is that they have Planck scale horizon area and thus it is not clear that the supergravity approximation can be trusted. Also one can not clearly separate between the picture of microstates being horizonless configurations extending all the way up to where would be a horizon or Planck size corrections around the singularity [43].

Three-charge black holes have macroscopic horizons and thus the study of three-charge systems is of crucial importance in the support of the ideas of this program. As we mentioned earlier various microstate solutions have been constructed for BPS three-charge black holes and we present some of them in this chapter in section 3.3. However the counting is far from complete. For three charge systems one needs $S \sim Q^{3/2}$. The microstate geometries constructed so far account only for $Q^{5/4}$ [44, 45]. The entropy counting is expected to be completed by a conjectured object, the *superstratum*, which seems to provide the dominant semiclassical contribution to the entropy of three-charge systems [46]. The superstratum is a generalization of the supertube for three-charge geometries and some first steps towards its construction are the topic of chapter 7.

3.1 The view from eleven dimensions

Black hole systems in five dimensions can be better understood from eleven-dimensional supergravity compactified on a T^6. So the spacetime splits as $\mathcal{M}^{4,1} \times T^6$. For the three charge system we consider three sets of M2 branes wrapping three orthogonal T^2's inside the T^6. While preserving the same supersymmetries one can add three more sets of M5 branes such that they wrap a T^4 orthogonal to the T^2 of each of the M2 branes and they also form a closed curve $y^\mu(\lambda)$ inside the spatial part of the five-dimensional spacetime. The curve $y^\mu(\lambda)$ is the same for all of the three sets of M5 branes. The above are summarized in Table

Brane	0	1	2	3	4	5	6	7	8	9	10
M2	$\updownarrow$	$\star$	$\star$	$\star$	$\star$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$
M2	$\updownarrow$	$\star$	$\star$	$\star$	$\star$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$
M2	$\updownarrow$	$\star$	$\star$	$\star$	$\star$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$y^\mu(\lambda)$			$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$y^\mu(\lambda)$			$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$y^\mu(\lambda)$			$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$

Table 3.1: M-theory brane configuration for three charges. Branes are extended along the $\leftrightarrow$ directions and smeared along $\updownarrow$. The M5 branes also extend along a closed curve $y^\mu(\lambda)$ in the 4d spatial part of spacetime. A star, $\star$, means that a brane is smeared along the $y^\mu(\lambda)$ profile and is pointlike on the other three directions. Table taken from [22].

3.1. The M2 branes contribute to the electric charges of the system. The M5 branes as magnetic duals of M2 branes provide magnetic charges, but since they form a closed curve in the five-dimensional spacetime they generate magnetic dipoles and higher moments. The metric corresponding to this brane configuration can be written as

$$\begin{aligned} ds_{11}^2 = ds_5^2 \quad &+ \quad \left(Z_2 Z_3 Z_1^{-2}\right)^{\frac{1}{3}} (dx_5^2 + dx_6^2) \\ &+ \quad \left(Z_1 Z_3 Z_2^{-2}\right)^{\frac{1}{3}} (dx_7^2 + dx_8^2) + \left(Z_1 Z_2 Z_3^{-2}\right)^{\frac{1}{3}} (dx_9^2 + dx_{10}^2)\,, \end{aligned} \tag{3.1}$$

where the five-dimensional space-time metric has the form:

$$ds_5^2 \;\equiv\; -\,(Z_1 Z_2 Z_3)^{-\frac{2}{3}} (dt + k)^2 + (Z_1 Z_2 Z_3)^{\frac{1}{3}}\; h_{\mu\nu} dy^\mu dy^\nu\,, \tag{3.2}$$

for some one-form field, k, defined upon the spatial section of this metric. Also the Maxwell three-form potential is given by

$$C^{(3)} = A^{(1)} \wedge dx_5 \wedge dx_6 \;+\; A^{(2)} \wedge dx_7 \wedge dx_8 \;+\; A^{(3)} \wedge dx_9 \wedge dx_{10}\,. \tag{3.3}$$

Since we want the metric to be locally asymptotic to flat $\mathbb{R}^{4,1} \times T^6$, we require

$$ds_4^2 \;\equiv\; h_{\mu\nu} dy^\mu dy^\nu \,, \tag{3.4}$$

to limit to an asymptotically flat metric at spatial infinity and we require the warp factors, Z_I, to limit to constants at infinity.

Since we have three sets of M2 branes the system is 1/8-BPS and the supersymmetry, ϵ should be annihilated by the following three projection conditions

$$\big(\mathbb{1} \;-\; \Gamma^{056}\big)\,\epsilon \;=\; \big(\mathbb{1} \;-\; \Gamma^{078}\big)\,\epsilon \;=\; \big(\mathbb{1} \;-\; \Gamma^{09\,10}\big)\,\epsilon \;=\; 0\,. \tag{3.5}$$

For the Clifford algebra in eleven dimensions we also have

$$\Gamma^{0123456789\,10} \;=\; \mathbb{1}\,. \tag{3.6}$$

Then the conditions (3.5) imply that

$$\big(\mathbb{1} \;-\; \Gamma^{1234}\big)\,\epsilon \;=\; 0\,. \tag{3.7}$$

Spinors are also related to the holonomy of the metric

$$[\,\nabla_\mu\,,\,\nabla_\nu\,]\,\epsilon \;=\; \tfrac{1}{4}\,R^{(4)}_{\mu\nu cd}\,\Gamma^{cd}\,\epsilon\,, \tag{3.8}$$

where $R^{(4)}_{\mu\nu cd}$ is the Riemann tensor of the four-dimensional base metric.
The holonomy of a general Euclidean four-metric is $SU(2) \;\times\; SU(2)$. If the holonomy lies

only in one of those $SU(2)$ factors [1] the metric is hyper-Kähler. Then the Riemann tensor can be either self-dual or anti-self-dual depending on which of the two $SU(2)$s the holonomy lies. We have

$$R^{(4)}_{abcd} \;=\; \pm\,\tfrac{1}{2}\,\varepsilon_{cd}{}^{ef}\,R^{(4)}_{abef}\,. \tag{3.9}$$

If the Riemann tensor is self-dual then (3.8) vanishes identically as a consequence of (3.7). This implies that because of (3.7) all the components of supersymmetry on which the non-trivial holonomy would act, vanish. As we will discuss in chapter 5, choosing an anti-self-dual Riemann tensor leads to a specific class of extremal non-supersymmetric solutions known as "almost-BPS". The main point is that, in addition to the three sets of M2 branes, the base metric is also $\frac{1}{2}$-BPS and for the system to be supersymmetric there should be a common subset of supersymmetries preserved by all four objects.

Upon reducing eleven dimensional supergravity on T^6, for the specific brane configuration we obtain five dimensional $\mathcal{N}=2$ ungauged supergravity coupled to two vector multiplets. The bosonic sector of this theory has three $U(1)$ gauge fields [2] $A^{(I)}$ and two scalar fields. The action is given by

$$S=\frac{1}{2\kappa_5}\int\sqrt{-g}\,d^5x\Big(R-\tfrac{1}{2}Q_{IJ}F^I_{\mu\nu}F^{J\mu\nu}-Q_{IJ}\partial_\mu X^I\partial^\mu X^J-\tfrac{1}{24}C_{IJK}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda\bar{\epsilon}^{\mu\nu\rho\sigma\lambda}\Big), \tag{3.10}$$

with $I,J=1,2,3$. The scalars, X^I, satisfy the constraint $X^1X^2X^3=1$ and are related to

[1] such metrics are called "half-flat"

[2] one gauge field is the graviphoton which together with the metric form the gravity multiplet. The vector multiplets contain one gauge field and one scalar

Z_I via:

$$X^1 = \left(\frac{Z_2\, Z_3}{Z_1^2}\right)^{1/3}, \quad X^2 = \left(\frac{Z_1\, Z_3}{Z_2^2}\right)^{1/3}, \quad X^3 = \left(\frac{Z_1\, Z_2}{Z_3^2}\right)^{1/3}. \tag{3.11}$$

The matrix that defines the kinetic terms can be written as:

$$Q_{IJ} \;=\; \frac{1}{2}\,\mathrm{diag}\left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\right). \tag{3.12}$$

The $F^I = dA^I$ are the field strengths and $C_{IJK} = |\varepsilon_{IJK}|$ are the intersection numbers of $(1,1)$ forms defined on the compactification manifold (T^6 in our case). From (3.1) and (3.11) we see that the X^I's represent the volume of the each of the three T^2's and that the constraint between the X^I, which can more generally be written as

$$\frac{1}{6} C_{IJK} X^I X^J X^K = 1\,, \tag{3.13}$$

states that the volume of the compactification manifold (that is T^6 in our case) remains fixed. It is also important to observe that the action has a Chern-Simons term and thus can support solutions of non-trivial topology. It has actually been shown in [40] that solutions with non trivial topology are the only way to circumvent "no-go" theorems of general relativity about the existence of non-singular soliton solutions. [3]. The microstate solutions we will display later in this chapter will be exactly of that type. For the gauge fields we adopt the floating brane ansatz [48]

$$A^{(I)} \;=\; -\,Z_I^{-1}\,(dt + k) + B^{(I)}\,. \tag{3.14}$$

This ansatz relates the M2 brane electrostatic potentials with the metric functions Z_I and

[3]for a similar construction in eleven dimensions check [47].

guarantees extremality. Thus the electrostatic repulsion balances the gravitational attraction and the M2 branes float without feeling any force. The $B^{(I)}$ are one-forms on the four-dimensional base space and carry the dipole magnetic charges of the M5 branes. We further define the magnetic field strengths as

$$\Theta^{(I)} \equiv dB^{(I)} . \tag{3.15}$$

The most general supersymmetric configuration is then obtained by solving the BPS equations [49]:

$$\begin{aligned} \Theta^{(I)} &= \star_4 \Theta^{(I)} , && (3.16) \\ \nabla^2 Z_I &= \frac{1}{2} C_{IJK} \star_4 (\Theta^{(J)} \wedge \Theta^{(K)}) , && (3.17) \\ dk + \star_4 dk &= Z_I \Theta^{(I)} , && (3.18) \end{aligned}$$

where $\star_4$ is the Hodge dual taken with respect to the four-dimensional metric $h_{\mu\nu}$ and ∇^2 is the four-dimensional Laplacian. These equations solve the equations of motion of eleven-dimensional supergravity and are sequentially linear. The latter means that the above equations are a linear system if solved in the order presented above. Then the non-linear term of one equation can be realized as a source term by using the solution of the previous equation.

3.2 Gibbons-Hawking metrics

As we discussed in section 3.1 supersymmetry requires the four-dimensional base metric to be hyper-Kähler. Gibbons-Hawking metrics are the most general hyper-Kähler metrics with a tri-holomorphic U(1) isometry, which means that the $U(1)$ preserves all three complex structures of the hyper-Kähler metric. These metrics provide examples of asymptotically locally Euclidean (ALE) and asymptotically locally flat (ALF) spaces which are asymptotic to $\mathbb{R}^4/\mathbb{Z}_n$ and $\mathbb{R}^3 \times S^1$ respectively. ALF metrics, because of the compact direction, provide a smooth way to transition between a five-dimensional and a four-dimensional interpretation of the solutions. The general form of Gibbons-Hawking metrics is

$$h_{\mu\nu} dx^\mu dx^\nu \;=\; V^{-1} \left(d\psi + \vec{A} \cdot d\vec{y} \right)^2 \;+\; V \left(dx^2 + dy^2 + dz^2 \right), \tag{3.19}$$

where we write $\vec{y} = (x, y, z)$. The latter is a $U(1)$ fibration over a flat $\mathbb{R}^3$ base. The functions V and A depend only on the $\mathbb{R}^3$ coordinates. The function V is harmonic and is related to A by

$$\vec{\nabla} \times \vec{A} \;=\; \vec{\nabla} V \,. \tag{3.20}$$

It is also convenient to introduce a set of frames

$$\hat{e}^1 \;=\; V^{-\frac{1}{2}} \left(d\psi \;+\; A \right), \qquad \hat{e}^{a+1} \;=\; V^{\frac{1}{2}} \, dy^a \,, \quad a = 1, 2, 3 \,, \tag{3.21}$$

from which we construct two sets of two-forms:

$$\Omega_\pm^{(a)} \;\equiv\; \hat{e}^1 \wedge \hat{e}^{a+1} \;\pm\; \tfrac{1}{2} \, \epsilon_{abc} \, \hat{e}^{b+1} \wedge \hat{e}^{c+1} \,, \qquad a = 1, 2, 3 \,. \tag{3.22}$$

The two-forms, $\Omega_-^{(a)}$, are anti-self-dual, closed and non-normalizable and are the three Kähler forms of the base. The forms, $\Omega_+^{(a)}$, are self-dual and we will use them later.

The general form of V for multiple sources located on $\mathbb{R}^3$ at $\vec{y}_j$ is

$$V = h \ + \ \sum_{j=1}^{N} \frac{q_j}{r_j} \,, \tag{3.23}$$

where $r_j = |\vec{y} - \vec{y}_j|$. If we consider an axisymmetric configuation of charges located on the z-axis at $\vec{y}_j = (0, 0, a_j)$ we find

$$A = \sum_{j=1}^{N} \left(q_j \, \frac{r\cos\theta - a_j}{r_j} \ + \ c_j \right) d\phi \,, \tag{3.24}$$

where (θ, ϕ) are spherical polar coordinates and c_j is a constant related to the position of Dirac strings. By choosing c_j we can cancel the string along the positive or negative z-axis. The constant c_j can be absorbed in the $d\psi$ part of the metric (3.19). Usually one takes $q_j \geq 0$ so that the metric is always positive definite, but in a bit we will see that we can relax this condition.

Before we go any further it is useful to mention two very common metrics in their Gibbons-Hawking form. For $V = \frac{1}{r}$ the metric is that of flat $\mathbb{R}^4$ and for $V = h + \frac{q}{r}$ the metric is Taub-NUT. In general when $h = 0$ the metric is asymptotic to $\mathbb{R}^4/\mathbb{Z}_{q_T}$, with $q_T = \sum_{j=1}^{N} q_j$ and when $h \neq 0$ it is asymptotic to $\mathbb{R}^3 \times S^1$. For $q_T = 1$ and $h = 0$ the metric at infinity goes to $\mathbb{R}^4$.

When we approach one of the singular points $\vec{y}_i$ we have $V \sim \frac{q_i}{r_i}$ which looks like it is becoming singular. However at the vicinity of this point and after changing coordinates to

$r^2 = \frac{\rho^2}{4}$ the metric becomes

$$ds_4^2 \sim d\rho^2 + \rho^2 \, d\Omega_3^2 \,, \qquad (3.25)$$

where $d\Omega_3^2$ is the metric on $S^3/\mathbb{Z}_{|q_j|}$. This means that $q_j \in \mathbb{Z}$. If $|q_j| = 1$ the space looks locally like $\mathbb{R}^4$ but if $|q_j| \neq 1$ there is an orbifold singularity. These are benign in string theory and thus, we will consider such backgrounds as regular. Thus we see that at each point the ψ-fiber pinches off smoothly giving a non-singular metric.

We can now see that Gibbons-Hawking metrics are ideal for creating spacetime foam free of singularities and thus are good candidates as base metrics for microstate geometries. However if we want asymptotically the metric to be $\mathbb{R}^4$ the constraints we have imposed on the charges q_i are too stringent. Specifically we asked that $q_i \geq 0$, $q_i \in \mathbb{Z}$ and that $q_T = 1$. Obviously the only way that these conditions can be satisfied is if we have only one center with charge $q = 1$, that is if our space was $\mathbb{R}^4$ from the beginning. There is actually a theorem that states that any a regular, Riemannian, hyper-Kähler metric that is asymptotic to flat $\mathbb{R}^4$ is necessarily flat $\mathbb{R}^4$ globally. The way to go around this theorem is to allow the charges q_i to be negative so that the metric is ambipolar i.e. it can change signature from $(+,+,+,+)$ to $(-,-,-,-)$. The latter may seem unphysical, but we have to keep in mind that the Gibbons-Hawking metric is a base metric and not our spacetime metric. The complete spacetime metric (3.2) has warp factors Z_I which as we will see guarantee that the signature of the spacetime metric does not change.

As we discussed the ψ-fiber pinches off smoothly at each point $\vec{y}_i$ of the metric thus creating a series of homological two-cycles Δ_{ij}. This is depicted in Fig. 3.1. Because of

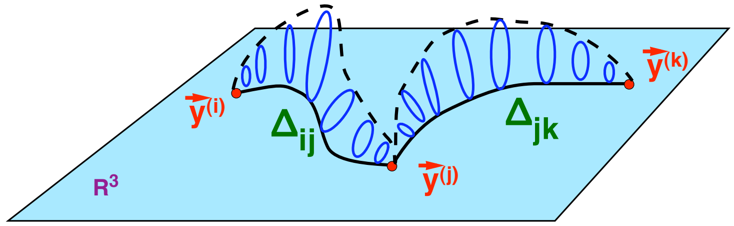


Figure 3.1: This figure is taken from [22]. It shows two non-trivial two- cycles of the Gibbons-Hawking geometry. The $U(1)$ fiber pinches off at the sources of the potential V and while sweeping across them creates homological two-cycles.

these homological two-spheres we can construct dual cohomological fluxes as

$$\Theta \;\equiv\; \sum_{a=1}^{3} \big(\partial_a \big(V^{-1}\, K\big)\big)\, \Omega_+^{(a)} \,. \tag{3.26}$$

This is a self-dual two-form (because of $\Omega_+^{(a)}$) and in order for it to be closed we need the function K to be harmonic, that is $\nabla^2 K = 0$. Thus a general solution for K is

$$K \;=\; k_\infty \;+\; \sum_{j=1}^{N} \frac{k_j}{r_j} \,. \tag{3.27}$$

In general one can distribute the sources k_j anywhere in the $\mathbb{R}^3$ base and that would correspond to generic solutions of black holes and black rings. However for non-singular solutions we need the fluxes Θ to be regular. Thus the positions of the charges of V should match with those of K and if $h = 0$ then one must also impose $k_\infty = 0$ for Θ to remain finite at infinity. The two-form Θ is also square integrable, that is it satisfies $\int \Theta \wedge \Theta < \infty$.

A local potential B such that $\Theta = dB$ is given by

$$B \;\equiv\; V^{-1}\, K\, (d\psi \;+\; A) \;+\; \vec{\xi} \cdot d\vec{y} \,, \tag{3.28}$$

with $\vec{\xi}$ being a vector potential for magnetic monopoles located at the singular points of K. Thus it satisfies

$$\vec{\nabla} \times \vec{\xi} \;=\; -\vec{\nabla} K \,. \tag{3.29}$$

In a similar fashion as in A we have

$$\xi = -\sum_{j=1}^{N} \left(k_j \, \frac{r \cos\theta - a_j}{r_j} \;+\; c_j \right) d\phi \,, \tag{3.30}$$

and it has Dirac strings. To complete expression for B (3.28) however should be free of Dirac-strings. To check that we consider what happens to B as we approach $\vec{y}_i$. Then the circles swept by ψ and ϕ are shrinking to zero size and the string singularities near $\vec{y}_i$ are of the form

$$B \;\sim\; \frac{h_i}{q_i} \left(d\psi + q_i \left(\frac{(z-a)}{r_i} + c_i \right) d\phi \right) \;-\; h_i \left(\frac{(z-a)}{r_i} + c_i \right) d\phi \;\sim\; \frac{h_i}{q_i} \, d\psi \,. \tag{3.31}$$

Hence the vector, $\vec{\xi}$, in (3.28) cancels the string singularities in the $\mathbb{R}^3$ and the singular components of B point along the $U(1)$ fiber of the Gibbons-Hawking metric. The magnetic flux through a cycle is given by

$$\Pi_{ij} \;\equiv\; \frac{1}{4\,\pi} \int_{\Delta_{ij}} \Theta \;=\; \left(\frac{k_j}{q_j} \;-\; \frac{k_i}{q_i} \right) . \tag{3.32}$$

We see that we have constructed fluxes with non singular sources which yet amount to a non-zero total flux through a Gaussian surface along a homological two-cycle. When we have singular sources, integrating over a small sphere $S^2_\epsilon \subset \mathbb{R}^3$ around the singular point $\vec{y}_j$ we get

$$\int_{\epsilon^2} \Theta \;=\; -\,4\pi k_j\,. \tag{3.33}$$

3.3 Solutions on a Gibbons-Hawking base

As we saw with a Gibbons-Hawking base there is a natural way to write down self-dual two-forms Θ that satisfy (3.16). Thus the BPS equations admit the following solution

$$\begin{aligned}
\Theta^{(I)} &= \partial_\alpha \left(V^{-1} K^I\right) \Omega_+^{(\alpha)}\,, & (3.34)\\
Z_I &= \frac{1}{2} C_{IJK} \frac{K^J K^K}{V} + L_I\,, & (3.35)\\
k &= \mu\,(d\psi + A) + \omega\,, & (3.36)\\
\mu &= \frac{1}{6} C_{IJK} \frac{K^I K^J K^K}{V^2} + \frac{1}{2V} \sum_I K^I L_I + M\,, & (3.37)
\end{aligned}$$

with the one-form ω satisfying the equation

$$\vec{\nabla} \times \vec{\omega} \;=\; (V \vec{\nabla} \mu \;-\; \mu \vec{\nabla} V) \;-\; V \sum_{I=1}^{3} Z_I\, \vec{\nabla} \left(\frac{K^I}{V} \right), \tag{3.38}$$

which can also be written as

$$\vec{\nabla} \times \vec{\omega} \;=\; V \vec{\nabla} M \;-\; M \vec{\nabla} V \;+\; \tfrac{1}{2}\, (K^I \vec{\nabla} L_I - L_I \vec{\nabla} K^I)\,. \tag{3.39}$$

Consequently the solution is specified in terms of the eight harmonic functions K^I, L_I, V and M. The integrability condition for (3.39) is the fact that the divergence of both sides vanish, because the functions K^I, L_I, M and V are harmonic on $\mathbb{R}^3$. The solution is also invariant under the following gauge transformations

$$
\begin{aligned}
K^I &\to K^I + c^I V\,, \\
L_I &\to L_I - C_{IJK}\, c^J K^K - \tfrac{1}{2} C_{IJK}\, c^J c^K V\,, \\
M &\to M - \tfrac{1}{2} c^I L_I + \frac{1}{12} C_{IJK} \left(V c^I c^J c^K + 3 c^I c^J K^K \right)\,,
\end{aligned}
\tag{3.40}
$$

where the c^I are three arbitrary constants. This gauge invariance exists for any C_{IJK}, not only for those coming from reducing M-theory on T^6 and we will present a generalization of that for four charges in Chapter 4.

3.3.1 Regularity of the solution

To guarantee the physical content of the solution one has to make sure it is free of pathologies like closed timelike curves (CTCs). One takes t to be constant and examines spacelike slices of the metric. From the compact T^6 directions we get the condition $Z_I Z_J > 0$. The rest of the metric can be arranged as

$$
ds_5^2|_{t=ct.} = \frac{I_4}{W^4 V^2} \Big(d\psi + A - \frac{\mu V^2}{I_4}\, \omega \Big)^2 + W^2 V \Big(r^2 \sin^2\theta d\phi^2 - \frac{\omega^2}{I_4} \Big) + W^2 V (dr^2 + r^2 d\theta^2)\,, \tag{3.41}
$$

where

$$
I_4 \equiv Z_1 Z_2 Z_3 V - \mu^2 V^2\,, \qquad W \equiv (Z_1 Z_2 Z_3)^{1/6}\,. \tag{3.42}
$$

Thus the absence of CTCs requires that the following conditions must be globally satisfied

$$
I_4 \geq 0\,, \qquad W^2 V \geq 0\,, \qquad I_4 r^2 \sin^2\theta d\phi^2 \geq \omega^2. \tag{3.43}
$$

In general one would have to numerically check the above conditions over the whole range of coordinates. From the last condition one it can be seen that at the poles $\theta = 0, \pi$ there is a danger for closed timelike curves unless ω vanishes at these points.

3.3.2 Smooth solutions

In this section we are going to examine smooth solutions with an ambipolar multi-centered Gibbons-Hawking metric. As we discussed these metrics allow us to write down multi-centered metrics with $\mathbb{R}^4$ asymptotics as well as non-zero fluxes sourced by non-singular sources. These constructions are examples of a geometric transition, a mechanism in string theory that allows singularities to be resolved. The main idea behind geometric transition is that if we consider a stack of branes that wraps a circle at weak coupling, as we increase the coupling the circle collapses but instead of forming a singularity the brane sources are replaced by topological cycles which now source the charges. In our construction the M5 branes wrap a closed curve $y(\lambda)$ inside $\mathbb{R}^4$ which is topologically a circle. These are replaced by fluxes on topological two-cycles (also known as bubbles), which in turn occur by nucleating additional centers with opposite charges $\pm q$ in the base metric. This for the case of a black ring is depicted in Fig. 3.2

For the functions Θ, Z_I and μ to be regular as we approach one of the centers ($r_j \to 0$) we need the sources of the harmonic functions K^I, L_I and M to coincide with those of V. That is

$$K^I = k^I_\infty + \sum_{j=1}^{N} \frac{k^I_j}{r_j}, \quad L^I = l^I_\infty + \sum_{j=1}^{N} \frac{l^I_j}{r_j}, \quad M = m_\infty + \sum_{j=1}^{N} \frac{m_j}{r_j}. \qquad (3.44)$$

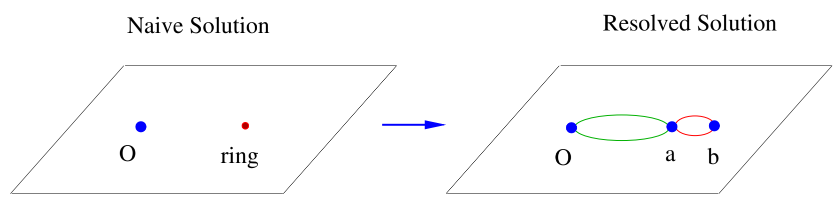


Figure 3.2: *Geometric transition for a black ring*: The singular black ring geometry is displayed on the left. This is resolved through a geometric transition by the geometry on the right where two bubbles occur by a pair of Gibbons-Hawking centers nucleated at a and b. The ring and the bubbles wrap the ψ-fiber of the base metric which in this picture is suppressed. Thus the ring looks like a point and the bubbles flat in the $\mathbb{R}^3$ part of base metric depicted here.

The total flux of $\Theta^{(I)}$, through the two-cycle Δ_{ij} is

$$\Pi^{(I)}_{ij} \;=\; \left(\frac{k^I_j}{q_j} - \frac{k^I_i}{q_i}\right), \qquad 1 \leq i,j \leq N\,. \tag{3.45}$$

To have asymptotically five-dimensional Minkowski spacetime we need $h = 0$ and thus $k^I_\infty = 0$ as well. Also we want $(Z_1 Z_2 Z_3)^{-1} \to 1$ and $\mu \to 0$ as $r \to \infty$. So,

$$l^1_\infty l^2_\infty l^3_\infty = 1\,, \qquad m_\infty = -\tfrac{1}{2}\, q_T^{-1} \sum_{j=1}^{N} \sum_{I=1}^{3} k^I_j\,. \tag{3.46}$$

Regularity at the centers

Furthermore canceling the singularities of Z_I and μ near the centers requires

$$l^I_j \;=\; -\tfrac{1}{2}\, C_{IJK}\, \frac{k^J_j\, k^K_j}{q_j}\,, \qquad j = 1, \dots, N\,, \tag{3.47}$$

$$m_j \;=\; \tfrac{1}{12}\, C_{IJK} \frac{k^I_j\, k^J_j\, k^K_j}{q_j^2} \;=\; \tfrac{1}{2}\, \frac{k^1_j\, k^2_j\, k^3_j}{q_j^2}\,, \qquad j = 1, \dots, N\,. \tag{3.48}$$

Since Z_I and μ are regular near the centers, as can be seen from (3.2) the absence of closed timelike curves requires $\mu \to 0$ at $r_j \to 0$. The same condition guarantees the absence of Dirac-strings from ω. The latter can be seen from (3.38) since μ, Z_I and K_I/V go to finite values at the limit $r_j \to 0$, the only term that can lead to Dirac strings is μdV. Thus from the requirement $\mu \to 0$ as $r_j \to 0$ we take the bubble equations which are necessary integrability conditions for the regularity of the solution.

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \Pi_{ij}^{(1)}\, \Pi_{ij}^{(2)}\, \Pi_{ij}^{(3)}\, \frac{q_i\, q_j}{r_{ij}} \;=\; -\,2\left(m_\infty\, q_i \;+\; \tfrac{1}{2}\sum_{I=1}^{3} k_i^I\right), \tag{3.49}$$

where $r_{ij} = |\vec{y}_i - \vec{y}_j|$ is the distance between two centers.

Regularity at the critical surfaces

Since the solution is ambipolar there are regions where $V < 0$ and $V = 0$ surfaces. Thus we have to check the metric and the fields of the solution in the neighborhood of $V = 0$. The tori warp factors contain the same power of Z_I s in the numerator and the denominator and thus are regular at $V = 0$. It is also easy to check from (3.41) that the five-dimensional part of the metric is regular at $V = 0$ since I_4, $(Z_1Z_2Z_3)^{1/3}V$ and μV^2 remain finite. Thus the metric and its inverse are regular at $V = 0$. For $V < 0$ we need again $I_4 \geq 0$ and $W^2V \geq 0$. That is satisfied since if we focus near $V = 0$ the warp factor W^2V is $\left(K_1^2K_2^2K_3^2\right)^{1/3}$ which is positive. Regarding the other fields of the solution it is simple to see that no pathologies are hidden in A_I and ω as $V = 0$. Actually for the gauge fields near $V = 0$ we get $A_I \sim 0$ for $I = 1, 2, 3, 4$.

3.4 Supertubes

In the previous section we set up the formalism that describes three-charge solutions. Here we are going to discuss about supertubes, which being a particular category of two-charge solutions can be described by the previous formalism by setting one of the charges to zero. In their original inception [50], [51] supertubes are realized as D-brane bound states in the IIA frame. The electric charges correspond to F1 strings and D0 branes dissolved into a tubular D2 brane with the support of angular momentum. The brane can form and arbitrary closed curve $\vec{F}(\sigma)$ in the target space and thus it gives rise to a dipole charge. Locally a supertube looks like a flat D2-brane and is 1/2-BPS. However globally the supersymmetry projectors depend on the direction of the curve $\vec{F}(\sigma)$ wrapped by the D2-brane and they have only 1/4-BPS supersymmetries in common.

Supertubes are smooth supergravity solutions at D1D5 frame [52, 53]. The have D1, D5 electric charges and a Kaluza Klein monopole (KKM) dipole magnetic charge. For this reason supertubes have been used to account for the entropy of two charge black holes. The entropy comes by counting the possible shapes $\vec{F}(\sigma)$ of the supertube profile and all other oscillation modes after quantization [12], [41], [42].

From the M-theory point of view (as shown in Table 3.1) the electric charges come from two M2 branes wrapping two orthogonal T^2's and the dipole charge from an M5 brane wrapping both of the previous T^2's. Also from the three gauge fields $A^{(I)}$ we keep only the one that appears in the three-form potential $C^{(3)}$ (3.3) with legs on the T^2 orthogonal to the other two T^2's of the M2 branes. Thus one can see that by choosing different combinations of T^2's one can construct three different species of supertubes. For example the harmonic functions for a supertube with an $A^{(3)}$ gauge field located at position $\vec{R}$ inside $\mathbb{R}^3$ are

$$
\begin{aligned}
&L_1 = l^1_\infty + \frac{Q_1}{4r_R}\,, \quad L_2 = l^1_\infty + \frac{Q_2}{4r_R}\,, \quad L_3 = l^3_\infty\,, \\
&K_1 = K_2 = 0\,, \quad K_3 = \frac{k_3}{r_R}\,, \quad M = m_\infty + \frac{m}{r_R}\,,
\end{aligned}
\tag{3.50}
$$

where $r_R = |\vec{y} - \vec{R}|$. Let us also assume that the supertube is in a Taub-NUT base space and thus $V = h + \frac{q}{R}$ where $R = |\vec{R}|$. Then the above functions describe a circular supertube which wraps the ψ-fiber.

To have $I_4 \geq 0$, since the electric potentials Z_I are not sourced at the origin we need $\mu \to 0$ as $r \to 0$. Thus

$$
m_\infty = -\frac{m}{R}\,. \tag{3.51}
$$

Also be requiring that ω has no Dirac strings we get

$$
2mV_R = l^3_\infty k_3\,, \tag{3.52}
$$

where $V_R = \left(h + \frac{q}{R}\right)$.

As we see the supertube solution is singular in five dimensions as we approach the limit $r_R \to 0$. Thus with a series of U-dualities let us switch to the IIB frame where the solution is regular [4].

The electric charges of the solution correspond to D1, D5 branes and momentum wrapping

[4] A display of the three charge system in various frames and of the appropriate U-dualities can be found in [44]

the following directions.

$$N_1 : D1(z) \quad N_2 : D5(5678z) \quad N_3 : P(z)\,, \tag{3.53}$$

and the dipole magnetic charges come from D1, D5 branes and KKM's as follows

$$n_1 : D5(y5678) \quad n_2 : D1(y) \quad N_3 : KKM(y5678z)\,, \tag{3.54}$$

where $z \equiv x_{10}$ and we compactified from eleven dimensions down to ten along the direction x_9. The metric is

$$\begin{aligned} ds^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}\left(dt+k\right)^2 + \sqrt{Z_1 Z_2}ds_4^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\left(dz + A^{(3)}\right)^2 \\ + \sqrt{\frac{Z_1}{Z_2}}\left(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2\right)\,. \end{aligned} \tag{3.55}$$

As we can see we have a compactification on T^4 and thus the solution is essentially six-dimensional. The direction z is an additional S^1 compared to the five-dimensional metric (3.2) and the gauge field $A^{(3)}$ fibered on it resolves the five-dimensional singularity. To establish regularity near the supertube we examine for potential singularities along the direction it wraps, that is the ψ-fiber. Thus collecting the $(d\psi + A)^2$ from (3.55) we get

$$\frac{1}{V^2\sqrt{Z_1 Z_2}}\left(Z_1 Z_2 V - 2\mu V K_3 + Z_3 K_3^2\right)\,, \tag{3.56}$$

which in combination with (3.52) for regularity as $r_R \to 0$ gives

$$m = \frac{Q_1 Q_2}{32 k_3} \, . \tag{3.57}$$

Equation (3.57) together with (3.52) fixes the location of the supertube in terms of its electric and dipole magnetic charges.

Finally it should be mentioned that supertubes in Taub-NUT are related to smooth multi-centered solutions in Taub-NUT through a spectral flow transformation [54]. Spectral flow from the six-dimensional perspective is nothing else but a coordinate transformation that mixes the periodic coordinate in the base (ψ in our case) with the Kaluza-Klein coordinate z. From the five-dimensional perspective this amounts to exchanging a singular two-charge solution (i.e. a supertube) with a smooth three-charge ambipolar one.

By using several spectral flows one can transform a singular five-dimensional solution consisting of multiple supertubes of different species, to a smooth, horizonless ambipolar one. A multiple species supertube solution is not smooth in six dimensions since by oxidizing the five-dimensional metric one can promote only one the gauge fields $A^{(I)}$ to a KKM (3.55). Thus only one species of supertubes can be regular at a time. However one can use spectral flow to transform this particular species to a smooth ambipolar solution. Then after a series of T-dualities one can uplift to M-theory, then reduce to IIA along a different circle and after another series of T-dualities end up in a D1D5P frame where now a different field $A^{(J)}$ acts as a KKM. As a result in this frame a different species of supertube becomes a smooth solution. Supertubes of the previous species are not affected by this process as they have become smooth ambipolar geometries. By repeating this process one can transform a solution with any species of supertubes into a smooth one. The cost of this procedure is that with every spectral flow the four-dimensional base metric becomes more complicated as one nucleates pairs of charges $\pm q$. For BPS solutions the base metric resulting after spectral flow is always of the Gibbons-Hawking type. For the almost-BPS solutions we will examine in chapter 5 one ends up with a more general electro-vac background [48, 55] .

Chapter 4

BPS geometries with four charges

The content of this chapter comes from [18] which is one of my sole author papers and is based on the results of [23].

4.1 Motivation

In the previous chapter we examined the three charged system and the relevant microstate geometries. Here, following similar steps, we are going to present the four charge system and the relevant smooth geometries associated with it. In addition to constructing microstate geometries with ambipolar Gibbons-Hawking spaces we examine a new regular supertube solution with three electric and one dipole magnetic charge that seems to exist in this framework.

The eleven-dimensional supergravity description of the four-charge system and its reduction to $\mathcal{N} = 2$ five-dimensional supergravity coupled to *three* vector multiplets was given in [23]. The authors motivated by the CFT analysis of [56] present a black ring solution with four charges[1]. The solution has more hair and, although it has four electric and four dipole magnetic charges, it is still 1/8-BPS. The similarity of the equations and the black ring solution between [23] and [22] makes it natural to provide a microstate description by using ambipolar Gibbons-Hawking metrics. As it was mentioned in [23], these microstates are dual to states of CFT with central charge different from the one of the D1-D5 CFT which is dual to the three-charge system. Finding the subset of microstates within the current formalism, that are dual to D1-D5 CFT might provide a hint for the additional required states to account for the black ring entropy.

4.2 The hairier BPS solution

4.2.1 General form of the solution

Once again we start be describing the system from eleven dimensions. The additional charge is encoded in the function Z_4. The eleven-dimensional metric is

$$\begin{aligned} ds_{11}^2 = &- \left(\frac{\alpha}{Z_1 Z_2 Z_3}\right)^{2/3} (dt+k)^2 + \left(\frac{Z_1 Z_2 Z_3}{\alpha}\right)^{1/3} ds_4^2 \\ &+ \alpha^{2/3} \left(Z_1 Z_2 Z_3\right)^{1/3} \left[\frac{dw_1 d\bar{w}_1}{Z_1} + \frac{dw_2 d\bar{w}_2}{Z_2} + \frac{dw_3 d\bar{w}_3}{\alpha Z_3} + \frac{Z_4}{Z_1 Z_2}(dw_1 d\bar{w}_2 + dw_2 d\bar{w}_1)\right] , \end{aligned} \tag{4.1}$$

where $\alpha = \left(1 - \frac{Z_4^2}{Z_1 Z_2}\right)^{-1}$ and ds_4^2 is a hyper-Kahler base metric.

[1] A general description of a black rings in $\mathcal{N} = 2$ supergravity with n vector multiplets has been given in [57, 58].

The three-form potential is as follows

$$A = \left(-\frac{\alpha}{Z_1}(dt+k)+B_1\right)\wedge\frac{dw_1\wedge d\bar{w}_1}{-2i}+\left(-\frac{\alpha}{Z_2}(dt+k)+B_2\right)\wedge\frac{dw_2\wedge d\bar{w}_2}{-2i}+ \\ \left(-\frac{1}{Z_3}(dt+k)+B_3\right)\wedge\frac{dw_3\wedge d\bar{w}_3}{-2i}+\left(-\frac{\alpha Z_4}{Z_1 Z_2}(dt+k)+B_4\right)\wedge\frac{dw_1\wedge d\bar{w}_2+dw_2\wedge d\bar{w}_1}{-2i}\,. \tag{4.2}$$

Alternatively we can write

$$A = A^1\wedge\frac{dw_1\wedge d\bar{w}_1}{-2i}+A^2\wedge\frac{dw_2\wedge d\bar{w}_2}{-2i}+A^3\wedge\frac{dw_3\wedge d\bar{w}_3}{-2i}+A^4\wedge\frac{dw_1\wedge d\bar{w}_2+dw_2\wedge d\bar{w}_1}{-2i}\,, \tag{4.3}$$

with the obvious definition of the gauge fields A^I, $I = 1,2,3,4$. Also

$$w_1 = x_5 - ix_6\,,\quad w_2 = x_7 + ix_8\,,\quad w_3 = x_{10} + ix_9\,, \tag{4.4}$$

are the complex torus coordinates. In terms of brane content, compared to the three-charge case presented in Table 3.1, we have an additional M2 brane along the directions x_6 and x_7 and an additional M5 brane on the T^4 orthogonal to those directions. The BPS equations that need to be solved are

$$\begin{aligned}
&\Theta_I = *_4\Theta_I\,,\ I = 1,2,3,4\,,\\
&d*_4 dZ_1 = -\Theta_2\wedge\Theta_3\,,\quad d*_4 dZ_2 = -\Theta_1\wedge\Theta_3\,,\\
&d*_4 dZ_3 = -\Theta_1\wedge\Theta_2+\Theta_4\wedge\Theta_4\,,\quad d*_4 dZ_4 = -\Theta_3\wedge\Theta_4\,,\\
&dk + *_4 dk = Z_1\Theta_1+Z_2\Theta_2+Z_3\Theta_3-2Z_4\Theta_4\,,
\end{aligned} \tag{4.5}$$

where $\Theta_I = dB_I$. As in the three-charge system the equations are sequentially linear.

By choosing a Gibbons-Hawking base metric the general form of the solution [23] is

$$
\begin{aligned}
&Z_1 = L_1 + \frac{K_2 K_3}{V}\,, \quad Z_2 = L_2 + \frac{K_1 K_3}{V}\,, \\
&Z_3 = L_3 + \frac{K_1 K_2 - K_4^2}{V}\,, \quad Z_4 = L_4 + \frac{K_3 K_4}{V}\,, \\
&k = \left(M + \frac{L_1 K_1 + L_2 K_2 + L_3 K_3 - 2 L_4 K_4}{2V} + \frac{(K_1 K_2 - K_4^2) K_3}{V^2} \right) (d\psi + A) + \omega\,, \\
&*_3\, d\omega = V dM - M dV + \frac{1}{2} \left(\sum_{i=1}^{3} K_i dL_i - L_i dK_i \right) - (K_4 dL_4 - L_4 dK_4)\;,
\end{aligned}
\tag{4.6}
$$

where we expanded $k = \mu(d\psi + A) + \omega$. Also

$$
\Theta^I = \sum_{a=1}^{3} \partial_a \left(\frac{K^I}{V} \right) \Omega_+^a\,,\; I = 1, 2, 3, 4\,, \tag{4.7}
$$

where Ω_+^a are the Gibbons-Hawking self dual two-forms (3.22). Closure of the Maxwell fields, $d\Theta^I = 0$, implies that the functions, K^I are harmonic

$$
\nabla^2 K^I = 0\,. \tag{4.8}
$$

Thus for the local potential B_I we find

$$
B_I = \frac{K_I}{V}(d\psi + A) + \vec{\xi}_I \cdot \vec{y}\,, \tag{4.9}
$$

where $\vec{\nabla} \times \vec{\xi}_I = -\vec{\nabla} K_I$, $\vec{y} = (x, y, z)$ and $\vec{\nabla}$ is with respect to $\mathbb{R}^3$.

The solution displayed above remains invariant under the following gauge transformations

$$\begin{aligned}
&K_I \to K_I + c_I V\,, \qquad I = 1,2,3,4\,,\\
&L_I \to L_I - C_{IJK}c^J K^K - \frac{1}{2}C_{IJK}c^J c^K V + \delta_{I,3}\left(2c_4K_4 + c_4^2 V\right)\,, \qquad I,J,K = 1,2,3\,,\\
&L_4 \to L_4 - c_3c_4V - K_3c_4 - c_3K_4\,,\\
&M \to M - \frac{1}{2}c^I L_I + \frac{1}{12}C_{IJK}\left(c^Ic^Jc^KV + 3c^Ic^JK^K\right)\\
&+ c_4L_4 - c_3c_4K_4 - \frac{1}{2}c_4^2K_3 - \frac{1}{2}c_4^2c_3V\,,\; I,J,K = 1,2,3\,,
\end{aligned} \tag{4.10}$$

where $C_{IJK} = |\epsilon_{IJK}|$ and c_I are constants.

4.2.2 Regularity

Regularity of the metric (4.1) from the tori gives

$$Z_2Z_3 > 0\,,\; Z_1Z_3 > 0\,,\; Z_1Z_2 > Z_4^2\,,\; Z_4Z_3 > 0\,. \tag{4.11}$$

These conditions guarantee that $\alpha > 0$.

By completing the square with respect to $d\psi$ the five-dimensional part of the metric becomes

$$ds_5^2 = \left(\frac{\alpha}{Z_1Z_2Z_3}\right)^{2/3}\frac{I_4}{V^2}\left(d\psi + A - \frac{\mu V^2}{I_4}(dt+\omega)\right)^2 + \left(\frac{\alpha}{Z_1Z_2Z_3}\right)^{-1/3}\frac{V}{I_4}\left(I_4 ds_3^2 - (dt+\omega)^2\right)\,, \tag{4.12}$$

where

$$
\begin{aligned}
I_4 &= \alpha^{-1} Z_1 Z_2 Z_3 V - \mu^2 V^2 \Rightarrow \\
I_4 &= \frac{1}{2} \sum_{I<J=1}^{3} K_I K_J L_I L_J - \frac{1}{4} \sum_{I=1}^{3} K_I^2 L_I^2 + V \left(L_1 L_2 - L_4^2 \right) L_3 \\
&+ \left(K_1 L_1 + K_2 L_2 - K_3 L_3 \right) K_4 L_4 - K_4^2 L_1 L_2 - K_1 K_2 L_4^2 \\
&- 2M \left(K_1 K_2 - K_4^2 \right) K_3 - MV \left(\sum_{I=1}^{3} K_I L_I - 2 K_4 L_4 \right) - M^2 V^2 \,.
\end{aligned}
\tag{4.13}
$$

The function I_4 is invariant under the gauge transformations (4.10).

The absence of CTC's requires

$$
(Z_1 Z_2 Z_3)^{1/3} V \geq 0 \,, \quad I_4 \geq 0 \,. \tag{4.14}
$$

Writing $ds_3^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$ it can be seen from (4.12) that at the poles $\theta = 0, \pi$ there is the additional danger of closed timelike curves unless ω vanishes at these points.

4.3 Bubbling

Our purpose is to resolve the singularity and construct smooth microstate geometries by allowing the multi-centered Gibbons-Hawking metric to be ambipolar. For the functions characterizing the solution we write the ansatz

$$\begin{aligned} V &= h + \sum_{j=0}^{N} \frac{q_j}{r_j}\,, \quad L^I = l^I_\infty + \sum_{j=0}^{N} \frac{l^I_j}{r_j}\,, \\ K^I &= k^I_\infty + \sum_{j=0}^{N} \frac{k^I_j}{r_j}\,, \quad M = m_\infty + \sum_{j=0}^{N} \frac{m_j}{r_j}\,, \end{aligned} \tag{4.15}$$

with $I = 1, 2, 3, 4$. We also set as $r_j = |\vec{y} - \vec{y}_j|$ the distance from the j^{th} center located at $\vec{y}_j$. The charges q_j can be positive or negative integers with the requirement that $\sum_{j=0}^{N} q_j = q_{tot} = 1$, for the space to be asymptotically Minkowski.

4.3.1 Flat asymptotics

To have five-dimensional Minkowski spacetime at infinity we need $h = 0$, which means that $k^I_\infty = 0$ as well. Also from the metric we see we must required that $\frac{\alpha}{Z_1 Z_2 Z_3} \to 1$ and $\mu \to 0$ as $r \to \infty$. So,

$$l^3_\infty \left(l^1_\infty l^2_\infty - (l^4_\infty)^2 \right) = 1\,, \quad m_\infty = -\frac{\sum_{j=0}^{N} \left(\sum_{I=1}^{3} l^I_\infty k^I_j - 2 l^4_\infty k^4_j \right)}{2 q_{tot}}\,. \tag{4.16}$$

An obvious choice for flat asymptotics is $l^2_\infty = l^3_\infty = l^4_\infty = 1$ and $l^1_\infty = 2$. The choice $l^1_\infty = l^2_\infty = l_{\infty_4} = 1$ and $l^4_\infty = 0$ might be interesting since it resembles the three-charge case (3.46), but as it can be seen from (4.16) and (4.20) it may also disentangle some interesting physical sectors which involve the fourth dipole charge k_4.

4.3.2 Regularity at the critical surfaces

Since the solution is ambipolar there are regions where $V < 0$ and $V = 0$ surfaces. Thus we have to check the metric and the fields of the solution in the neighborhood of $V = 0$.

The tori warp factors and the function α contain the same power of Z_I's in the numerator and the denominator and thus are regular at $V = 0$. It is also easy to check from (4.12) that the five-dimensional part of the metric is regular at $V = 0$ since I_4 , $(Z_1 Z_2 Z_3)^{1/3} V$ and μV^2 remain finite. Thus the metric and its inverse are regular at $V = 0$.

For $V < 0$ we need again $I_4 \geq 0$. That can be seen from (4.12), since if we focus near $V = 0$ the warp factor $\left(\frac{\alpha}{Z_1 Z_2 Z_3}\right)^{-1/3} V$ is $((K_1 K_2 - K_4^2)^2 K_3^2)^{1/3}$ which is positive.

Regarding the other fields of the solution from (4.2) and (4.6) it is simple to see that no pathologies are hidden in A_I and ω as $V = 0$. Actually for the gauge fields we get $A_I \sim 0$ for $I = 1, 2, 3, 4$ in the same manner as the previous chapter.

4.3.3 Regularity at the centers

We also have to check that the solution is regular as one approaches the Gibbons-Hawking centers $r_j \to 0$. In order to cancel the singularities in the functions Z_I we require

$$\begin{aligned} l_j^1 &= -\frac{k_j^2 k_j^3}{q_j}\,, \quad l_j^2 = -\frac{k_j^1 k_j^3}{q_j}\,, \\ l_j^3 &= \frac{(k_j^4)^2 - k_j^1 k_j^2}{q_j}\,, \quad l_j^4 = -\frac{k_j^3 k_j^4}{q_j}\,, \end{aligned} \tag{4.17}$$

and cancelling the singularities in μ requires,

$$m_j = \frac{k_j^3}{2q_j^2}\left(k_j^1 k_j^2 - (k_j^4)^2\right) . \tag{4.18}$$

The magnetic fields strengths Θ_I are regular since the singularities of the functions V and K_I coincide. This suggests that if $h = 0$ and our base space is flat $\mathbb{R}^4$ for Θ_I to be regular we should also have $k_\infty^I = 0$, $I = 1, 2, 3, 4$.

Since Z_I and μ are finite as $r_j \to 0$, then from (4.1) the absence of closed timelike curves requires $\mu \to 0$ at this limit. Also from (4.5) there is the danger of Dirac-strings in ω as there are $d\frac{1}{r_j}$ terms in the right hand side of the equation. Another way to write the equation for ω is:

$$*_3\, d\omega = V d\mu - \mu dV - V\left(\sum_{I=1}^{3} Z_I d\left(\frac{K^I}{V}\right) - 2Z_4 d\left(\frac{K^4}{V}\right)\right) . \tag{4.19}$$

Also, because μ, Z_I and K_I/V go to finite values at the limit $r_j \to 0$ the only term that can lead to Dirac strings in (4.19) is μdV. Thus the absence of both Dirac strings and closed timelike curves requires that $\mu \to 0$ as $r_j \to 0$. From this requirement we take the bubble equations which are necessary integrability conditions for the regularity of the solution.

$$\sum_{j=0, j\neq i}^{N}\left(\left(\left(\Pi_{ij}^1\Pi_{ij}^2 - (\Pi_{ij}^4)^2\right)\Pi_{ij}^3\right)\frac{q_i q_j}{r_{ij}}\right) = -2m_\infty q_i - \sum_{I=1}^{3} l_\infty^I k_i^I + 2l_\infty^4 k_i^4 . \tag{4.20}$$

where $\Pi_{ij}^I = \frac{k_j^I}{q_j} - \frac{k_i^I}{q_i}$ are the magnetic fluxes running through the two-cycle formed between the centers i and j and $r_{ij} = |\vec{y}_i - \vec{y}_j|$ the distance between them.

4.3.4 Solving for ω

For ω we find

$$\vec{\omega} = \frac{1}{4}\sum_{i,j=0}^{N} q_i q_j \left(\Pi^1_{ij}\Pi^2_{ij} - (\Pi^4_{ij})^2\right)\Pi^3_{ij}\vec{\omega}_{ij}\,, \tag{4.21}$$

where

$$\omega_{ij} = -\frac{x^2+y^2+(z-a+r_i)(z-b-r_j)}{(a-b)r_i r_j}d\phi_{ij}\,, \tag{4.22}$$

and we have set the z-axis along the two points i and j so that $\vec{y}_i = (0,0,a)$, $\vec{y}_j = (0,0,b)$ and $a > b$. The angle ϕ_{ij} is the azimuthal angle of the (i,j) coordinate system with z-axis passing through the points i and j. The functions ω_{ij} vanish along the z-axis and thus have no dirac string singularities. They satisfy the equation,

$$\vec{\nabla}\times\vec{\omega}_{ij} = \frac{1}{r_i}\vec{\nabla}\frac{1}{r_j} - \frac{1}{r_j}\vec{\nabla}\frac{1}{r_i} + \frac{1}{r_{ij}}\left(\vec{\nabla}\frac{1}{r_i} - \vec{\nabla}\frac{1}{r_j}\right)\,. \tag{4.23}$$

Taking $\vec{\nabla}\times\vec{\omega}$ and using (4.21) together with (4.23) and the regularity equations (4.20), (4.17), (4.18) we obtain the right hand side of the last equation in (4.6).

4.3.5 Asymptotic charges

The electric charges $\tilde{Q}_I$ are given by the asymptotic behaviour of the electric potentials Z_I at infinity as follows

$$Z^I \sim l^I_\infty + \frac{\tilde{Q}^I}{4r} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \quad r\to\infty\,. \tag{4.24}$$

Thus by expanding Z_I and making use of (4.15) and (4.17) we get

$$\begin{aligned}
\tilde{Q}^1 &= -4\sum_{j=0}^{N}\frac{\tilde{k}_j^2\tilde{k}_j^3}{q_j}\,, \quad \tilde{Q}^2 = -4\sum_{j=0}^{N}\frac{\tilde{k}_j^1\tilde{k}_j^3}{q_j}\,,\\
\tilde{Q}^3 &= -4\sum_{j=0}^{N}\frac{\tilde{k}_j^1\tilde{k}_j^2-(\tilde{k}_j^4)^2}{q_j}\,, \quad \tilde{Q}^4 = -4\sum_{j=0}^{N}\frac{\tilde{k}_j^3\tilde{k}_j^4}{q_j}\,,
\end{aligned} \tag{4.25}$$

where the quantities $\tilde{k}_j^I$ are invariant under the gauge transformations (4.10),

$$\tilde{k}_j^I = k_j^I - q_j\sum_{i=0}^{N}k_i^I\,, \quad I = 1,2,3,4\,. \tag{4.26}$$

As it was mentioned in [23] the D1-D5 CFT can be obtained by setting $\tilde{Q}_4 = 0$. It can be seen from (4.25) there is a variety of possibilities of achieving that without setting $k_j^4 = 0$. Each one of these choices though should be checked for consistency with the regularity constraints of the solution. Studies of the bubbled equations of the STU model suggest when one of the asymptotic charges is being set to zero the solution becomes pathological. However, the extra freedom of parameters of this model may allow such a choice.

The angular momentum can be derived from the expansion

$$k \sim \frac{1}{16r}\left((J_1+J_2)+(J_1-J_2)cos\theta\right)d\psi\,, \tag{4.27}$$

and we get

$$J_R = J_1 + J_2 = 8\sum_{j=0}^{N}\frac{(k_j^1k_j^2-(k_j^4)^2)k_j^3}{q_j^2}\,, \tag{4.28}$$

$$\vec{J}_L = \vec{J}_1 - \vec{J}_2 = \sum_{i,j=0,j\neq i}^{N} J_{L_{ij}} \,, \tag{4.29}$$

where

$$J_{L_{ij}} = -8 \left(\left(\left(\Pi^1_{ij}\Pi^2_{ij} - (\Pi^4_{ij})^2\right) \Pi^3_{ij} \right) q_i q_j \frac{(\vec{y}_i - \vec{y}_j)}{r_{ij}} \right) \,, \tag{4.30}$$

is the angular momentum flux vector associated with the ij^{th} bubble and in the derivation of it we used (4.20). Comparing with the already known bubbled geometries [22, 59] we see that most our results can be obtained from the old ones by making the substitution $\Pi^1_{ij}\Pi^2_{ij} \to \Pi^1_{ij}\Pi^2_{ij} - (\Pi^4_{ij})^2$ and $k_1k_2 \to k_1k_2 - k_4^2$. There is no need so make such a substitution for the electric charges as they have been dissolved into the magnetic fluxes (4.17).

4.4 A three charge supertube in IIB

We start with a solution with three electric charges Q_1, Q_2, Q_4 and one dipole magnetic charge k_3 which we call a three charge supertube. The solution has a tubular shape since it wraps around the Gibbons-Hawking fiber ψ. It can be directly obtained from the one in [23] by setting $Q_3 = 0$ and $k_1 = k_2 = k_4 = 0$. For more generality we are going to assume $V = h + \frac{q}{r}$. The functions describing the solution are as follows

$$\begin{aligned} &L_1 = l^1_\infty + \frac{Q_1}{4r_R}\,, \quad L_2 = l^2_\infty + \frac{Q_2}{4r_R}\,, \quad L_3 = l^3_\infty\,, \quad L_4 = l^1_\infty + \frac{Q_4}{4r_R}\,, \\ &K_1 = K_2 = K_4 = 0\,, \quad K_3 = \frac{k_3}{r_R}\,, \quad M = m_\infty + \frac{m}{r_R}\,, \end{aligned} \tag{4.31}$$

where $r_R = \sqrt{r^2 + R^2 - 2rR\cos\theta}$ and the supertube is positioned at distance R from the origin along the positive z-axis.

Since the electric potentials Z_I are not sourced at the origin, to keep $I_4 \geq 0$ as $r \to 0$ we need $\mu \to 0$ at this limit. Consequently

$$m_\infty = -\frac{m}{R} \,. \tag{4.32}$$

By canceling the remaining terms in the right hand side of the ω equation in (4.6) so that it looks like (4.23) we get the condition for the absence of closed timelike curves

$$2mV_R = l_\infty^3 k_3 \,, \tag{4.33}$$

where $V_R = \left(h + \frac{q}{R}\right)$.

Once again the supertube solution is singular in five dimensions. However it is smooth in six dimensions. Under a series of U-dualities [23] we can go to the IIB frame [2]. The metric in the IIB frame is

$$ds^2 = \frac{\alpha}{\sqrt{Z_1 Z_2}} \left(-\frac{1}{Z_3}(dt+k)^2 + Z_3 \left(dz - \frac{dt+k}{Z_3} + B_3 \right)^2 \right) + \sqrt{Z_1 Z_2} ds_4^2 + \sqrt{\frac{Z_1}{Z_2}} ds_{T^4}^2 \,, \tag{4.34}$$

where $z \equiv x_{10}$.

The ten-dimensional metric is split between between a six-dimensional part and four dimensions compactified in T^4. The six-dimensional metric can be obtained from the five-dimensional one by promoting one of the gauge fields to a Kaluza-Klein coordinate. This field is A^3 in our case and when (4.36) holds, resolves the singularity in five dimensions. To

[2] The U-dualities that take us from a nice M-theory frame to a D1D5 frame are different compared to the three charge case [44].

examine regularity along the supertube as $r_R \to 0$ we look for potential singularities along the fiber. Thus collecting all the $(d\psi + A)^2$ terms from (4.34) we obtain

$$\frac{\alpha}{V^2\sqrt{Z_1 Z_2}}\left(\frac{Z_1 Z_2 V}{\alpha} - 2\mu V K_3 + Z_3 K_3^2\right) . \tag{4.35}$$

Thus for regularity as $r_R \to 0$ and from 4.33 we need

$$m = \frac{Q_1 Q_2 - Q_4^2}{32 k_3} . \tag{4.36}$$

Equation (4.36) together with (4.33) guarantees the regularity of the solution and fixes the location of the supertube in terms of its electric and dipole magnetic charges. Once again, by substituting in (4.36) the combination $Q_1Q_2 - Q_4^2 \to Q_1Q_2$ we get the regularity condition the STU model supertube. Since we have already set $k_4 = 0$, to get a supertube dual to a D1-D5 CFT state we need $Q_4 = 0$ which takes us back to the STU model supertube. Supertubes correspond to unbound states in the dual CFT. Thus probably only bound states in the new CFT may lead us back to the D1-D5 sector. It would be interesting to find the combinations of generalized supertubes that achieve the latter.

4.5 Concluding remarks

Using the M-theory framework we have obtained microstate geometries corresponding to the black rings presented in [23]. The off-diagonal term in the supergravity gauge field gives an additional flux, Π^4_{ij}, in the bubble equations, which dissolves the fourth electric charge and resolves the singularity associated with it. A smooth supertube solution with three electric charges and one dipole magnetic charge has been shown to exist in this framework as well. Most of the old regularity equations of the three-charge system can be rewritten for the case of four charges by replacing the quadratic $X_1X_2 \to X_1X_2 - X_4^2$ where X_I some parameters of the solution. It is interesting that the fourth flux, Π^4_{ij}, couples only to Π^3_{ij}. This reflects the fact that even in the M-theory frame the extra $U(1)$ gauge field is not in equal footing with the previous three. This should be related to the geometry being 1/8-BPS in spite of having four charges. The quadratic $Q_X \equiv X_1X_2 - X_4^2$ also appears in [23] and originates from the intersection numbers C_{IJK} which occur after truncating eleven-dimensional supergravity down to five dimensions. We have,

$$C_{IJK} = |\epsilon_{IJK}| \ , \ I,J,K = 1,2,3 \quad , \quad C_{344} = -2 \tag{4.37}$$

with all the rest being zero. The cubic invariant factorizes into the quadratic Q_X as:

$$\frac{1}{6}C_{IJK}X^IX^JX^K = \left(X^1X^2 - (X^4)^2\right)X^3 = Q_XX^3 \tag{4.38}$$

The latter constraint defines the symmetric space $SO(1,1)\otimes(SO(1,2)/SO(2))$, which is the scalar manifold of $\mathcal{N}=2$ supergravity coupled to three vector multiplets . This space is one of the many possible scalar manifolds that occur after truncating eleven dimensional supergravity to five dimensions with $\mathcal{N}=2$ supersymmetry [60, 61]. By exploring further down this road, there may be more general families of 1/8-BPS black hole hair which allow further generalizations of the quadratic in terms of cubic and other symplectic invariants.

The algebraic similarities with the three-charge case are so many that it would be straightforward to perform the analysis done for the BPS case in [54, 62] and for the non-BPS case and in [17, 63]. The three charge supertube solution we presented makes use of the field A^3 as a Kaluza-Klein coordinate to oxidize the five-dimensional metric. Because of the symmetry that exists in the three-charge model it is trivial to oxidize the metric with any of the gauge fields A^I. We can try doing the same in this case and then find a connection between generalized supertubes and bubbled geometries by using spectral flow transformations [54] or some generalized version of them. This could lead to a larger family of microstate geometries being constructed. Then one can see how much entropy these solutions take by putting supertubes in ambipolar base spaces and exploring the entropy enhancement mechanism [62]. For the non-BPS case we can break supersymmetry by reversing the holonomy of the background with respect to the duality of the magnetic field strengths Θ^I and construct multicenter non-BPS solutions [63]. Then in the spirit of [17] it would be interesting to examine how the fourth charge affects the tolerance of the non-BPS microstate solution to supersymmetry breaking. There might be the case that one can use the fourth dipole charge to dilute the holonomy of the background, which breaks the supersymmetry, while keeping the values of the other electric and magnetic charges in a region that was previously excluded.

Finally, exploring the work of [56] it would be interesting to make the connection between the microstates at strong gravitational coupling and the states of the dual CFT. The four charge solutions are dual to a CFT with central charge [23] :

$$c \sim \tilde{Q}_1 \tilde{Q}_2 - \tilde{Q}_4^2 \tag{4.39}$$

This CFT is still unknown and we believe that this and subsequent work may shed more light towards its nature.

Chapter 5

Almost-BPS vs. BPS

In this chapter we present the material of [17] and some parts of [21] which refer to the generic structure of almost-BPS solutions. [17] is work I did together with Nicholas Warner.

5.1 Motivation

Much of the progress on microstate geometries has also centered around BPS solutions because the supersymmetry of such backgrounds greatly simplifies the equations of motion. However, there has also been significant progress in constructing large families of non-BPS extremal solutions using "Almost-BPS" and "floating-brane" techniques [48, 63–65]. These constructions use the simplifications provided by supersymmetric systems and yet break the supersymmetry in a very controlled manner, typically using the holonomy of a background metric. By using this approach one can readily reproduce, and then dramatically extend, the known extremal families of black-hole and black-ring solutions and one can find whole new families of solutions.

As yet, there are still rather few known non-BPS microstate geometries (some examples can be found in [66–68]). However, we believe that the systematic construction of non-BPS microstate geometries is, at present, primarily limited by the technical complexity of such solutions rather than by some strong physical limitation on their existence. On the other hand, it is a very interesting and important physical question to investigate whether the breaking of supersymmetry does lead to limitations on the solutions, or on the moduli space of such solutions. One of the purposes of this chapter is to study precisely this phenomenon for a very simple microstate geometry corresponding to an "Almost-BPS" black ring.

The simplicity of the supersymmetry-breaking mechanism in "Almost-BPS" solutions provides an ideal laboratory for studying this problem because one can start with supersymmetric configurations and then turn on the supersymmetry breaking very slowly. The microstate geometry that we will consider here will be a simple example of a "bubbled black ring," in which the original, singular charge sources have been replaced by smooth, cohomological fluxes supported on non-trivial cycles, or bubbles. This results in a smooth, horizonless geometry that looks exactly like a black ring until one gets very close to the would-be horizon where the geometry caps off smoothly. The bubbling process is, by itself, $\frac{1}{8}$-BPS, preserving four supersymmetries locally, but in the Almost-BPS solution these supersymmetries are broken by the holonomy of the background. We find that, as the supersymmetry breaking holonomy gets stronger in the vicinity of the bubbled black ring, the possible locations of the bubbles becomes progressively more limited and, in our simple example, one of the bubbles is required to become progressively smaller. We also see that if the supersymmetry-breaking holonomy becomes too large then a physical solution ceases to exist[1]. We also contrast this with the corresponding supersymmetric, BPS bubbled black ring and see that no such restrictions occur.

Thus we find that, compared to the BPS solution, the Almost-BPS object has a "gap" in its configuration space and the size of the gap increases with the strength of the supersymmetry breaking to a point where no physical solution exists.

While we are studying an extremely simple example in this chapter, there are broader conclusions for microstate geometries. The first and most evident is that supersymmetry breaking can restrict the configuration and moduli spaces. Second, the restriction emerges from a competition between the supersymmetry breaking scale, which manifests itself, in our example, through the curvature scale and the scale of the fluxes in the bubbled geometries. We find that the bubbles persist if the fluxes are large enough. Conversely, one might expect that if one could "dilute" the supersymmetry breaking scale then bubbles with smaller fluxes would persist. In our example, the supersymmetry breaking is generated by the holonomy of a Taub-NUT space and this supersymmetry breaking curvature can be diluted by passing to a multi-Taub-NUT. The requirement that some bubbles have to be small might also be interpreted as requiring a solution to form "locally supersymmetric" clusters of bubbles. The latter conclusions are rather speculative given the simplicity of our example but it does suggest some very interesting generalizations of our work here.

In Section 5.2 we outline the supergravity theory that we will study and give the equations that define a BPS and non-BPS systems of interest. In Section 5.3 we specialize to a Taub-NUT background with three supertubes and we discuss why and how this represents a

[1]Specifically, there is no solution that does not have closed time-like curves in the neighborhood of the bubbles.

geometric microstate of a black hole in four dimensions. We examine the regularity conditions and the requirements that there are no closed time-like curves (CTC's) and obtain the "bubble equations" or "integrability conditions" that constrain the supertube locations for both BPS and non-BPS solutions. In Section 5.4 we examine the *scaling solutions* in which the supertubes come very close together in the base geometry. This limit corresponds to the opening of a deep black-hole, or black-ring, throat in the physical geometry and as a result, in this limit the microstate geometry looks more and more like a black object. To clarify the differences between the BPS and non-BPS systems and analyze the moduli space of solutions in more detail, we make some further simplifications to the system of charges in the later parts of Section 5.4. We also solve the bubble equations in a flat space limit and exhibit two branches of solutions that will be part of our analysis of the more general solutions.

Sections 5.5 and 5.6 contain a careful analysis of the families of BPS and non-BPS solutions. We first linearize the bubble equations around the coincidence limit of the points and show how the two branches of solutions persist in both the BPS and non-BPS solutions. To find the additional restrictions imposed by the non-BPS system it is easier to reverse the usual approach and, instead of fixing charges, we fix the positions of the supertubes and solve the bubble equations for the charges. We find that there are always physically sensible charges that solve the BPS bubble equations for any supertube location. However, the non-BPS system has a non-trivial discriminant that restricts the locations of the supertubes: We find a "forbidden region," or "gap," in the non-BPS moduli space.

Section 5.7 contains our conclusions and discussion of the implications of our work. Some technical aspects as well as the asymptotic structure and charges of our solutions have been relegated to appendix B.

5.2 The supergravity equations

In this section we will describe the almost-BPS together with the BPS system. For the BPS case we are going to use a slightly different notation compared to Chapter 3 in order to be able to compare and differentiate with the almost-BPS case.

The simplest way to describe the solutions of interest is to work in M-theory, with the metric

$$\begin{aligned} ds^2_{11} &= -\left(Z_1 Z_2 Z_3\right)^{-\frac{2}{3}} (dt+k)^2 + \left(Z_1 Z_2 Z_3\right)^{\frac{1}{3}} \, ds_4^2 \\ &+ \left(Z_2 Z_3 Z_1^{-2}\right)^{\frac{1}{3}} (dx_5^2 + dx_6^2) + \left(Z_1 Z_3 Z_2^{-2}\right)^{\frac{1}{3}} (dx_7^2 + dx_8^2) + \left(Z_1 Z_2 Z_3^{-2}\right)^{\frac{1}{3}} (dx_9^2 + dx_{10}^2)\,, \end{aligned} \tag{5.1}$$

where ds_4^2 is a four-dimensional hyper-Kähler metric. The three-form potential is given by:

$$\mathcal{A} = A^{(1)} \wedge dx_5 \wedge dx_6 + A^{(2)} \wedge dx_7 \wedge dx_8 + A^{(3)} \wedge dx_9 \wedge dx_{10}\,, \tag{5.2}$$

where the Maxwell fields are required to obey the “floating brane Ansatz” [48]:

$$A^I \;=\; -\,\varepsilon\, Z_I^{-1}\,(dt+k) + B^{(I)}\,, \tag{5.3}$$

and where $\varepsilon \;= \pm 1$ and $B^{(I)}$ is a “magnetic” vector potential on the base, ds_4. We further define the field strengths:

$$\Theta^{(I)} \;\equiv\; dB^{(I)}\,. \tag{5.4}$$

It was shown in [48] that if one chooses the base to be merely Ricci-flat and solves the linear system of equations (for a fixed choice of $\varepsilon = \pm 1$):

$$\Theta^{(I)} = \varepsilon *_4 \Theta^{(I)} , \tag{5.5}$$

$$\widehat{\nabla}^2 Z_I = \tfrac{1}{2} \varepsilon C_{IJK} *_4 [\Theta^{(J)} \wedge \Theta^{(K)}] , \tag{5.6}$$

$$dk + \varepsilon *_4 dk = \varepsilon Z_I \Theta_I , \tag{5.7}$$

then one obtains solutions to the complete equations of motion of M-theory. In particular one obtains a BPS solution [49, 69, 70] by requiring that the duality structure of the $\Theta^{(I)}$ matches that of the Riemann tensor of the base. That is, one obtains BPS solutions by taking $\varepsilon = +1$ and choosing the base metric to be self-dual hyper-Kähler, or by taking $\varepsilon = -1$ and choosing the base metric to be anti-self-dual hyper-Kähler. Almost-BPS solutions [63–65] are also obtained by using hyper-Kähler base metrics but supersymmetry is broken by mismatching the duality structure of the $\Theta^{(I)}$ and that of the Riemann tensor of the base.

Here we will work with the Taub-NUT metric:

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V\,(dx^2 + dy^2 + dz^2) , \tag{5.8}$$

with

$$V = h + \frac{q}{r} , \tag{5.9}$$

where $r^2 \equiv \vec{y} \cdot \vec{y}$ with $\vec{y} \equiv (x, y, z)$. We will also frequently use polar coordinates: $x =$

$r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$ and $z = r\cos\theta$. The metric (5.8) is hyper-Kähler if

$$\vec{\nabla} V \;=\; \pm\, \vec{\nabla} \times \vec{A}\,, \tag{5.10}$$

where ∇ denotes the flat derivative of $\mathbb{R}^3$. For (5.9) we get

$$A = \;\pm\, q\, \cos\theta\, d\phi\,, \tag{5.11}$$

where we chose the integration constant to vanish. We use the orientation with $\epsilon_{1234} = +1$ and then choosing the positive sign in (5.10) results in a the Riemann tensor that is self-dual while choosing the negative sign makes the Riemann tensor anti-self-dual.

Henceforth within this chapter, we will follow the conventions of [17] and take

$$\varepsilon \;=\; +1\,, \tag{5.12}$$

and then the BPS solutions and almost-BPS solutions correspond to choosing the $+$ or $-$ sign, respectively, in (5.10) and (5.11).

The Riemann-squared invariant for this metric is:

$$R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma} \;=\; \frac{24\, h^2\, q^2}{(q + h\, r)^6}\,, \tag{5.13}$$

which measures the strength of supersymmetry breaking in the non-BPS solutions.

5.3 Some three-charge multi-supertube solutions in five dimensions

5.3.1 Supertubes as microstate geometries

We are going to consider a system of three different species of supertube in the simplest of hyper-Kähler base metrics, namely the Taub-NUT background. In our formulation, a "type I" supertube corresponds to allowing an isolated, singular magnetic source for $\Theta^{(I)}$ with singular electric sources in Z_J and Z_K (I, J, K all distinct) at the same location. Such a solution is, of course, singular in five dimensions. However, one can uplift this to the six dimensions using the vector potential, $A^{(I)}$, as a Kaluza-Klein field and the resulting geometry is then completely regular $D1$-$D5$ supertube geometry in the IIB duality frame (Section 3.4). The singular sources in five dimensions become smooth magnetic flux on a three-dimensional bubble in six dimensions.

The obvious problem is that one can only perform this Kaluza-Klein uplift with one species of supertube and thus a solution with three different species of supertube will always have singularities in six dimensions. Given that one can always resolve any single supertube singularity one might, quite reasonably, take the attitude that a multi-species supertube background should be considered to be a microstate geometry. On the other hand, there is better way to show that such a viewpoint is correct: One can use spectral flow [48, 54] to show that a multi-species supertube solution can be used to generate a physically equivalent and truly non-singular microstate geometry.

More specifically, once one has resolved one species of supertube by uplifting to six dimensions, one can perform a coordinate transformation in the six-dimensional solution and reduce back down to a *smooth* five-dimensional geometry in which the Taub-NUT space has been replaced by a more complicated base geometry. For BPS solutions this simply modifies the potential that appears in the Gibbons-Hawking base [54] but for non-BPS solutions the four-dimensional base is replaced by a generically more complicated electro-vac background [48]. The important point is that, from the five-dimensional perspective, spectral flow represents a highly non-trivial transformation of the solution space in which singular supertube configurations are replaced by smooth fluxes on new two-dimensional cycles in the four dimensional base space. Moreover, spectral flow can be done successively with each different species of supertube. The only cost is that with each spectral flow, the base geometry and fluxes become more complicated, particularly for non-BPS solutions. Indeed, the result of a sequence of three spectral flows of an Almost-BPS solution was obtained indirectly in [55] by performing six T-dualities.

Thus, the important point is that a multi-species supertube solution, while singular in five dimensions, can always be transformed into a *physically equivalent*[2] smooth, horizonless solution in five dimensions. For Almost-BPS systems, the resulting geometry is typically very complicated and so it is much easier to work with the multi-species supertube solution, as we will here, because it encodes the physically equivalent, albeit rather more complicated, true microstate geometry.

[2]It is equivalent because spectral flows are induced by coordinate transformations in six dimensions.

5.3.2 BPS and almost-BPS solutions

On a GH base space the magnetic field strengths are given by the two-forms:

$$\Theta^I_\pm \;\equiv\; -\sum_{a=1}^{3} (\partial_a P_\pm)\,\Omega^{(a)}_\pm\,, \tag{5.14}$$

are then harmonic if and only if

$$P_+ \;=\; V^{-1}\,K^I_+ \quad \text{or} \quad P_- \;=\; K^I_-\,; \qquad\qquad \nabla^2 K^I_\pm \;=\; 0\,, \tag{5.15}$$

where ∇^2 denotes the Laplacian on $\mathbb{R}^3$.

Then the vector potentials, $B^{(I)}_\pm$ in (3.14) are given by:

$$B^{(I)}_\pm \;=\; P^{(I)}_\pm(d\psi + A) \;+\; \vec{\xi}^{(I)}_\pm \cdot d\vec{y}\,, \qquad P^{(I)}_+ \;=\; V^{-1}\,K^I_+\,; \quad P^{(I)}_- \;=\; K^I_-\,, \tag{5.16}$$

$$\nabla^2 K^I_\pm \;=\; 0\,, \qquad \vec{\nabla}\times\vec{\xi}^{(I)}_+ \;=\; -\,\vec{\nabla} K^I_+\,, \qquad \vec{\nabla}\times\vec{\xi}^{(I)}_- \;=\; K^I_-\vec{\nabla}V - V\vec{\nabla}K^I_-\,, \tag{5.17}$$

where the $\pm$ corresponds to the choice in (5.10) and hence to BPS or almost-BPS respectively.

A "type I" supertube, $I = 1, 2, 3$, has a singular magnetic source for $B^{(I)}$ and singular electric sources for Z_J and Z_K, where I, J, K are all distinct. We study an axisymmetric supertube configuration with one supertube of each type on the positive z-axis and thus we take harmonic functions:

$$K^I_\pm \;=\; \frac{k^\pm_I}{r_I}\,, \tag{5.18}$$

where

$$r_J \;\equiv\; \sqrt{x^2 + y^2 + (z - a_J)^2}\,. \tag{5.19}$$

Without loss of generality we will assume that

$$a_1 \;>\; a_2 \;>\; a_3 > 0\,. \tag{5.20}$$

We then have

$$\xi_+^{(I)} \;=\; -\,k_I^+\,\frac{(r\cos\theta - a_I)}{r_I}\,d\phi\,, \qquad \xi_-^{(I)} \;=\; -\,\frac{k_I^-}{a_I\,r_I}\,\big[\,q\,(r - a_I\cos\theta) + h\,a_I\,(r\cos\theta - a_I)\,\big]\,d\phi\,, \tag{5.21}$$

and we may then write

$$B_+^{(I)} \;=\; \frac{K_+^I}{V}\,\Big[\,d\psi + \frac{q\,a_I}{r}\,d\phi - h\,(r\cos\theta -\; a_I)\,d\phi\,\Big]\,, \tag{5.22}$$

$$B_-^{(I)} \;=\; \frac{k_I^-}{a_I\,r_I}\,\Big[\,a_I\,d\psi - q\,r\,d\phi - h\,a_I\,(r\cos\theta - a_I)\,d\phi\,\Big]\,. \tag{5.23}$$

One can measure the local dipole strength by taking the integral of $\Theta^{(I)}$ over a small sphere, $S^2_\epsilon \subset \mathbb{R}^3$, around the singular point and one finds:

$$\int_{S^2_\epsilon} \Theta^{(I)} \;=\; \begin{cases} -4\,\pi\,k_j^+ & \text{for BPS} \\ -4\,\pi\,\big(h + \frac{q}{a_j}\big)\,k_j^- & \text{for almost-BPS} \end{cases} \tag{5.24}$$

For the two-sphere at infinity one has

$$\int_{S^2_\infty} \Theta_+ \;=\; -\,2\,\pi\,k_j^+\,, \qquad\qquad \int_{S^2_\infty} \Theta_- \;=\; +\,2\,\pi\,h\,k_j^-\,. \tag{5.25}$$

The difference between (5.24) and (5.25) arises because S^2_ϵ and S^2_∞ are not homologous and

Θ_- has a non-trivial flux through the non-compact 2-cycle running defined by (r, ψ) for $0 \leq r < \infty$.

Hence, the quantized dipole charges are

$$\widehat{k}_j \equiv \Big(h + \frac{q}{a_j}\Big)\, k_j^- \tag{5.26}$$

in the almost-BPS solutions.

It was found in [63] that this definition of dipole charges was also very natural because it was an essential step in bringing the expression for the horizon area, J_4, of non-BPS black rings into its canonical form.

For BPS supertubes one can write down the complete solution using the results of Section 3 and for non-BPS supertubes one can use the results in [63]. In particular, one has, for the BPS supertubes:

$$Z_1 = 1 + \frac{Q_2^{(1)}}{4\, r_2} + \frac{Q_3^{(1)}}{4\, r_3} + \frac{k_2^+\, k_3^+}{(h + \frac{q}{r})\, r_2\, r_3}\,, \tag{5.27}$$

$$Z_2 = 1 + \frac{Q_1^{(2)}}{4\, r_1} + \frac{Q_3^{(2)}}{4\, r_3} + \frac{k_1^+\, k_3^+}{(h + \frac{q}{r})\, r_1\, r_3}\,, \tag{5.28}$$

$$Z_3 = 1 + \frac{Q_1^{(3)}}{4\, r_1} + \frac{Q_2^{(3)}}{4\, r_2} + \frac{k_1^+\, k_2^+}{(h + \frac{q}{r})\, r_1\, r_2}\,. \tag{5.29}$$

while for the non-BPS supertubes one has:

$$Z_1 \;=\; 1 \,+\, \frac{Q_2^{(1)}}{4\,r_2} \,+\, \frac{Q_3^{(1)}}{4\,r_3} \,+\, \Big(h + \frac{q\,r}{a_2\,a_3}\Big)\,\frac{k_2^-\,k_3^-}{r_2\,r_3}\,, \tag{5.30}$$

$$Z_2 \;=\; 1 \,+\, \frac{Q_1^{(2)}}{4\,r_1} \,+\, \frac{Q_3^{(2)}}{4\,r_3} \,+\, \Big(h + \frac{q\,r}{a_1\,a_3}\Big)\,\frac{k_1^-\,k_3^-}{r_1\,r_3}\,, \tag{5.31}$$

$$Z_3 \;=\; 1 \,+\, \frac{Q_1^{(3)}}{4\,r_1} \,+\, \frac{Q_2^{(3)}}{4\,r_2} \,+\, \Big(h + \frac{q\,r}{a_1\,a_2}\Big)\,\frac{k_1^-\,k_2^-}{r_1\,r_2}\,. \tag{5.32}$$

The $Q_j^{(I)}$ define the local electric charge source of species I at point j. For the BPS solutions these also give the electric charges at infinity, but for the non-BPS solutions there is also a contribution from the dipole-dipole interaction term.

As usual one writes the Ansatz for the angular-momentum vector, k:

$$k \;=\; \mu\,(d\psi + A) \;+\; \omega \tag{5.33}$$

and one can solve (5.7) for μ and ω. The expressions for these functions are completely explicit and details may be found in Chapter 3 and [63]. For BPS solutions, μ is given by

$$\mu \;=\; \tfrac{1}{6}\,V^{-2}\,C_{IJK}\,K_+^I K_+^J K_+^K \;+\; \tfrac{1}{2}\,V^{-1}\,K_+^I L_I^+ \;+\; M_+\,, \tag{5.34}$$

where M_+ is another harmonic function which we will take to be

$$M_+ \;=\; m_\infty^+ \;+\; \frac{m_0^+}{r} \;+\; \sum_{j=1}^{3} \frac{m_j^+}{r_j}\,. \tag{5.35}$$

Thus for the system we are studying,

$$\mu \;=\; \frac{k_1^+ k_2^+ k_3^+}{r_1 r_2 r_3 V^2} \;+\; \frac{1}{2V}\left(\frac{k_1^+}{r_1}\left(1+\frac{Q_2^{(1)}}{4r_2}+\frac{Q_3^{(1)}}{4r_3}\right) \;+\; \frac{k_2^+}{r_2}\left(1+\frac{Q_1^{(2)}}{4r_1}+\frac{Q_3^{(2)}}{4r_3}\right)\right. \tag{5.36}$$

$$\left. +\; \frac{k_3^+}{r_3}\left(1+\frac{Q_1^{(3)}}{4r_1}+\frac{Q_2^{(3)}}{4r_2}\right)\right) \;+\; M_+\,. \tag{5.37}$$

For non-BPS solutions the expression for μ is rather more complicated:

$$\mu \;=\; \sum_I k_I^- \mu_I^{(1)} \;+\; h\sum_I\sum_{j\neq I}\frac{Q_j^{(I)}k_I^-}{4}\mu_{Ij}^{(3)} \;+\; q\sum_I\sum_{j\neq I}\frac{Q_j^{(I)}k_I^-}{4}\mu_{Ij}^{(5)} \tag{5.38}$$

$$+\; k_1^- k_2^- k_3^-\left(h^2\mu^{(6)}+q^2\mu^{(7)}+qh\mu^{(8)}\right) \;+\; \mu^{(9)}\,, \tag{5.39}$$

where, following [63], the $\mu^{(j)}$ are defined by:

$$\mu_I^{(1)} \;=\; \frac{1}{2r_I}\,, \qquad \mu_{Ij}^{(3)} \;=\; \frac{1}{2Vr_I r_j}\,, \qquad \mu_{Ij}^{(5)} \;=\; \frac{r^2+a_I a_j - 2a_I r cos\theta}{2Va_I(a_j-a_I)rr_I r_j}\,, \tag{5.40}$$

$$\mu^{(6)} \;=\; \frac{1}{Vr_1r_2r_3}\,, \qquad \mu^{(7)} \;=\; \frac{rcos\theta}{Va_1a_2a_3r_1r_2r_3}, \tag{5.41}$$

$$\mu^{(8)} \;=\; \frac{r^2(a_1+a_2+a_3)+a_1a_2a_3}{2Vra_1a_2a_3r_1r_2r_3}\,, \qquad \mu^{(9)} \;=\; \frac{M_-}{V}\,. \tag{5.42}$$

The remaining details of the solution, including the expressions for M_- and ω are given in Appendix B , where we discuss the connection of the three supertube solution with the one of a black ring.

5.3.3 Constituent charges

One of our primary purposes in this Chapter will be to compare "the same" BPS and non-BPS supertube configurations and the corresponding solution spaces. There are, however, two natural notions of being "the same:" one can either arrange to have the configurations made out of the same number and type of branes or one can arrange the configurations to have the same bulk charges measured at infinity. We will adopt the former perspective primarily because it seems more physically in keeping with the idea that we are taking some otherwise supersymmetric collection of branes and using the holonomy of the background to break the supersymmetry and then studying the effects on the physics of the solution. The charges and angular momenta measured at infinity will thus rather naturally depend upon the supersymmetry breaking process.

On a more practical level, we want to understand and elucidate the effects of supersymmetry breaking on the possible geometric transitions to bubbled microstate geometries. The system of equations and constraints is far simpler to understand in terms of local quantities whereas the asymptotic charges not only depend upon more complicated algebraic combinations of these local charges but also depend upon the geometric layout. Since we are going to find limitations on the geometric layout as a result of supersymmetry breaking, the study of the effects of supersymmetry breaking as a function of asymptotic charges becomes a formidably entangled problem. It is thus far easier to work with fixed constituent charges.

As we saw from equation (5.24), the local dipole charges are determined by k_j^+ for BPS supertubes and by $\hat{k}_j$ for non-BPS supertubes. We also noted that if one tries to compare BPS and non-BPS black rings then it is the "effective charges," $\hat{k}_j$, of the non-BPS object that replace the dipole charges, k_j^+, of the BPS object in canonical physical quantities like the horizon area [63]. More directly, the strength of the divergence of the Maxwell field measures the local *constituent* charge of the object in terms of the underlying branes and this is why these quantities naturally appear in formulae that determine horizon areas and entropies. Thus the same supertube configuration is obtained by fixing the $\hat{k}_j$'s of the non-BPS configuration to the same values as the k_j^+'s for the BPS configuration.

It is also evident from (5.25) that fixing the constituent charges results in different charges measured at infinity but, as we remarked in Section 5.3.2, this difference in charge is related to non-localizable fluxes through non-compact cycles. It is also worth noting that the relationship between local and asymptotic charges is just as much an issue for the electric charges because the BPS electric charges measured at infinity arising from (5.27)–(5.29) depend solely upon the $Q_j^{(I)}$ whereas the non-BPS electric charges measured at infinity arising from (5.30)–(5.32) involve dipole-dipole interactions and depend upon the geometric details.

Another advantage of fixing the constituent magnetic charges of a configuration is that the corresponding local electric charges are then easy to identify and fix. For example, for BPS supertubes, (5.27) shows that as $r_2 \to 0$ one has:

$$Z_1 \;\sim\; \frac{1}{4\,r_2}\left[\,Q_2^{(1)} + \frac{k_2^+\,k_3^+}{(h+\frac{q}{a_2})\,|a_2-a_3|}\,\right], \tag{5.43}$$

whereas for non-BPS supertubes (5.30) yields:

$$Z_1 \;\sim\; \frac{1}{4\,r_2}\left[\,Q_2^{(1)} + \Big(h+\frac{q}{a_3}\Big)\,\frac{k_2^-\,k_3^-}{|a_2-a_3|}\,\right] \;=\; \frac{1}{4\,r_2}\left[\,Q_2^{(1)} + \frac{\hat{k}_2\,\hat{k}_3}{(h+\frac{q}{a_2})\;|a_2-a_3|}\,\right]. \tag{5.44}$$

The electric charge thus has a pure source contribution, defined by the $Q_j^{(I)}$'s, and a part that comes from the magnetic dipole-dipole interactions. More importantly, the local electric charges arising from the magnetic dipole-dipole interactions are identical between BPS and non-BPS solutions if one fixes $\hat{k}_j$ of the non-BPS solutions to the values of k_j^+ in the BPS solutions. Thus the constituent charges of the BPS system are completely determined by $(Q_j^{(I)}, k_j^+)$ and by $(Q_j^{(I)}, \hat{k}_j)$ for the non-BPS system and it is these sets of charges that are to be identified in order to get "the same" underlying local configuration. In terms of string theory, this amounts to requiring the local brane constituents to be identical between BPS and non-BPS system.

5.3.4 Supertube regularity

Supertubes are not regular in five dimensions but a "type I" supertube can be made regular in six dimensions, in the IIB frame by taking its Maxwell field and realizing it in terms of geometry as a Kaluza-Klein field. For the type 3 supertube, the six-dimensional metric in IIB frame can be written as:

$$ds_6^2 \;=\; -\frac{1}{Z_3\sqrt{Z_1 Z_2}}\,(dt+k)^2 + \sqrt{Z_1 Z_2}\;ds_4^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\,(dz + A^{(3)})^2\,. \tag{5.45}$$

where $A^{(3)}$ is the gauge potential defined in (3.14). There are two things that need to be verified for supertube regularity: One must first ensure that there are no divergent terms along the ψ-fiber and then one must remove any CTC's associated with Dirac strings. Once one has done this the metric is completely regular at the supertube (Section 3.4).

The regularity conditions

Collecting all the $(d\psi + A)^2$ terms in (5.45):

$$(Z_1 Z_2)^{-\frac{1}{2}} V^{-2} \left[Z_3 \, (V P^3_\pm)^2 \; - \; 2\mu \, V^2 P^3_\pm \; + \; Z_1 Z_2 V \right] \, (d\psi + A)^2 \,, \tag{5.46}$$

where $P^3_+ \; = \; V^{-1} K^3_+$ for BPS supertubes and $P^3_- \; = \; K^3_-$ for non-BPS supertubes. For regularity as $r_3 \to 0$, one must have:

$$\lim_{r_3 \to 0} \, r_3^2 \left[\, Z_3 \, (V P^3_\pm)^2 \; - \; 2\mu \, V^2 P^3_\pm \; + \; Z_1 Z_2 V \, \right] \; = \; 0 \,. \tag{5.47}$$

To determine the condition for no Dirac strings it is simplest to look at the equation for ω. For BPS solutions one has:

$$\vec\nabla \times \vec\omega \; = \; (V \vec\nabla \mu \; - \; \mu \vec\nabla V) \; - \; V \sum_{I=1}^{3} Z_I \, \vec\nabla \left(\frac{K^I_+}{V} \right) , \tag{5.48}$$

while for non-BPS solutions one has:

$$\vec\nabla \times \vec\omega \; = \; - \vec\nabla (V\mu) \; + \; V \sum_{I=1}^{3} Z_I \, \vec\nabla K^I_- \,. \tag{5.49}$$

To avoid Dirac strings at $r_3 = 0$ one must have no terms that limit, as $r_3 \to 0$, to a constant multiplet of $\vec\nabla(\frac{1}{r_3})$ in the source on the right-hand side. This means that one must have

$$\lim_{r_3 \to 0} \, r_3 \left[\mu \; - \; Z_3 \, P^3_\pm \right] \; = \; 0 \,. \tag{5.50}$$

One can then use either (5.47), (5.50) or some combination of them, to fix m_3 in (5.35). The other, independent condition, is then most easily expressed by eliminating μ from (5.47) using (5.50) to obtain:

$$\lim_{r_3 \to 0} r_3^2 \left[V Z_1 Z_2 \; - \; Z_3 \, (V P_\pm^3)^2 \right] \; = \; 0 \,. \tag{5.51}$$

Regularity of the other supertubes in their IIB frames imposes conditions parallel to (5.50) and (5.51) as $r_1, r_2 \to 0$.

As was noted in [63], a rather technical calculation involving the explicit forms of μ shows that the m_j are fixed to be:

$$m_1^\pm \; = \; \frac{Q_1^{(2)} \, Q_1^{(3)}}{32 \, k_1^\pm} \,, \qquad m_2^\pm \; = \; \frac{Q_2^{(1)} \, Q_2^{(3)}}{32 \, k_2^\pm} \,, \qquad m_3^\pm \; = \; \frac{Q_3^{(1)} \, Q_3^{(2)}}{32 \, k_3^\pm} \,, \tag{5.52}$$

in both the BPS and non-BPS solutions. Regularity and the absence of Dirac strings at the origin imposes conditions on μ and thus fixes the parameters m_0 and m_∞ in the function M defined in (5.35) . For the BPS solution, regularity requires that μ be finite as $r \to 0$ and the absence of Dirac strings at the origin imposes the stronger requirement that $\mu \to 0$ as $r \to 0$. We therefore find that for the BPS solution we must impose:

$$m_0^+ \; = \; 0 \,, \qquad m_\infty^+ \; = \; - \sum_{i=1}^{3} \frac{m_i^+}{a_i} \,. \tag{5.53}$$

The result for the non-BPS solution is rather less edifying and its general form may be found in [63]. The details for the three supertube system will be given in Appendix B.

The bubble equations

Of central importance here are the other regularity conditions (5.51), and the similar conditions for $r_j \to 0$ in general, because these produce the bubble equations, or integrability conditions that constrain the locations of the supertubes in terms of the charges. Define the symplectic inner products, $\Gamma^+_{ij} = -\Gamma^+_{ji}$ and $\widehat{\Gamma}_{ij} = -\widehat{\Gamma}_{ji}$ by:

$$\Gamma^+_{12} \;=\; k^+_1\, Q^{(1)}_2 - k^+_2\, Q^{(2)}_1\,, \qquad \Gamma^+_{13} \;=\; k^+_1\, Q^{(1)}_3 - k^+_3\, Q^{(3)}_1\,, \qquad \Gamma^+_{23} \;=\; k^+_2\, Q^{(2)}_3 - k^+_3\, Q^{(3)}_2\,, \tag{5.54}$$

and

$$\widehat{\Gamma}_{12} \;=\; \hat{k}_1\, Q^{(1)}_2 - \hat{k}_2\, Q^{(2)}_1\,, \qquad \widehat{\Gamma}_{13} \;=\; \hat{k}_1\, Q^{(1)}_3 - \hat{k}_3\, Q^{(3)}_1\,, \qquad \widehat{\Gamma}_{23} \;=\; \hat{k}_2\, Q^{(2)}_3 - \hat{k}_3\, Q^{(3)}_2\,, \tag{5.55}$$

where the $\hat{k}_j$ are the effective dipole charges defined in (5.26). Then the bubble equations for the BPS supertubes may be written:

$$\frac{\Gamma^+_{12}}{|a_1 - a_2|} \;+\; \frac{\Gamma^+_{13}}{|a_1 - a_3|} \;=\; \frac{1}{4}\, \frac{Q^{(2)}_1\, Q^{(3)}_1}{k^+_1} \Big(h + \frac{q}{a_1} \Big) \;-\; 4\, k^+_1\,, \tag{5.56}$$

$$\frac{\Gamma^+_{21}}{|a_1 - a_2|} \;+\; \frac{\Gamma^+_{23}}{|a_2 - a_3|} \;=\; \frac{1}{4}\, \frac{Q^{(1)}_2\, Q^{(3)}_2}{k^+_2} \Big(h + \frac{q}{a_2} \Big) \;-\; 4\, k^+_2\,, \tag{5.57}$$

$$\frac{\Gamma^+_{31}}{|a_1 - a_3|} \;+\; \frac{\Gamma^+_{32}}{|a_2 - a_3|} \;=\; \frac{1}{4}\, \frac{Q^{(1)}_3\, Q^{(2)}_3}{k^+_3} \Big(h + \frac{q}{a_3} \Big) \;-\; 4\, k^+_3\,. \tag{5.58}$$

while the bubble equations for the non-BPS supertubes are:

$$\frac{\widehat{\Gamma}_{12}}{|a_1 - a_2|} + \frac{\widehat{\Gamma}_{13}}{|a_1 - a_3|} = \frac{1}{4} \frac{Q_1^{(2)} Q_1^{(3)}}{\hat{k}_1} \Big(h + \frac{q}{a_1} \Big) - 4 \hat{k}_1 - \epsilon_{123} \widehat{Y} , \tag{5.59}$$

$$\frac{\widehat{\Gamma}_{21}}{|a_1 - a_2|} + \frac{\widehat{\Gamma}_{23}}{|a_2 - a_3|} = \frac{1}{4} \frac{Q_2^{(1)} Q_2^{(3)}}{\hat{k}_2} \Big(h + \frac{q}{a_2} \Big) - 4 \hat{k}_2 - \epsilon_{213} \widehat{Y} , \tag{5.60}$$

$$\frac{\widehat{\Gamma}_{31}}{|a_1 - a_3|} + \frac{\widehat{\Gamma}_{32}}{|a_2 - a_3|} = \frac{1}{4} \frac{Q_3^{(1)} Q_3^{(2)}}{\hat{k}_3} \Big(h + \frac{q}{a_3} \Big) - 4 \hat{k}_3 - \epsilon_{312} \widehat{Y} , \tag{5.61}$$

where

$$\widehat{Y} \equiv \frac{4\, h\, q\, k_1^- k_2^- k_3^-}{a_1 a_2 a_3} = \frac{4\, h\, q\, \hat{k}_1 \hat{k}_2 \hat{k}_3}{(q + h a_1)(q + h a_2)(q + h a_3)} , \tag{5.62}$$

and

$$\epsilon_{ijk} \equiv \frac{(a_i - a_j)\,(a_i - a_k)}{|a_i - a_j|\,|a_i - a_k|} . \tag{5.63}$$

Thus the BPS and non-BPS bubble equations are nearly identical, except for the additional term defined by $\widehat{Y}$, if one replaces the k_j of the BPS configuration with the $\hat{k}_j$ of the non-BPS configuration. This is because the bubble equations depend upon the local constituent charges of the underlying branes.

It is also very interesting to note that $\widehat{Y}^2$ is proportional to the geometric mean of the curvature invariant, (5.13), evaluated at the supertubes. Thus $\widehat{Y}$ directly measures the strength of the supersymmetry breaking in the vicinity of the supertubes. Our purpose now is to see how this affects the space of solutions to these equations.

5.3.5 The Minkowski-space limit

If one sets $h = 0$ then the metric (5.8) becomes that of flat Minkowski space modded out by $\mathbb{Z}_q$. In this limit, the BPS and almost-BPS solutions must become the same because both solutions are necessarily BPS since the base is flat.

It is evident that the potential functions, (5.27)–(5.29) and (5.30)–(5.32), are identical for $h = 0$. The metric may then be written as

$$ds_4^2 \;=\; q\left[\frac{dr^2}{r} \;+\; r\left(\left(\frac{d\psi}{q}\right)^2 \;+\; d\theta^2 \;+\; d\phi^2 \;\pm\; \frac{2}{q}\cos\theta\, d\psi\, d\phi\right)\right], \tag{5.64}$$

which takes a more canonical form if one sets $r = \frac{1}{4}\rho^2$. The $\pm$ sign reflects the BPS, or almost-BPS description (5.11) but this is a coordinate artifact.

There are several obvious ways of mapping the BPS description of the metric onto the almost-BPS description but one can most easily see the correct choice by looking at the magnetic fields, $B_\pm^{(I)}$, for $h = 0$. Indeed, from (5.23) one sees that the substitution

$$\psi \;\to\; -\,q\,\phi\,, \qquad \phi \;\to\; \frac{1}{q}\,\psi\,, \qquad k_I^+ \;\to\; \widehat{k}_I \tag{5.65}$$

maps $B_+^{(I)} \to B_-^{(I)}$. This transformation also interchanges the choice of signs in (5.64).

In Appendix B we also verify that the transformation (5.65), for $h = 0$, also maps the angular momentum vector in the BPS description to that of almost-BPS description.

5.4 Scaling solutions

One chooses boundary conditions with Z_I going to constants at infinity so that the space-time is asymptotically flat and with a Taub-NUT background, the non-compact space time is $\mathbb{R}^{3,1}$, or four-dimensional, at large scales. In order for the microstate geometry to look like a black hole (or black ring) at larger scales, all the multi-centered parts of the solution must cluster to look like a concentrated object and around this cluster there must be an "intermediate region" in which the warp factors (or electrostatic potential functions), Z_I, behave as:

$$Z_I \sim \frac{Q_I}{r_c}\,, \tag{5.66}$$

where r_c is the radial coordinate measured from the center of the cluster. This intermediate region then defines the black-hole (or black-ring) throat and since the radial part of the metric behaves like $\frac{dr_c}{r_c}$ in this region, the distance diverges logarithmically as the cluster gets more and more tightly packed. In the intermediate region, the physical metric approaches that of $AdS_3 \times S^2$ and the area of the black-hole-like throat is then determined by the Q_I in (5.66) and is finite if all of the Q_I are non-zero. Thus an apparently singular coincidence limit in the base geometry is not singular in the full geometry. On the contrary, it represents the physically most interesting limit in which a finite-sized black-hole throat opens up and all the microstate details then cut off the throat and resolve the geometry at an arbitrarily depth set by the (small) size of the cluster.

For regular, bubbled geometries in five dimensions, the Z_I functions are finite everywhere and so there is a very important scaling limit that must be taken in order for the black-hole throat to open up in the proper manner [71, 72]. For the supertube backgrounds, the divergence of the Z_I at the supertubes, (5.27)–(5.32), automatically guarantees the correct behavior, (5.66), of the Z_I as one approaches a cluster. It is interesting to recall that spectral flow does not modify the physics of a solution and yet with three spectral flows [48, 54] one can convert all three supertube species into pure geometric bubbles in five-dimensions in which the Z_I's remain finite at the geometric centers. Thus the simple beauty of using three supertubes is that one gets the microstate geometry of a black hole of finite horizon area simply by arranging the clustering of the supertubes.

We therefore wish to study the scaling limit of our solutions and see when the two sets of bubble equations, (5.56)–(5.58) or (5.59)–(5.61), allow the supertubes to form an arbitrarily tight cluster.

5.4.1 Clustered supertubes

The first thing to note about the bubble equations is that if one adds them then the left-hand sides cancel and so the sum of the right-hand sides must be zero. For a cluster, one has $a_j \to R$ for some fixed R, and so one finds the "radius relation" for the cluster, which is to be identified with the radius relation of the black ring that asymptotically our three supertube system looks like. This determines the location, R, of the cluster in terms of the charges and for BPS solutions it is a simple linear equation in R:

$$\left[\frac{Q_1^{(2)}\, Q_1^{(3)}}{k_1} + \frac{Q_2^{(1)}\, Q_2^{(3)}}{k_2} + \frac{Q_3^{(1)}\, Q_3^{(2)}}{k_3} \right] \left(h + \frac{q}{R} \right) \;=\; 16\, (k_1^+ + k_2^+ + k_3^+)\,, \qquad (5.67)$$

while for the non-BPS solution one has[3]

$$\left[\frac{Q_1^{(2)} Q_1^{(3)}}{\hat{k}_1} + \frac{Q_2^{(1)} Q_2^{(3)}}{\hat{k}_2} + \frac{Q_3^{(1)} Q_3^{(2)}}{\hat{k}_3} \right] \Big(h + \frac{q}{R} \Big) \; = \; 16 \, (\hat{k}_1 + \hat{k}_2 + \hat{k}_3) \; + \; \frac{16 \, h \, q \, \hat{k}_1 \hat{k}_2 \hat{k}_3}{(q + hR)^3} \,, \tag{5.68}$$

with

$$\hat{k}_i \; \approx \; \Big(h + \frac{q}{R} \Big) \, k_j^- \,. \tag{5.69}$$

Whether one writes (5.68) in terms of k_j^- or $\hat{k}_j$, the non-BPS radius relation is a cubic in R.

It is also obvious that the left-hand sides of the bubble equations for clusters are potentially divergent while the right-hand sides are finite. This means that one must either have $\Gamma_{ij} \to 0$ or, more interestingly, one can have Γ_{ij} finite but with scaling arranged so that the divergences cancell on the left-hand sides of the bubble equations. We will focus on arrangements of charges that allow the latter but do not necessarily exclude the former. In particular, we will consider configurations for which

$$|a_1 - a_2| \; \sim \; \lambda \, \Gamma_{12} \,, \qquad |a_2 - a_3| \; \sim \; \lambda \, \Gamma_{23} \,, \qquad |a_1 - a_3| \; \sim \; - \lambda \, \Gamma_{13} \,, \tag{5.70}$$

for some *small* parameter, λ. In particular, for $\lambda > 0$, this means that $\Gamma_{12}, \Gamma_{23} > 0$ and $\Gamma_{13} < 0$. This means that for a scaling solution satisfying (5.70) one must have:

$$\Gamma_{12} \; + \; \Gamma_{23} \; + \; \Gamma_{13} \; = \; 0 \,. \tag{5.71}$$

There are other permutations that yield scaling solutions but our purpose here is not to make an exhaustive classification but to study the differences between BPS and non-BPS

[3]Note that for any ordering of points, the sum of the three ϵ_{ijk}'s defined in (5.63) is always 1.

solutions. We will thus make exactly the same assumptions for the $\widehat{\Gamma}_{ij}$ and for the ordering of non-BPS supertube configurations.

5.4.2 A simplified system

From our discussion is Section 5.3.3, to get the same supertube configurations we need to identify $(Q_j^{(I)}, k_j^+)$ for the BPS system with $(Q_j^{(I)}, \hat{k}_j)$ for the non-BPS system. To simplify things still further, we will take all the dipole charges to be exactly the same. This means we take $k_j^+ = d, j = 1, 2, 3$ for BPS solutions and $\hat{k}_j = d, j = 1, 2, 3$ for non-BPS solutions and treat d as a fixed dipole field strength in both instances. With this choice (5.71) becomes

$$Q_2^{(1)} \;-\; Q_1^{(2)} \;+\; Q_3^{(1)} \;-\; Q_1^{(3)} \;+\; Q_3^{(2)} \;-\; Q_2^{(3)} \;=\; 0\,, \tag{5.72}$$

for both the BPS and non-BPS systems.

While (5.72) means that there are five possible independent charges, we will keep things very simple by passing to the three-parameter subspace defined by:

$$Q_2^{(1)} \;=\; Q_2^{(3)} \;=\; \alpha\,, \qquad Q_3^{(1)} \;=\; Q_1^{(2)} \;=\; \beta\,, \qquad Q_1^{(3)} \;=\; Q_3^{(2)} \;=\; \gamma\,. \tag{5.73}$$

With this choice one has, for the BPS system, as $r \to \infty$:

$$Z_1 \;\sim\; \frac{\alpha + \beta}{4\,r}\,, \qquad Z_2 \;\sim\; \frac{\beta + \gamma}{4\,r}\,, \qquad Z_3 \;\sim\; \frac{\alpha + \gamma}{4\,r}\,, \tag{5.74}$$

which means that α, β and γ can be used to parametrize three independent electric charges at infinity. The corresponding asymptotics are a little more complicated for the non-BPS system but the conclusion is still the same.

With these choices, the bubble equations reduce to

$$\frac{\alpha-\beta}{a_1-a_2} - \frac{\gamma-\beta}{a_1-a_3} = \frac{1}{4}\frac{\beta\,\gamma}{d^2}\Big(h+\frac{q}{a_1}\Big) - 4 - Y\,, \tag{5.75}$$
$$-\frac{\alpha-\beta}{a_1-a_2} + \frac{\gamma-\alpha}{a_2-a_3} = \frac{1}{4}\frac{\alpha^2}{d^2}\Big(h+\frac{q}{a_2}\Big) - 4 + Y\,, \tag{5.76}$$
$$\frac{\gamma-\beta}{a_1-a_3} - \frac{\gamma-\alpha}{a_2-a_3} = \frac{1}{4}\frac{\beta\,\gamma}{d^2}\Big(h+\frac{q}{a_3}\Big) - 4 - Y\,, \tag{5.77}$$

where $Y=0$ for the BPS system and

$$Y = \widetilde{Y} \equiv \frac{4\,h\,q\,d^2}{(q+ha_1)(q+ha_2)(q+ha_3)} = \frac{\widehat{Y}}{d}\,, \tag{5.78}$$

for the non-BPS system. Note that for $\lambda>0$ the scaling conditions (5.70) are equivalent to $\gamma>\alpha>\beta$. As we noted earlier, the only difference between the BPS bubble equations and the non-BPS bubble equations is a source term related to the background curvature that is doing the supersymmetry breaking.

It is also useful to rewrite these equations by multiplying by the obvious common denominators. One the obtains

$$\begin{aligned}
-(\gamma-\alpha)\,a_1+(\gamma-\beta)\,a_2-(\alpha-\beta)\,a_3 &= (a_1-a_2)(a_1-a_3)\bigg(\frac{1}{4}\frac{\beta\,\gamma}{d^2}\Big(h+\frac{q}{a_1}\Big)-4-Y\bigg)\\
&= -(a_1-a_2)(a_2-a_3)\bigg(\frac{1}{4}\frac{\alpha^2}{d^2}\Big(h+\frac{q}{a_2}\Big)-4+Y\bigg)\\
&= (a_1-a_3)(a_2-a_3)\bigg(\frac{1}{4}\frac{\beta\,\gamma}{d^2}\Big(h+\frac{q}{a_3}\Big)-4-Y\bigg)\,.
\end{aligned} \tag{5.79}$$

5.4.3 The solution in flat, cylindrical geometry

To understand the branches of the solution, it is very instructive to start by solving the equations with $q = 0$ and subsequently study what happens as q is reintroduced. Setting $q = 0$ reduces the geometry of the base space to a flat cylinder, $\mathbb{R}^3 \times S^1$, where the radius of the circle is set by the value of h. The BPS and non-BPS bubble equations become identical and trivially solvable.

First one should note that because the cylinder is translationally invariant, it is only the differences $(a_i - a_j)$ that are going to have physical meaning. Taking the sum of the bubble equations yields

$$h \;=\; \frac{48\, d^2}{\alpha^2 + 2\,\beta\,\gamma}\,. \tag{5.80}$$

The radii of the supertubes are determined by their charges and this identity forces the supertube radii to match that of the S^1 of the cylinder.

The system of equations (5.79) has three branches of solution. Taking the difference between (5.75) and (5.77) one easily finds that there is a set of solutions that has a_2 mid-way between a_1 and a_3:

$$(a_1 - a_2) \;=\; (a_2 - a_3) \;=\; \frac{(\alpha^2 + 2\,\beta\,\gamma)}{8\,(\alpha^2 - \beta\,\gamma)}\,(2\,\alpha - \beta - \gamma)\,, \tag{5.81}$$

We call this the *symmetric branch.*

The other two branches allow the a_i to be placed at any position but the charges are constrained accordingly. That is, we can solve (5.79) for α, β and γ in terms of the a_i and we find either

$$\alpha \;=\; \beta \;=\; \gamma \;=\; \pm\,\frac{4d}{\sqrt{h}}\,, \tag{5.82}$$

or

$$\alpha = \pm\frac{4d}{\sqrt{h}}\,, \qquad \beta = \pm\frac{4d|a_1-a_2|}{\sqrt{h}|a_2-a_3|}\,, \qquad \gamma = \pm\frac{4d|a_2-a_3|}{\sqrt{h}|a_1-a_2|}\,. \tag{5.83}$$

We refer to the latter as the *geometric branch* because the ratios of charges are related to ratios of separations and because the charge α is the geometric mean of β and γ:

$$\alpha^2 - \beta\,\gamma \;=\; 0\,. \tag{5.84}$$

As we will see, the solution given by (5.82) extends, for $q \neq 0$, to a scaling solution with $\alpha \to \frac{1}{2}(\beta+\gamma)$ in the scaling limit, $a_i - a_j \to 0$. We will therefore refer to such solutions as the *arithmetic branch*. The three branches of solution obviously meet when all the points and charges coincide.

One should also recall that we required:

$$\gamma \;>\; \alpha \;>\; \beta\,, \qquad a_1 \;>\; a_2 \;>\; a_3\,, \tag{5.85}$$

Combining this with (5.83) one finds the additional condition

$$a_2 - a_3 \;>\; a_1 - a_2 \tag{5.86}$$

on the geometric branch. To arrange the supertube geometry so that $a_2 - a_3 < a_1 - a_2$ we just have to flip the signs in (5.70), which also flips the order of the charges $\gamma \;<\; \alpha \;<\; \beta$.

This transformation leaves (5.71) unaffected.

It is evident that from (5.82) and (5.83) that, for $q = 0$, the fluxes, Γ_{ij}, vanish on the arithmetic branch and remain finite on the geometric branch. More generally, for the scaling limit with $q \neq 0$, we will see in Section 5.6.2 that one typically has $\Gamma_{ij} \to 0$ on the arithmetic branch while Γ_{ij} remains finite on the geometric branch. In addition, one can only have $\alpha = \beta = \gamma$ if $q = 0$ or $q \to \infty$.

5.5 Linearizing the bubble equations

To define the linearization of the bubble equations, we first define the parameter, R, that sets the location at which all the supertubes would coinicide. This is given by (5.67) or (5.68) for the BPS and non-BPS systems respectively. These collapse to:

$$\frac{1}{4}\Big(\frac{\alpha^2 + 2\,\beta\,\gamma}{d^2}\Big)\Big(h + \frac{q}{R}\Big) \;=\; 12 \;+\; Y_0\,, \tag{5.87}$$

where $Y_0 = 0$ for the BPS system while for the non-BPS system one has:

$$Y_0 \;=\; \frac{4\,h\,q\,d^2}{(q + hR)^3}\,. \tag{5.88}$$

Having thus defined R, we now consider a solution that is very close to this limit point and introduce a small parameter, λ, that is *defined* by setting $a_2 = (1+\lambda)R$. One also assumes that $(a_i - a_j) \sim \mathcal{O}(\lambda)$. A quick examination of the bubble equations shows that at first order one must have $(a_1 - a_2) = \lambda\mu(\alpha - \beta)$ and $(a_2 - a_3) = \lambda\mu(\gamma - \alpha)$ for some parameter, μ. Thus we are led to the expansion:

$$\begin{aligned} a_1 &= (1+\lambda)\,R \;+\; \lambda\,\mu\,(\alpha-\beta) \;+\; \lambda^2\,\mu^2\,x_1\,, && (5.89)\\ a_2 &= (1+\lambda)\,R\,, && (5.90)\\ a_3 &= (1+\lambda)\,R \;-\; \lambda\,\mu\,(\gamma-\alpha) \;+\; \lambda^2\,\mu^2\,x_3\,, && (5.91) \end{aligned}$$

for some parameters, λ, μ, x_1 and x_3. One then finds that (5.79) is trivially satisfied at zeroeth and first order in λ and the first, non-trivial set of equations emerges at second order:

$$\begin{aligned} -(\gamma-\alpha)\,x_1-(\alpha-\beta)\,x_3 &= (\alpha-\beta)(\gamma-\beta)\Big(\frac{1}{4}\,\frac{\beta\,\gamma}{d^2}\Big(h+\frac{q}{R}\Big)-4-Y_0\Big)\\ &= -(\alpha-\beta)(\gamma-\alpha)\Big(\frac{1}{4}\,\frac{\alpha^2}{d^2}\Big(h+\frac{q}{R}\Big)-4+Y_0\Big)\\ &= (\gamma-\alpha)(\gamma-\beta)\Big(\frac{1}{4}\,\frac{\beta\,\gamma}{d^2}\Big(h+\frac{q}{R}\Big)-4-Y_0\Big)\,. \qquad (5.92) \end{aligned}$$

The equality of the three right-hand-sides of these equations gives two conditions, one of them is the defining "radius relation" for R, given by (5.87). For the BPS system, the second condition reduces to:

$$(\alpha-\beta)\,(2\alpha-\beta-\gamma)\,(\alpha^2-\beta\,\gamma) \;=\; 0\,, \qquad (5.93)$$

while for the non-BPS system it becomes:

$$(\alpha-\beta)\,(2\alpha-\beta-\gamma)\,\big((\alpha^2+\beta\,\gamma)\,(q+h\,R)-32\,R\,d^2\big) \;=\; 0\,. \qquad (5.94)$$

Both BPS and non-BPS systems admit the arithmetic branch, which naively involves setting $\alpha = \frac{1}{2}(\beta+\gamma)$, but remembering that we are making a series expansion, the arithmetic branch is defined more precisely by taking:

$$\alpha \;=\; \frac{1}{2}\,(\beta + \gamma) \;+\; \mathcal{O}(\lambda)\,. \tag{5.95}$$

The geometric branch involves taking

$$\alpha^2 \;=\; \beta\,\gamma \;+\; \mathcal{O}(\lambda)\,, \tag{5.96}$$

for the BPS system and

$$(\alpha^2 + \beta\,\gamma)\,(q + h\,R) - 32\,R\,d^2 \;+\; \mathcal{O}(\lambda) \;=\; 0\,. \tag{5.97}$$

for the non-BPS system.

For both systems, the right-hand side of (5.92) vanishes on the geometric branch and one has:

$$\frac{x_1}{x_3} \;=\; -\,\frac{(\alpha - \beta)}{(\gamma - \alpha)}\,, \tag{5.98}$$

One can then use the fact that, to leading order, the right-hand side of (5.92) vanishes to rewrite the condition (5.98) as

$$\alpha^2 \;-\; \beta\,\gamma \;=\; -\,\frac{32hqd^4}{(q + h\,R)^3} \;+\; \mathcal{O}(\lambda) \tag{5.99}$$

which shows, more explicitly, how the BPS and non-BPS geometric branches differ.

One can expand to higher orders and obtain expressions for λ and for how the $\mathcal{O}(\lambda)$ terms in (5.95), (5.96) or (5.97) relate to the values of x_1, x_3 and higher order corrections. We will not pursue this here but instead we will move on to discuss the restrictions placed upon the non-BPS space of solutions. To that end, we note that (5.97) is linear in R and is trivially solvable to yield:

$$R \;=\; \frac{q\,(\alpha^2+\beta\,\gamma)}{32\,d^2 - h\,(\alpha^2+\beta\,\gamma)} \,+\, \mathcal{O}(\lambda)\,. \tag{5.100}$$

One can now substitute this in the defining relation, (5.87), for R and the result is a complicated polynomial relationship between α, β, γ and d that is required for the non-BPS system. This replaces the simple constraint (5.96) on the charges that arises in the BPS system. The non-trivial constraint on the charges for the non-BPS system is a quartic in α^2 and $\beta\gamma$ and reduces to:

$$(\alpha^2+\beta\,\gamma)\,\big(h\,(\alpha^2+\beta\,\gamma) - 32\,d^2\big)^3 \;-\; 32768\,q^2\,d^4\,(\alpha^2-\beta\,\gamma) \;=\; 0\,. \tag{5.101}$$

We will not analyze this further here, but we will examine the solution space in more generality in the next section and we will show that there are very non-trivial constraints on the solution space for the non-BPS system.

5.6 The solution spaces in terms of charges

5.6.1 The quartic constraint

To find restrictions on the solution space and charges it turns out to be easier to reverse the usual perspective and treat the bubble equations (5.75)–(5.77) as constraints that fix α, β and γ for given locations, a_j, of the supertubes[4].

To do this most efficiently, one solves equations (5.75) and (5.77) for α and γ. Substituting this into (5.76) generates quartic in β that is required to vanish. This quartic is extremely complicated and details are given in Appendix B. On the other hand, in the scaling limit the quartic simplifies dramatically. To that end, we substitute

$$a_1 \;=\; a_3 \;+\; \lambda\, y_1\,, \qquad a_2 \;=\; a_3 \;+\; \lambda\, y_2 \tag{5.102}$$

and expand the quartic to leading order in small λ. We find that the result starts at $\mathcal{O}(\lambda^4)$ and is actually a quadratic in β^2: The β and β^3 terms actually vanish as $\mathcal{O}(\lambda^6)$. Dropping some overall factors, this quadratic, at leading order in λ, is:

$$\begin{aligned} P_4 \;&=\; (q+ha_3)^2\, y_2^2\, \beta^4 \;+\; 16\, a_3^2\, d^4\, (Y+4)^2\, (y_1-y_2)^2 \\ &\quad +\; 4\, a_3\, d^2\, (q+ha_3)\, \big((y_1^2+2\, y_1 y_2 - 2 y_2^2) Y - 4\, (y_1^2 - 2\, y_1 y_2 + 2\, y_2^2)\big)\, \beta^2 \end{aligned} \tag{5.103}$$

[4]It is partially for this reason that we chose to reduce the configuration to three independent charge parameters.

The discriminant of this quadratic is:

$$\Delta \;=\; 16\, a_3^2\, d^4\, (q + h a_3)^2\, y_1^2\, (4 - Y)\, \big(4\,(y_1 - 2\, y_2)^2 - (y_1^2 + 4\, y_1 y_2 - 4\, y_2^2)\, Y\big)\,. \qquad (5.104)$$

First observe that for $Y = 0$ one has

$$\Delta \;=\; 256\, a_3^2\, d^4\, (q + h a_3)^2\, y_1^2\, (y_1 - 2\, y_2)^2\,, \qquad (5.105)$$

which is a perfect square and hence the quadratic always has roots. Indeed, the quadratic has roots:

$$\beta^2 \;=\; \frac{16\, a_3\, d^2}{(q + h a_3)}\,, \qquad \beta^2 \;=\; \frac{16\, a_3\, d^2}{(q + h a_3)\, y_2^2}\, (y_1 - y_2)^2\,. \qquad (5.106)$$

For the non-BPS system one has, from (5.78):

$$Y \;=\; \widetilde{Y} \;=\; \frac{4\, h\, q\, d^2}{(q + h a_3)^3} \;+\; \mathcal{O}(\lambda)\,. \qquad (5.107)$$

Define

$$\mu \;\equiv\; \frac{y_2}{y_1}\,, \qquad \Lambda \;\equiv\; \frac{h\, q\, d^2}{(q + h a_3)^3}\,, \qquad f(\mu) \;\equiv\; \frac{(1 - 2\,\mu)^2}{(1 + 4\,\mu - 4\,\mu^2)}\,, \qquad (5.108)$$

then the discriminant is non-negative if either

$$\Lambda \;\leq\; 1\,, \qquad f(\mu) \;\geq\; \Lambda\,, \qquad (5.109)$$

or

$$\Lambda \;\geq\; 1\,, \qquad\qquad f(\mu) \;\leq\; \Lambda\,. \tag{5.110}$$

However, since this is a quadratic in β^2, one must also show that there are positive roots. Since the coefficient of β^4 and β^0 are non-negative, it follows that the roots of the polynomial are either both positive or both negative. The sign of these roots is determined by the sign of the coefficient of β^2. This coefficient may be written:

$$-\,2\,a_3\,d^2\,(q+ha_3)\,\big(y_1^2\,(4-Y)\;+\;\big(4\,(y_1-2\,y_2)^2-(y_1^2+4\,y_1y_2-4\,y_2^2)Y\big)\big)\,, \tag{5.111}$$

which is negative in the region defined by (5.109) and positive for (5.110). This means that the solutions for β^2 are *both* positive and hence there are four real roots in the region defined by (5.109) while (5.110) corresponds to four purely imaginary roots for β and is, therefore, unphysical. Since we want all the asymptotic charges to be positive, (5.74), will generically mean we want to focus on the solutions with $\beta > 0$.

Thus we conclude that the physical branches are defined by (5.109) and there are always two positive, real solutions for β in this range. Indeed, these solutions must correspond to the non-BPS extensions of the geometric and arithmetic branches. We therefore focus on this domain.

In order to preserve the proper ordering of the points, (5.85), one must have $0 < \mu < 1$ and in this range one has $0 < f(\mu) < 1$ however there are then “forbidden regions” for μ in

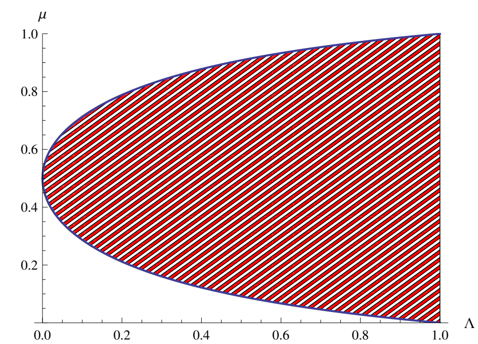


Figure 5.1: *This shows the possible locations of the non-BPS supertubes, as determined by $\mu \equiv y_2/y_1$, as a function of the supersymmetry-breaking parameter, Λ. The shaded area depicts the forbidden region.*

which (5.109) is not satisfied. These regions are defined by:

$$\frac{1}{2} - \sqrt{\frac{\Lambda}{2\,(1+\Lambda)}} \quad < \quad \mu \quad < \quad \frac{1}{2} + \sqrt{\frac{\Lambda}{2\,(1+\Lambda)}} \tag{5.112}$$

and they are depicted in Fig. 5.1. Inside these regions the values of β are all complex and no real, physical solution exists.

The simplest way to describe the moduli space of physical, non-BPS solutions is to start at $q = 0 \Rightarrow \Lambda = 0$ and slowly turn on the supersymmetry breaking parameter, q. From this we learn several things. First, the critical control parameter is d/ℓ, where ℓ is the curvature length scale at the location of the ring. (Flat space has $\ell = \infty$.) For a given ring dipole charge, the first inequality in (5.109) tells us that a solution will cease to exist if the curvature gets too strong at the ring. The second inequality then tells us that if the curvature is in the range that allows solutions then μ must either be close to zero or it must be close to 1 and $\mu = \frac{1}{2}$ is always excluded. In other words, for the non-BPS solution with intermediate curvatures, the middle point (a_2) must be close to one or other of the outer points (a_1 and

a_3) and as $\Lambda \to 1$ one must have $a_2 \to a_1$ or $a_2 \to a_3$.

Therefore, if one imagines turning on the curvature, or supersymmetry breaking, slowly from flat space by increasing q from zero, one starts from a situation where all points in the range $a_1 > a_2 > a_3$ are allowed but as the curvature increases the forbidden zone grows, driving a_2 to one or other end of the interval and then, when the curvature reaches a critical value, the scaling solutions cease to exist altogether. Supersymmetry breaking thus forces two of the supertubes to come together and merge if the supersymmetry breaking becomes strong enough. For our choice of scaling conditions (5.70), to keep the proper order of charges we need $a_2 \to a_1$ so that (5.86) is satisfied. Once again for the case $a_2 \to a_3$, the order of charges has to flip and so we also need to flip the signs in the scaling conditions (5.70).

5.6.2 Some numerical examples

To illustrate the foregoing analysis, we fix the supertube dipole moments and positions and fix the geometry at infinity and then examine the space of solutions for the charges α, β and γ as we vary q. Note that for $q = 0$ and for $q \to \infty$ the geometry is flat and hence supersymmetric. In both of these limits we expect the BPS and non-BPS solutions to coincide.

We choose the following values of parameters

$$d = 1\,, \quad h = 1\,, \quad a_3 = 1\,, \quad |a_2 - a_3| = 2 \times 10^{-8}\,, \quad |a_1 - a_2| = 1 \times 10^{-8}\,. \tag{5.113}$$

One should note this corresponds to having $\mu = 2/3$ in (5.108) and that this lies outside the forbidden region for small q. Indeed, this configuration lies in the forbidden region for

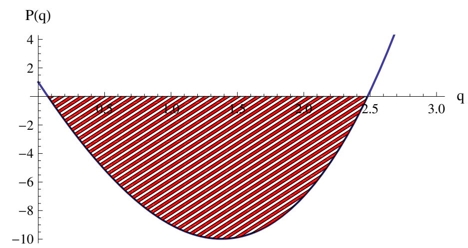


Figure 5.2: *The cubic inequality (5.114) forbids non-BPS solutions for a range of q determined by the shaded region of this graph.*

$\Lambda > \frac{1}{17}$ and with the choices (5.113), this corresponds to

$$P(q) \;\equiv\; (q+1)^3 \;-\; 17\,q \;<\; 0\,. \tag{5.114}$$

This cubic inequality is depicted in Fig. 5.2 and it forbids a range of q that is approximately given by $0.07258 < q < 2.4808$ [5]. The solution exists for smaller or larger values of q.

In Figures 5.3 and 5.4 we plot the solution for α, β and γ of the exact system of the bubble equations. We generate these solutions using the reduction of the bubble equations provided by (A.69) and (A.70).

The BPS and non-BPS solutions both exhibit arithmetic and geometric branches and, as one would expect, the corresponding BPS and non-BPS solutions are very similar when the NUT charge is very small or very large. On the arithmetic branch the charges seem to be identical but a closer examination reveals that the fluxes, Γ_{ij}, are very small, but non-zero. Indeed, near the region $q = 1$ for the BPS solution they are of the order 10^{-6}. We show this

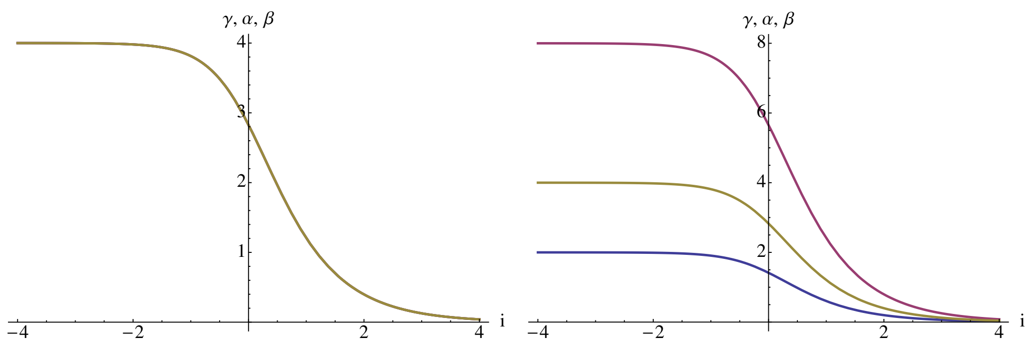
[5] The NUT charge, q, is, of course, quantized and is required to be a non-negative integer and so the intermediate real values of q are not physical. On the other hand, the detailed restrictions on q are a direct consequence of choosing $h = a_3 = 1$ and different choices of these parameters will scale the allowed ranges of q and these ranges can certainly be arranged to contain physically sensible, positive integer values for q on both sides of the inequality.

detail of the BPS arithmetic branch in Fig. 5.5. As one approaches $q \to 0$ or $q \to \infty$ the fluxes limit to exactly zero, consistent with (5.82).

For the non-BPS solutions we observe a "gap" in the solution space, corresponding to the forbidden region identified earlier. In this gap the values of the charges β and γ become complex. Somewhat surprisingly, the charge α, which is approximately the geometric or arithmetic average of the other two charges, remains real for all values of q. At leading order (5.95) and (5.96) then imply that β and γ are complex conjugates of one another but this is not true in general as can be seen at linear order from (5.98) and at all orders from (A.70).

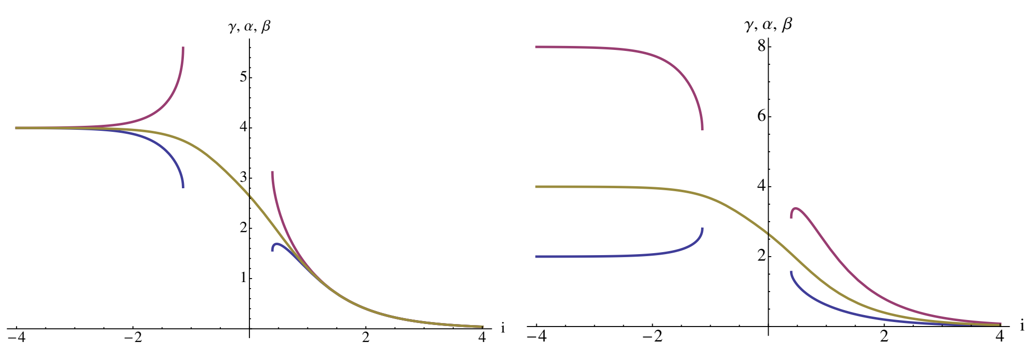
For completeness we also plot the discriminant of the full quartic in Fig. 5.6 and Fig. 5.7. The discriminant of a quartic with real coefficients is positive when either all roots are real or all roots are complex. The discriminant is negative when there are two real roots and a pair of complex conjugate roots. Fig. 5.6 shows that the BPS discriminant is always strictly positive. Since one has real values for the charges at $q = 0$, it follows that by continuously deforming q one will always have real roots. The non-BPS discriminant is never negative but there are two points where it vanishes. This is precisely where the four roots change from all being real to all being complex. The intermediate positive regime corresponds to the region where all roots are complex. The points where the discriminant vanishes coincide with the boundaries of the forbidden region and match the results from (5.114) within the given accuracy.

Finally we note that as $q \to \infty$ all the electric charges of the supertube go to zero. The fact that this must happen is an immediate consequence of the radius relations, (5.67) and (5.68). We have fixed the dipole moment and the location of the supertubes and as we increase q the circumference of the $U(1)$ fiber at a fixed location goes to zero. The only way such fixed supertubes can have a circumference that limits to zero is if the electric charges of the supertubes vanish as well.



(a) BPS Arithmetic Branch (b) BPS Geometric Branch

Figure 5.3: *The charges α, β, γ for the BPS system as a function of $i = log_{10}q$. These two graphs show the arithmetic and geometric branches of the solution. Note that the charges are almost identical on the arithmetic branch.*



(a) non-BPS Arithmetic Branch (b) non-BPS Geometric Branch

Figure 5.4: *The charges α, β, γ for the non-BPS system as a function of $i = log_{10}q$. Again we show the arithmetic and geometric branches of the solution. There is a gap in solution space where β and γ become complex. The arithmetic and geometric branches connect precisely at the gap.*

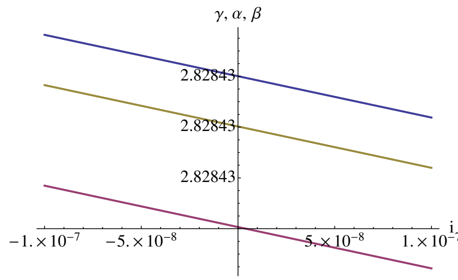
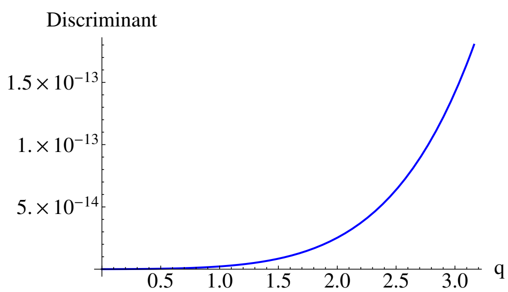
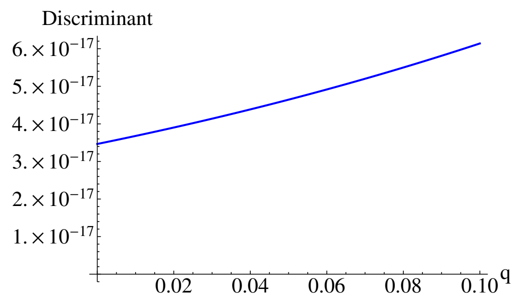


Figure 5.5: *A close-up of Fig. 5.3(a) near the region $i = 0$ ($q = 1$). Note that the fluxes are very small but non-zero.*



(a) BPS Discriminant vs. q

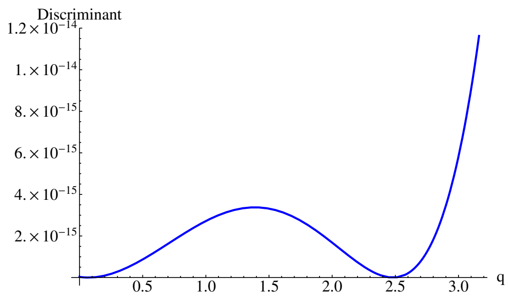


(b) BPS Discriminant detail near $q = 0$

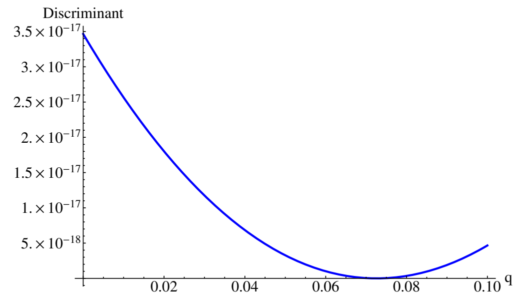
Figure 5.6: *The BPS discriminant is always positive.*

5.7 Concluding remarks

The non-BPS solutions obtained using the "floating brane" Ansatz have greatly enriched the families of known extremal solutions. The fact that these solutions are also determined by solving linear systems of equations has also enabled one to generate families of multi-centered solutions. To date, the majority of such solutions have involved black holes or multi-centered black rings, however, it is evident from the work presented here that one can use this approach to generate interesting families of non-BPS scaling microstate geometries. By using multiple species of supertubes one can generate scaling geometries whose appearance at large scales



(a) non-BPS Discriminant vs q



(b) non-BPS Discriminant detail near $q = 0$

Figure 5.7: *The non-BPS discriminant is non-negative but has two zeroes that define the edges of the forbidden region, or gap. These zeroes closely match the approximation given in Fig. 5.2.*

exactly matches that of a black hole or black ring and yet, via spectral flow, these solutions represent true microstate geometries in that they are smooth, horizonless and have a long AdS throat.

As we have shown here, one of the interesting new features of these non-BPS microstate geometries is that the breaking of supersymmetry places restrictions on the moduli space of solutions, compared to the analogous BPS solutions. If the supersymmetry breaking scale is large in the vicinity of the supertubes then a smooth solution does not exist and when the supersymmetry breaking is sufficiently small then there are still bounds upon the allowed moduli.

The restrictions on parameters that are imposed by supersymmetry breaking emerge in a very interesting manner: The equations of motion can be solved for all values of the parameters and the restrictions then emerge from the bubble equations, which require that there are no closed timelike curves near the supertube. While it has not been shown in general, there is good evidence that the bubble equations can be thought of as a condition of finding a local minimum of a potential for the supertube interaction. That is, the interactions between supertubes involve non-trivial forces between their magnetic dipoles and electric charges and the requirement of a static solution forces them to find an equilibrium configuration. More generally, a dynamic solution will involve some kind of oscillation around this equilibrium configuration[73, 74]. One way one can interpret the results presented here is that the supersymmetry-breaking holonomy modifies these forces and the potential and restricts the range of equilibrium configurations and indeed wipes out the minima if the supersymmetry breaking is too strong.

There is also a rather natural aspect to the restriction placed upon the bubbles, discussed in Section 5.6.1, that shows that supersymmetry breaking forces two of the supertubes to come together. We have described the supersymmetry breaking in terms of the supertube supersymmetry being broken by holonomy, but the supersymmetry breaking is a rather more democratic between the elements of the solution. One should recall that each supertube is a $\frac{1}{2}$-BPS state and that the Taub-NUT background is similarly a $\frac{1}{2}$-BPS state. Any three out of the four of these elements will preserve four supersymmetries and create a $\frac{1}{8}$-BPS state but the whole point of the non-BPS procedure is that all four elements, when put together, do not agree on which supersymmetries to preserve and thus the supersymmetry is thus completely broken and the control parameters are the separation of the different geometric elements. Thus the solution is essentially supersymmetric when there are only three elements involved in determining the background. Given a strong background holonomy, one can preserve some approximate supersymmetry locally by bringing two supertubes together while maintaining a much larger, finite separation from the third. Thus one can think of the restriction on the moduli space in terms of the solution trying to preserve approximate supersymmetry locally by forcing supertubes into clusters and excluding regions of moduli space in which all the supertubes are widely separated with large intermediate regions in which the supersymmetry is broken. Whether this perspective is born out in more general classes of solution remains to be seen but it is certainly worthy of further investigation.

More generally, even though we have studied an extremely simple example, it is very tempting to conjecture other implications for more general classes of non-BPS microstate geometries. It may be that the natural geometric elements of microstate geometries are $\frac{1}{2}$-BPS "atoms" that do not necessarily preserve any supersymmetry when combined together. The moduli space of such non-BPS solutions would then be restricted so that in each region some approximate supersymmetry would survive. This would necessitate that the "atoms" have strong enough local charges (like the dipole moment, d, in Λ in (5.108)) to open up a non-trivial moduli spaces in the presence of the other charges. This would also be balanced against the separation between the geometric elements. If one were to think of microstate geometries being nucleated in regions of high curvature then this would favor the initial formation of bubbles with larger dipole charges. As this happened, the nucleation of bubbles would involve diluting the curvature over a larger region thus enabling bubbles with smaller dipole moments to occur through either splitting of bubbles or nucleation. (In a Taub-NUT space, this curvature dilution would involve a transition to multi-centered Taub-NUT so that the background curvature would be distributed into a cloud around all the centers.) This is consistent with the belief, common to both the fuzzball proposal and ideas of emergent space-time, that the microstate structure of a black hole is not localized in a Planck-scale region around a classical singularity, but is, instead, smeared out over a region whose scale is set by the horizon area.

While the solutions we have considered here are far too simple to provide one with the general picture of non-BPS microstate geometries, we have shown that there are interesting families of non-BPS, scaling microstate geometries and we have shown how supersymmetry breaking can modify and restrict those families in very interesting ways that might naturally be characterized in terms of some locally approximate supersymmetry.

Chapter 6

Non-renormalization for almost-BPS

The contents of this chapter are taken from [21] which is work I did in collaboration with Iosif Bena, Andrea Puhm and Nicholas Warner.

6.1 Motivation

The physics of multi-center four-dimensional BPS solutions and of their five-dimensional counterparts [58, 59, 75–79] has been one of the keys that could unlock longstanding mysteries of black hole physics, such as the information paradox and the microscopic origin of the black-hole entropy. These solutions yield the easiest-to-construct black-hole microstate geometries [71, 72] and can be used to study the wall-crossing behavior of black-hole partition functions [80, 81]. They also provide the best-known examples of entropy enigmas [80, 82, 83].

Another key feature of BPS, multi-center solutions is that the equations controlling the positions of these centers, also known as the bubble equations, are not renormalized as one goes from weak to strong effective coupling. At weak coupling, these configurations are described by a supersymmetric quiver quantum mechanics (QQM) [76], and the equations determining the vevs of the QQM Coulomb-branch fields are the same as those determining the inter-center distances of the fully-back-reacted supergravity solution. This remarkable non-renormalization property has allowed one to compute the symplectic form, quantize the moduli space of supergravity solutions from the QQM perspective [84][1] and has given a clear mapping of some of the microscopic black-hole degrees of freedom to horizonless solutions that exist in the same regime of the moduli space as the classical black hole [85–87].

For BPS solutions one can also test this non-renormalization of the bubble equations by considering multi-center solutions in an intermediate region of the effective coupling, where some of the branes have back-reacted while others are treated as probes. More precisely, one can place a supertube in a multi-center solution and examine its supersymmetric minima. The equations that determine the positions of these minima can then be shown to be identical to the bubble equations [44]. Hence, one can recover the BPS bubble equations both at small effective coupling from the QQM, at intermediate coupling from the supertube DBI action, and at large effective coupling by asking that the fully back-reacted solution has no closed timelike curves.

Since the non-renormalization of the bubble equations is established by invoking quantities protected by supersymmetry, one might naively expect that the beautiful pieces of physics that are protected from renormalization in BPS systems would no longer survive in non-supersymmetric, multi-center solutions or in non-supersymmetric fuzzballs. The pur-

[1]This in turn has confirmed that quantum effects can wipe out macroscopically large regions of certain smooth low-curvature solutions [72, 84].

pose of this chapter is to demonstrate that, on the contrary, in certain classes of non-supersymmetric multi-center solutions - the so-called almost-BPS solutions [63–65] - the bubble equations are also protected when one goes from intermediate to strong coupling. This suggests that there are no quantum corrections despite the lack of supersymmetry.

This result is quite surprising because there is neither supersymmetry nor any other underlying symmetry that would prevent the bubble equations from receiving corrections and because the bubble equations of almost-BPS solutions contain complicated cubic combinations of the inter-center distances, whose coefficients do not *a priori* appear to be related to anything one can define in a quiver quantum mechanics. However, in retrospect, this result may not look so surprising: Emparan and Horowitz have shown in [88] that the entropy of certain extremal non-supersymmetric black holes can be calculated at weak effective coupling, and does not change as one increases this coupling. Since a certain subclass of almost-BPS solutions can be dualized to the black hole considered in [88], it seems plausible that whatever principle protects the quantum states calculated at weak coupling from being uplifted and disappearing at strong coupling also protects the bubble equations from receiving quantum corrections[2].

To establish that the almost-BPS bubble equations are not changing with the coupling, we consider an intermediate-coupling configuration where all the centers, except one, have back-reacted into an almost-BPS supergravity solution. One can then probe this solution with a supertube and use the (Dirac-Born-Infeld and Wess-Zumino) action to determine the equilibrium position of the probe. If one considers the back-reaction of the probe then

[2]It is also interesting to note that whatever underlying principle protects the states and bubble equations of almost-BPS system from being uplifted by quantum corrections can be even stronger than supersymmetry, which, for example, does not protect the degeneracy of the D1-D5-P system in the non-Cardy regime [83].

the supertube would become another center of the supergravity solution and its location is fixed by the bubble equations, which arise from requiring that there be no closed time-like curves. The comparison of these two physically distinct conditions, one in field theory on the brane and the other from its gravity dual, yields the test of non-renormalization between the intermediate and strong coupling regimes. At first glance, the equations look nothing like each other: both differ from the BPS bubble equations by extra cubic terms, but these cubic terms are not the same. However, the story is a bit more complicated: Unlike the BPS solution, the quantized electric charge of the supertube DBI action and the electric charge parameters in the supergravity harmonic functions are not the same. One can then ask whether there exists a relation between the DBI and the supergravity electric charges of the supertube that maps one set of bubble equations onto the other. One of the primary results of this chapter is to prove that such a relation exists, and write it down for the most general known almost-BPS solution in a single-center Taub-NUT space, involving an arbitrary collection of concentric supertubes and black rings.

The implications of this relation for the physics of almost-BPS solutions are quite significant. First, in a general supergravity solution that contains several supertubes, the quantized charges of these supertubes are not the obvious coefficients in the supergravity harmonic functions. The supergravity parameters are related to the quantized charges by certain shifts that come from dipole-dipole interaction and depend on the location of all the other centers of the solution. An interesting feature of these shifts is that they only depend on the dipole charges and positions of the centers that lie between location of the charge in question and the (supersymmetry-breaking) center of the Taub-NUT space.

For almost-BPS black rings the effect of the charge shift is more subtle because the black rings have more than one dipole charge, and hence the formulae that give the dipole-dipole contribution to the charge appear to be degenerate. However one can assemble a black ring by bringing together three supertubes with different kinds of dipole charges, and this will allow us to calculate the shifts between the black-ring supergravity and the quantized charges.

The second significant implication has to do with charge quantization, and solves a long-standing puzzle of the physics of almost-BPS black rings and of multi-center almost-BPS solutions. As one can see from [63] or from [17], the (quantized) asymptotic charge of such solutions is equal to the sum of the supergravity charges of the centers plus an extra dipole-dipole term that depends on the positions of the centers and the moduli of the solution. If the supergravity charges were equal to the quantized ones, this would have implied that the moduli-dependent contribution to the charges are also quantized, and hence multi-center almost-BPS solutions could only exist on special codimension-three slices of the moduli space (where the moduli-dependent contributions are integers). This would have been quite puzzling. The results presented here show that this does not happen. Upon using our formulas that relate the supergravity and the quantized charges, one finds that the asymptotic charge of the almost-BPS black ring is equal to the sum of the quantized charges of its centers, and hence the almost-BPS black ring as well as other multi-center almost-BPS solutions exists for any values of the moduli. If one keeps the quantized charges fixed and changes the moduli, the quantities that change are simply the supergravity charge parameters of the centers. Our result also implies that the $E_{7(7)}$ quartic invariant of the almost-BPS black ring of [65] needs to be rewritten in terms of the quantized charges, and also yields the relation between the “quantized” angular momentum of this black ring and the angular-momentum parameter that appears in the corresponding almost-BPS harmonic function.

A third, perhaps more unexpected consequence of our result is that if one probes a certain supergravity solution with a supertube and finds, say, two minima, one where the supertube is at the exterior of the other supertubes and one where it is at an intermediate position, these two minima do not correspond to two vacua in the same vacuum manifold of the bubble equations. Indeed, one can relate both minima of the supertube potential to almost-BPS supergravity solutions, and then use our recipe to compute the quantized charges of the centers of these two solutions. Since the shifts of the charges of the centers depend on the position of the tube, the quantized charges of the centers of the two resulting solutions will not be the same, and hence these solutions do not describe different arrangements of the same supertubes, and thus cannot be related by moving in the moduli space of solutions of a certain set of bubble equations. Instead they live in different superselection sectors.

In Section 6.2 we calculate and examine the action of supertube probes in these solutions. We show that one can reproduce the supergravity bubble equation of the outermost supertube by considering this supertube as a probe in the fully back-reacted solution formed by the other supertubes and find the equation that relates the supergravity charge parameters and the quantized charges of the probe. We then give a recipe to read off the quantized supertube charges in a general multi-center almost-BPS solution. In Section 6.3 we then show that one can also recover the bubble equations of all the other supertube centers by examining the minima of probe supertubes and relating their supergravity charges to their quantized charges. This demonstrates that all the supergravity data of a multi-center solution can be recovered from the action of supertube probes, and hence this data is not renormalized as one goes from weak to strong effective coupling. We also discuss in more detail the physics behind the shift needed to relate supergravity and quantized charges. Section 6.4 contains our conclusions and a discussion of further issues arising from this work.

6.2 Brane probes in almost-BPS solutions

6.2.1 Brane probes

We once again consider the system of three supertubes, one of each species, that we considered in the previous chapter. Without loss of generality we take the supertubes to be ordered as

$$0 < a_1 < a_2 < a_3 \,. \tag{6.1}$$

This is opposite compared to the order of the previous chapter, however all the results from the previous chapter we will use still hold in the same form.

To perform the probe calculation we only require a relatively simple result from the brane probe analysis of [44] or the more recent work of [89]. We need the so-called "radius relation" that determines the equilibrium position of a probe in a supergravity background.

One can use the M-theory frame or one can go back to the approach of [44] and work with three-charge solutions in the IIA frame in which the three electric charges, N_1, N_2 and N_3, of the solution correspond to[3]:

$$N_1 : \text{D0} \qquad N_2 : \text{F1}\,(z) \qquad N_3 : \text{D4}\,(5678), \tag{6.2}$$

where the numbers in the parentheses refer to spatial directions wrapped by the branes and

[3] For later convenience, we have permuted the labels relative to those of [44].

$z \equiv x^{10}$. The magnetic dipole moments of the solutions correspond to:

$$n_1 : \text{ D6}\,(y5678z) \qquad n_2 : \text{ NS5}\,(y5678) \qquad n_3 : \text{ D2}\,(yz)\,, \tag{6.3}$$

where y denotes the brane profile in the spatial base, $(x^1, \ldots, x^4)$. We will use a D2-brane probe, carrying electric D0 and F1 charges, $\widehat{Q}^{(1)}$ and $\widehat{Q}^{(2)}$, and with a D2-dipole moment, d_3. For the supertube worldsheet coordinates, ζ^μ, we use static gauge and allow the supertube to wind along both the ψ and ϕ angles of the base space according to:

$$x^0 = \zeta^0\,,\ z = \zeta^1\,,\ \psi = \nu_\psi \zeta^2\,,\ \phi = \nu_\phi \zeta^2\,. \tag{6.4}$$

For such a probe, sitting along the positive z-axis of the base metric, the radius relation is given by:

$$\left[\widehat{Q}^{(1)} + d_3\, \mathcal{B}^{(2)}\right] \left[\widehat{Q}^{(2)} + d_3\, \mathcal{B}^{(1)}\right] \;=\; d_3^2\, \frac{Z_3}{V}\, (\nu_\psi + q\nu_\phi)^2\,, \tag{6.5}$$

where $\mathcal{B}^{(I)}$ is the pullback of the vector field, $B^{(I)}$, onto the profile of the supertube on the spatial base:

$$\mathcal{B}^{(I)} \;=\; B^{(I)}_\mu\, \frac{\partial x^\mu}{\partial \zeta_2}\,. \tag{6.6}$$

Then, using (5.23), for the BPS solutions:

$$\mathcal{B}^{(I)} \;=\; \mathcal{B}^{(I)}_+ \;=\; \frac{k_I^+}{(h + \frac{q}{r})\, r\, r_I}\left[\,\nu_\psi\, r + q\, a_I\, \nu_\phi - h\, r\,(r\cos\theta - \, a_I)\, \nu_\phi\,\right], \tag{6.7}$$

while for non-BPS we have

$$\mathcal{B}^{(I)} \;=\; \mathcal{B}^{(I)}_{-} \;=\; \frac{k_I^-}{a_I\, r_I}\,\big[\, a_I\, \nu_\psi - q\, r\, \nu_\phi - h\, a_I\, (r\cos\theta - a_I)\, \nu_\phi \,\big]\,. \tag{6.8}$$

In deriving this radius relation we have set the worldvolume field strength $\mathcal{F}_{tz} = +1$, which is the choice one makes when placing supersymmetric probes in supersymmetric solutions, and one may ask whether this choice is justified for a non-BPS probe supertube, especially because this choice does not describe the metastable supertube minima of [89, 90]. There are two ways to see that this choice correctly reproduces the almost-BPS probe supertube minima. The first, and most direct, would be to evaluate the Hamiltonian derived in [89] in an almost-BPS solution, and to find directly that the minima of this Hamiltonian have $\mathcal{F}_{tz} = +1$. While he have not done the former computation, there is a simpler, second argument that shows what the outcome must be. Remember that, unlike non-extremal solutions, almost-BPS solutions have the mass and the charge equal, and the only way a supertube action can yield a minimum with the mass equal to the charge is if its worldvolume field strength satisfies $\mathcal{F}_{tz} = +1$.

6.2.2 Probing a supergravity solution

We now replace the third supertube in the solution of Section 5.3.2 by a probe: we set $k_3^{\pm} = Q_3^{(1)} = Q_3^{(2)} = 0$ and consider the action of a probe with charges $\widehat{Q}^{(1)}_{probe}$, $\widehat{Q}^{(2)}_{probe}$ and dipole charge d_3 at a location, a_3, on the z-axis and winding once around the ψ-fiber.

We then have:

$$\mathcal{B}_+^{(I)} \;=\; \frac{k_I^+}{(h+\frac{q}{a_3})\,|a_I-a_3|}\,, \qquad \mathcal{B}_-^{(I)} \;=\; \frac{k_I^-}{|a_I-a_3|}\,. \tag{6.9}$$

We know from the results of [44] that for the BPS choice the radius relation, (6.5), indeed yields the bubble equation (5.58). We therefore focus on the almost-BPS solution.

Inserting $\mathcal{B}^{(I)} = \mathcal{B}_-^{(I)}$ in (6.5) and using (6.9) one can rearrange the radius relation to give

$$\frac{1}{|a_1-a_3|}\Big[d_3\,Q_1^{(3)} \;-\; \Big(h+\frac{q}{a_3}\Big)\,k_1^-\,\widehat{Q}^{(1)}_{probe}\Big] \;+\; \frac{1}{|a_2-a_3|}\Big[d_3\,Q_2^{(3)} \;-\; \Big(h+\frac{q}{a_3}\Big)\,k_2^-\,\widehat{Q}^{(2)}_{probe}\Big] \tag{6.10}$$

$$= \Big(h+\frac{q}{a_3}\Big)\,\frac{\widehat{Q}^{(1)}_{probe}\,\widehat{Q}^{(2)}_{probe}}{d_3} \;-\; d_3 \;+\; \frac{q\,k_1^-\,k_2^-\,d_3}{|a_1-a_3|\,|a_2-a_3|}\,\Big(\frac{a_1\,a_2-a_3^2}{a_1\,a_2\,a_3}\Big)\,. \tag{6.11}$$

We now explicitly use the ordering (5.20) to write $|a_i - a_j| = a_i - a_j$ for $i > j$ and we use the definition (5.26) to obtain:

$$\begin{aligned}\frac{1}{|a_1-a_3|}\Big[d_3\,Q_1^{(3)} \;-\; \widehat{k}_1\,Q^{(1)}_{probe}\Big] \;+\; \frac{1}{|a_2-a_3|}\Big[d_3\,Q_2^{(3)} \;-\; \widehat{k}_2\,Q^{(2)}_{probe}\Big]&\\ = \Big(h+\frac{q}{a_3}\Big)\,\frac{Q^{(1)}_{probe}\,Q^{(2)}_{probe}}{d_3} \;-\; d_3 \;-\; \frac{h\,q\,k_1^-\,k_2^-\,d_3}{a_1a_2a_3\,(h+\frac{q}{a_3})}\,,&\end{aligned} \tag{6.12}$$

where we have introduced

$$Q^{(1)}_{probe} \;\equiv\; \widehat{Q}^{(1)}_{probe} \;-\; \frac{q\,\widehat{k}_2\,d_3}{a_2a_3\,(h+\frac{q}{a_2})(h+\frac{q}{a_3})}\,, \qquad Q^{(2)}_{probe} \;\equiv\; \widehat{Q}^{(2)}_{probe} \;-\; \frac{q\,\widehat{k}_1\,d_3}{a_1a_3\,(h+\frac{q}{a_1})(h+\frac{q}{a_3})}\,. \tag{6.13}$$

This exactly matches the bubble equation (5.61) provided one makes the identifications:

$$d_3 \;=\; \widehat{k}_3 \;=\; \Big(h + \frac{q}{a_3}\Big)\, k_3^- \,, \qquad Q^{(I)}_{probe} \;=\; Q^{(I)}_3 \,. \tag{6.14}$$

Hence, the action of a probe supertube that is placed at the outermost position of an almost-BPS solution that contains two other supertubes of different species can capture exactly the supergravity information about the location of this supertube. Since all the terms in the supertube bubble equations only contain two- and three-supertube interactions, we will see in Section 6.2.4 that this result can be straightforwardly generalized to a probe supertube placed at the exterior position of an almost-BPS solution containing an arbitrary number of supertubes of arbitrary species.

There is another more compact way to write the formula that gives the shift from the quantized charges to the BPS parameters

$$Q^{(1)}_{probe} \;=\; \widehat{Q}^{(1)}_{probe} \;-\; \frac{q\,\widehat{k}_2\,\widehat{k}_3}{a_2\, a_3 V_2 V_3}\,, \qquad Q^{(2)}_{probe} \;=\; \widehat{Q}^{(2)}_{probe} \;-\; \frac{q\,\widehat{k}_1\,\widehat{k}_3}{a_1\, a_3 V_1 V_3}\,. \tag{6.15}$$

6.2.3 Interpretation of the charge shift

Given that we have matched the probe calculation to the supergravity one by postulating the charge shift above, it is legitimate to ask whether this charge shift has any direct physical interpretation apart from the fact that it maps the probe result onto the supergravity result. We will argue in the remaining part of the chapter that this charge shift encodes very non-trivial properties of almost-BPS solutions.

We begin by comparing almost-BPS solutions to BPS multi-center solutions (for which there is no charge shift [44]) in the limits when the two kinds of solutions become identical. We will then explain how this charge shift solves an old puzzle about the relation between quantized charges and moduli in almost-BPS solutions

Mapping BPS to non-BPS solutions

There are two ways to turn an almost-BPS solution into a BPS one. The first is to set the Taub-NUT charge, q, to zero (and obtain black strings and supertubes extended along the S^1 of $\mathbb{R}^3 \times S^1$), and the second is to set the constant, h, in the Taub-NUT harmonic function equal to zero. In the first limit ($q = 0$), the charge shifts automatically vanish, consistent with the result of [44].

However, for $h = 0$, the shift does not vanish and this may seem a little puzzling. To understand the origin of the shift we should remember that, in the $h = 0$ limit, the BPS and the almost-BPS solutions are identical up to the coordinate transformation (5.65). Hence, a supertube wrapping the ψ-fiber once in the almost-BPS solution and a supertube wrapping the ψ-fiber once in the BPS writing of the same solution wrap different circles. Indeed, the supertube that wraps the ψ-fiber once in the almost-BPS solution has $(\nu_\psi, \nu_\phi) = (1, 0)$. Then the transformation (5.65) means that in the BPS writing of the solution this object is a supertube that wraps the ϕ-fiber $\frac{1}{q}$ times, and hence it has $(\nu_\psi, \nu_\phi) = (0, \frac{1}{q})$.

Clearly two supertubes with different windings are not the same object, and the only way their radius relations can be the same is if their charges are different. To be more precise, when $h = 0$, (6.7) reduces to:

$$\mathcal{B}_+^{(I)}(\nu_\psi, \nu_\phi) \;=\; \frac{k_I^+}{q\,(a_3 - a_I)}\,[\nu_\psi\, a_3 + \nu_\phi\, q\, a_I] \;=\; \frac{k_I^+\,(\nu_\psi + q\,\nu_\phi)\,a_3}{q\,(a_3 - a_I)} \;-\; k_I^+\,\nu_\phi\,, \tag{6.16}$$

where we have used (6.1). The first term is identical for either choice of winding numbers: $(\nu_\psi, \nu_\phi) = (1, 0)$ and $(\nu_\psi, \nu_\phi) = (0, \frac{1}{q})$, and this makes sense because, for either choice, the Taub-NUT fiber is being wrapped once. However, the last term is different, which indicates that the almost-BPS supertube also wraps the Dirac string of the background magnetic flux. Hence, if the supertubes with $(\nu_\psi, \nu_\phi) = (1, 0)$ and $(\nu_\psi, \nu_\phi) = (0, \frac{1}{q})$ are to have the same radius their charges must be related by a shift. Using the charge identification $k_I^+ \to \widehat{k}_I$ in (5.65) and using (5.26) for $h = 0$, one sees that this charge shift is simply:

$$\widehat{Q}^{(1)}_{(1,0)} \;=\; \widehat{Q}^{(1)}_{(0,\frac{1}{q})} \;-\; \frac{d_3\,\widehat{k}_2}{q}\,, \qquad \widehat{Q}^{(2)}_{(1,0)} \;=\; \widehat{Q}^{(2)}_{(0,\frac{1}{q})} \;-\; \frac{d_3\,\widehat{k}_1}{q}\,. \tag{6.17}$$

which exactly matches the shift in (6.13) for $h = 0$. Hence the shift between the almost-BPS supergravity charge and the almost-BPS quantized charge is the same as the shift from the BPS quantized charge to the almost-BPS quantized charge. This implies that the BPS quantized and supergravity charges are the same as the almost-BPS supergravity charge.

Another way to understand this result is to remember that one can consider a solution with a probe supertube of a certain charge wrapping a Dirac string, and take away the Dirac string by a large gauge transformation. As explained in [44], this changes the charges of the supertube. If one does this in our situation, and takes the Dirac string away from the location of the probe supertube, both the BPS and the almost-BPS supertubes will have the same charge and wrapping and will become the same object. Clearly, upon back-reaction the supergravity charges of the two will be the same. Hence, the reason why the quantized charge of the almost-BPS supertube is shifted from its supergravity charge is because, when written as a probe in $\mathbb{R}^4$, this supertube wraps a Dirac string non-trivially.

Note that when $h = 0$ this shift involves only integers, but for a generic almost-BPS solution this shift depends also on the moduli. As we will explain in section 6.3.3 this happens because almost-BPS solutions have an additional magnetic flux on the non-compact cycle extending to infinity.

A puzzle about charges and its resolution

Once one understands how the quantized charges of the centers of an almost-BPS solution are related to the supergravity charge parameters one can re-examine the problem of moduli-dependence of the asymptotic charge of almost-BPS solutions. Recall that for a BPS solution in Taub-NUT the warp factors go asymptotically like

$$\begin{aligned} Z^{(1)} &\sim \left[Q_2^{(1)} + Q_3^{(1)}\right]\frac{1}{r}\,, \qquad Z^{(2)} \sim \left[Q_1^{(2)} + Q_3^{(2)}\right]\frac{1}{r}\,, \\ Z^{(3)} &\sim \left[Q_1^{(3)} + Q_2^{(3)}\right]\frac{1}{r}\,, \end{aligned} \tag{6.18}$$

which confirms the fact that the asymptotic charges are simply the sum of the quantized charges of the individual component supertubes. However, for an almost-BPS solution one has

$$\begin{aligned} Z^{(1)} &\sim \left[Q_2^{(1)} + Q_3^{(1)} + \tfrac{q\,\widehat{k}_2\,\widehat{k}_3}{a_2\,a_3 V_2 V_3}\right]\frac{1}{r}\,, \qquad Z^{(2)} \sim \left[Q_1^{(2)} + Q_3^{(2)} + \frac{q\,\widehat{k}_1\,\widehat{k}_3}{a_1\,a_3 V_1 V_3}\right]\frac{1}{r}\,, \\ Z^{(3)} &\sim \left[Q_1^{(3)} + Q_2^{(3)} + \tfrac{q\,\widehat{k}_1\,\widehat{k}_2}{a_1\,a_2 V_1 V_2}\right]\frac{1}{r}\,. \end{aligned} \tag{6.19}$$

This appears to imply that if one changes the moduli of a solution continuously, while keeping the charges of the centers fixed, the asymptotic charges will change continuously. If this were true, an almost-BPS solution would have non-quantized asymptotic charges for generic values of the moduli, and would only exist on certain submanifolds of the moduli space where the asymptotic charges are integer. Needless to say, this would be rather peculiar.

Our results show that this does not, in fact, happen. If one expresses the supergravity charge in terms of quantized charges, the moduli-dependent term drops out and the asymptotics become identical to the one in (6.18). Hence, the asymptotic charges of almost-BPS solutions are also equal to the sum of the quantized charges of the centers, and the quantization of charges does not restrict the the moduli space of supertube locations.

To see this we can make the following Gedankenexperiment: we start with a single supertube of species $I = 1$ at position a_1, for which the supergravity charges are the same as the quantized charges

$$Q_1^{(2)} = \widehat{Q}_1^{(2)}\,, \qquad Q_1^{(3)} = \widehat{Q}_1^{(3)}\,. \tag{6.20}$$

We then bring in a second supertube of species $I = 2$ at position a_2. The supergravity charge $Q_2^{(3)}$ of this supertubes is shifted from its quantized charge $\widehat{Q}_2^{(3)}$ because of the presence of the first supertube, while its other charge remains the same:

$$Q_2^{(1)} = \widehat{Q}_2^{(1)}\,, \qquad Q_2^{(3)} = \widehat{Q}_2^{(3)} - \frac{q\widehat{k}_1^{(1)}\widehat{k}_2^{(2)}}{a_1 a_2 V_1 V_2}\,, \tag{6.21}$$

The asymptotic charges $\widehat{\mathcal{Q}}_{12}^{(I)}$ of the resulting solution are given by the (sum of the) quantized charges, $\widehat{\mathcal{Q}}_{12}^{(1)} = \widehat{Q}_2^{(1)} = Q_2^{(1)}$, $\widehat{\mathcal{Q}}_{12}^{(2)} = \widehat{Q}_1^{(2)} = Q_1^{(2)}$ and

$$\widehat{\mathcal{Q}}_{12}^{(3)} = \widehat{Q}_1^{(3)} + \widehat{Q}_2^{(3)} = Q_1^{(3)} + Q_2^{(3)} + \frac{q\widehat{k}_1^{(1)}\widehat{k}_2^{(2)}}{a_1 a_2 V_1 V_2}\,, \tag{6.22}$$

which agrees with the coefficient of r^{-1} in $Z^{(3)}$ in (6.19). After back-reacting the second supertube we bring in a third supertube of species $I = 3$ at position a_3 whose supergravity charges after back-reaction are shifted with respect to their quantized charges

$$Q_3^{(1)} = \widehat{Q}_3^{(1)} - \frac{q\widehat{k}_2^{(2)}\widehat{k}_3^{(3)}}{a_2 a_3 V_2 V_3}\,, \qquad Q_3^{(2)} = \widehat{Q}_3^{(2)} - \frac{q\widehat{k}_1^{(1)}\widehat{k}_3^{(3)}}{a_1 a_2 V_1 V_2}\,, \tag{6.23}$$

because of the other supertubes. The asymptotic charges $\widehat{\mathcal{Q}}_{123}^{(I)}$ of the solution with all three supertubes back-reacted are $\widehat{\mathcal{Q}}_{123}^{(3)} = \widehat{\mathcal{Q}}_{12}^{(3)} \sim r\ Z^{(3)}$ and

$$\widehat{\mathcal{Q}}_{123}^{(1)} = \widehat{\mathcal{Q}}_{12}^{(1)} + \widehat{Q}_3^{(1)} = Q_2^{(1)} + Q_3^{(1)} + \frac{q\widehat{k}_2^{(2)}\widehat{k}_3^{(3)}}{a_2 a_3 V_2 V_3} \sim r\ Z^{(1)}\,, \tag{6.24}$$

$$\widehat{\mathcal{Q}}_{123}^{(2)} = \widehat{\mathcal{Q}}_{12}^{(2)} + \widehat{Q}_3^{(2)} = Q_1^{(2)} + Q_3^{(2)} + \frac{q\widehat{k}_1^{(1)}\widehat{k}_3^{(3)}}{a_1 a_3 V_1 V_3} \sim r\ Z^{(2)}\,. \tag{6.25}$$

As advertised, these charges match the coefficients of r^{-1} in the $Z^{(I)}$. Hence, while the shift between the quantized and supergravity charges might have seemed surprising at first, it represents the missing ingredient necessary to relate the asymptotic charge of a multi-center solution to those of the centers.

One can also use these equations to determine the shift between the quantized and supergravity charges for an almost-BPS black ring in Taub-NUT [65]. Although black rings have no DBI description, one can make a black ring by bringing together three supertubes with three different types of dipole charges. Since in this process both the supergravity and the quantized charges are preserved, the shifts of the black ring charges will be the shifts of the charges of the composing supertubes. For a Taub-NUT almost-BPS black ring with dipole charges $\widehat{k}^{(I)}$ at position a_1 these shifts are:

$$Q^{(I)} = \widehat{Q}^{(I)} - |\epsilon_{IJK}| \frac{q\widehat{k}^{(J)}\widehat{k}^{(K)}}{a_1^2 V_1^2}\,, \tag{6.26}$$

where capital Latin indices are not summed. Note that there are $n!$ ways of making a black ring starting with n supertubes, corresponding to the different relative orderings of these supertubes, and for each ordering the charge shifts of various supertubes are different, but the sum of all these charge shifts (which gives the black ring charge shift) is always the same.

We are now ready to further use this result and the iterative procedure outlined above in order to unambiguously determine the relation of the supergravity charge parameters and the quantized charges in a solution with an arbitrary number of centers.

6.2.4 Generalization to many supertubes

We can generalize the calculation in the previous section to an almost-BPS solution containing an arbitrary number of colinear supertubes. Without loss of generality we consider a solution containing $i-1$ supertubes at positions

$$0 < a_1 < a_2 < ..... < a_{i-1}\,, \tag{6.27}$$

and we bring a probe supertube of species I (wrapping the tori T^2_J and T^2_K in (5.1)) to the point a_i on the z-axis, at the outermost position with respect to the other supertubes ($a_i > a_{i-1}$). This probe has electric charges $\widehat{Q}_i^{(J)}$, $\widehat{Q}_i^{(K)}$ and dipole charge $d_i^{(I)}$. Starting from the radius relation:

$$\left[\widehat{Q}_i^{(J)} + d_i^{(I)}\mathcal{B}_i^{(K)}\right]\left[\widehat{Q}_i^{(K)} + d_i^{(I)}\mathcal{B}_i^{(J)}\right] = (d_i^{(I)})^2\frac{Z_i^{(I)}}{V_i} \tag{6.28}$$

with

$$\mathcal{B}_i^{(L)} = \mathcal{B}_-^{(L)}(r = a_i) = \sum_l \frac{k_l^{-(L)}}{|a_l - a_i|} = \sum_l \frac{\widehat{k}_l^{(L)}}{|a_l - a_i|\, V_l} \quad , \qquad V_i = h + \frac{q}{a_i}\,, \tag{6.29}$$

and

$$Z_i^{(I)} = Z^{(I)}(r = a_i) = 1 + \sum_j \frac{Q_j^{(I)}}{|a_i - a_j|} + |\epsilon_{IJK}|\sum_{j,k}\left(h + \frac{qa_i}{a_ja_k}\right)\frac{k_j^{-(J)}k_k^{-(K)}}{|a_i - a_j||a_i - a_k|}\,, \tag{6.30}$$

we obtain the shift of charges by matching this relation to the bubble equation for the back-reacted probe center (that can be derived from the results of [63]):

$$\begin{aligned}\Big\{\sum_j \frac{1}{|a_i - a_j|}\left[d_i^{(I)}Q_j^{(I)} - \widehat{k}_j^{(J)}Q_i^{(J)}\right] + \sum_k \frac{1}{|a_i - a_k|}\left[d_i^{(I)}Q_k^{(I)} - \widehat{k}_k^{(K)}Q_i^{(K)}\right]\Big\}\\ = \frac{Q_i^{(J)}Q_i^{(K)}}{d_i^{(I)}}\left(h + \frac{q}{a_i}\right) - d_i^{(I)} - |\epsilon_{IJK}|\frac{hqd_i^{(I)}}{a_i\left(h + \frac{q}{a_i}\right)}\sum_{j,k}\epsilon_{ijk}\frac{k_j^{-(J)}k_k^{-(K)}}{a_ja_k}\,,\end{aligned} \tag{6.31}$$

where there is no summation over capital Latin indices[4]. We find that, upon identifying $d_i^{(I)} = \widehat{k}_i^{(I)} = k_i^{-(I)}V_i$, the shift between the supergravity charge parameters of the supertube

[4]Note that the terms involving $|\epsilon_{IJK}|$ in $Z^{(I)}$ and the bubble equation (6.31) differ by a factor 1/2 from the usual form (see, for example, [63]) where *there is* summation over I, J, K.

and its quantized charges are

$$Q_i^{(J)} = \widehat{Q}_i^{(J)} - |\epsilon_{JKI}| \sum_k \frac{q\widehat{k}_k^{(K)}\widehat{k}_i^{(I)}}{a_k a_i V_k V_i} \tag{6.32}$$

and similarly for $Q_i^{(K)}$, where there is no summation over capital Latin indices. In equations (6.31)–(6.32) the indices j, k run over the positions of supertubes of species J, K respectively.

One can now use this formula and find the charge shifts for all the centers of a certain solution, by constructing it recursively by bringing in probe supertubes. From the previous analysis it is clear that the relation between the supergravity and the quantized charges of a given center in the interior of the solution does not change as one brings more and more supertubes at the outermost positions. Hence, the shifts of the charges of a given supertube center only depend on the locations and dipole charges of the centers that are between this supertube center and the Taub-NUT center, but not on the locations or charges of the centers at its exterior:

$$Q_i^{(J)} = \widehat{Q}_i^{(J)} - |\epsilon_{JKI}| \sum_{k<i} \frac{q\widehat{k}_k^{(K)}\widehat{k}_i^{(I)}}{a_k a_i V_k V_i}. \tag{6.33}$$

It is not hard to see that the asymptotic charge of the solution (which can be read off from the asymptotics of $Z^{(I)}$) is now the sum of the quantized charges of all the centers

$$\widehat{\mathcal{Q}}_\infty^{(I)} = \sum_j Q_j^{(I)} + |\epsilon_{IJK}| \sum_{j,k} \frac{q\widehat{k}_j^{(J)}\widehat{k}_k^{(K)}}{a_j a_k V_j V_k} = \sum_i \widehat{Q}_i^{(I)}. \tag{6.34}$$

One can also extend this calculation to describe a configuration containing an arbitrary collection of concentric supertubes and black rings, by constructing the black rings from three different species of supertubes. The general formula relating the supergravity charge parameters to the quantized charges is

$$Q_i^{(J)} = \widehat{Q}_i^{(J)} - |\epsilon_{JKI}| \sum_{k<i} \frac{q\widehat{k}_k^{(K)}\widehat{k}_i^{(I)}}{a_k a_i V_k V_i} - |\epsilon_{JKI}| \frac{q\widehat{k}_i^{(K)}\widehat{k}_i^{(I)}}{a_i^2 V_i^2}\,, \tag{6.35}$$

where once again the capital latin indices are not being summed.

6.3 Extracting the complete supergravity data from supertubes

We have seen that if one considers an axially-symmetric brane configuration and brings in a brane probe along the axis from one side of the configuration then the radius relation of that probe exactly reproduces the bubble equation for the charge center that would replace the brane probe in a fully back-reacted supergravity configuration. This correspondence requires the charge shifts described in Section 6.2. Since we are considering only the action of the probe, there is no immediate way in which this action could directly yield the fully back-reacted bubble equations for the other centers in response to the probe. That is, given the n-supertube solution, the probe radius relation for the $(n+1)^{\text{st}}$ supertube yields the exact supergravity bubble equation for that supertube. However, when back-reacted, the probe supertube must introduce modifications to the bubble equations for the other supertubes and these are not given directly by the computations described above.

In contrast, the analysis of brane probes in BPS solutions [44] required no charge shifts and the bubble equations contained only two-body interactions of the form $\Gamma_{ij}|\vec{y}_i - \vec{y}_j|^{-1}$ as in (5.54)–(5.58). In that solution, the action of the probe was used to read off all the two-by-two terms between the probe and the centers, and thus derive the bubble equations for *all* the centers. However, for almost-BPS solutions the bubble equations for a certain center contain complicated three-body terms, which cannot be read off from the action describing the location of another center. However, there is still a (more complicated) way to recover the full supergravity data from DBI actions: given a solution with $(n+1)$ supertubes one can examine all the ways of extracting one supertube and treating it as probe in the background sourced by the others. As we will show below, this yields all the bubble equations of the solution.

6.3.1 Reconstructing the bubble equations from probes

For non-BPS supertubes there are three elements for the inductive modification of the bubble equations as one goes from n to $(n+1)$ supertubes:

(i) Compute the charge shifts of all probe charges.

(ii) Write the left-hand sides of the bubble equations with all two-body interactions, $\Gamma_{ij}|a_i - a_j|^{-1}$.

(iii) Compute and include all the supersymmetry breaking terms, Y, as in (5.62) and include them with the correct relative sign on the right-hand sides of the bubble equations as in (5.59)–(5.61).

If one assembles the configuration by bringing in the charges successively along the symmetry axis from the right then the charge shifts are given by (6.32). The two-body interactions are fixed iteratively exactly as in the BPS solution. These terms are also fixed by the consistency condition that the sum of all the bubble equations must be identically zero. The new feature is that we need to specify the algorithm for (iii): We need to generalize the supersymmetry breaking terms and how they are to be included in the bubble equations. It is clear that the bubble equation for a supertube of type I will have an interaction, Y, of the form (5.62) with every pair of supertubes of species J, K where I, J and K are all distinct. The issue is to introduce these terms with the correct signs in each bubble equation. One can do this by making the more formal induction using all the ways of extracting one supertube and treating it is a probe. However, for simplicity, we will start by restricting our attention to configurations in which the probe is always the outermost supertube. As we will see in the next section, the charge shifts are modified when the probe is some other supertube in the configuration and so the discussion is a little more complicated. We will begin by completing the discussion of Section 6.2.4 by giving a recipe for the supersymmetry breaking terms when the probe is the outermost supertube and we will explain its derivation below.

Suppose we have an n-supertube system and we bring in, as a probe, a supertube of species I then the bubble equations for a center of type J at position a_j must include an extra supersymmetry breaking term, $Y^{(J,I)}_{j,(n+1)}$, that is to be subtracted from the right-hand side. These terms are given by:

$$\begin{aligned} Y^{(J,I)}_{j,(n+1)} &= |\epsilon_{IJK}| \sum_k \frac{(a_j - a_{n+1})\,(a_j - a_k)}{|a_j - a_{n+1}||a_j - a_k|} \frac{hq d^{(I)}_{n+1} \hat{k}^{(J)}_j \hat{k}^{(K)}_k}{a_{n+1} a_j a_k V_{n+1} V_j V_k} = \\ &= -|\epsilon_{IJK}| \sum_k \frac{(a_j - a_k)}{|a_j - a_k|} \frac{hq d^{(I)}_{n+1} \hat{k}^{(J)}_j \hat{k}^{(K)}_k}{a_{n+1} a_j a_k V_{n+1} V_j V_k} \end{aligned} \tag{6.36}$$

where, for fixed I and J, the summation over k runs only over supertubes of species K, with I, J, K all distinct. In the last equality we used that $a_{n+1} > a_j$ for $j = 1, 2, ..., n$. Thus, as we assemble the system by successively bringing supertubes from the right, for every new supertube added we have to subtract (6.36) from the right hand side of the bubble equations of the previously back-reacted system.

6.3.2 Assembling colinear supertubes in general

The general bubble equations (6.31) for a system of n colinear supertubes contain supersymmetry breaking terms, Y, that come with a relative sign given by by ϵ_{ijk} defined in (5.63)[5]. In the procedure discussed in Section 3, we always had $\epsilon_{ijk} = +1$ since the probe was always being placed in the outermost position of the back-reacted geometry. However, if the probe is of species I and located at a general position, a_i, then we find that its radius relation can be mapped directly onto is supergravity bubble equation if one uses the following more general relation between supergravity and quantized brane charges:

$$Q_i^{(J)} = \widehat{Q}_i^{(J)} - |\epsilon_{JKI}| \sum_k \frac{a_i - a_k}{|a_i - a_k|} \frac{q \widehat{k}_k^{(K)} \widehat{k}_i^{(I)}}{a_k a_i V_k V_i} , \tag{6.37}$$

where, once again, the capital latin indices are not being summed and the sum over k runs over supertubes of species K. Thus for $a_i > a_k$ the shift terms get subtracted from the quantized charge as before, but for $a_i < a_k$ the shift terms have to be added! We also find that if one uses the more general shifts in (6.37) then the radius relation also exactly reproduces precisely the correct supersymmetry breaking terms, Y, in the bubble equations.

[5]This ϵ_{ijk} should not be confused with the factors of $|\epsilon_{IJK}|$ that give the triple intersection number of the dipole charges.

As a result, by using (6.37), we can recreate the whole set of bubble equations of the system by the following iterative procedure. Consider a $(n+1)$-supertube system with preassigned order

$$0 < a_1 < a_2 < ..... < a_{n-1} < a_{n+1}\,. \tag{6.38}$$

Now consider each supertube, in turn, as a probe in the supergravity background of the remaining n and imagine placing the probe in its assigned position, (6.38), to recreate the full $(n+1)$-supertube system. To be more explicit, start by taking the supertube at a_{n+1} out of the supergravity system and treat it as a probe being placed at position a_{n+1}. This generates the bubble equation for the center at a_{n+1} as described earlier. Next, imagine the supergravity system where the supertube at position a_n has been removed and replaced by a probe placed at position a_n. The radius relation of this probe produces exactly the bubble equation for the center at a_n provided one uses the charge shifts defined in (6.37). One can then repeat this iterative procedure (illustrated in Fig. 6.1) for each of the $n+1$ centers of the system.

As we noted earlier, for a probe of species I, the bubble equations for the center of type J will involve a supersymmetry breaking term, $Y^{(J,I)}_{j,(n+1)}$, that is a sum over all centers of species K (with I, J, K distinct). The importance of this general iterative procedure is that it generates all these terms in the bubble equations in exactly the correct form and yields the formula (6.36) for a general position, a_i, of the probe:

$$\mathrm{Y}^{(J,I)}_{j,i} \;=\; |\epsilon_{IJK}| \sum_k \frac{(a_j - a_i)\,(a_j - a_k)}{|a_j - a_i||a_j - a_k|} \, \frac{hq d^{(I)}_{n+1} \hat{k}^{(J)}_j \hat{k}^{(K)}_k}{a_{n+1} a_j a_k V_{n+1} V_j V_k}\,. \tag{6.39}$$

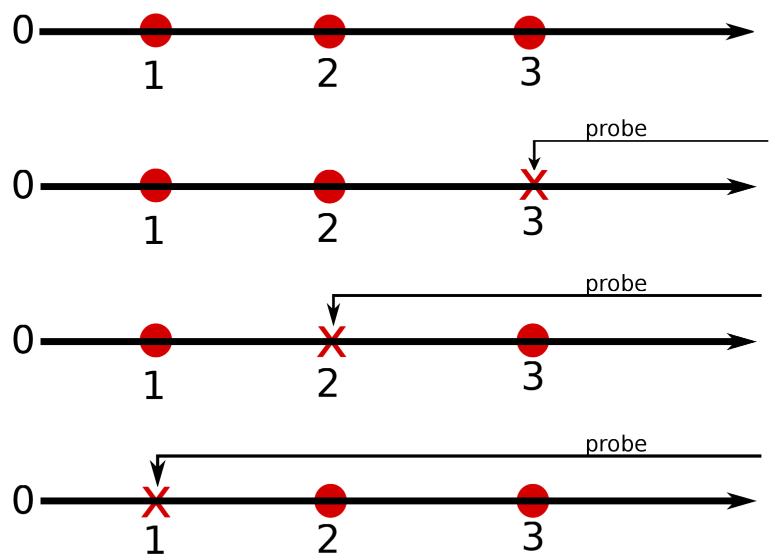


Figure 6.1: *The first graph represents the three-supertube back-reacted geometry while the three below represent the iterative procedure by which one removes each center of the system and treats it as a probe in the background of the remaining two supergravity supertubes. Because supertubes cross over each other we need the formula (6.37) to generate the shift with the correct sign.*

6.3.3 Topology, charge shifts and back-reacting probes

To understand the charge shifts and their dependence on the location of the supertubes it is important to recall the geometric structure underlying the almost-BPS supergravity solutions. In an appropriate duality frame[6] a solution containing multiple supertubes becomes smooth, and all the charges come from fluxes wrapping topologicaly non-trivial cycles. These cycles can be described in terms of $U(1)$ fibers over paths that end at the locations of the supertubes. The supertube locations are precisely where one of the $U(1)$ fibers pinches off and this, in turn, defines cycles and their intersections. The magnetic dipole charges of the supertube then correspond to cohomological fluxes through these cycles. Hence, in this

[6]This can be realized by performing three generalized spectral flows on almost-BPS solutions [55].

particular duality frame the back-reaction of a supertube corresponds to blowing up a new cycle and replacing a singular magnetic source with a cohomological flux. If one starts out with a particular supergravity solution and makes a particular choice of homology basis, then introducing a new supertube and back-reacting it will involve blowing up a new cycle and perhaps pinching off other cycles in order to achieve this. Hence, this will involve generically a reshuffling of the homology basis.

The presence of the Chern-Simons term in the electromagnetic action means that the interaction of pairs of magnetic charges can source electric charges and so the change of homology basis arising from the back-reaction of a supertube can lead to shifts of electric charges. Because the magnetic charges on the compact cycles are quantized and the magnetic contributions to electric charges are determined though the intersection form, one would expect all charge shifts to be quantized. This is, indeed, precisely what one finds in all the BPS solutions: the shifts, if they are non-zero, are indeed quantized. What distinguishes the non-BPS solutions is that there is a non-vanishing, normalizable flux on a non-compact cycle that extends to infinity and this flux depends upon the supergravity parameters and moduli. It is the interactions between the fluxes on this non-compact cycle that leads to moduli-dependence of the shifts.

One can recast this geometric picture in more physical terms through a careful examination of Dirac strings. Because a supergravity supertube carries a magnetic charge, the supertube comes with Dirac strings attached (these are, of course, an artifact of trying to write a vector potential for a topologically non-trivial flux). A supertube wraps a $U(1)$ fiber in the background and to define its configuration properly one must specify precisely which Dirac strings are being wrapped by the supertube. This defines the dipole-dipole interaction between the probe and the background[7] and the difference between wrapping and not wrapping will appear as a shift of the electric charges that it contributes to the supergravity solution.

If the supertubes are all colinear then one can choose all the Dirac strings to follow the axis of symmetry out to infinity. One can then set up a configuration in which an outermost probe supertube wraps the Dirac strings of all the other supertubes, and hence its charges get shifted. On the other hand the tubes at the interior of this configuration will not feel the Dirac string of the outermost supertube, and hence the relationship between their supergravity charge and their quantized charge does not shift. In the duality frame where the multi-supertube solution is smooth, this corresponds to blowing up a homology cycle at the outer edge of the original configuration.

If one were to place the probe supertube so that it is the closest to the Taub-NUT center, its Dirac string would generically affect all the other supertubes, and would change the relation between quantized and supergravity charges. This corresponds to blowing up a different new homology element and reshuffling the homology basis. Hence, if one keeps the supergravity charges of the back-reacting supertubes fixed, as one must do in a probe approximation, bringing a supertube to a point that is closer to the center of Taub-NUT than the other supertubes will change the quantized charges of these supertubes. Thus, the resulting configuration will not have the same quantized charges on the centers as when the supertube is at the outermost location, and hence belongs to different sector of the multi-centered solutions. There are, of course, similar consequences to bringing the probe

[7]This wrapping choice determines the new homology element and its intersections with other other elements of homology.

supertube to some point in the middle of the back-reacted supertube centers.

One can also take a more pragmatic perspective and try to understand the charge shifts by compactifying the multi-supertube solutions to obtain a multi-center almost-BPS solution in four dimensions. The five-dimensional, smooth solutions we discuss here are singular at the supertube centers in four dimensions, but this does not impede the calculation of the conserved charges at the singularities. As explained in [91], these conserved charges differ in general from the supergravity charge parameters by dipole-dipole position-dependent shifts similar to the ones found here, and it would be interesting to see whether one can reproduce our results using the four-dimensional KK reduction formulae of [91] (starting, for example, from equation (266) on page 55).

6.3.4 Quantized charges, supergravity parameters and probes

As we have seen in the previous section, bringing a supertube to some point in the middle of a multi-center supertube solution changes the relation between the quantized charges and the supergravity parameters of the supertube centers at its exterior. Hence, all the probe calculations that describe supertubes at interior locations in a multi-center solution are not self-consistent, because the quantized charges of the other centers before bringing in the probe are not the same as the quantized charges with the probe inside. One can think of this as coming from the fact that all probe supertubes come with Dirac strings attached, and when these Dirac strings touch the other centers, the relation between the supergravity and quantized charges of these centers change. Thus, the only probe supertube that one can bring without shifting everybody else's quantized charges is one that lays at the outermost position and whose Dirac string extends away from the supertubes and towards infinity.

An immediate corollary of this is that if one wants to calculate the amplitude for a supertube to tunnel to a vacuum across another center, this calculation cannot be done using the DBI action of that supertube and treating it as a probe in a fixed supergravity background. Instead one would have to change the solution as the probe moves around, and this cannot be done off-shell. One can wonder whether there exists *any* way to compute this tunneling amplitude. Again, the correct probe for this would be a supertube *with Dirac strings attached*, such that one would change the supergravity charges of the back-reacted solution as one moves the probe around, in such a way that the quantized charges stay the same. Unfortunately, there is no known action for such a probe, so probing the interior of a multi-supertube solution with a probe supertube does not correspond to a physical process. Hence, the calculation we performed in Section 6.3.2, where we took the supergravity parameters to be fixed and treated the background of n supertubes merely as one would treat any other supergravity background ignoring the details of how it might have been assembled from other supertubes, should be interpreted as a formal calculation, which does not correspond to a physical process, but which does however reproduce the bubble equations of all the interior supertube centers. It would be clearly interesting to understand the reason for this.

It is also important to stress out that this charge shift subtlety only affects the validity of the probe calculation when the other centers are supertubes or black rings, that interact directly with the Dirac string of the probe. If the other centers are bubbled Gibbons-Hawking centers, where the geometry is smooth, the presence of a Dirac string does not change their four-dimensional charge parameters nor the fluxes wrapping the corresponding cycle in the five-dimensional solution. Hence, one can use the probe supertube action to calculate tunneling probabilities of metastable supertubes in bubbling geometries [89, 90]. Similarly, the description of multi-center non-BPS solutions using supergoop methods [92], being intrinsically non-gravitational, is not affected by this subtlety.

6.4 Concluding remarks

We have shown that a brane probe in an almost-BPS background of supertubes can capture the complete supergravity data of the background in which the probe becomes a fully back-reacted source. A similar result was established for BPS solutions in [44] but it is rather surprising that this can also be achieved for almost-BPS solutions in which the supersymmetry is broken, albeit in a rather mild manner. It is surprising because going from a probe to a back-reacted supergravity solution represents going from weak to strong coupling in the field theory on the brane and when supersymmetry is broken one would expect the parameters of this field theory to be renormalized. Our result therefore suggests that, even though supersymmetry is broken, the couplings of the field theory that govern the probe location are still protected. This does not mean that the theory is unchanged relative to its BPS counterpart: there are terms in the probe action that come from the supersymmetry breaking and these exactly reproduce the corresponding terms found in the supergravity solution.

We suspect that the non-renormalization of the relevant parts of the probe action arises from the rather special form of the supersymmetry breaking: the probe is supersymmetric with respect to every other center in the supergravity solution taken individually and the supersymmetry breaking arises from the fact that these supersymmetries are incompatible between multiple centers. Indeed the supersymmetry breaking terms depend on the product of the charges and dipole charges of three or more centers, and do not have the "two-body" structure of the terms that appears in the BPS bubble equations. It would be interesting to try to extend our analysis to the other known class of non-BPS extremal multi-center solutions, the interacting non-BPS black holes [93, 94], and see if the bubble equations of these solutions are also not renormalized when one goes from weak to strong effective coupling.

One of the new features of our analysis is that the supergravity charge parameters and the quantized charges of the supertubes need to be shifted relative to one another in order to match the supergravity bubble equations and the probe radius relations. By taking a flat-space limit, we saw that part of this difference was related to the choice of how the supertube wraps Dirac strings, or equivalently, how the cycle is blown up after back-reaction. This accounts for a quantized shift but in a Taub-NUT background the shift is no longer obviously quantized because it depends upon moduli. Quantized shifts have also been encountered and understood in the study of black rings [44, 95, 96] but it would be very nice to understand how to extract the quantized charges, and hence the more general moduli-dependent shifts, via a pure supergravity calculation. A good starting point may be to use the four-dimensional reduction formulas of [91] as well as a judicious accounting of the integer charge shifts caused by Dirac string, to try to reproduce the shifts we find.

Our results also suggest further interesting supergravity calculations. For rather mysterious reasons, explicit solutions of the almost-BPS equations still elude us for non-axi-symmetric configurations. Such solutions are extremely simple in the BPS system but for the almost-BPS system even the three-centered, non-axi-symmetric solution is not known. The most one could get is an implicit solution in terms of integrals that one can evaluate asymptotically [91]. Finding the three-centered solution is particularly important because one can then use this solution and the methods of [63] to generate the general, non-axi-symmetric, multi-centered solutions. A two-centered solution is, of course, trivially axi-symmetric and it is easy to introduce a probe in any location. It would therefore be very interesting to see if such a probe could be used to gain insight into the structure of the full supergravity solution.

Given that our results suggest that there are non-renormalization theorems that protect the bubble equations of almost-BPS multi-center solutions, it would be extremely interesting to investigate whether almost-BPS solutions could be described in the regime of parameters where none of the centers is back-reacted, using a suitable generalization of quiver quantum mechanics of [76]. Clearly, this generalization will have to account for the supersymmetry-breaking terms, which is highly non-trivial: these terms involve three- and four-center interactions, and at first glance no theory of strings on multiple D-branes appears capable of producing such a term. It would be very nice to understand how the charge shifts and supersymmetry breaking terms emerge from the field theory on the branes and whether the effective number of hypermultiplets may be different from the number of hypermultiplets appearing in the Lagrangian. This could be the effect of some of these hypermultiplets becoming massive and no longer contributing to the low energy dynamics. It is thus not too difficult to imagine that our results might be derivable from a field theory analysis and we intend to investigate this further in the future.

Chapter 7

Multi-Superthreads and Supersheets

The material of this chapter is taken from [19, 20]. The latter ([20]) is one of my sole author papers and is based on work I did in collaboration with Benjamin Niehoff and Nicholas Warner in [19].

7.1 Motivation

In the previous chapters we examined BPS and non-BPS systems in five dimensions and presented solutions which constitute microstate geometries for black holes and black rings. These geometries are a semi-classical coarse-grained description of the microstates of black holes but the hope is they will at least account for the correct scaling of the entropy with respect to the charges, which for 1/8-BPS systems in five dimensions is $S \sim Q^{3/2}$. The ambipolar geometries of chapter 3 through various entropy enhancement mechanisms [44, 62] can account for $Q^{5/4}$ of the entropy which is certainly less than required.

The expectation is that for 1/8-BPS systems the dominant semi-classical contribution to their microstate structure will be given by conjectured microstate geometries called superstrata [46, 97], in the same way that supertubes accounted for the entropy of 1/4-BPS systems. These *superstrata* depend upon several functions of two variables and generalize supertubes in several important ways. Supertubes are supergravity backgrounds that carry two electric charges and one magnetic dipole charge and, in the IIB duality frame, are completely smooth solutions [52, 53]. It is such geometries that lie at the heart of Mathur's original fuzzball proposal for the microstate structure of two-charge black holes (see, for example, [12, 98]). The conjectured superstratum carries *three* electric charges and three dipole charges, two of which are independent, and is described by an arbitrary, $(2+1)$-dimensional world-surface in six space-time dimensions.

The argument for the existence of the superstratum has its origins in earlier work [99] that suggested that one should be able to make two independent supertube transitions to produce new BPS solutions that carry three electric charges, two magnetic dipole charges and depend upon functions of two variables. It was originally believed that such objects would be non-geometric and have spatial co-dimension two, but it was shown in [97] that if one does this in the proper manner for the D1-D5-P system in IIB supergravity then the result will not only be a *geometric* BPS object with co-dimension *three* but one that is also completely smooth. Indeed, very near the superstratum the geometry approaches that of the supertube and so the smoothness follows directly from that of the supertube geometry. Thus the superstratum provides a new microstate *geometry* of co-dimension three that carries three electric charges, two independent magnetic dipole charge and depends upon several functions of two variables.

While the arguments given in [97] for the existence of the superstratum are fairly compelling, it still remains to construct one explicitly and thereby establish its existence beyond all doubt. An interesting development is that the BPS equations for six-dimensional, minimal $\mathcal{N}=1$ supergravity [100] coupled to an anti-self-dual tensor multiplet [101] were shown to be linear[1] [102]. This not only provides a huge simplification in solving these equations but it also enables one to use superposition to obtain multi-component solutions and, more abstractly, analyze the moduli spaces of such solutions. It is not only anticipated that this will lead to interesting new developments in the study of black-hole microstate geometries but that it will also lead to interesting new results for holography on $AdS_3 \times S_3$ geometries.

The fact that the BPS equations in six dimensions are linear gives one hope that the explicit supergravity solution may just be within reach (although it will still be extremely complicated). The construction in [97] has the virtue that it lays out a sequence of steps, via two supertube transitions, to arrive at the superstratum and so a possible route to making a superstratum might be to replicate these intermediate steps in a series of progressively more complicated but exact supergravity solutions. Indeed some initial progress in this direction was achieved in [102] where the D1-D5-P system was pushed through the first supertube transition to obtain a new three-charge, two-dipole charge[2] generalized supertube with an arbitrary profile as a function of one variable. We will refer to such a solution as a *superthread*.

The next step towards a superstratum, which will be the subject of this chapter, requires

[1]As with the corresponding result in five dimensions [22, 49], the equations that determine the spatial base geometry are still non-linear.

[2]The two dipole charges in this solution are related to one another and so, to get to the superstratum, a further independent dipole charge must be added via a second supertube transition [97].

the construction of a multi-superthread solution that could then be smeared to a continuum and thus obtain a three-charge solution with a two-dimensional spatial profile that is a function of two variables, namely a *supersheet*. This solution will still be singular and, like the standard supertube, will only become regular after the second supertube transition in which a Kaluza-Klein monopole is combined with the smearing. This last step is probably going to be the most difficult and will not be addressed here.

In [102] the step to the multi-superthread was only achieved for the highly restricted situation in which each thread was given exactly the same profile with a rigid translation to each distinct center. The smearing of such a multi-threaded solution will thus produce two-dimensional surface that is determined by a several functions of *one* variable, namely the smearing density and the original superthread profile functions. (See Fig. 7.1). To get a surface that is truly a generic function of two variables one must find the multi-superthread solution in which the threads at each center have independent profile functions so that, in the continuum limit, one obtains a one-parameter family of curves and hence a surface swept out by a generic function of two variables (See Fig. 7.2). The purpose of this chapter is to find this general a multi–thread solution. The difficulty that we overcome here is that multiple superthreads with different profiles have highly non-trivial shape-shape interactions and we show exactly how these contribute to the angular momentum and local momentum charge densities. We also generalize our results to concentric multi-supersheet solutions.

For different choices of the profile functions we perform the smearing integrals and produce supersheets which do not depend on the sixth dimension and are essentially five-dimensional as well as corrugates multi-supersheet solutions which are genuinely six-dimensional. Specifically we consider coaxial multi-supersheet geometries such that the shape of each supersheet is fixed but its scale varies within the compactification direction. They represent a new class of six-dimensional solutions that are by construction free of Dirac-strings. In five dimensions the cancellation of Dirac strings is given by integrability conditions which for single objects appear as a radius relation that gives the position of the object in terms of its charges, while for multiple objects they express the interactions between them in terms of magnetic fluxes. Thus it is interesting to explore how these interactions get embedded in the solution in the six-dimensional geometry and allow us in the future to construct microstate geometries in six dimensions. Analyzing the structure of these objects will also help us understand more about the superstratum which is also a genuine six-dimensional geometry. A novel feature of these new geometries is that, because of their fluctuation in the sixth dimension, under certain conditions they can touch or even intersect through each other.

It should, of course, be stressed that even though our solutions represent only a step towards the ultimate goal of the superstratum, the multi-superthread solutions presented here are completely new BPS solutions that are interesting in their own right.

In section 7.2 we briefly summarize the linear BPS system we must solve, briefly summarize the superthread of [97] and then present the new family of multi-superthread solutions. In section 7.3 we take the continuum limit and right the solution as a single supersheet. We also perform the smearing integrals that give a straight supersheet and a corrugated supersheet with an arbitrary periodic function describing the fluctuations in the compact direction. In section 7.4 we generalize our formalism to multi-supersheet solutions. We construct multiple corrugated supersheets with independent and arbitrary oscillation profiles along the compact direction. We derive the local conditions that need to be satisfied for two supersheets to touch or intersect and by a specific choice of examples we provide a numerical analysis of the global regularity conditions as well. In section 7.5, based on results of section 7.3, we give general arguments about the structure of six-dimensional solutions and a possible perturbative approach towards constructing black geometries as well as the superstratum. Finally we display our conclusions in section 7.6.

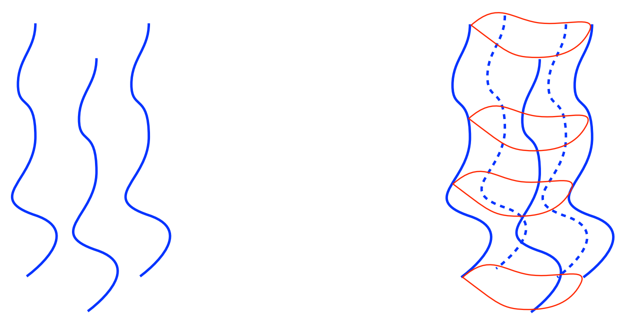


Figure 7.1: *Multi-thread solution in which all the threads are parallel. When smeared the sheet profile is described by a product of functions of one variable: the original thread profile and the thread densities.*

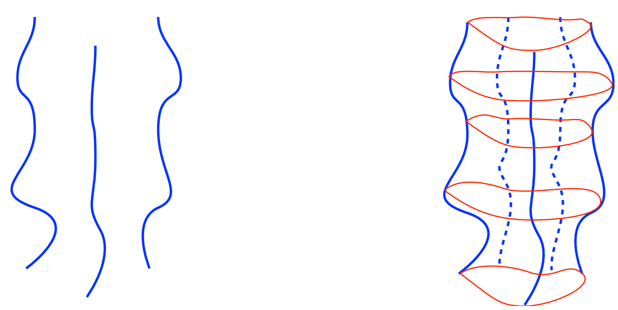


Figure 7.2: *Multi-thread solution in which all the threads have independent profiles. When smeared the sheet profile is described by generic functions of two variables.*

7.2 Solving the BPS equations

7.2.1 The BPS equations

The six-dimensional metric has the form:

$$ds^2 = 2H^{-1}(dv+\beta)\big(du+\omega \; + \; \tfrac{1}{2}\,\mathcal{F}\,(dv+\beta)\big) \; - \; H\,ds_4^2\,. \tag{7.1}$$

where the metric on the four-dimensional base satisfies some special conditions that will not be relevant here because we are going to take ds_4^2 to be the flat metric on $\mathbb{R}^4$. The coordinate, v, has period L and the metric functions generically depend upon both v and the $\mathbb{R}^4$ coordinates, $\vec{x}$. We are also going to take the trivial fibration by setting $\beta = 0$, which means, in particular, that there will be no Kaluza-Klein monopoles in the solution. The spatial part of the metric is simply flat $\mathbb{R}^4 \times S^1$. Also because $\beta = 0$, although the inhomogeneous terms in the BPS equations as we will see contain v derivatives, the differential operators in the left hand side contain no v derivatives. The latter means that because of linearity, the solutions we find, although v-dependent, can be thought to be assembled by constant v sections of the full solution.

With these choices, the first step in constructing a solution is to determine the harmonic functions that encode the D1 and D5 charges:

$$*_4\,\tilde{d}\,*_4\,\tilde{d}\,Z_a \;=\; -\,\nabla^2 Z_a \;=\; 0\,, \qquad i = 1,2\,, \tag{7.2}$$

where $\tilde{d}$ is the exterior derivative and ∇^2 is the Laplacian on $\mathbb{R}^4$. The metric function, H, is then given by $H = \sqrt{Z_1 Z_2}$ and one then must find self-dual Maxwell fields

$$\Theta_a \;=\; *_4\, \Theta_a\,, \qquad i = 1, 2\,, \tag{7.3}$$

that satisfy

$$\tilde{d}\Theta_1 \;=\; \tfrac{1}{2}\, \partial_v \big[\, *_4 \,(\tilde{d} Z_2)\big]\,, \qquad \tilde{d}\Theta_2 \;=\; \tfrac{1}{2}\, \partial_v \big[\, *_4 \,(\tilde{d} Z_1)\big]\,. \tag{7.4}$$

These Maxwell fields determine the magnetic components of the fluxes in six dimensions and thus the magnetic dipole D1 and D5 charges.

The angular momentum vector is obtained by solving

$$(1 + *_4)\, \tilde{d}\omega \;=\; 2\,(Z_1 \Theta_1 + Z_2 \Theta_2)\,. \tag{7.5}$$

and the last metric function is determined via:

$$*_4\, \tilde{d}\, *_4\, \tilde{d}\, \mathcal{F} \;=\; -\nabla^2 \mathcal{F} \;=\; 2\, *_4\, \tilde{d}\, *_4\, \dot{\omega} - 2(\dot{Z}_1 \dot{Z}_2 + Z_1 \ddot{Z}_2 + \ddot{Z}_1 Z_2) + 4\, *_4\, (\Theta_1 \wedge \Theta_2)\,. \tag{7.6}$$

This last function encodes the momentum (P) charge of the solution.

7.2.2 The new solutions

The first steps in our new solution directly parallel those of [102]. The harmonic functions, Z_i, are sourced on the thread profiles, $\vec{F}^{(p)}(v)$:

$$Z_i = 1 \;+\; \sum_{p=1}^{n} \frac{Q_{i\,p}}{|\vec{x} - \vec{F}^{(p)}(v)|^2}\,, \tag{7.7}$$

where we have required that $Z_i \to 1$ at infinity so that the metric is asymptotically Minkowskian. The Maxwell fields, Θ_i, that solve (7.3) are simply given by:

$$\Theta_i \;=\; \tfrac{1}{2}\,(1+*_4)\,\tilde{d}\left(\sum_{p=1}^{n}\frac{Q_{i\,p}\,\dot{F}^{(p)}_m dx^m}{|\vec{x}-\vec{F}^{(p)}(v)|^2}\right). \tag{7.8}$$

As noted in [102], the magnetic dipoles of this solution may be thought of as being defined by

$$\vec{d_1} \;=\; Q_1\dot{\vec{F}}(v)\,, \qquad \vec{d_2} \;=\; Q_2\dot{\vec{F}}(v)\,, \tag{7.9}$$

and they satisfy the constraint that is familiar from the five-dimensional, generalized supertube [103–105]:

$$Q_1\left|\vec{d_2}\right| \;=\; Q_2\left|\vec{d_1}\right|. \tag{7.10}$$

This means that even though the solution has two dipole charges, only one of them is independent of the other charges.

To write the solution for the angular momentum vector and the third function, $\mathcal{F}$, it is useful to define:

$$\vec{R}^{(p)} \;\equiv\; \vec{x} \;-\; \vec{F}^{(p)}(v)\,, \qquad R_p \;\equiv\; \left|\vec{R}^{(p)}\right| \;\equiv\; \left|\vec{x} \;-\; \vec{F}^{(p)}(v)\right|, \tag{7.11}$$

and for each p and q, introduce the anti-self-dual two form area element:

$$\mathcal{A}^{(p,q)}_{ij} \;\equiv\; R^{(p)}_i R^{(q)}_j - R^{(p)}_j R^{(q)}_i \;-\; \varepsilon^{ijk\ell} R^{(p)}_k R^{(q)}_\ell\,, \tag{7.12}$$

where $\varepsilon^{1234} = 1$. The angular momentum vector can be written in three pieces:

$$\omega \; = \; \omega_0 \; + \; \omega_1 \; + \; \omega_2 \,. \tag{7.13}$$

where the first two parts are very similar to the those in [102]:

$$\begin{aligned}
\omega_0 \; &= \; \sum_{i=1}^{2} \sum_{p=1}^{n} \frac{Q_{i\,p}\, \dot{F}_m^{(p)} dx^m}{|\vec{x} - \vec{F}^{(p)}(v)|^2} \,, \\
\omega_1 \; &= \; \frac{1}{2} \sum_{p,q=1}^{n} (Q_{1\,p}\, Q_{2\,q} + Q_{2\,p}\, Q_{1\,q})\, \frac{\dot{F}_m^{(p)} dx^m}{R_p^2\, R_q^2} \,.
\end{aligned} \tag{7.14}$$

The last part of the solution, ω_2, is part of our new result and arises from the interaction between non-parallel threads:

$$\omega_2 \; = \; \frac{1}{4} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} (Q_{1p} Q_{2q} + Q_{2p} Q_{1q})\, \frac{\left(\dot{F}_i^{(p)} - \dot{F}_i^{(q)}\right)}{\left|\vec{F}^{(p)} - \vec{F}^{(q)}\right|^2} \left\{ \left(\frac{1}{R_p^2} - \frac{1}{R_q^2} \right) dx^i - \frac{2}{R_p^2 R_q^2}\, \mathcal{A}_{ij}^{(p,q)}\, dx^j \right\}. \tag{7.15}$$

From this one can easily verify that

$$\vec{\nabla} \cdot \vec{\omega} \; = \; -\, \partial_v (Z_1 Z_2) \,, \tag{7.16}$$

which means that the equation for $\mathcal{F}$ simplifies to

$$\begin{aligned}
\nabla^2 \mathcal{F} \; &= \; -\,2 \left[\dot{Z}_1 \dot{Z}_2 \; + \; *_4 \left(\Theta_1 \wedge \Theta_2 \right) \right] \\
&= \; -\,4 \sum_{p,q=1}^{n} (Q_{1p} Q_{2q} + Q_{2p} Q_{1q})\, \frac{1}{R_p^4 R_q^4} \left[(\vec{R}^{(p)} \cdot \vec{R}^{(q)}) \left(\dot{\vec{F}}^{(p)} \cdot \dot{\vec{F}}^{(q)} \right) \; - \; \dot{\vec{F}}^{(p)i} \dot{\vec{F}}^{(q)j}\, \mathcal{A}_{ij}^{(p,q)} \right].
\end{aligned} \tag{7.17}$$

This can be solved by the somewhat obvious guess:

$$
\begin{aligned}
\mathcal{F} \;=\; & -4 \;-\; 4\sum_{p=1}^{n} \frac{Q_{3\,p}}{R_p^2} \;-\; \frac{1}{2}\sum_{p,q=1}^{n} \frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{R_p^2 R_q^2}\Big(\dot{\vec{F}}^{(p)}\cdot\dot{\vec{F}}^{(q)}\Big) \\
& + \sum_{\substack{p,q=1\\ p\neq q}}^{n} (Q_{1p}Q_{2q}+Q_{2p}Q_{1q})\,\frac{1}{R_p^2 R_q^2}\,\frac{\dot{F}_i^{(p)}\,\dot{F}_j^{(q)}\,\mathcal{A}_{ij}^{(p,q)}}{\big|\vec{F}^{(p)}-\vec{F}^{(q)}\big|^2}\,,
\end{aligned}
\tag{7.18}
$$

where the first two terms represent particular choices for the harmonic pieces of $\mathcal{F}$. In normalizing these harmonic pieces we have kept in mind the fact that dimensional reduction to five space-time dimensions yields $\mathcal{F} = -4Z_3$, where Z_3 determines the third electric charge of the solution and is on the same footing (in five dimensions) as Z_1 and Z_2. The terms in ω and $\mathcal{F}$ that contain $\mathcal{A}_{ij}^{(p,q)}$ express the non-trivial interaction between non-parallel superthreads. These terms vanish for solutions with multiple threads of parallel profiles, $\vec{F}(v)$, and hence did not appear in [102].

Finally, there are also possible harmonic pieces that can be added to the angular momentum vector, ω. To define these, introduce the following self-dual harmonic forms on $\mathbb{R}^4$:

$$
\begin{aligned}
\Omega_+^{(1)} &= dx^1\wedge dx^2 \;+\; dx^3\wedge dx^4\,, \\
\Omega_+^{(2)} &= dx^1\wedge dx^3 \;-\; dx^2\wedge dx^4\,, \\
\Omega_+^{(1)} &= dx^1\wedge dx^4 \;+\; dx^2\wedge dx^3\,.
\end{aligned}
\tag{7.19}
$$

Then the following are zero modes of the equation (7.5) that defines ω:

$$
\omega_{harm} \;=\; \sum_{a=1}^{3}\sum_{p=1}^{n} \frac{1}{R_p^4}\, J_p^{(a)}(v)\,\Omega_{+ij}^{(a)}\,R^{(p)i}dx^j\,,
\tag{7.20}
$$

where the $J_p^{(a)}(v)$ are v-dependent angular momentum densities. The one-form in (7.20) is sourced along the profile of the superthread. Moreover, one can easily verify that:

$$*_4\, \tilde{d} *_4 \omega_{harm} \;=\; 0\,, \tag{7.21}$$

and so this induces no additional contribution to $\mathcal{F}$ in (7.6).

7.2.3 Regularity and the near-thread limit

The six-dimensional metric we are considering is:

$$ds^2 \;=\; 2(Z_1 Z_2)^{-\frac{1}{2}} dv \left(du + \omega \;+\; \tfrac{1}{2}\,\mathcal{F}\, dv\right) \;-\; 2(Z_1 Z_2)^{\frac{1}{2}}\, |d\vec{x}|^2\,. \tag{7.22}$$

Regularity requires that $Z_1 Z_2 > 0$ and we will ensure this by taking

$$Q_{1p}\,,\, Q_{2p} \;\geq\; 0\,, \tag{7.23}$$

for all p. Moreover, if one sets all displacements to zero except along the circular fiber parametrized by v then the metric collapses to $ds^2 = (Z_1 Z_2)^{-\frac{1}{2}} \mathcal{F}\, dv^2$, which means that one must require

$$-\,\mathcal{F} \;\geq\; 0 \tag{7.24}$$

everywhere if one is to avoid closed timelike curves. The expression for $\mathcal{F}$ in (7.18) is somewhat complicated but the condition (7.24) can generically be satisfied if one takes Q_{3p} to be positive and large enough. We will discuss this further below.

The near-thread limit is going to be singular because it is locally a three-charge, two-dipole charge object. However we must also ensure that there are no closed time-like curves (CTC's) near the superthreads. To that end we collect all the divergent and finite parts of the metric in the limit $R_p \to 0$:

$$
\begin{aligned}
Z_i &\sim \frac{Q_{ip}}{R_p^2} \;+\; Q_{ip} \;+\; \sum_{q\neq p} \frac{Q_{iq}}{R_{pq}^2} \;+\; \mathcal{O}(R_p)\,, \qquad i=1,2\,,\\
\mathcal{F} &\sim -\frac{Q_{1p}\,Q_{2p}}{R_p^4}\,\big|\dot{\vec{F}}^{(p)}\big|^2 \;-\; \frac{1}{R_p^2}\bigg[4\,Q_{3p} \;+\; \sum_{q\neq p} \frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{F_{pq}^2}\,\big(\dot{\vec{F}}^{(p)}\cdot\dot{\vec{F}}^{(q)}\big)\bigg] \;+\; \mathcal{O}(1)\,,\\
\omega &\sim -\frac{Q_{1p}\,Q_{2p}}{R_p^4}\,\big(\dot{\vec{F}}^{(p)}\cdot d\vec{x}\,\big) \;+\; \frac{1}{R_p^3}\sum_{a=1}^{3} J_p^{(a)}(v)\,\Omega^{(a)}_{+ij}\,\widehat{R}^{(p)i}\,dx^j\\
&\quad + \frac{1}{R_p^2}\bigg[(Q_{1p}+Q_{2p}) \;+\; \sum_{q\neq p}\frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{F_{pq}^2}\bigg]\big(\dot{\vec{F}}^{(p)}\cdot d\vec{x}\,\big) \;+\; \mathcal{O}\Big(\frac{1}{R_p}\Big)\,.
\end{aligned}
\tag{7.25}
$$

where we have included the harmonic pieces, (7.20), of ω and where

$$
F_{pq}^2 \;\equiv\; \big|\vec{F}^{(p)}-\vec{F}^{(q)}\big|^2\,, \qquad \widehat{R}^{(p)} \;\equiv\; \frac{\vec{R}^{(p)}}{R_p}\,. \tag{7.26}
$$

Setting $du=0$, one finds, at leading order as $R_p\to 0$,

$$
ds^2 \;\sim\; \frac{\sqrt{Q_{1p}Q_{2p}}}{R_p^2}\left[\big|\dot{\vec{F}}^{(p)}\big|^2\left(dv - \frac{\dot{\vec{F}}^{(p)}\cdot d\vec{x}}{\big|\dot{\vec{F}}^{(p)}\big|^2}\right)^2 \;+\; dx_\perp^2\right]. \tag{7.27}
$$

where

$$
dx_\perp^2 \;\equiv\; |d\vec{x}|^2 \;-\; \frac{\big|\dot{\vec{F}}^{(p)}\cdot d\vec{x}\big|^2}{\big|\dot{\vec{F}}^{(p)}\big|^2}\,, \tag{7.28}
$$

which is the spatial metric in $\mathbb{R}^4$ perpendicular to the tangent, $\dot{\vec{F}}^{(p)}$, to the superthread. The asymptotic metric (7.27) is manifestly positive but not positive-definite: There is a null direction along the supertube. That is, the leading order terms vanish precisely if one takes

$$d\vec{x} \;=\; \dot{\vec{F}}^{(p)}\, d\lambda\,, \qquad dv \;=\; \big|\dot{\vec{F}}^{(p)}\big|\, d\lambda\,, \tag{7.29}$$

for some infinitesimal displacement, $d\lambda$.

For this displacement one finds a leading order term coming from the harmonic pieces of ω:

$$ds_\lambda^2 \;=\; \frac{d\lambda^2}{R_p}\,\frac{1}{\sqrt{Q_{1p}Q_{2p}}}\,\sum_{a=1}^{3} J_p^{(a)}(v)\,\Omega_{+ij}^{(a)}\,\widehat{R}^{(p)i}\,\dot{F}^{(p)j} \tag{7.30}$$

If one looks in the direction $R_i^{(p)} \sim -\sum_{a=1}^{3} J_p^{(a)}(v)\,\Omega_{+ij}^{(a)}\,\dot{F}^{(p)j}$ one finds that ds_λ^2 is negative and proportional to $\sum_{a=1}^{3}(J_p^{(a)}(v))^2\big|\dot{\vec{F}}^{(p)}\big|^2$. Thus for a superthread with $\dot{\vec{F}}^{(p)} \neq 0$ one can only avoid CTC's if one sets

$$J_p^{(a)}(v) \;=\; 0 \tag{7.31}$$

that is, the harmonic pieces, (7.20), produce CTC's and so must be discarded. The complete physical solution is thus given by $\omega_0 + \omega_1 + \omega_2$ defined in (7.14) and (7.15).

An important consequence of this analysis is that the angular momentum vector is completely determined by the electric charges and profiles of the configuration. This is slightly different from the five-dimensional solutions in which one has independent choices of harmonic functions in the angular momentum vectors and the angular momenta are then fixed in terms of the charges and positions of the sources via bubble equations, or integrability conditions, that remove CTC's. For the six-dimensional solutions presented here one fixes charges, positions and profiles and the angular-momentum vector is adjusted automatically: there are no bubble equations.

Having now killed the leading order of the metric along the displacement (7.29) it turns out that there is a finite order piece. As $R_p \to 0$ the metric becomes:

$$\begin{aligned} ds_\lambda^2 \;=\; & \frac{d\lambda^2}{\sqrt{Q_{1p}Q_{2p}}}\Big[4\,Q_{3p} - \big|\dot{\vec{F}}^{(p)}\big|^2 (Q_{1p}+Q_{2p}) \\ & \qquad - \sum_{q\neq p} \frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{F_{pq}^2}\,\dot{\vec{F}}^{(p)}\cdot\big(\dot{\vec{F}}^{(p)}-\dot{\vec{F}}^{(q)}\big)\Big] \;+\; \mathcal{O}(R_p). \end{aligned} \tag{7.32}$$

Again, to avoid closed timelike curves we require that the quantity in brackets be non-negative, which is equivalent to asking that

$$-\,\mathcal{F} \;\geq\; \dot{F}_i^{(p)}\,\omega_i \tag{7.33}$$

near each thread. Hence the positivity of ds_λ^2 in (7.32) places a lower bound on each of the charges Q_{3p}:

$$Q_{3p} \;\geq\; \frac{1}{4}\big|\dot{\vec{F}}^{(p)}\big|^2(Q_{1p}+Q_{2p}) \;+\; \frac{1}{4}\sum_{q\neq p}\frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{F_{pq}^2}\,\dot{\vec{F}}^{(p)}\cdot\big(\dot{\vec{F}}^{(p)}-\dot{\vec{F}}^{(q)}\big)\,. \tag{7.34}$$

The individual bounds for each p depend upon the detailed geometric layout of the threads but if one sums over all the threads then one obtains a global bound upon the total charges:

$$\sum_{p=1}^{n} Q_{3p} \;\geq\; \frac{1}{4}\sum_{p=1}^{n}\big|\dot{\vec{F}}^{(p)}\big|^2(Q_{1p}+Q_{2p})\,. \tag{7.35}$$

The origins of these bounds can be understood in terms of “charges dissolved in flux”

[49]. From (7.9) one sees that the right-hand sides of (7.34) and (7.35) can be thought of as the dipole-dipole interactions that give rise to an effective electric contribution to the Kaluza-Klein charge described by $\mathcal{F}$. As we will describe below, the harmonic charge term, described by Q_{3p} in $\mathcal{F}$, is the charge measured at infinity and so these bounds mean that the only physically sensible solutions are those in which one does indeed correctly account, at infinity, for the charge coming dipole-dipole interactions.

7.2.4 Asymptotic charges

The electric charges measured at infinity come from the asymptotic forms of Z_1, Z_2 and $Z_3 \equiv -\frac{1}{4}\mathcal{F}$. From the leading ($\mathcal{O}(R^{-2})$) terms in (7.7) and (7.18) one can easily read off the D1, D5, and P charges:

$$\text{D1:} \ \sum_p Q_{1p}, \qquad \text{D5:} \ \sum_p Q_{2p} \qquad \text{P:} \ \sum_p Q_{3p}. \tag{7.36}$$

The terms in the tensor, $\mathcal{A}_{ij}^{(p,q)}$, defined (7.12) do not contribute in $\mathcal{F}$ because $\vec{R}^{(p)}$ and $\vec{R}^{(q)}$ become nearly parallel at large distances and so this term vanishes at leading order.

The asymptotic form of ω can be massaged into

$$
\begin{aligned}
\omega \sim \frac{1}{R^2} \sum_p \big(Q_{1p} + Q_{2p}\big)\, \dot{\vec{F}}^{(p)} \cdot \mathrm{d}\vec{x} + \frac{2}{R^4} \sum_p \big(Q_{1p} + Q_{2p}\big)\big(\vec{R}\cdot \vec{F}^{(p)}\big)\, \dot{\vec{F}}^{(p)} \cdot \mathrm{d}\vec{x} \\
+ \frac{1}{2}\frac{1}{R^4} \sum_{\substack{p,q \\ p\neq q}} \frac{Q_{1p}Q_{2q} + Q_{2p}Q_{1q}}{F_{pq}^2} R^i \Big[F_i^{(pq)} \big(\dot{\vec{F}}^{(pq)} \cdot \mathrm{d}\vec{x}\big) \\
- \dot{F}_i^{(pq)} \big(\vec{F}^{(pq)} \cdot \mathrm{d}\vec{x}\big) + \varepsilon_{ijk\ell} F_j^{(pq)} \dot{F}_k^{(pq)} \, \mathrm{d}x^\ell \Big] \\
+ \frac{1}{2}\frac{1}{R^4} \vec{R}\cdot \mathrm{d}\vec{x} \sum_{\substack{p,q \\ p\neq q}} \frac{Q_{1p}Q_{2q} + Q_{2p}Q_{1q}}{F_{pq}^2} \big(\vec{F}^{(pq)} \cdot \dot{\vec{F}}^{(pq)}\big),
\end{aligned}
\tag{7.37}
$$

where $\vec{F}^{(pq)} \equiv \vec{F}^{(p)} - \vec{F}^{(q)}$. The first term falls of as R^{-1} and is perhaps somewhat unexpected. Mathematically it arises through the contribution of the constant terms in the Z_i to the source for ω in (7.5). These source terms mean that, to leading order, $(1+*_4)\tilde{d}\omega$ limits to $2(\Theta_1+\Theta_2)$ and thus ω inherits an asymptotic behavior given by the vector fields in parentheses in (7.8). In five dimensions, $(\Theta_1 + \Theta_2)$ falls off faster and leads to standard expansions for angular momenta in ω. The presence of the $\mathcal{O}(R^{-1})$ terms in six-dimensions comes because of the v-dependent sources in (7.4). The fact that this term is a total v-derivative means it will always vanish when we reduce to five dimensions. This is because, in order to reduce to five dimensions, the sources must be smeared in a way that kills all v dependence; hence the unusual $\mathcal{O}(R^{-1})$ term disappears and one recovers the standard behavior of five-dimensional solutions. We will illustrate this in the next section.

Physically, the $\mathcal{O}(R^{-1})$ terms represent a *linear* momentum for the configuration. The somewhat unusual feature of the six-dimensional linear system is that all the equations are solved on a constant-v slice and that, for a given value of v, the solution is insensitive to the configuration at other values of v and so, slice-by-slice, the solution sees the superthread as indistinguishable from the thread that carries a linear momentum. It is only when one smears the solution along a closed profile that the solution combines different sections of the solution with different orientations so that the leading momentum behavior cancels and leaves one with a more standard angular momentum.

The second term is in (7.37) is purely rotational, and expresses the difference $J_T \equiv J_1 - J_2$. The third term is the potential of a purely anti-self-dual 2-form, and so it expresses the sum $J_1 + J_2$. The last term is a total derivative, and may be viewed as pure gauge.

7.3 Single supersheets

7.3.1 General supersheets

It is straightforward to take the continuum limit of the multi-superthread solution. The set of profiles, $\vec{F}^{(p)}(v)$, are replaced by a function of two variables, $\vec{F}(\sigma, v)$, the discrete charges, $Q_{i\,p}$, are replaced by density functions, $\rho_i(\sigma)$ and the sums are replaced by integrals. Thus we have

$$Z_i = 1 \; + \; \int_0^{2\pi} \frac{\rho_i(\sigma)\, d\sigma}{|\vec{x} - \vec{F}(\sigma, v)|^2}\,, \tag{7.38}$$

$$\Theta_i \;=\; \tfrac{1}{2}\,(1 + *_4)\, \tilde{d} \left(\int_0^{2\pi} \frac{\rho_i(\sigma)\, \partial_v \vec{F}(\sigma, v) \cdot d\vec{x}}{|\vec{x} - \vec{F}(\sigma, v)|^2}\, d\sigma \right), \tag{7.39}$$

where we have chosen to normalize the smearing over the interval $[0, 2\pi]$. Following (7.11) and (7.12) we define

$$\vec{R}(\sigma) \equiv \vec{x} - \vec{F}(\sigma, v), \qquad R(\sigma) \equiv \left|\vec{R}(\sigma, v, \vec{x})\right|, \tag{7.40}$$

and the tensor

$$\mathcal{A}_{ij}(\sigma_1, \sigma_2) \equiv R_i(\sigma_1)R_j(\sigma_2) - R_j(\sigma_1)R_i(\sigma_2) - \varepsilon^{ijk\ell} R_k(\sigma_1)R_\ell(\sigma_2), \tag{7.41}$$

With these definitions, the rest of the continuum solution can be written

$$\begin{aligned}
\omega_0 &= \sum_{i=1}^{2} \int_0^{2\pi} \frac{\rho_i(\sigma)\,\partial_v \vec{F}(\sigma, v) \cdot d\vec{x}}{|\vec{x} - \vec{F}(\sigma, v)|^2}\, d\sigma, \\
\omega_1 &= \frac{1}{2} \int_0^{2\pi}\int_0^{2\pi} (\rho_1(\sigma_1)\rho_2(\sigma_2) + \rho_2(\sigma_1)\rho_1(\sigma_2))\, \frac{\partial_v \vec{F}(\sigma_1, v) \cdot d\vec{x}}{R(\sigma_1, v, \vec{x})^2\, R(\sigma_2, v, \vec{x})^2}\, d\sigma_1 d\sigma_2,
\end{aligned} \tag{7.42}$$

$$\begin{aligned}
\omega_2 = \frac{1}{4} \int_0^{2\pi}\int_0^{2\pi} & (\rho_1(\sigma_1)\rho_2(\sigma_2) + \rho_2(\sigma_1)\rho_1(\sigma_2))\, \frac{\big(\partial_v F_i(\sigma_1, v) - \partial_v F_i(\sigma_2, v)\big)}{\left|\vec{F}(\sigma_1, v) - \vec{F}(\sigma_2, v)\right|^2} \\
& \left\{ \left(\frac{1}{R(\sigma_1)^2} - \frac{1}{R(\sigma_2)^2} \right) dx^i - \frac{2}{R(\sigma_1)^2 R(\sigma_2)^2}\, \mathcal{A}_{ij}(\sigma_1, \sigma_2)\, dx^j \right\} d\sigma_1 d\sigma_2,
\end{aligned} \tag{7.43}$$

$$
\begin{aligned}
\mathcal{F} \;=\; & -4 \;-\; 4\int_0^{2\pi} \frac{\rho_3(\sigma)}{R(\sigma)^2}\, d\sigma \\
& - \int_0^{2\pi}\!\!\int_0^{2\pi} \big(\rho_1(\sigma_1)\rho_2(\sigma_2) + \rho_2(\sigma_1)\rho_1(\sigma_2)\big)\; \frac{1}{R(\sigma_1)^2 R(\sigma_2)^2} \\
& \qquad \left[\frac{1}{2}\big(\partial_v \vec{F}(\sigma_1, v)\big)\cdot\big(\partial_v \vec{F}(\sigma_2, v)\big) \;-\; \frac{\partial_v F_i(\sigma_1, v)\,\partial_v F_j(\sigma_2, v)\;\; \mathcal{A}_{ij}(\sigma_1,\sigma_2)}{\big|\vec{F}(\sigma_1, v) - \vec{F}(\sigma_2, v)\big|^2}\right] d\sigma_1 d\sigma_2\,.
\end{aligned}
\tag{7.44}
$$

The integrals for ω_2 and $\mathcal{F}$ have potential singularities at the coincidence limits, $\sigma_1 = \sigma_2$, with a double pole coming from the denominator factor of $|\vec{F}(\sigma_1, v) - \vec{F}(\sigma_2, v)|^2$. However, the tensor $\mathcal{A}_{ij}$ has a simple zero as $\sigma_1 \to \sigma_2$ and this skew tensor is further contracted with factors that have simple zeroes in the coincidence limit. Thus there is also a double zero in the numerator leading to a finite contribution in the coincidence limit.

While we have smeared the multi-superthread solution into a single supersheet, it is also clear that one can smear the multi-superthread solutions into multiple supersheets and such solutions will be given by straightforward generalizations of (7.38)–(7.44).

Finally, we note that one can, of course, recover the multi-superthread solutions from this continuum solution by replacing the density functions, ρ_a, by sums over delta functions:

$$
\rho_a(\sigma) \;=\; \sum_{j=1}^{N} Q_{a\,p}\,\delta\big(\sigma - \sigma^{(p)}\big)\,, \qquad a = 1,2,3\,, \tag{7.45}
$$

and where the individual profile functions are specified by the sampled values of $\vec{F}(\sigma, v)$:

$$
\vec{F}^{(p)}(v) \;=\; \vec{F}\big(\sigma^{(p)}, v\big)\,. \tag{7.46}
$$

7.3.2 The five-dimensional generalized supertube as a supersheet

The supersheets described above are sourced by sheet profiles described by arbitrary functions of *two* variables and are thus much more general than previously-known solutions. However, it is worthwhile to smear our solutions in a more trivial way in order to see exactly how five-dimensional solutions emerge. Therefore we give an example that produces a v-independent sheet profile, allowing us to reduce on the v fiber and obtain a standard five-dimensional solution.

A useful, non-trivial way to accomplish this is to choose any profile $\vec{F}(\sigma)$ in $\mathbb{R}^4$, and define $\vec{F}(\sigma, v) = \vec{F}(\sigma + \kappa v)$. The result should then be a solution of the standard, linear BPS system in five dimensions [22, 49]. One should also directly recover physical constraints like radius relations. For simplicity, we will take the charge densities to be constant and we will smear a simple helical configuration that will produce a cylinder along v and a ring in $\mathbb{R}^2 \subset \mathbb{R}^4$:

$$\vec{F}(\sigma, v) \;=\; \Big(0, 0, a\cos(\kappa v + \sigma), a\sin(\kappa v + \sigma)\Big)\,, \tag{7.47}$$

where κ and a are constants with $\kappa = \frac{2n\pi}{L}$, for $n \in \mathbb{Z}$. Each thread will have a constant charge distribution, given by

$$\rho_i(\sigma) \equiv \frac{Q_i}{2\pi}. \tag{7.48}$$

To carry out the integrals (7.38), (7.42), (7.43), (7.44), it is easiest to work in polar coordinates on $\mathbb{R}^2 \times \mathbb{R}^2$ given by:

$$x^1 = \eta\cos\phi, \qquad x^2 = \eta\sin\phi, \qquad x^3 = \zeta\cos\psi, \qquad x^4 = \zeta\sin\psi. \tag{7.49}$$

Note that $R^2 = \eta^2 + \zeta^2$. From these coordinates we can easily go to spherical coordinates by defining $\eta = R\cos\theta$ and $\zeta = R\sin\theta$. Then, for example, we obtain Z_1 by integrating

$$\begin{aligned} Z_1 &= 1 + \frac{Q_1}{2\pi}\int_0^{2\pi} d\sigma\, \frac{1}{\eta^2+\zeta^2+a^2-2a\zeta\cos(\sigma+\kappa v)} \\ &= 1 + \frac{Q_1}{\sqrt{(\eta^2+\zeta^2+a^2)^2-4a^2\zeta^2}}. \end{aligned} \tag{7.50}$$

The rest of the integrals are tedious, but straightforward. The result is

$$Z_{1,2} = 1 + \frac{Q_{1,2}}{\Sigma}, \qquad \mathcal{F} = -4 - \frac{4\,Q_3}{\Sigma} - \kappa^2\, Q_1 Q_2\, \frac{1}{\Sigma}\bigg(\frac{\eta^2+\zeta^2}{\Sigma} - 1\bigg), \tag{7.51}$$

$$\omega = \frac{\kappa}{2}\,(Q_1+Q_2)\bigg(\frac{\eta^2+\zeta^2+a^2}{\Sigma} - 1\bigg)\, d\psi + \kappa\, Q_1 Q_2\, \frac{1}{\Sigma^2}\,\big(\eta^2\, d\phi + \zeta^2\, d\psi\big), \tag{7.52}$$

where we have defined

$$\Sigma \equiv \sqrt{(\eta^2+\zeta^2+a^2)^2-4a^2\zeta^2} \;=\; \sqrt{(R^2+a^2)^2-4a^2R^2\sin^2\theta}. \tag{7.53}$$

At infinity, these behave as

$$Z_{1,2} \;\sim\; 1 + \frac{Q_{1,2}}{R^2}\,, \qquad\qquad Z_3 \;=\; -\tfrac{1}{4}\,\mathcal{F} \;\sim\; 1 + \frac{Q_3}{R^2} \tag{7.54}$$

$$\omega \;\sim\; \frac{\kappa}{R^2}\big[\big((Q_1+Q_2)\,a^2 + Q_1Q_2\big)\sin^2\theta\, d\psi + Q_1Q_2\,\cos^2\theta\, d\phi\big] \tag{7.55}$$

$$= \;\frac{1}{2\,R^2}\big[J_1\,\sin^2\theta\, d\psi \;+\; J_2\,\cos^2\theta\, d\phi\big]\,, \tag{7.56}$$

where the five-dimensional angular momentum vector, k, is related to ω via $\omega = 2k$. This explains the factor of 2 in (7.56).

This solution corresponds, as expected, to the three-charge, two-dipole-charge generalized supertube [105], with charges Q_1, Q_2, Q_3, and dipole charges

$$q^1 \equiv -\frac{\kappa Q_2}{2}, \qquad q^2 \equiv -\frac{\kappa Q_1}{2}, \qquad q^3 \equiv 0. \tag{7.57}$$

We define $\widetilde{Q}_3$ as

$$\widetilde{Q}_3 \;\equiv\; Q_3 - \tfrac{1}{4}\,\kappa^2 Q_1 Q_2 \tag{7.58}$$

Note that $\widetilde{Q}_3$ is the constituent electric charge while the charge measured at infinity, Q_3, also contains the charge arising from the dipole-dipole interaction.

From (7.56) one can read off the angular momenta and one can also check that the radius relation:

$$J_T \;\equiv\; J_1 - J_2 \;=\; -\tfrac{1}{2}\,\kappa\, a^2\,(Q_1 + Q_2) \;=\; (q^1 + q^2 + q^3)\, a^2 \tag{7.59}$$

is satisfied automatically.

The condition that one has $\mathcal{F} \leq 0$ globally implies that $\widetilde{Q}_3 \geq 0$ and hence:

$$Q_3 \;\geq\; \tfrac{1}{4}\,\kappa^2\, Q_1 Q_2 \;=\; q^1\, q^2 \tag{7.60}$$

This is simply the continuum analog of (7.35).

Near the ring, we find that to avoid CTC's one must have:

$$Q_1 Q_2 \left(\widetilde{Q}_3 - \tfrac{1}{4}\,\kappa^2 a^2 (Q_1 + Q_2)\right) \;\geq\; 0\,, \tag{7.61}$$

and hence

$$\widetilde{Q}_3 \;\geq\; \tfrac{1}{4}\,\kappa^2 a^2\,(Q_1+Q_2) \;=\; \tfrac{1}{2}\,\kappa\,J_T\,. \tag{7.62}$$

This is not quite the same as the continuum limit of (7.34) because the latter bound was derived assuming that R_q remained finite as $R_p \to 0$ whereas the continuum limit gets other important terms in from the coincidence limits when two threads approach one another. This is evident from the fact that the general integrals in Section 7.3 are finite in the coincidence limit but the continuum limit of (7.34) involves a divergent integral.

We have thus recovered one of the standard five-dimensional solutions. The process of obtaining a solution in five dimensions usually involves choosing some harmonic functions and then adjusting the coefficients so as to avoid closed timelike curves. These choices are already implicit in our six-dimensional solution and emerge directly in the smeared solution.

7.3.3 A corrugated supersheet

In the previous calculation by considering superthreads of helical profile the smearing integrals for a single supersheet were explicitly calculated. Although the profile functions depended on both v and σ the resulting supersheet was independent of v and it matched a special class of already known five-dimensional solutions [103], [104], [105]. Here we want to extend these results by considering a slightly more general choice of profile functions so that the resulting supersheet can also fluctuate in the coordinate v. Thus, we choose

$$\vec{F}(v,\sigma) = (A(\lambda v)\cos(\kappa v+\sigma)\,,\; A(\lambda v)\sin(\kappa v+\sigma)\,,\; 0\,,\; 0). \tag{7.63}$$

We once again choose helical profile functions, but now the radius A, instead of being constant, is an arbitrary function of v. The resulting supersheet will have a circular profile in $\mathbb{R}^4$ for a specific value of v, with varying circle radius as we move along the v direction. The constants λ and κ are of the form $2\pi n/L$, where n is an integer, and can in principle be independent. Although our results are for a circular profile we believe that they can be generalized to any closed, non-intersecting curve by constructing the appropriate Green's functions.

For the $\mathbb{R}^4$ base space with take double polar coordinates

$$x_1 = \eta \cos\psi\,,\; x_2 = \eta \sin\psi\,,\; x_3 = \zeta \cos\phi\,,\; x_4 = \zeta \sin\phi, \tag{7.64}$$

from which we can go to spherical coordinates by substituting

$$\eta = r\sin\theta\,,\; \zeta = r\cos\theta. \tag{7.65}$$

The solution

To describe a single supersheet we need to calculate the integrals (7.85) with no summations and the latin indices removed. We also consider a constant charge distribution

$$\rho_m(\sigma) = \frac{Q_m}{2\pi}. \tag{7.66}$$

The integrals are easily calculated by going to the complex plane and summing the residues of the poles that are within the unit circle. The double integrals transform to a double complex integral and the residues of the first complex integration act as integrand functions for the second complex integral. Then for the functions describing the solution we find

$$Z_1 = 1 + \frac{Q_1}{\Sigma}\,,\ Z_2 = 1 + \frac{Q_2}{\Sigma}, \tag{7.67}$$

where

$$\Sigma = \sqrt{(A^2+\eta^2+\zeta^2)^2 - 4A^2\eta^2} = \sqrt{(A^2+r^2)^2 - 4A^2r^2\sin^2\theta}, \tag{7.68}$$

is the position of the supersheet in $\mathbb{R}^4$ and is now v dependent via the function $A(\lambda v)$. Similarly we have

$$\mathcal{F} = -4 - \frac{4Q_3A^2 - Q_1Q_2\left(\lambda^2\acute{A}^2 + \kappa^2A^2\right)}{A^2\Sigma} - Q_1Q_2\frac{\left(\lambda^2\acute{A}^2 + \kappa^2A^2\right)(\eta^2+\zeta^2)}{A^2\Sigma^2}. \tag{7.69}$$

For the angular momentum one obtains

$$\begin{aligned}\omega = &\frac{(Q_1+Q_2)}{2\eta A}\left(-1 + \frac{A^2+\eta^2+\zeta^2}{\Sigma}\right)\left(\kappa A\eta d\psi + \lambda\acute{A}d\eta\right) + \\ &+ Q_1Q_2\frac{\eta}{A\Sigma^2}\left(\eta\kappa A d\psi + \lambda\acute{A}d\eta\right) + Q_1Q_2\frac{\zeta}{A\Sigma^2}\left(\zeta\kappa A d\phi + \lambda\acute{A}d\zeta\right),\end{aligned} \tag{7.70}$$

where

$$\partial_v A = \dot{A} = \acute{A}\lambda. \tag{7.71}$$

It is interesting to observe that the functions (7.69) and (7.70) , because of linearity of the BPS equations, can be considered as the “superposition of modes” occurring from two distinct cases of supersheets: one made from superthreads of helical profile with constant radius ($\lambda = 0$) and another made of straight superthreads with a corrugated profile ($\kappa = 0$). Thus corrugated and helical modes of the solution are independent, originating from the fact that the profile function factorizes and each mode is sourced by a different factor of (7.63).

The $\mathbb{R}^4$ coordinates on which the angular momentum one form ω has legs depend on $\dot{F}_i dx^i$, (7.85). Thus the helical mode generates the components $d\psi$ and $d\phi$. The corrugated mode generates new components $d\eta$ and $d\zeta$ along radial directions, expressing that the $\mathbb{R}^4$ circular profile changes radius as we move along v.

Regularity and asymptotic charges

Before examining our solution for closed timelike curves there are some restrictions to be placed on the function A. In general there is the possibility of antipodal points of the $\mathbb{R}^4$ circular profile to intersect over each other at specific values of $v = v_0$. For our choice of profile functions this happens exactly when $A(\lambda v_0) = 0$. Then the functions $\mathcal{F}$ and ω diverge. Thus we need

$$A(\lambda v) \neq 0 \quad \forall v \in [0, L]. \tag{7.72}$$

For example for the simple choice $A = b + a\cos(\lambda v)$ we need $b > a$. However, since A is a periodic function of v there will be points $v = v_1$ such that $\dot{A}(\lambda v_1) = 0$. For these values of v the solution and its physical analysis exactly matches the one for the supersheet where A is constant. That might lead to an interesting perturbative approach in exploring the superstratum by expanding around the points $\dot{A}(\lambda v_1) = 0$ where the superstratum should match the v-independent supertube solution. Thus the superstratum can possibly be realized as additional perturbation modes in the supertube solution around these points. We further comment on this idea in section 7.5. A different perturbative approach to the superstratum

[3], which differs from what we discuss here has recently appeared in [106].

We read the asymptotic charges of the solution from the expansion of Z_1, Z_2, $\mathcal{F}$ and ω for $r \to \infty$. The asymptotic electric charges are

$$Q_{1,\infty} = Q_1 \, , \; Q_{2,\infty} = Q_2 \, , \; Q_{3,\infty} = Q_3 \tag{7.73}$$

and from the expansion of ω we get

$$\omega \sim \frac{1}{r^2}\left(\Big(\kappa A^2(Q_1+Q_2)+\kappa Q_1 Q_2\Big)\sin^2\theta d\psi + \kappa Q_1 Q_2 \cos^2\theta d\phi + \frac{1}{2}(Q_1+Q_2)\lambda A\dot{A}\sin(2\theta)d\theta\right). \tag{7.74}$$

Thus, in contrast to the case that the supersheet is v-independent there is an additional term proportional to $\dot{A}$ along the θ direction.

To examine the solution for closed timelike curves it is useful to rewrite the metric (7.1) by completing the squares

$$ds^2 = H^{-1}\mathcal{F}\left(dv+\beta+\mathcal{F}^{-1}(du+\omega)\right)^2 - H^{-1}\mathcal{F}^{-1}\left(du+\omega\right)^2 - H ds_4^2. \tag{7.75}$$

Taking a slice of u =constant the absence of closed timelike curves requires that the following conditions hold globally

$$\mathcal{F} \leq 0, \tag{7.76}$$

$$ds_4^2 + \frac{\omega^2}{H^2\mathcal{F}} \geq 0. \tag{7.77}$$

[3] The superstratum discussed in [106] would be a generalization of the three electric one, magnetic dipole charge supertube presented in [18].

From (7.76) for $r \to 0$ we get

$$Q_3 \geq \frac{Q_1 Q_2 \left(\lambda^2 \acute{A}^2 + \kappa^2 A^2\right)}{4A^2}. \tag{7.78}$$

The charge Q_3 should be big enough so that is greater than the right hand side of (7.78) for every value of v. Since Q_3 enters $\mathcal{F}$ as a harmonic term, the charge Q_3 can be an arbitrary function of v without affecting the solution. Consequently by defining

$$Q_3(v) = \widehat{Q}_3 \frac{\left(\lambda^2 \acute{A}^2 + \kappa^2 A^2\right)}{\kappa^2 A^2}, \tag{7.79}$$

we get a result similar to the five dimensional case

$$\widehat{Q}_3 \geq \frac{1}{4} \kappa^2 Q_1 Q_2, \tag{7.80}$$

where $\widehat{Q}_3$ is an effective v-independent charge. From (7.77) by taking the near supersheet limit $\Sigma \to 0$ we obtain from the leading order term

$$\frac{1}{\kappa^2 A^2 + \lambda^2 \acute{A}^2} \left(A\kappa dr - \acute{A}\lambda r \sin^2\theta d\psi - \acute{A}\lambda r \cos^2\theta d\phi\right)^2 + r^2 \sin^2\theta \cos^2\theta \left(d\psi + d\phi\right)^2 + r d\theta^2, \tag{7.81}$$

which is always positive. Suppose we have $\dot{A}(\lambda v_1) = 0$ at some point v_1, then we can make the leading order term vanish by choosing, consistently with the $\Sigma \to 0$ limit, $r \to A$, $\theta \to \pi/2$. By considering the next to leading order term in the expansion we get the constraint given for a non-corrugated supersheet, which is

$$Q_1 Q_2 \left(\widetilde{Q}_3(v_1) - \frac{1}{4}\kappa^2 A(\lambda v_1)(Q_1 + Q_2) \right) \geq 0. \tag{7.82}$$

Thus

$$\widetilde{Q}_3(v_1) \geq \frac{1}{4}\kappa^2 A(\lambda v_1)(Q_1 + Q_2), \tag{7.83}$$

where we defined

$$\widetilde{Q}_3(v) = Q_3 - \frac{Q_1 Q_2 \left(\lambda^2 \acute{A}^2 + \kappa^2 A^2 \right)}{4A^2}. \tag{7.84}$$

Also in the limit $\Sigma \to 0$ the metric function blows up and thus the corrugated supersheet geometry is singular. This is consistent with the five-dimensional non-corrugated supersheet picture and the comment in the beginning of section 7.2.1 that because $\beta = 0$ these geometries can be thought as a collection of slices of constant v. Thus for every value of v we have a singularity with radial profile in $\mathbb{R}^4$ and the collection of all the different v-slices creates a singular six-dimensional geometry.

7.4 Multi-Supersheets

7.4.1 General multi-supersheets

Here we give the immediate generalization for multiple supersheets with arbitrary and independent two-dimensional profiles. Thus for a multi-supersheet solution the profile functions $\vec{F}_p(v)$ become functions of two variables $\vec{F}_I(\sigma^{(I)}, v)$ and the discrete charges Q_{mp} are being replaced by density functions $\rho^{(I)}(\sigma^{(I)})$. In the above we used capital latin indices to separate between different supersheets of the solution. Then we have

$$
\begin{aligned}
Z_m &= 1 \;+\; \sum_I \int_0^{2\pi} \frac{\rho_m^{(I)}(\sigma^{(I)})\, d\sigma^{(I)}}{|\vec{x} - \vec{F}_I(\sigma^{(I)}, v)|^2}, \\
\mathcal{F} \;&=\; -4 \;-\; \sum_I 4 \int_0^{2\pi} \frac{\rho_3^{(I)}(\sigma^{(I)})}{R^{(I)}(\sigma^{(I)})^2}\, d\sigma^{(I)} \\
&- \sum_{I,J} \int_0^{2\pi} \int_0^{2\pi} (\rho_1^{(I)}(\sigma_1^{(I)})\rho_2^{(J)}(\sigma_2^{(J)}) + \rho_2^{(I)}(\sigma_1^{(I)})\rho_1^{(J)}(\sigma_2^{(J)}))\; \frac{1}{R^{(I)}(\sigma_1^{(I)})^2 R^{(J)}(\sigma_2^{(J)})^2} \\
&\left[\frac{1}{2} \left(\partial_v \vec{F}_I(\sigma_1^{(I)}, v)\right) \cdot \left(\partial_v \vec{F}_J(\sigma_2^{(J)}, v)\right) \;-\; \frac{\partial_v F_{I,i}(\sigma_1^{(I)}, v)\, \partial_v F_{J,j}(\sigma_2^{(J)}, v)\;\; \mathcal{A}_{ij}^{(IJ)}(\sigma_1^{(I)}, \sigma_2^{(J)})}{\left|\vec{F}_I(\sigma_1^{(I)}, v) - \vec{F}_J(\sigma_2^{(J)}, v)\right|^2} \right] d\sigma_1^{(I)} d\sigma_2^{(J)}, \\
\omega_0 \;&=\; \sum_I \sum_{m=1}^{2} \int_0^{2\pi} \frac{\rho_m^{(I)}(\sigma^{(I)})\, \partial_v \vec{F}_I(\sigma^{(I)}, v) \cdot d\vec{x}}{|\vec{x} - \vec{F}_I(\sigma^{(I)}, v)|^2}\, d\sigma^{(I)}, \\
\omega_1 \;&=\; \frac{1}{2} \sum_{I,J} \int_0^{2\pi} \int_0^{2\pi} \frac{(\rho_1^{(I)}(\sigma_1^{(I)})\rho_2^{(J)}(\sigma_2^{(J)}) + \rho_2^{(I)}(\sigma_1^{(I)})\rho_1^{(J)}(\sigma_2^{(J)}))\partial_v \vec{F}_I(\sigma_1^{(I)}, v) \cdot d\vec{x}}{R^{(I)}(\sigma_1^{(I)}, v, \vec{x})^2\, R^{(J)}(\sigma_2^{(J)}, v, \vec{x})^2}\, d\sigma_1^{(I)} d\sigma_2^{(J)}, \\
\omega_2 \;&=\; \frac{1}{4} \sum_{I,J} \int_0^{2\pi} \int_0^{2\pi} (\rho_1^{(I)}(\sigma_1^{(I)})\rho_2^{(J)}(\sigma_2^{(J)}) + \rho_2^{(I)}(\sigma_1^{(I)})\rho_1^{(J)}(\sigma_2^{(J)}))\, \frac{\left(\partial_v F_{I,i}(\sigma_1^{(I)}, v) - \partial_v F_{J,i}(\sigma_2^{(J)}, v)\right)}{\left|\vec{F}_I(\sigma_1^{(I)}, v) - \vec{F}_J(\sigma_2^{(J)}, v)\right|^2} \\
&\left\{ \left(\frac{1}{R^{(I)}(\sigma_1^{(I)})^2} - \frac{1}{R^{(J)}(\sigma_2^{(J)})^2} \right) dx^i \;-\; \frac{2}{R^{(I)}(\sigma_1^{(I)})^2 R^{(J)}(\sigma_2^{(J)})^2}\, \mathcal{A}_{ij}^{IJ}(\sigma_1^{(I)}, \sigma_2^{(J)})\, dx^j \right\} d\sigma_1^{(I)} d\sigma_2^{(J)},
\end{aligned}
\tag{7.85}
$$

where we define

$$
\begin{aligned}
\vec{R}^{(I)}(\sigma^{(I)}) \;&\equiv\; \vec{x} \;-\; \vec{F}_I(\sigma^{(I)}, v), \\
\mathcal{A}_{ij}^{IJ}(\sigma_1^{(I)}, \sigma_2^{(J)}) \;&\equiv\; R_i^{(I)}(\sigma_1^{(I)}) R_j^{(J)}(\sigma_2^{(J)}) - R_j^{(I)}(\sigma_1^{(I)}) R_i^{(J)}(\sigma_2^{(J)}) \;-\; \varepsilon^{ijk\ell} R_k^{(I)}(\sigma_1^{(I)}) R_\ell^{(J)}(\sigma_2^{(J)}).
\end{aligned}
\tag{7.86}
$$

The capital latin indices in the integration variables $\sigma_i^{(I)}$ of (7.85), (7.86) are dummy indices and can be removed. For the single integrals this is a trivial observation, while for the double integrals it is based on the symmetry of the integrand in exchanging $I \leftrightarrow J$. The structure of the equations (7.85) indicates that a mult-supersheet solution will consist of two parts. First, there will be a summation over individual supersheets of different radii and profiles coming from single summation terms and from double summations when $I = J$. Secondly there will be interaction terms between the different supersheets coming from double summations when $I \neq J$. Furthermore, because superthreads interact in pairs so will the resulting supersheets. Thus the solution naturally decomposes as

$$\begin{aligned} Z_i &= 1 + \sum H_i^{(I)}, \\ \omega &= \sum_I \omega^{(I)} + \sum_{I \neq J} \omega^{(I,J)}, \\ \mathcal{F} &= -4 + \sum_I \mathcal{F}^{(I)} + \sum_{I \neq J} \mathcal{F}^{(I,J)}. \end{aligned} \tag{7.87}$$

Because the integrands of ω_2 and $\mathcal{F}$ are symmetric in exchanging $I \leftrightarrow J$, we have $\omega_2^{(I,J)} = \omega_2^{(J,I)}$ and $\mathcal{F}^{(I,J)} = \mathcal{F}^{(J,I)}$. So we can further reduce (7.87) to

$$\begin{aligned} Z_i &= 1 + \sum H_i^{(I)}, \\ \omega &= \sum_I \omega^{(I)} + \sum_{I<J} \left(\omega_1^{(I,J)} + \omega_1^{(J,I)} + 2\omega_2^{(I,J)} \right), \\ \mathcal{F} &= -4 + \sum_I \mathcal{F}^{(I)} + 2 \sum_{I<J} \mathcal{F}^{(I,J)}. \end{aligned} \tag{7.88}$$

7.4.2 Corrugated multi-supersheets

Here we calculate the smearing integrals (7.85) for corrugated multi-supersheets. Generalizing (7.63) we separate the superthreads at different sets A_I with profile functions at each set

$$\vec{F}_I(v,\sigma^{(I)}) = (A_I(\lambda_I v)\cos(\kappa_I v + \sigma^{(I)})\,,\; A_I(\lambda_I v)\sin(\kappa_I v + \sigma^{(I)})\,,\; 0\,,\; 0), \tag{7.89}$$

with $A_I < A_J$ for $I < J \quad \forall v \in [0, L]$.

To each set of threads we assign a constant charge distribution given by

$$\rho_i^{(I)}(\sigma^{(I)}) = \frac{Q_i^{(I)}}{2\pi}. \tag{7.90}$$

The solution

These supersheets have concentric circular profiles in $\mathbb{R}^4$ with radii $A_I(\lambda_I v)$ which fluctuate as we move along the v coordinate. In general they are also non-parallel as each one has its own function $A_I(\lambda_I v)$ that oscillates along v. By using (7.88) we write the solution as a combination of individual supersheets and interaction terms. For the individual supersheets terms we have

$$
\begin{aligned}
H_m^{(I)} &= \frac{Q_m^{(I)}}{\Sigma_I}, \\
\mathcal{F}^{(I)} &= -\frac{4Q_3^{(I)}A_I^2 - Q_1^{(I)}Q_2^{(I)}\left(\lambda_I^2 \acute{A_I}^2 + \kappa_I^2 A_I^2\right)}{A_I^2\Sigma_I} - Q_1^{(I)}Q_2^{(I)}\frac{\left(\lambda_I^2\acute{A_I}^2 + \kappa_I^2 A_I^2\right)(\eta^2+\zeta^2)}{A_I^2\Sigma_I^2}, \\
\omega^{(I)} &= \frac{(Q_1^{(I)}+Q_2^{(I)})}{2\eta A_I}\left(-1 + \frac{A_I^2+\eta^2+\zeta^2}{\Sigma_I}\right)\left(\kappa_I A_I \eta d\psi + \lambda_I \acute{A}_I d\eta\right) + \\
&+ Q_1^{(I)}Q_2^{(I)}\frac{\eta}{A_I\Sigma_I^2}\left(\eta\kappa_I A_I d\psi + \lambda_I\acute{A}_I d\eta\right) + Q_1^{(I)}Q_2^{(I)}\frac{\zeta}{A_I\Sigma_I^2}\left(\zeta\kappa_I A_I d\phi + \lambda_I \acute{A}_I d\zeta\right)
\end{aligned}
\tag{7.91}
$$

and for the interaction terms

$$
\begin{aligned}
\mathcal{F}^{(I,J)} &= -\frac{4Q_3^{(I,J)}}{\Sigma_I} - \frac{Q_1^{(I)}Q_2^{(J)}+Q_2^{(I)}Q_1^{(J)}}{2}\left(\lambda_I\lambda_J\acute{A}_I\acute{A}_J + \kappa_I\kappa_J A_I A_J\right)\left(-\frac{1}{A_IA_J\Sigma_I} + \frac{\eta^2+\zeta^2}{A_IA_J\Sigma_I\Sigma_J}\right), \\
\omega_1^{(I,J)} &= \frac{Q_1^{(I)}Q_2^{(J)}+Q_2^{(I)}Q_1^{(J)}}{4\eta A_J\Sigma_I\Sigma_J}\left(A_J^2+\eta^2+\zeta^2-\Sigma_J\right)\left(\kappa_J A_J\eta d\psi + \lambda_J\acute{A}_J d\eta\right), \\
\omega_2^{(I,J)} &= \left(Q_1^{(I)}Q_2^{(J)}+Q_2^{(I)}Q_1^{(J)}\right)\Bigg(\left(-G_1^{(I,J)}\kappa_I + G_2^{(I,J)}\kappa_J\right)d\psi + \\
&+ \left(-\frac{\acute{A}_I}{A_I\eta}G_1^{(I,J)}\lambda_I + \frac{\acute{A}_J}{A_J\eta}G_2^{(I,J)}\lambda_J\right)d\eta + \\
&+ \left(L_1^{(I,J)}A_IA_J(\kappa_I+\kappa_J) + L_2^{(I,J)}(\kappa_I A_I^2 + \kappa_J A_J^2)\right)d\phi + \\
&+ \frac{1}{\zeta}\left(L_1^{(I,J)}(\lambda_I A_J\acute{A}_I + \lambda_J A_I\acute{A}_J) + L_2^{(I,J)}(\lambda_I A_I\acute{A}_I + \lambda_J A_J\acute{A}_J)\right)d\zeta\Bigg),
\end{aligned}
\tag{7.92}
$$

where as in (7.88) we require $I < J$ and we also define

$$\Sigma_I = \sqrt{(A_I^2 + \eta^2 + \zeta^2)^2 - 4A_I^2\eta^2} = \sqrt{(A_I^2 + r^2)^2 - 4A_I^2 r^2 \sin^2\theta} \tag{7.93}$$

and we have introduced the functions

$$\begin{aligned}
G_1^{(I,J)} &= \frac{(A_I^2 + \Sigma_J + \zeta^2)(A_I^2 - \Sigma_I + \zeta^2) + \eta^2(2\zeta^2 - 2A_I^2 + \Sigma_J - \Sigma_I) + \eta^4}{8(A_I^2 - A_J^2)\Sigma_I\Sigma_J}, \\
G_2^{(I,J)} &= \frac{A_J^4 + 2A_I^2\Sigma_J - (\Sigma_I - \eta^2 - \zeta^2)(\Sigma_J + \eta^2 + \zeta^2) - A_J^2(\Sigma_I + \Sigma_J + 2\eta^2 - 2\zeta^2)}{8(A_I^2 - A_J^2)\Sigma_I\Sigma_J}, \\
L_1^{(I,J)} &= \frac{A_I A_J \zeta^2 \left(A_I^2(\Sigma_J - 2\eta^2) + (\Sigma_I + \Sigma_J)(\eta^2 + \zeta^2) + A_J^2(\Sigma_I + 2\eta^2)\right)}{(A_I^2 - A_J^2)\Sigma_I\Sigma_J(A_J^2 - A_I^2 + \Sigma_I + \Sigma_J)(A_I^2(\Sigma_J - \eta^2 - \zeta^2) + A_J^2(\Sigma_I + \eta^2 + \zeta^2))}, \\
L_2^{(I,J)} &= \frac{\zeta^2 \left((A_I^2 - A_J^2)\Sigma_I\Sigma_J - (A_I^2 + A_J^2)(A_I^2 + \eta^2 + \zeta^2)\Sigma_J\right)}{2(A_I^2 - A_J^2)\Sigma_I\Sigma_J(A_J^2 - A_I^2 + \Sigma_I + \Sigma_J)(A_I^2(\Sigma_J - \eta^2 - \zeta^2) + A_J^2(\Sigma_I + \eta^2 + \zeta^2))} - \\
&- \frac{\zeta^2 \left((A_J^2 + \eta^2 + \zeta^2)\left(A_J^2(\Sigma_I + \eta^2 + \zeta^2) + A_I^2(A_J^2 + \Sigma_I - \eta^2 - \zeta^2) - A_I^4\right)\right)}{2(A_I^2 - A_J^2)\Sigma_I\Sigma_J(A_J^2 - A_I^2 + \Sigma_I + \Sigma_J)(A_I^2(\Sigma_J - \eta^2 - \zeta^2) + A_J^2(\Sigma_I + \eta^2 + \zeta^2))}.
\end{aligned} \tag{7.94}$$

Although the integrands of the interaction terms $\omega_2^{(I,J)}$ and $\mathcal{F}^{(I,J)}$ are symmetric under $I \leftrightarrow J$, the functions that occur after smearing are not. This is due to the fact that while calculating the integrals we had to make use of $A_I < A_J$ for $I < J$. Our solutions match the already studied five-dimensional solutions provided we set all $\lambda_I = 0$. In the five-dimensional case however one writes down the solution with freely adjustable parameters and positions and the interaction terms are symmetric under $I \leftrightarrow J$. Then one has to separately solve integrability conditions (also called bubble equations), coming from requiring the absence of Dirac strings, that relate the positions of the objects with their electric and dipole magnetic charges. Here the solution is by construction free of Dirac strings and that is reflected in the lack of symmetry under $I \leftrightarrow J$.

To this end it is useful to provide additional details on how the asymmetry in $I \leftrightarrow J$ arises from the multi-supersheet integrals (7.85). To calculate any double integral expressing the interactions between the supersheets I and J we go to the complex plane by introducing complex variables $z = e^{i\sigma_1}$, $w = e^{i\sigma_2}$. We perform a double complex integration where the residues of the first integral become the integrand of the second. At each integral we pick the poles that are within the unit circle. The denominator of the integrand contains the factor $(A_I z - A_J w)(A_J z - A_I w)$, coming from the term $|\vec{F}_I(\sigma_1, v) - \vec{F}_J(\sigma_2, v)|^2$ in (7.85). As a result when performing the first integral (either over z or w) we need to know which of the two ratios $\frac{A_I}{A_J}$ or $\frac{A_J}{A_I}$ is less than one, to pick the pole within the unit circle, and thus the ordering of the supersheets matters.

7.4.3 Regularity and asymptotic charges

Once again we have to verify that our solution is free of closed timelike curves (7.76), (7.77). For each of the terms of the solution representing an individual supesheet the analysis of section 7.3.3 can be repeated in exactly the same manner and the conditions (7.76) and (7.77) give exactly the conditions (7.78) - (7.84) with the capital latin index I included to specify the supersheet we are referring to. In the multi-supersheet case though one has to examine the interaction terms as well. Imposing

$$\mathcal{F}^{(I,J)} \leq 0, \tag{7.95}$$

we get from (7.92) the additional constraint

$$Q_3^{(I,J)} \geq \left(Q_1^{(I)}Q_2^{(J)} + Q_2^{(I)}Q_1^{(J)}\right)\frac{\lambda_I\lambda_J\acute{A}_I\acute{A}_J + \kappa_I\kappa_J A_I A_J}{8A_I A_J}. \tag{7.96}$$

Similarly to (7.79) we can define an effective v-independent interaction charge $\widehat{Q}_3^{(I,J)}$ such that

$$Q_3^{(I,J)}(v) = \widehat{Q}_3^{(I,J)}\frac{\lambda_I\lambda_J\acute{A}_I\acute{A}_J + \kappa_I\kappa_J A_I A_J}{\kappa_I\kappa_J A_I A_J}. \tag{7.97}$$

Then (7.96) reduces to the condition we would take for the five-dimensional case ($\lambda_I = \lambda_J = 0$)

$$\widehat{Q}_3^{(I,J)} \geq \left(Q_1^{(I)}Q_2^{(J)} + Q_2^{(I)}Q_1^{(J)}\right)\frac{\kappa_I\kappa_J}{8}. \tag{7.98}$$

One should also verify that the global condition (7.77) is satisfied by numerically examining the space between the different supersheets for specific choices of the functions $A_I(v)$. We partially perform this analysis later and find there are areas of the parameter space of solutions for which (7.77) is satisfied. In general, because of the correspondence to the five-dimensional case, if the supersheets are well separated and the oscillations along v are small we expect the solution to be regular.

The asymptotic charges of the solution, in addition to being the sum of the individual supersheet pieces, will also get contributions from the interaction terms. The asymptotic electric charges are

$$Q_{1,\infty} = \sum_I Q_1^{(I)}\,,\; Q_{2,\infty} = \sum_I Q_2^{(I)}\,,\; Q_{3,\infty} = \sum_I Q_3^{(I)} + 2\sum_{I<J} Q_3^{(I,J)}. \tag{7.99}$$

By expanding ω we get

$$\omega \sim \frac{1}{r^2}\left(\sum_I J^{(I)} + \sum_{I<J} J^{(I,J)}\right), \tag{7.100}$$

where

$$\begin{aligned} J^{(I)} &= \left(\kappa_I A_I^2 (Q_1^{(I)} + Q_2^{(I)}) + \kappa_I Q_1^{(I)} Q_2^{(I)}\right)\sin^2\theta d\psi + \kappa_I Q_1^{(I)} Q_2^{(I)} \cos^2\theta d\phi + \\ &+ \frac{1}{2}(Q_1^{(I)} + Q_2^{(I)})\lambda_I A_I \acute{A}_I \sin(2\theta) d\theta, \\ J^{(I,J)} &= \left(Q_1^{(I)} Q_2^{(J)} + Q_2^{(I)} Q_1^{(J)}\right)\kappa_J \left(\sin^2\theta d\psi + \cos^2\theta d\phi\right). \end{aligned} \tag{7.101}$$

The contributions to the angular momentum that come from the interaction pieces neither depend on $\dot{A}$ nor do they have a $d\theta$ component. Thus the interaction terms in the asymptotic charges for corrugated multi-supersheets are the same with those for solutions with trivial reduction to five dimensions. It will be interesting to examine whether this holds when we include the third dipole charge i.e. $\beta \neq 0$.

Having fully described the solution there is an interesting coincidence limit where all the supersheets become identical

$$A_I \to A\,,\ \lambda_I \to \lambda\,,\ \kappa_I \to \kappa\,,\ Q_i^{(I)} \to Q_i\,,\ Q_3^{(I,J)} \to Q_3, \tag{7.102}$$

for all the N supersheets of the solution and for $i = 1, 2, 3$. Then we obtain the single supersheet solution of section 7.3.3 and its asymptotic charges with

$$Q_1 \to NQ_1\,,\ Q_2 \to NQ_2\,,\ Q_3 \to N^2 Q_3. \tag{7.103}$$

7.4.4 Touching, intersecting & regularity

Although so far we have strictly imposed the condition $A_I < A_J \quad \forall v \in [0, L]$, the functions describing the solution are regular in the limit $A_I \to A_J$. The individual supersheet terms $H^{(I)}$, $\mathcal{F}^{(I)}$, $\omega^{(I)}$ and the interaction terms $\mathcal{F}^{(I,J)}$, $\omega_1^{(I,J)}$ are trivially regular in this limit. For the terms appearing in $\omega_2^{(I,J)}$ the functions $G_1^{(I,J)}$ and $G_2^{(I,J)}$ are regular while $L_1^{(I,J)}$ and $L_2^{(I,J)}$ are singular. However one can observe from $\omega_2^{(I,J)}$ in (7.92) that in the limit $A_I \to A_J$ the $d\phi$ and $d\zeta$ components factorize and the quantity $L_1^{(I,J)} + L_2^{(I,J)}$ is regular as well. Thus it seems that since supersheets can be realized as a collection of different v-slices we can have different supersheets touching or intersecting (fig.7.3) through each other at specific values of $v = v_2$ such that $A_I(\lambda_I v_2) = A_J(\lambda_J v_2)$.

For touching supersheets we should by definition have

$$A_I(\lambda_I v_2) = A_J(\lambda_J v_2) \quad , \quad \dot{A}_I(\lambda_I v_2) = \dot{A}_J(\lambda_J v_2). \tag{7.104}$$

Then continuity of the solution requires that taking the limit (7.104) on the result (7.92), (7.94) should match the result of the smearing integral for $v = v_2$ where (7.104) holds before smearing. Then one gets that the helical winding numbers of the two supersheets should match i.e. $\kappa_I = \kappa_J$. Overall the conditions for two supersheets touching at $v = v_2$ are

$$\kappa_I = \kappa_J \, , \, A_I(\lambda_I v_2) = A_J(\lambda_J v_2) \, , \, \dot{A}_I(\lambda_I v_2) = \dot{A}_J(\lambda_J v_2). \tag{7.105}$$

The somewhat notorious case of supersheets intersecting through each other such that $A_I(\lambda_I v_2) = A_J(\lambda_J v_2)$ needs some additional attention, as the ordering of the supersheets changes. As we mentioned at the end of section 7.4.2 not only the functions $\mathcal{F}^{(I,J)}$ and $\omega_2^{(I<J)}$ describing the interactions between different supersheets are not symmetric under $I \leftrightarrow J$, but also the ordering of the supersheets matters in calculating the integrals. Reminding ourselves that our solution is a collection of different constant v slices the previous issue could easily be resolved. One would have to calculate the smearing integrals of the interaction at two different areas of the coordinate v. For $v < v_2$ we have $A_I < A_J$ and the result would be the functions $\mathcal{F}^{(I,J)}$ and $\omega_2^{(I,J)}$ as given by (7.92), (7.94). For $v > v_2$ we would have $A_I > A_J$ and thus

$$\begin{aligned}\widetilde{\mathcal{F}}^{(I,J)} &= \mathcal{F}^{(J,I)},\\ \widetilde{\omega}_2^{(I,J)} &= \omega_2^{(J,I)}.\end{aligned} \tag{7.106}$$

Then we should at least require that the functions describing the supersheets are continuous at the intersection point $v = v_2$, which means we should demand

$$\begin{aligned}\lim_{v\to v_2}\left(\mathcal{F}^{(I,J)} - \widetilde{\mathcal{F}}^{(I,J)}\right) &= 0,\\ \lim_{v\to v_2}\left(\omega_2^{(I,J)} - \widetilde{\omega}_2^{(I,J)}\right) &= 0.\end{aligned} \tag{7.107}$$

Using (7.106) we find for the difference of the functions

$$
\begin{aligned}
&\lim_{v\to v_2}\left(\mathcal{F}^{(I,J)}-\widetilde{\mathcal{F}}^{(I,J)}\right)=4\frac{Q_3^{(J,I)}(v_2)-Q_3^{(I,J)}(v_2)}{\Sigma_I},\\
&\lim_{v\to v_2}\left(\omega_2^{(I,J)}-\widetilde{\omega}_2^{(I,J)}\right)=\lim_{v\to v_2}\left(Q_1^{(I)}Q_2^{(J)}+Q_2^{(I)}Q_1^{(J)}\right)\cdot\\
&\cdot\Bigg(\left(-\left(G_1^{(I,J)}+G_2^{(J,I)}\right)\kappa_I+\left(G_2^{(I,J)}+G_1^{(J,I)}\right)\kappa_J\right)d\psi+\\
&+\left(-\frac{\acute{A}_I}{A_I\eta}\left(G_1^{(I,J)}+G_2^{(J,I)}\right)\lambda_I+\frac{\acute{A}_J}{A_I\eta}\left(G_2^{(I,J)}+G_1^{(J,I)}\right)\lambda_J\right)d\eta\Bigg).
\end{aligned}
\tag{7.108}
$$

Thus taking the limit $v\to v_2$, for (7.107) to hold we need

$$
Q_3^{(I,J)}(v_2)=Q_3^{(J,I)}(v_2)\,,\;\kappa_I=\kappa_J\,,\;\dot{A}_I(\lambda_I v_2)=\dot{A}_J(\lambda_J v_2)\,,\;A_I(\lambda_I v_2)=A_J(\lambda_J v_2). \tag{7.109}
$$

The conditions (7.109) suggest that the supersheets can only intersect through a point at

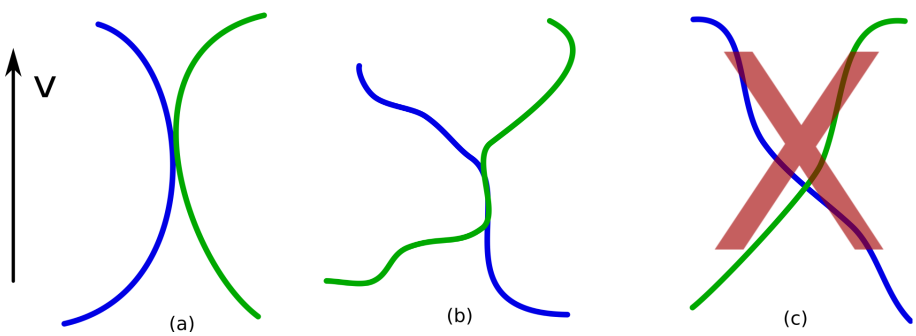


Figure 7.3: *Sections of different supersheets (blue & green): (a) touching supersheets, (b) supersheets intersecting by touching tangentially, (c) supersheets cannot intersect without touching tangentially.*

which they tangentially touch each other. It is interesting to observe that both intersecting and touching require the helical winding numbers κ_I, κ_J to be the same. In both cases the requirement of a local condition (continuity) gives a constraint on the global parameters κ. This constraint (along with $\dot{A}_I = \dot{A}_J$ for intersecting) can be understood as follows: for the supersheets to touch or intersect, the superthreads they consist of should, at the touching or intersection point $v = v_2$, be able to be realized as constituents of the same supersheet. Another way of realizing the constraints (7.105), (7.109) is through the angular momentum of the supersheets (7.100), (7.101). The parameters κ_I generate the ψ, ϕ components of the angular momentum and $\dot{A}_I$ the θ one. Thus for the supersheets to touch or intersect regularity requires that at the point of touching they should rotate in the same manner.

One could also argue that the cases of supersheets touching and tangentially intersecting are not essentially different since the regularity conditions for these two situations are essentially the same and one could change from one situation to the other after a piecewise relabeling of the supersheets whenever they come into contact. Such a procedure would of course change the $A(v)$ part of the profile functions of the superthreads and hence the supersheets, but that is safe to do since our solutions can be realized as a collection of different v slices. Thus as far as the constraints (7.109) are satisfied any pair of touching supersheets with some profile functions can be realized as a pair of tangentially intersecting supersheets with different profile functions and vice versa. Up to the change in the profile functions, both of these solutions are being described by (7.91), (7.92) and (7.94).

Another remark is that because the five-dimensional bubble equations are encoded in the ordering of the supersheets and since at a touching or intersection point the ordering becomes degenerate, the constraints (7.105), (7.109) are basically the six-dimensional conditions for the absence of Dirac strings when different supersheets touch or intersect.

7.4.5 Numerics on global regularity conditions

So far we have examined the case of well separated multi-supersheets as well as the situation at which they touch or intersect. In our analysis we mainly focused in the local regularity conditions that these solutions have to satisfy. However one should also check the global constraints (7.76), (7.77) which come from the absence of closed timelike curves. Here by considering specific examples of two concentric supersheets we perform a partial numerical analysis of these constraints and observe that at the usual areas of danger, our solutions pass the test. We will focus in the area between the supersheets and examine the phase space of solutions as the two supersheets approach or as we vary the parameters κ_1, κ_2. Regarding (7.77) we will examine the $d\psi^2$ part of it, which matches the five dimensional global regularity condition [22]. In all of the cases we will have a non-corrugated supersheet of constant radius A_1 and we choose

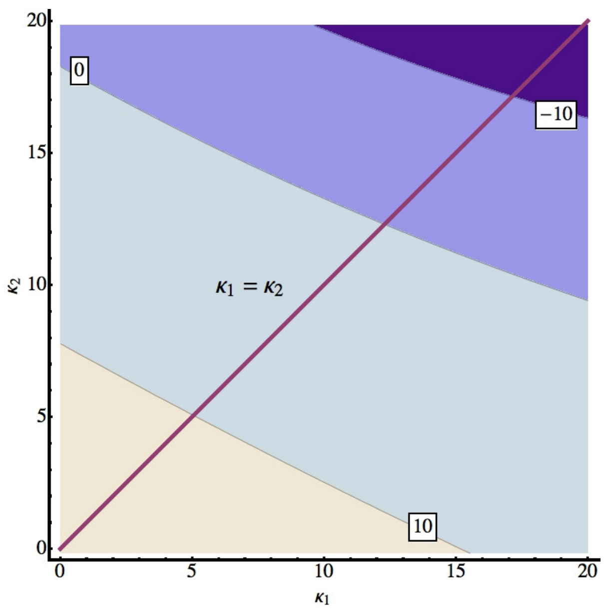
$$Q_1^{(1)} = Q_2^{(1)} = Q_1^{(2)} = Q_2^{(2)} = 1\,,\; Q_3^{(1)} = Q_3^{(2)} = Q_3^{(21)} = 50\,,\; \theta = \frac{\pi}{2}. \tag{7.110}$$

For distinct or touching supersheets we will choose

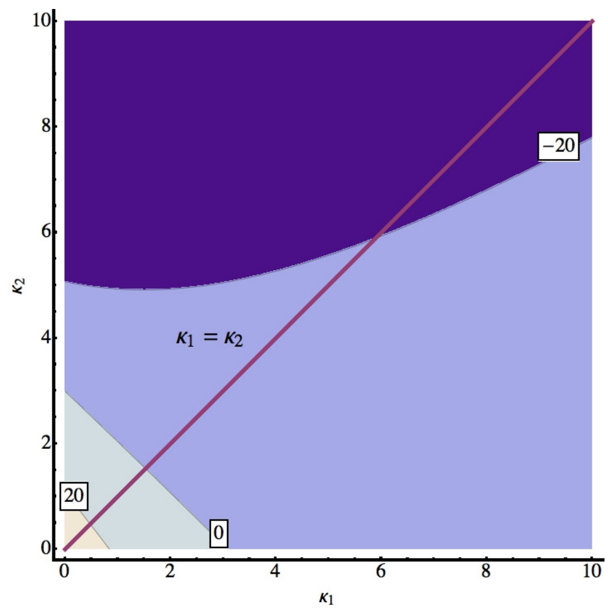
$$A_2 = 6 + \sin v, \tag{7.111}$$

which means that $0 < A_1 \leq 5$ with the supersheets touching for $A_1 = 5$. For $A_1 < 5$ we will examine the area for $v = \frac{3\pi}{2}$ where the supersheets A_1 and A_2 are closest to each other and we will choose the radial distance to be in the middle of the two supersheets $r = \frac{5+A_1}{2}$. The results are shown in figures (7.4(a)), (7.4(b)) and (7.5) in terms of contour plots of (7.77).

In fig.(7.4(a)) we have chosen $A_1 = 2$. This corresponds to the case where the supersheets are well separated. We see that there is enough parameter space for κ_1, κ_2 before closed timelike curves appear. In fig.(7.4(b)) we have $A_1 = 4.9$ and the allowed parameter space for κ_1, κ_2 is significantly smaller. This is displayed better in fig.(7.5) where we have set $\kappa_1 = \kappa_2$ (the condition for touching) and observe how the allowed values decrease as the supersheets approach.



(a) $A_1 = 2$ Well separated supersheets



(b) $A_1 = 4.9$ Supersheets close to each other

Figure 7.4: *Contour plot of the $d\psi^2$ component of the left hand side of (7.77) for $\theta = \frac{\pi}{2}$ shows the allowed values of κ_1, κ_2 for separated supersheets in the middle of the area between them. The left hand side of (7.77) has to be greater or equal to zero. The allowed values of κ_1 and κ_2 reduce as the supersheets approach.*

For touching supersheets we have $A_1 = 5$ and $\kappa_1 = \kappa_2 = \kappa$. We set $v = \frac{3\pi}{2} + 0.1$ and examine the area between the two supersheets near the touching point as we vary κ and the

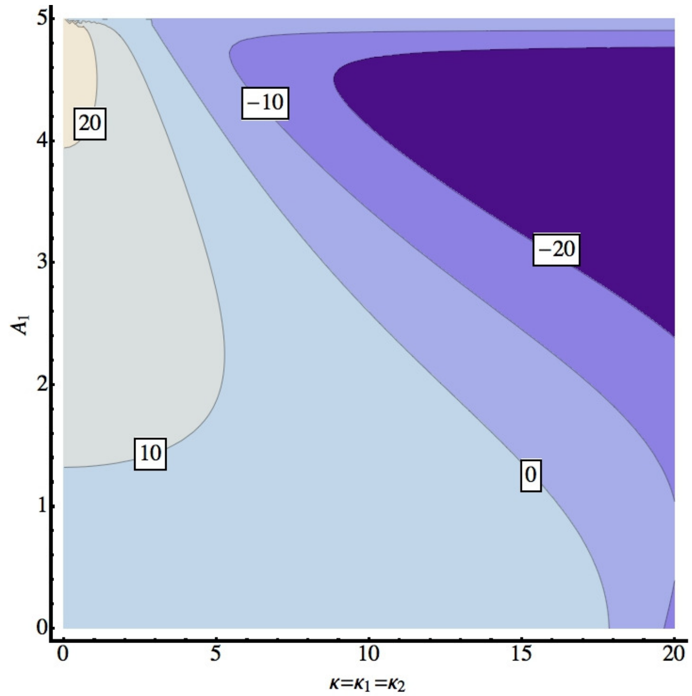


Figure 7.5: *Contour plot of the $d\psi^2$ component of the left hand side of (7.77) for $\theta = \frac{\pi}{2}$ in the middle of the area between two supersheets with same helical mode κ. The left hand side of (7.77) has to be greater or equal to zero. The allowed values of κ reduce as the supersheets approach.*

radius r. The results are displayed in fig.(7.6). Once again we see there are ample values for κ so that (7.77) is satisfied.

For intersecting supersheets we choose

$$A_1 = 5\,,\; A_2 = 5 + \left(\sin\left(v + \frac{\pi}{2}\right)\right)^3 . \tag{7.112}$$

The function A_2 has inflection points for $v = (2n{+}1)\frac{\pi}{2}$, where $n \in \mathbb{Z}$. At these points $A_1 = A_2$ and that's when the two supersheets intersect. As in the case of touching supersheets we

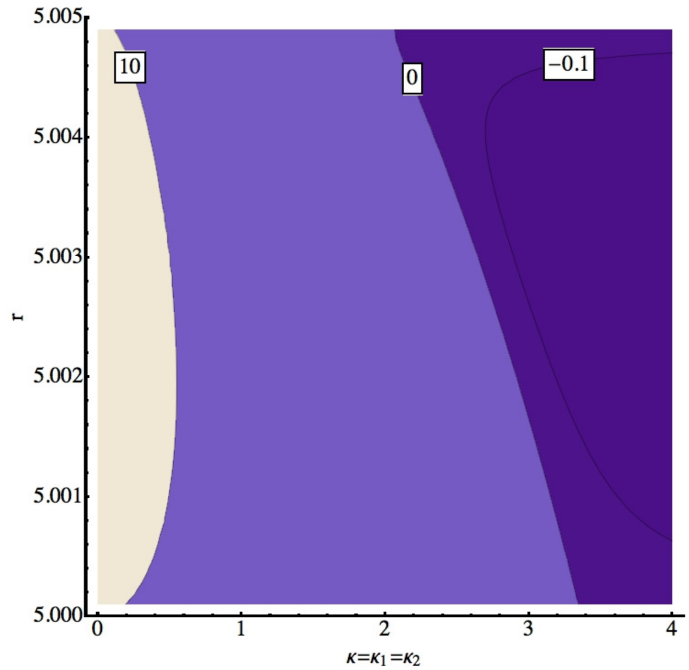


Figure 7.6: *For touching supersheets there are allowed values of κ for all radial distances r in the area between them next to the touching point.*

are going to choose $v = \frac{3\pi}{2} + 0.1$, so that we are near the intersection point, and examine the area between A_1 and A_2. As we observe from fig.(7.7) there are allowed values of κ from (7.77) and thus the solution exists.

The condition (7.76) is trivially satisfied in the areas we examined so far. Examining (7.76) near the origin we also need (7.78), (7.96). Also from examining (7.77) near the supersheets when $\dot{A}_I = 0$ we additionally get (7.83). All of these conditions (7.78), (7.96), (7.83) set finite upper bounds to the values of κ_1, κ_2. Upper bounds to the values of κ_1, κ_2 have also been found in the numerical analysis presented in this section. Thus the conditions (7.78), (7.96), (7.83) together with examining (7.77) in the area between the supersheets will have a common set of allowed values for κ_1, κ_2.

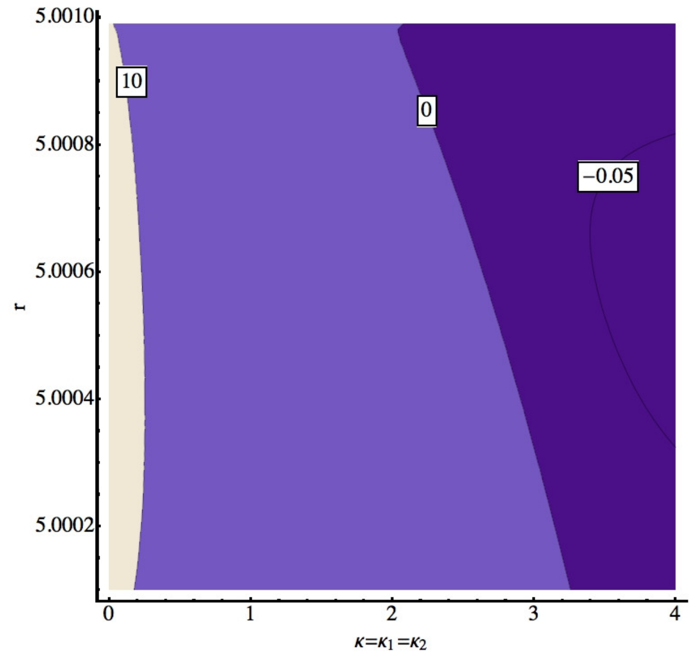


Figure 7.7: *For intersecting supersheets there are allowed values of κ for all radial distances r in the area between them next to the intersection point.*

7.5 Suggestions on the addition of KKM

The geometries we have studied so far are restricted to setting $\beta = 0$ and hence there is no third dipole charge. The addition of β is needed to construct not only generic black hole solutions, but also the superstratum. Here we use the results of this chapter to argue about certain aspects of the solution that include β [4]. We find that our arguments are consistent with previous analysis regarding the superstratum [97].

[4]Geometries with all three electric and dipole magnetic charges (they have $\beta \neq 0$ but still $\dot{\beta} = 0$) have been constructed in [107] for a Kähler base space. In contrast to five dimensions, in six dimensions the BPS conditions do not require the base space to be hyper-Kähler [100, 102] and indeed the more general solutions we are after may require a more general class of metrics. However it is still interesting to examine which genuinely six-dimensional geometries can be constructed with a hyper-Kähler base.

The difficulty arises because the BPS equation [102] for β is non-linear

$$\mathcal{D}\beta = *_4\mathcal{D}\beta, \tag{7.113}$$

where

$$D\Phi = \tilde{d}\Phi - \beta \wedge \dot{\Phi} \tag{7.114}$$

and $\tilde{d}$ is the exterior derivative with respect to the four-dimensional base space. When β is v-independent (7.113) becomes linear and reduces to the equation known from linearity in five dimensions

$$\tilde{d}\beta = *_4\tilde{d}\beta. \tag{7.115}$$

In section 7.3.3 we observed that because of the periodicity of v, there should be values of $v = v_1$ such that $\dot{A}(\lambda v_1) = 0$. At these points our solution directly reduces to the non-corrugated supersheet which is essentially five-dimensional. Thus for any generic genuinely six-dimensional solution there are points along the compactification circle such that the solution looks five-dimensional. Consequently, we should be able to explore six-dimensional solutions by adding appropriate perturbations around the points $v = v_1$ to already known geometries from five dimensions. This should be allowed based on the superposition of corrugated and helical modes we constructed in section 7.3.3. At $v = v_1$ we should simultaneously have

$$\dot{A}(v_1) = 0\,,\ \dot{\beta}(v_1) = 0, \tag{7.116}$$

with $A(v)$ representing the radius of the profile in the four-dimensional base metric.Thus by expanding around $v = v_1$, β should at least have similar linear order dependence on v with A.

The superstratum, since it is a smooth geometry, is expected to be much more constrained compared to generic black geometries and one would have to add appropriate perturbations in a very precise manner. Indeed, the only known D1-D5 geometry that has all of its electromagnetic sources at a single point, is smooth in six dimensions and can be reduced to five is the supertube. The supertube carries D1, D5 electric charges and KKM dipole charge. The superstratum is expected to have all three electric (D1-D5-P) and magnetic dipole (d1-d5-KKM) charges. Thus for $v = v_1$ the superstratum should look exactly like a supertube aligned along the v direction and because of the smoothness of the solution we should simultaneously have

$$\begin{aligned} &\dot{A}(v_1) = 0\,,\ \dot{\beta}(v_1) = 0, \\ &P(v_1) = 0\,,\ d1(v_1) = d5(v_1) = 0. \end{aligned} \tag{7.117}$$

The additional requirement on the charges of the object is consistent with the supersymmetry analysis in [97]. The D1-D5 system is being placed along the v direction and is being given momentum P vertically with respect to the branes. This gives a system with charges $Q_1 = Q_{D1}$, $Q_2 = Q_{D5}$ and angular momentum $J = P$. To generate dipole charge and momentum along v one gives a tilt of angle α to the D1-D5 system with respect to the v direction. Then

$$d1 = Q_{D1}\sin\alpha\,, d5 = Q_{D5}\sin\alpha\,,\ P_v = P\sin\alpha. \tag{7.118}$$

Consequently, the conditions (7.117) on the charges can be simultaneously satisfied provided the tilting angle α is a function of v such that

$$\alpha = \alpha(v)\,,\ \alpha(v_1) = n\pi, \tag{7.119}$$

where n is an integer. The dependence of the tilting angle with respect to v is an essential part of the second supertube transition [97] which is in turn required to generate the superstratum.

7.6 Concluding remarks

The BPS equations in six-dimensional, minimal $\mathcal{N} = 1$ supergravity coupled to one tensor multiplet have been shown to be a linear system [102] once an appropriate base geometry has been determined. This allows one to use superposition to create a wide variety of solutions and such solutions could lead to interesting new developments in the study of black hole microstate geometries, as well has holography on $AdS_3 \times S_3$. It has also been conjectured [97] that a new class of BPS microstate geometries, *superstrata*, may exist. Such objects carry three electric charges and two *independent* dipole charges, depend on arbitrary functions of two variables and are expected to be *regular* solutions in the IIB duality frame. They are thus a sheetlike, three-charge generalization of the supertube. The fact that they depend upon functions of two variables suggests that they should be able to store large amounts of entropy in their shape modes, indeed the superstrata microstate geometries are expected to give the dominant semi-classical contribution to the entropy of the three-charge system [46]. The results we present here are a very significant step in that direction.

The non-trivial aspect of our new solutions is that they take into account the shape-

shape interactions of the separate superthreads. It was evident in [102] that superthreads interact non-trivially with one another when the threads have different profiles and so the completely general multi-superthread was not constructed. Indeed, as depicted in Fig. 7.1, the multi-centered solutions found in [102] only involves parallel threads, shifted by rigid translation in $\mathbb{R}^4$. Such solutions can only be smeared together into a sheet depending on arbitrary functions of *one* variable with one set of functions describing the thread profile and another defining the smearing densities. To get a solution that is genuinely a function of two variables by smearing, it is essential to construct the multi-superthread solution in which all the threads can have independent profiles and so the smeared threads yields a thread-density profile, $\vec{F}(\sigma, v)$. This is depicted in Fig. 7.2.

In this chapter we have analyzed the effect of this shape-shape interaction and presented the general solution with multiple threads of completely arbitrary and independent shapes at each center. These solutions were then smeared to obtain new solutions sourced by a two-dimensional sheet of completely arbitrary profile, described by arbitrary functions of *two* variables.

We also checked our results against a known five-dimensional solution by taking a simple helical profile and smearing it to a cylindrical sheet and dimensionally reducing. We thus recovered the generalized supertube solution with three-charges and two-dipole charges [103–105]. We found that CTC conditions, like the radius relation, which usually require an additional constraint on the five dimensional solution, emerge automatically from our six-dimensional solutions.

Furthermore we described multi-supersheet solutions with D1-D5-P electric charges and d1-d5 magnetic dipoles, which have a non-trivial dependence on the compactification direction. The supersheet profile functions were chosen so that they are the product of a helical mode and an arbitrary corrugated mode. Then we saw that after smearing the resulting supersheet is basically the superposition of the two modes.When multiple supersheets are present they interact in pairs and we observed that the absence of integrability conditions is encoded as a lack of symmetry in exchanging the two supersheets in the functions describing the interaction. Before smearing the integrands are symmetric in exchanging the supersheets, but while performing the integral one has to choose an order to decide which poles are within the unit circle. The asymptotic charges get contributions from the interaction terms between the supersheets, but interestingly enough these extra terms do not get affected by the non-trivial dependence along the compactification direction. Thus the contribution of the interaction to the asymptotic charges is essentially five-dimensional.

A new feature is that different supersheets can under certain conditions touch and intersect through each other without the occurrence of closed timelike curves. The local regularity conditions (originating from the absence of Dirac strings in five dimensions) require that the supersheets come in contact tangentially and also have the same helical mode windings κ_I. These conditions essentially guarantee that the superthreads making the two supersheets can at the point of contact realize themselves as constituents of a single supersheet as well as that touching and intersecting supersheets are equivalent up to an appropriate relabeling of the profile functions. These features are a consequence of the fact that our solutions can be realized as a sequence of different slices of constant v and it would be interesting to examine whether such features hold in more general six-dimensional solutions with $\beta \neq 0$.

Another interesting aspect of these solutions is that because the direction v is periodic there are values v_1 of v where the first derivative of the radius of the four-dimensional profile $A(v)$ vanishes and thus the solution looks five-dimensional. This, together with the superposition of corrugated and helical modes, gives hope that one will be able to add perturbative modes on already known five-dimensional solutions to generate a richer class of solutions. We should note however that when $\beta \neq 0$ there might be a wider class of solutions such that there is no superposition between helical and corrugated modes.

The solutions presented in this chapter are completely new geometries and are interesting in their own right as three-charge solutions sourced by arbitrary two-dimensional surfaces. To obtain the superstratum we will need to do exactly what we have achieved here but with an additional KKM magnetic charge smeared along the profile thereby providing the required second *independent* dipole moment [97].

Chapter 8

Conclusions

In this thesis we employed the tools of supergravity in order to study the microscopic structure of supersymmetric and non-supersymmetric black holes at strong gravitational coupling. In supergravity the microstates can be realized as smooth horizonless geometries that have the same asymptotic structure and charges as a black hole. They offer a coarse-grained description of the black hole phase space which, in general, is expected to consist of brane configurations with non-geometric description [99, 108].

These solutions fit into the physical description of the fuzzball proposal according to which black holes should be thought of as horizonless star-like objects consisting of strings and branes extending all the way up to the scale of the horizon. This picture offers the necessary $\mathcal{O}(1)$ corrections required to extract information from the black hole [13] and allows for unitary scattering.

The goal of such a program is to create enough microstate geometries in order to at least account for the correct scaling of the black hole entropy with respect to the charges. This will eventually yield some form of a semiclassical description of black hole microstates. For BPS solutions this has almost been achieved, while for non-BPS solutions much more work is needed.

For BPS solutions we used the formalism of supertubes and ambipolar Gibbons-Hawking spaces to extend the construction of three-charge smooth solutions in five dimensions to 1/8-BPS systems with four charges. Also we constructed singular six-dimensional solutions and examined their properties. Our solutions are constructed by combining multiple superthreads of different shapes. The latter are one-dimensional singular objects with three electric and two dipole magnetic charges. Upon smearing the superthreads we got supersheets which are two-dimensional surfaces depending on profile functions of two variables. These, in addition to being new interesting solutions, are the first step towards constructing the superstratum. The latter is a conjectured six-dimensional smooth geometry which is expected to give the dominant semiclassical contribution to the entropy of 1/8-BPS black holes. To go from a supersheet to the superstratum one would need to add the third dipole magnetic charge in the solution which would play the role of KKM and smooth the geometry in a similar fashion with supertubes. The difference between supertubes and supersheets is that the first depend on profile functions of one variable, while the latter also have momentum modes and thus depend on profile functions of two variables.

For non-BPS solutions we examined a special class of extremal solutions with broken supersymmetry called almost-BPS. The main characteristic of these solutions is that the supersymmetry is broken by the holonomy of the background and thus one has a controlled way to break supersymmetry and study its effects. To that end we used a multi-species supertube solution to construct an example of a scaling microstate geometry for an almost-

BPS black ring. Examining the local regularity conditions, which control the positions of the supertubes with respect to their charges and guarantee the absence of Dirac strings and CTC's, we studied the moduli space of the solutions and compared it with its BPS counterpart. We found that almost-BPS microstate geometries are indeed allowed, but they are more restricted compared to the BPS case. Also we established evidence indicating that almost-BPS solutions are subject to a non-renormalization theorem despite the breaking of supersymmetry. By going at an intermediate gravitational coupling and treating one of almost-BPS centers as a probe in the background of the rest we showed that the probe action reproduces local regularity conditions as obtained in supergravity.

Looking Ahead

For supersymmetric solutions the steps that need to be taken are quite clear. The superstratum is the smooth 1/8-BPS geometry that needs to be found. An example of a smooth geometry that is a function of two variables has recently been constructed in [109] verifying thus the existence of the superstratum. However the solution describes only certain oscillation modes and these at the decoupling limit. Thus the quest for a generic superstratum solution with flat asymptotics is still open. Once this solution has been obtained the next step would be to quantize the oscillation modes of the superstratum and indeed verify that it gives $S \sim Q^{3/2}$ as according to the analysis of [46]. It is interesting that the solution found in [109] makes use of of the four charge formalism presented in chapter 4, since it simplifies the sources of the six-dimensional equations. It may be important to further elucidate the role of this system in constructing smooth geometries in six dimensions and specify the superstrata modes that correspond to the topological limit where the central charge is zero. In general, in addition to constructing the superstratum, another direction that would be interesting is to find new six-dimensional black hole geometries.

For non-BPS microstates the situation is less under control. For almost BPS black holes we showed that there is a systematic way to construct microstate geometries as well as the existence of a non-renormalization theorem. It will be useful to try extending these results in other classes of extremal non-supersymmetric systems [91, 93, 94, 110]. It is important to mention that some extremal non-BPS black holes in five dimensions [68] were found to be BPS in six dimensions [107]. It is important to investigate whether the same holds for other non-BPS systems like almost-BPS. Thus one could see whether the existence of non-renormalization is because the system is secretly supersymmetric or it can indeed be a property of systems with broken supersymmetry.

Of course the holy grail of the microstate geometry program would be to find smooth geometries that would provide the entropy for a Schwarzschild (or Kerr) black hole. For non-extremal black holes we are still lacking of a systematic way in constructing microstate geometries and only a few isolated examples exist [66, 111–115]. The authors in [116] may have made a first step towards a systematic way of constructing non-extremal microstate geometries. The presence of the above examples shows that the existence of smooth solutions is not only a characteristic of supersymmetric solutions. The difficulty is mainly technical since non-extremal black holes emit Hawking radiation and thus the relevant microstates are time dependent. Studies for near-extremal black holes have been done with the use of brane probes and have uncovered some the relevant physics [89, 90, 117]. Thus a more tractable problem may be to construct microstates for near-extremal systems by adding perturbations to already known BPS solutions.

Overall we can say that microstate geometries are in good standing in explaining BPS black holes and are taking their first steps towards non-extremal ones. Explaining the physics of black holes will eventually enhance our understanding of the physical world at a fundamental level. We can only hope that the contents of this thesis will provide inspiration for additional research on that direction. Finally overfilled with inspiration let us vent and close with an *Ode to the Black Hole*:

Oh black hole,
overall,
with no distinction
you accept us all.

Our information irretrievably lost,
your horizon once across,
there is no way out,
even for the brave most.

Shall we your eternal embrace to evade
only in thermal radiation escape?

Where are your microstates?
What are you made of?
The mystery remains
as the years drive off.

Complementarity and firewalls,

you deserve all that buzz,

but truly you may be

purely fuzz.

Bibliography

[1] K. Schwarzschild, "On the gravitational field of a mass point according to Einstein's theory", ArXiv Physics e-prints (1999), eprint: `physics/9905030`.

[2] P. O. Mazur, "Black hole uniqueness theorems", arXiv:`hep-th/0101012 [hep-th]`.

[3] J. D. Bekenstein, "Black holes and entropy", Phys. Rev. D **7**, 2333–2346 (1973).

[4] C. B. Bardeen, J. M. and S. W. Hawking, "The four laws of black hole mechanics", Commun. Math. Phys. **31**, 161–170 (1973).

[5] S. W. Hawking, "Black hole explosions?", Nature **238**, 30–31 (1974).

[6] S. W. Hawking, "Particle creation by black holes", Comm. Math. Phys. **43**, 199–220 (1975).

[7] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity", Int.J.Theor.Phys. **38**, 1113–1133 (1999), arXiv:`hep-th/9711200 [hep-th]`.

[8] E. Witten, "Anti-de Sitter space and holography", Adv.Theor.Math.Phys. **2**, 253–291 (1998), arXiv:`hep-th/9802150 [hep-th]`.

[9] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity”, Phys.Rept. **323**, 183–386 (2000), arXiv:hep-th/9905111 [hep-th].

[10] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy”, Phys.Lett. **B379**, 99–104 (1996), arXiv:hep-th/9601029 [hep-th].

[11] C. G. Callan and J. M. Maldacena, “D-brane approach to black hole quantum mechanics”, Nucl.Phys. **B472**, 591–610 (1996), arXiv:hep-th/9602043 [hep-th].

[12] S. D. Mathur, “The Fuzzball proposal for black holes: An Elementary review”, Fortsch.Phys. **53**, 793–827 (2005), arXiv:hep-th/0502050 [hep-th].

[13] S. D. Mathur, “The Information paradox: A Pedagogical introduction”, Class.Quant.Grav. **26**, 224001 (2009), arXiv:0909.1038 [hep-th].

[14] D. N. Page, “Average entropy of a subsystem”, Phys.Rev.Lett. **71**, 1291–1294 (1993), arXiv:gr-qc/9305007 [gr-qc].

[15] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, “Black Holes: Complementarity or Firewalls?”, JHEP **1302**, 062 (2013), arXiv:1207.3123 [hep-th].

[16] A. Almheiri, D. Marolf, J. Polchinski, D. Stanford, and J. Sully, “An Apologia for Firewalls”, JHEP **1309**, 018 (2013), arXiv:1304.6483 [hep-th].

[17] O. Vasilakis and N. P. Warner, “Mind the Gap: Supersymmetry Breaking in Scaling, Microstate Geometries”, JHEP **1110**, 006 (2011), arXiv:1104.2641 [hep-th].

[18] O. Vasilakis, “Bubbling the Newly Grown Black Ring Hair”, JHEP **1205**, 033 (2012), arXiv:1202.1819 [hep-th].

[19] B. E. Niehoff, O. Vasilakis, and N. P. Warner, “Multi-Superthreads and Supersheets”, JHEP **1304**, 046 (2013), arXiv:1203.1348 [hep-th].

[20] O. Vasilakis, “Corrugated Multi-Supersheets”, JHEP **1307**, 008 (2013), arXiv:1302.1241 [hep-th].

[21] I. Bena, A. Puhm, O. Vasilakis, and N. P. Warner, “Almost BPS but still not renormalized”, JHEP **1309**, 062 (2013), arXiv:1303.0841 [hep-th].

[22] I. Bena and N. P. Warner, “Black holes, black rings and their microstates”, Lect.Notes Phys. **755**, 1–92 (2008), arXiv:hep-th/0701216 [hep-th].

[23] S. Giusto and R. Russo, “Adding new hair to the 3-charge black ring”, Class.Quant.Grav. **29**, 085006 (2012), arXiv:1201.2585 [hep-th].

[24] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (2003).

[25] H. Reissner, “ber die eigengravitation des elektrischen feldes nach der einsteinschen theorie”, Annalen der Physik **355**, 106–120 (1916).

[26] G. Nordström, “On the energy of the gravitational field in einstein’s theory”, Proc. Kon. Ned. Akad. Wet. **20**, 1238–1245 (1918).

[27] I. Robinson, “A Solution of the Maxwell-Einstein Equations”, Bull.Acad.Pol.Sci.Ser.Sci.Math.Ast **7**, 351–352 (1959).

[28] B. Bertotti, “Uniform electromagnetic field in the theory of general relativity”, Phys. Rev. **116**, 1331–1333 (1959).

[29] S. Majumdar, “A class of exact solutions of Einstein’s field equations”, Phys.Rev. **72**, 390–398 (1947).

[30] A. Papapetrou, Proc. Roy. Irish Acad. **A51** (1947).

[31] G. Gibbons and C. Hull, "A Bogomolny Bound for General Relativity and Solitons in N=2 Supergravity", Phys.Lett. **B109**, 190 (1982).

[32] J. D. Bekenstein, "Bekenstein-Hawking entropy", Scholarpedia **3**, 7375 (2008).

[33] R. M. Wald, "The thermodynamics of black holes", Living Rev.Rel. **4**, 6 (2001), arXiv:`gr-qc/9912119 [gr-qc]`.

[34] M. Aizenman and E. H. Lieb, "The third law of thermodynamics and the degeneracy of the ground state for lattice systems", J. Stat. Phys. **24**, 279–297 (1981).

[35] R. M. Wald, "The 'Nernst theorem' and black hole thermodynamics", Phys.Rev. **D56**, 6467–6474 (1997), arXiv:`gr-qc/9704008 [gr-qc]`.

[36] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string* (1998).

[37] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond* (1998).

[38] C. V. Johnson, *D-branes* (2003).

[39] J. L. Cardy, "Operator Content of Two-Dimensional Conformally Invariant Theories", Nucl.Phys. **B270**, 186–204 (1986).

[40] G. Gibbons and N. Warner, "Global structure of five-dimensional fuzzballs", Class.Quant.Grav. **31**, 025016 (2014), arXiv:`1305.0957 [hep-th]`.

[41] B. C. Palmer and D. Marolf, "Counting supertubes", JHEP **0406**, 028 (2004), arXiv:`hep-th/0403025 [hep-th]`.

[42] K. Skenderis and M. Taylor, "The fuzzball proposal for black holes", Phys.Rept. **467**, 117–171 (2008), arXiv:`0804.0552 [hep-th]`.

[43] F. Chen, B. Michel, J. Polchinski, and A. Puhm, "Journey to the Center of the Fuzzball", JHEP **1502**, 081 (2015), arXiv:1408.4798 [hep-th].

[44] I. Bena, N. Bobev, C. Ruef, and N. P. Warner, "Supertubes in Bubbling Backgrounds: Born-Infeld Meets Supergravity", JHEP **0907**, 106 (2009), arXiv:0812.2942 [hep-th].

[45] N. P. Warner, "Microstate Geometries and Entropy Enhancement", Prog.Theor.Phys.Suppl. **177**, 228–246 (2009), arXiv:0810.2596 [hep-th].

[46] I. Bena, M. Shigemori, and N. P. Warner, "Black-Hole Entropy from Supergravity Superstrata States", JHEP **1410**, 140 (2014), arXiv:1406.4506 [hep-th].

[47] P. A. Haas, "Smarr's Formula in Eleven-Dimensional Supergravity", (2014), arXiv:1405.3708 [hep-th].

[48] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, "Supergravity Solutions from Floating Branes", JHEP **1003**, 047 (2010), arXiv:0910.1860 [hep-th].

[49] I. Bena and N. P. Warner, "One ring to rule them all ... and in the darkness bind them?", Adv.Theor.Math.Phys. **9**, 667–701 (2005), arXiv:hep-th/0408106 [hep-th].

[50] D. Mateos and P. K. Townsend, "Supertubes", Phys.Rev.Lett. **87**, 011602 (2001), arXiv:hep-th/0103030 [hep-th].

[51] R. Emparan, D. Mateos, and P. K. Townsend, "Supergravity supertubes", JHEP **0107**, 011 (2001), arXiv:hep-th/0106012 [hep-th].

[52] O. Lunin and S. D. Mathur, "AdS / CFT duality and the black hole information paradox", Nucl.Phys. **B623**, 342–394 (2002), arXiv:hep-th/0109154 [hep-th].

[53] O. Lunin, J. M. Maldacena, and L. Maoz, "Gravity solutions for the D1-D5 system with angular momentum", (2002), arXiv:hep-th/0212210 [hep-th].

[54] I. Bena, N. Bobev, and N. P. Warner, "Spectral Flow, and the Spectrum of Multi-Center Solutions", Phys.Rev. **D77**, 125025 (2008), arXiv:0803.1203 [hep-th].

[55] G. Dall'Agata, S. Giusto, and C. Ruef, "U-duality and non-BPS solutions", JHEP **1102**, 074 (2011), arXiv:1012.4803 [hep-th].

[56] S. Giusto, R. Russo, and D. Turton, "New D1-D5-P geometries from string amplitudes", JHEP **1111**, 062 (2011), arXiv:1108.6331 [hep-th].

[57] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, "Supersymmetric black rings and three-charge supertubes", Phys.Rev. **D71**, 024033 (2005), arXiv:hep-th/0408120 [hep-th].

[58] J. P. Gauntlett and J. B. Gutowski, "General concentric black rings", Phys.Rev. **D71**, 045002 (2005), arXiv:hep-th/0408122 [hep-th].

[59] P. Berglund, E. G. Gimon, and T. S. Levi, "Supergravity microstates for BPS black holes and black rings", JHEP **0606**, 007 (2006), arXiv:hep-th/0505167 [hep-th].

[60] A. Cadavid, A. Ceresole, R. D'Auria, and S. Ferrara, "Eleven-dimensional supergravity compactified on Calabi-Yau threefolds", Phys.Lett. **B357**, 76–80 (1995), arXiv:hep-th/9506144 [hep-th].

[61] G. Papadopoulos and P. Townsend, "Compactification of D = 11 supergravity on spaces of exceptional holonomy", Phys.Lett. **B357**, 300–306 (1995), arXiv:hep-th/9506150 [hep-th].

[62] I. Bena, N. Bobev, C. Ruef, and N. P. Warner, “Entropy Enhancement and Black Hole Microstates”, Phys.Rev.Lett. **105**, 231301 (2010), arXiv:0804.4487 [hep-th].

[63] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, “Multi-Center non-BPS Black Holes: the Solution”, JHEP **0911**, 032 (2009), arXiv:0908.2121 [hep-th].

[64] K. Goldstein and S. Katmadas, “Almost BPS black holes”, JHEP **0905**, 058 (2009), arXiv:0812.4183 [hep-th].

[65] I. Bena, G. Dall’Agata, S. Giusto, C. Ruef, and N. P. Warner, “Non-BPS Black Rings and Black Holes in Taub-NUT”, JHEP **0906**, 015 (2009), arXiv:0902.4526 [hep-th].

[66] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, “A (Running) Bolt for New Reasons”, JHEP **0911**, 089 (2009), arXiv:0909.2559 [hep-th].

[67] N. Bobev and C. Ruef, “The Nuts and Bolts of Einstein-Maxwell Solutions”, JHEP **1001**, 124 (2010), arXiv:0912.0010 [hep-th].

[68] N. Bobev, B. Niehoff, and N. P. Warner, “Hair in the Back of a Throat: Non-Supersymmetric Multi-Center Solutions from Káhler Manifolds”, JHEP **1110**, 149 (2011), arXiv:1103.0520 [hep-th].

[69] J. B. Gutowski and H. S. Reall, “General supersymmetric AdS(5) black holes”, JHEP **0404**, 048 (2004), arXiv:hep-th/0401129 [hep-th].

[70] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, “All supersymmetric solutions of minimal supergravity in five- dimensions”, Class.Quant.Grav. **20**, 4587–4634 (2003), arXiv:hep-th/0209114 [hep-th].

[71] I. Bena, C.-W. Wang, and N. P. Warner, "Mergers and typical black hole microstates", JHEP **0611**, 042 (2006), arXiv:hep-th/0608217 [hep-th].

[72] I. Bena, C.-W. Wang, and N. P. Warner, "Plumbing the Abyss: Black ring microstates", JHEP **0807**, 019 (2008), arXiv:0706.3786 [hep-th].

[73] D. Marolf and A. Virmani, "A Black hole instability in five dimensions?", JHEP **0511**, 026 (2005), arXiv:hep-th/0505044 [hep-th].

[74] I. Bena, C.-W. Wang, and N. P. Warner, "Sliding rings and spinning holes", JHEP **0605**, 075 (2006), arXiv:hep-th/0512157 [hep-th].

[75] F. Denef, "Supergravity flows and D-brane stability", JHEP **0008**, 050 (2000), arXiv:hep-th/0005049 [hep-th].

[76] F. Denef, "Quantum quivers and Hall / hole halos", JHEP **0210**, 023 (2002), arXiv:hep-th/0206072 [hep-th].

[77] B. Bates and F. Denef, "Exact solutions for supersymmetric stationary black hole composites", JHEP **1111**, 127 (2011), arXiv:hep-th/0304094 [hep-th].

[78] I. Bena and N. P. Warner, "Bubbling supertubes and foaming black holes", Phys.Rev. **D74**, 066001 (2006), arXiv:hep-th/0505166 [hep-th].

[79] A. Saxena, G. Potvin, S. Giusto, and A. W. Peet, "Smooth geometries with four charges in four dimensions", JHEP **0604**, 010 (2006), arXiv:hep-th/0509214 [hep-th].

[80] F. Denef and G. W. Moore, "Split states, entropy enigmas, holes and halos", JHEP **1111**, 129 (2011), arXiv:hep-th/0702146 [HEP-TH].

[81] J. Manschot, B. Pioline, and A. Sen, "Wall Crossing from Boltzmann Black Hole Halos", JHEP **1107**, 059 (2011), arXiv:1011.1258 [hep-th].

[82] J. P. Gauntlett and J. B. Gutowski, "Concentric black rings", Phys.Rev. **D71**, 025013 (2005), arXiv:hep-th/0408010 [hep-th].

[83] I. Bena, B. D. Chowdhury, J. de Boer, S. El-Showk, and M. Shigemori, "Moulting Black Holes", JHEP **1203**, 094 (2012), arXiv:1108.0411 [hep-th].

[84] J. de Boer, S. El-Showk, I. Messamah, and D. Van den Bleeken, "Quantizing N=2 Multicenter Solutions", JHEP **0905**, 002 (2009), arXiv:0807.4556 [hep-th].

[85] I. Bena, M. Berkooz, J. de Boer, S. El-Showk, and D. Van den Bleeken, "Scaling BPS Solutions and pure-Higgs States", JHEP **1211**, 171 (2012), arXiv:1205.5023 [hep-th].

[86] S.-J. Lee, Z.-L. Wang, and P. Yi, "Quiver Invariants from Intrinsic Higgs States", JHEP **1207**, 169 (2012), arXiv:1205.6511 [hep-th].

[87] J. Manschot, B. Pioline, and A. Sen, "From Black Holes to Quivers", JHEP **1211**, 023 (2012), arXiv:1207.2230 [hep-th].

[88] R. Emparan and G. T. Horowitz, "Microstates of a Neutral Black Hole in M Theory", Phys.Rev.Lett. **97**, 141601 (2006), arXiv:hep-th/0607023 [hep-th].

[89] I. Bena, A. Puhm, and B. Vercnocke, "Metastable Supertubes and non-extremal Black Hole Microstates", JHEP **1204**, 100 (2012), arXiv:1109.5180 [hep-th].

[90] I. Bena, A. Puhm, and B. Vercnocke, "Non-extremal Black Hole Microstates: Fuzzballs of Fire or Fuzzballs of Fuzz ?", JHEP **1212**, 014 (2012), arXiv:1208.3468 [hep-th].

[91] G. Bossard, “Octonionic black holes”, JHEP **1205**, 113 (2012), arXiv:1203.0530 [hep-th].

[92] D. Anninos, T. Anous, F. Denef, G. Konstantinidis, and E. Shaghoulian, “Supergoop Dynamics”, JHEP **1303**, 081 (2013), arXiv:1205.1060 [hep-th].

[93] G. Bossard and C. Ruef, “Interacting non-BPS black holes”, Gen.Rel.Grav. **44**, 21–66 (2012), arXiv:1106.5806 [hep-th].

[94] G. Bossard and S. Katmadas, “Duality covariant non-BPS first order systems”, JHEP **1209**, 100 (2012), arXiv:1205.5461 [hep-th].

[95] B. de Wit and S. Katmadas, “Near-Horizon Analysis of D=5 BPS Black Holes and Rings”, JHEP **1002**, 056 (2010), arXiv:0910.4907 [hep-th].

[96] K. Hanaki, K. Ohashi, and Y. Tachikawa, “Comments on charges and near-horizon data of black rings”, JHEP **0712**, 057 (2007), arXiv:0704.1819 [hep-th].

[97] I. Bena, J. de Boer, M. Shigemori, and N. P. Warner, “Double, Double Supertube Bubble”, JHEP **1110**, 116 (2011), arXiv:1107.2650 [hep-th].

[98] B. D. Chowdhury and A. Virmani, “Modave Lectures on Fuzzballs and Emission from the D1-D5 System”, (2010), arXiv:1001.1444 [hep-th].

[99] J. de Boer and M. Shigemori, “Exotic branes and non-geometric backgrounds”, Phys.Rev.Lett. **104**, 251603 (2010), arXiv:1004.2521 [hep-th].

[100] J. B. Gutowski, D. Martelli, and H. S. Reall, “All Supersymmetric solutions of minimal supergravity in six- dimensions”, Class.Quant.Grav. **20**, 5049–5078 (2003), arXiv:hep-th/0306235 [hep-th].

[101] M. Cariglia and O. A. Mac Conamhna, "The General form of supersymmetric solutions of N=(1,0) U(1) and SU(2) gauged supergravities in six-dimensions", Class.Quant.Grav. **21**, 3171–3196 (2004), arXiv:hep-th/0402055 [hep-th].

[102] I. Bena, S. Giusto, M. Shigemori, and N. P. Warner, "Supersymmetric Solutions in Six Dimensions: A Linear Structure", JHEP **1203**, 084 (2012), arXiv:1110.2781 [hep-th].

[103] I. Bena and P. Kraus, "Three charge supertubes and black hole hair", Phys.Rev. **D70**, 046003 (2004), arXiv:hep-th/0402144 [hep-th].

[104] I. Bena, "Splitting hairs of the three charge black hole", Phys.Rev. **D70**, 105018 (2004), arXiv:hep-th/0404073 [hep-th].

[105] I. Bena, P. Kraus, and N. P. Warner, "Black rings in Taub-NUT", Phys.Rev. **D72**, 084019 (2005), arXiv:hep-th/0504142 [hep-th].

[106] S. Giusto and R. Russo, "Perturbative superstrata", Nucl.Phys. **B869**, 164–188 (2013), arXiv:1211.1957 [hep-th].

[107] N. Bobev, B. E. Niehoff, and N. P. Warner, "New Supersymmetric Bubbles on $AdS_3 \times S^3$", JHEP **1210**, 013 (2012), arXiv:1204.1972 [hep-th].

[108] J. de Boer and M. Shigemori, "Exotic Branes in String Theory", Phys.Rept. **532**, 65–118 (2013), arXiv:1209.6056 [hep-th].

[109] I. Bena, S. Giusto, R. Russo, M. Shigemori, and N. P. Warner, "Habemus Superstratum! A constructive proof of the existence of superstrata", (2015), arXiv:1503.01463 [hep-th].

[110] G. Bossard and S. Katmadas, "Duality covariant multi-centre black hole systems", JHEP **1308**, 007 (2013), arXiv:1304.6582 [hep-th].

[111] V. Jejjala, O. Madden, S. F. Ross, and G. Titchener, "Non-supersymmetric smooth geometries and D1-D5-P bound states", Phys.Rev. **D71**, 124030 (2005), arXiv:hep-th/0504181 [hep-th].

[112] E. G. Gimon, T. S. Levi, and S. F. Ross, "Geometry of non-supersymmetric three-charge bound states", JHEP **0708**, 055 (2007), arXiv:0705.1238 [hep-th].

[113] S. Giusto, S. F. Ross, and A. Saxena, "Non-supersymmetric microstates of the D1-D5-KK system", JHEP **0712**, 065 (2007), arXiv:0708.3845 [hep-th].

[114] S. D. Mathur and D. Turton, "Oscillating supertubes and neutral rotating black hole microstates", JHEP **1404**, 072 (2014), arXiv:1310.1354 [hep-th].

[115] S. Banerjee, B. D. Chowdhury, B. Vercnocke, and A. Virmani, "Non-supersymmetric Microstates of the MSW System", JHEP **1405**, 011 (2014), arXiv:1402.4212 [hep-th].

[116] G. Bossard and S. Katmadas, "Floating JMaRT", (2014), arXiv:1412.5217 [hep-th].

[117] B. D. Chowdhury and B. Vercnocke, "New instability of non-extremal black holes: spitting out supertubes", JHEP **1202**, 116 (2012), arXiv:1110.5641 [hep-th].

[118] A. W. Peet, "TASI lectures on black holes in string theory", 353–433 (2000), arXiv:hep-th/0008241 [hep-th].

Appendix A

Units and Conventions

Let us summarize the conventions used in this thesis. We use the conventions of [40, 44]. One can also check [37, 118] for more details. Throughout we use $c = \hbar = 1$ and a mostly plus Lorentzian metric $(-, +, \ldots, +)$.

Strings, branes and Newton's constant

For the Planck length l_D and the Newton's constant G_D in D spacetime dimensions we have

$$16\pi G_D \equiv (2\pi)^{D-3}\, l_D^{D-2} \,. \tag{A.1}$$

For eleven and ten dimensions specifically we have,

$$16\pi G_{11} \equiv (2\pi)^8\, l_D^9 \,, \quad 16\pi G_{10} \equiv (2\pi)^7\, g_s^2 l_s^8 \,, \tag{A.2}$$

where $l_s = \sqrt{\alpha'}$ is the string length, g_s is the string coupling constant and we have $l_{10} = l_s g_s^{1/4}$.

Upon dimensional reduction we have

$$G_D = \frac{G_{11}}{(2\pi)^{11-D} V_{11-D}} \tag{A.3}$$

where V_{11-D} is the volume of the internal compactification manifold.

When compactifying M-theory in a circle of radius R for the resulting IIA theory we have

$$R = l_s g_s \,. \tag{A.4}$$

Thus one can show that

$$l_{11} = g^{1/3} l_s \tag{A.5}$$

The tensions of the extended objects in string and M-theory are:

$$T_{F1} = \frac{1}{2\pi\alpha'} \,, \quad T_{Dp} = \frac{1}{g_s (2\pi)^p l_s^{p+1}} \,, \quad T_{NS5} = \frac{1}{g_s^2 (2\pi)^5 l_s^6} \,, \tag{A.6}$$

$$T_{M2} = \frac{1}{(2\pi)^2 l_{11}^3} \,, \quad T_{M5} = \frac{1}{(2\pi)^5 l_1 1^6} \,. \tag{A.7}$$

Compactification from eleven to five dimensions

We use conventions such that

$$G_5 = \frac{\pi}{4} \,, \quad Q_I = N_I \,, \quad k_I = n_I \,, \tag{A.8}$$

where N_I, n_I are the numbers of M2 and M5 branes respectively and Q_I, k_I are the physical charges of the five-dimensional solution.

When compactifying M-theory on T^6 along the directions 5, 6, 7, 8, 9, 10 with each circle having radius R_i we have the following relations

$$G_5 = \frac{G_{11}}{(2\pi)^6 V_6} = \frac{\pi}{4}\frac{(l_{11})^9}{V_6}\,, \tag{A.9}$$

$$Q_1 = \frac{l_{11}^6}{R_7 R_8 R_9 R_{10}} N_1\,, \quad Q_2 = \frac{l_{11}^6}{R_5 R_6 R_9 R_{10}} N_2\,, \quad Q_3 = \frac{l_{11}^6}{R_5 R_6 R_7 R_8} N_3\,, \tag{A.10}$$

$$q_1 = \frac{l_{11}^3}{R_5 R_6} n_1\,, \quad q_2 = \frac{l_{11}^3}{R_7 R_8} n_2\,, \quad q_3 = \frac{l_{11}^3}{R_9 R_{10}} n_3\,. \tag{A.11}$$

Thus for our conventions (A.8) we choose all three T^2's to be of equal volume

$$R_5 R_6 = R_7 R_8 = R_9 R_{10} = l_{11}^3 \equiv g_s l_s^3\,, \tag{A.12}$$

and the volume of T^6 is fixed at

$$V_6 = R_5 R_6 R_7 R_8 R_9 R_{10} = l_{11}^9\,. \tag{A.13}$$

Conventions in the IIA frame

We compactify from M-theory down to IIA on the circle R_9. Thus $R_9 = g_s l_s = 1$ and from (A.12) $R_{10} = l_s^2$. Consequently, from (A.6), for the tensions used to describe supertubes in the D0-F1-D4 frame we have

$$T_{D_0} = 1\,, \quad 2\pi T_{F1} R_{10}\,, \quad 2\pi \frac{T_{D_2}}{T_{F1}} = 1\,. \tag{A.14}$$

Black hole asymptotics

The mass M, angular momentum J and charge Q of a black hole in D spacetime dimensions can be read of from the following expansions of the metric functions and the field strength

$$-g_{tt} = 1 - \frac{16\pi G_D}{(D-2)\,A_{D-2}}\frac{M}{r^{D-3}} + \dots\,, \tag{A.15}$$

$$g_{ti} = \frac{16\pi G_D}{A_{D-2}}\frac{x^j J_{ji}}{r^{D-1}} + \dots\,, \tag{A.16}$$

$$F_{tr} = (D-3)\,\frac{Q}{r^{D-2}} + \dots\,, \tag{A.17}$$

where A_{D-2} is the area of the $(D-2)$-sphere.

Appendix B

Details of the almost-BPS system

B.1 The angular momentum vector

The angular momentum vector, k, in the metric (5.1) is decomposed as in (5.33). Here we present the results for the functions μ and ω for both the BPS and the almost-BPS case. For the almost-BPS case we present the results in both conventions that is with $\epsilon = +1$ in (5.5) and the base metric anti-self-dual and $\epsilon = -1$ and the base metric being self-dual. The BPS results as usual correspond to $\epsilon = +1$ and a self-dual base metric. We also examine the relationship between BPS and almost-BPS descriptions for $h = 0$.

B.1.1 The function, μ

Here we write again the function μ that we provided in the main text in equations (5.34) - (5.40).

BPS

For BPS we have

$$\mu \;=\; \tfrac{1}{6}\,V^{-2}\,C_{IJK}\,K_+^I K_+^J K_+^K \;+\; \tfrac{1}{2}\,V^{-1}\,K_+^I L_I^+ \;+\; M_+\,, \tag{A.1}$$

where M_+ is a harmonic function which we will take to be

$$M_+ \;=\; m_\infty^+ \;+\; \frac{m_0^+}{r} \;+\; \sum_{j=1}^{3} \frac{m_j^+}{r_j} + m_{div}^+ r\cos\theta\,. \tag{A.2}$$

For reasons that will become apparent below, we have added an unphysical harmonic term that diverges at infinity. Thus for the system we are studying,

$$\mu \;=\; \frac{k_1^+ k_2^+ k_3^+}{r_1 r_2 r_3 V^2} \;+\; \frac{1}{2V}\left(\frac{k_1^+}{r_1}\left(1+\frac{Q_2^{(1)}}{4r_2}+\frac{Q_3^{(1)}}{4r_3}\right) \;+\; \frac{k_2^+}{r_2}\left(1+\frac{Q_1^{(2)}}{4r_1}+\frac{Q_3^{(2)}}{4r_3}\right)\right. \tag{A.3}$$

$$\left. +\; \frac{k_3^+}{r_3}\left(1+\frac{Q_1^{(3)}}{4r_1}+\frac{Q_2^{(3)}}{4r_2}\right)\right) \;+\; M_+\,. \tag{A.4}$$

Almost-BPS

For almost-BPS solutions the expression for μ is the same for both conventions

$$\mu \;=\; \sum_I k_I^- \mu_I^{(1)} \;+\; h\sum_I\sum_{j\neq I}\frac{Q_j^{(I)}k_I^-}{4}\mu_{Ij}^{(3)} \;+\; q\sum_I\sum_{j\neq I}\frac{Q_j^{(I)}k_I^-}{4}\mu_{Ij}^{(5)} \tag{A.5}$$

$$+\; k_1^- k_2^- k_3^-\left(h^2\mu^{(6)}+q^2\mu^{(7)}+qh\mu^{(8)}\right) \;+\; \mu^{(9)}\,, \tag{A.6}$$

where, following [63], the $\mu^{(j)}$ are defined as follows

$$\mu_I^{(1)} \;=\; \frac{1}{2r_I}\,, \qquad \mu_{Ij}^{(3)} \;=\; \frac{1}{2V r_I r_j}\,, \qquad \mu_{Ij}^{(5)} \;=\; \frac{r^2 + a_I a_j - 2a_I r cos\theta}{2V a_I (a_j - a_I) r r_I r_j}\,, \tag{A.7}$$

$$\mu^{(6)} \;=\; \frac{1}{V r_1 r_2 r_3}\,, \qquad \mu^{(7)} \;=\; \frac{r cos\theta}{V a_1 a_2 a_3 r_1 r_2 r_3}, \tag{A.8}$$

$$\mu^{(8)} \;=\; \frac{r^2(a_1 + a_2 + a_3) + a_1 a_2 a_3}{2V r a_1 a_2 a_3 r_1 r_2 r_3}\,, \qquad \mu^{(9)} \;=\; \frac{M_-}{V}\,. \tag{A.9}$$

$$M_- \;=\; m_\infty^- \;+\; \frac{m_0^-}{r} \;+\; \sum_{j=1}^{3} \frac{m_j^-}{r_j} \;+\; \frac{m_{div}^-}{r^2}\,\cos\theta\,. \tag{A.10}$$

Again we have added an unphysical harmonic term but this time it diverges at the origin.

Once again regularity at the supertubes requires the following for both classes of solution:

$$m_1^\pm \;=\; \frac{Q_1^{(2)}\,Q_1^{(3)}}{2\,k_1^\pm}\,, \qquad m_2^\pm \;=\; \frac{Q_2^{(1)}\,Q_2^{(3)}}{2\,k_2^\pm}\,, \qquad m_3^\pm \;=\; \frac{Q_3^{(1)}\,Q_3^{(2)}}{2\,k_3^\pm}\,. \tag{A.11}$$

B.1.2 The one-form, ω

BPS

For $a_i > a_j$ we define:

$$\omega_{ij} \;\equiv\; -\,\frac{(r^2\,\sin^2\theta + (r\cos\theta - a_i + r_i)(r\cos\theta - a_j - r_j))}{(a_i - a_j)\,r_i\,r_j}\,, \tag{A.12}$$

To define the BPS angular momentum vector, ω, it is also convenient to introduce:

$$\Delta_1 \;\equiv\; 2\left(h + \frac{q}{a_1}\right)m_1^+ \;-\; \frac{\Gamma_{12}^+}{(a_2-a_1)} \;-\; \frac{\Gamma_{13}^+}{(a_3-a_1)} \;-\; k_1^+\,, \tag{A.13}$$

$$\Delta_2 \;\equiv\; 2\left(h + \frac{q}{a_2}\right)m_2^+ \;+\; \frac{\Gamma_{12}^+}{(a_2-a_1)} \;-\; \frac{\Gamma_{23}^+}{(a_3-a_2)} \;-\; k_2^+\,, \tag{A.14}$$

$$\Delta_3 \;\equiv\; 2\left(h + \frac{q}{a_3}\right)m_3^+ \;+\; \frac{\Gamma_{13}^+}{(a_3-a_1)} \;+\; \frac{\Gamma_{23}^+}{(a_3-a_2)} \;-\; k_3^+\,. \tag{A.15}$$

Note that the bubble equations (5.56)–(5.58) and (A.11) imply that $\Delta_j = 0$, $j = 1, 2, 3$. We will not impose these conditions here and we will not remove Dirac strings more generally.

We then find that for the BPS solution, $\omega = \omega_\phi d\phi$, with

$$\begin{aligned}\omega_\phi \;=\; & \frac{1}{2}\left(\Gamma_{21}^+\,\omega_{21} + \Gamma_{31}^+\,\omega_{31} + \Gamma_{32}^+\,\omega_{32}\right) \;+\; q\sum_{i=1}^{3}\frac{m_i^+}{a_i\,r_i}(r\,\sin^2\theta + (r\cos\theta - a_i + r_i)(\cos\theta - 1) \\ & +\; \frac{1}{2}\sum_{i=1}^{3}\Delta_i\,\frac{r\cos\theta - a_i}{r_i} \;+\; \kappa^+ \;-\; \frac{1}{2}\left(\frac{\Gamma_{12}^+}{(a_1-a_2)} + \frac{\Gamma_{13}^+}{(a_1-a_3)} + \frac{\Gamma_{23}^+}{(a_2-a_3)}\right) \\ & +\; \left(q\sum_{i=1}^{3}\frac{m_i^+}{a_i} - h\,m_0^+\right)(1-\cos\theta) - q\,m_\infty^+\cos\theta \;+\; m_{div}^+(\tfrac{1}{2}hr^2 + qr)\sin^2\theta\,,\end{aligned} \tag{A.16}$$

where κ^+ is a constant of integration. After removing Dirac-strings and setting $m_{div}^+ = 0$ we get

$$\omega \;=\; \sum_{i=1}^{3} q\,m_i\,\omega_{i0} \;+\; \frac{1}{8}\sum_i \sum_{j,(a_j<a_i)} \Gamma_{ji}\,\omega_{ij} \tag{A.17}$$

The label $i = 0$ refers to the Taub-NUT center and, in particular, $a_0 = 0$, $r_0 = r$.

Almost-BPS

When using the conventions with $\epsilon = +1$ we have

$$
\begin{aligned}
\omega_\phi \;\; = \;\; & -\frac{1}{2} \, \sum_{i=1}^{3} \frac{k_i^-}{a_i \, r_i} \big(h \, a_i \, (r\cos\theta - a_i) + q \, (r - a_i \cos\theta) \big) \\
& -\frac{1}{2} \, \sum_{\substack{i,j=1 \\ j\neq i}}^{3} \frac{k_i^- Q_j^{(i)}}{a_i (a_i - a_j) \, r_i \, r_j} \big[h \, a_i \left(r^2 - (a_i + a_j) r\cos\theta + a_i a_j \right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad + q \left((r^2 + a_i a_j \cos\theta) - r(a_i \cos 2\theta + a_j) \right) \big] \\
& - \frac{k_1^- k_2^- k_3^-}{a_1 a_2 a_3 \, r_1 r_2 r_3} \big[\, q^2 \, r^2 \sin^2\theta \; + \; \tfrac{1}{2} \, h \, q \left(r^3 - (a_1 + a_2 + a_3) r^2 \cos\theta \right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad \left. + (a_1 a_2 + a_1 a_3 + a_2 a_3) r - a_1 a_2 a_3 \cos\theta \right) \big] \\
& - \kappa^- \; - \; \frac{m_{div}^-}{r} \sin^2\theta \; + \; m_0^- \cos\theta \; + \; \sum_{i=1}^{3} \frac{m_i^-}{r_i} (r\cos\theta - a_i) \, ,
\end{aligned}
\tag{A.18}
$$

where κ^- is another constant of integration.

When $\epsilon = -1$ we get

$$
\omega \;\; = \;\; \sum_I k_I \, \omega_I^{(1)} \; + \; h \sum_I \sum_{j \neq I} \frac{Q_j^{(I)} k_I}{4} \, \omega_{Ij}^{(3)} \; + \; q \sum_I \sum_{j\neq I} \frac{Q_j^{(I)} k_I}{4} \, \omega_{Ij}^{(5)} \tag{A.19}
$$

$$
+ \;\; k_1 k_2 k_3 \left(h^2 \omega^{(6)} + q^2 \omega^{(7)} + q h \omega^{(8)} \right) \; + \; \omega^{(9)} \, , \tag{A.20}
$$

where

$$\omega_I^{(1)} = \frac{h}{2}\frac{r\cos\theta - a_I}{r_I}d\phi + \frac{q}{2}\frac{r - a_I\cos\theta}{a_I r_I}\, d\phi\,, \tag{A.21}$$

$$\omega_{Ij}^{(3)} = \frac{r^2 + a_I a_j - (a_I + a_j) r\cos\theta}{2(a_I - a_j) r_I r_j}\, d\phi\,, \tag{A.22}$$

$$\omega_{Ij}^{(5)} = \frac{r\,(a_j + a_I\cos 2\theta) - (r^2 + a_I a_j)\cos\theta}{2a_I(a_j - a_I) r_I r_j}\, d\phi\,, \tag{A.23}$$

$$\omega^{(6)} = 0\,, \qquad \omega^{(7)} = \frac{r^2\sin^2\theta}{a_1 a_2 a_3 r_1 r_2 r_3}\, d\phi, \tag{A.24}$$

$$\omega^{(8)} = \frac{r^3 + r(a_1a_2 + a_2a_3 + a_1a_3) - (r^2(a_1 + a_2 + a_3) + a_1a_2a_3)\cos\theta}{2a_1a_2a_3r_1r_2r_3}\, d\phi, \tag{A.25}$$

$$\omega^{(9)} = \left(\kappa - m_0\cos\theta - \sum_i m_i\frac{r\cos\theta - a_i}{r_i}\right) d\phi\,, \tag{A.26}$$

where we also set $m^-_{div} = 0$. The absence of Dirac strings and CTC's requires that the bubble equations (5.59)–(5.61) are satisfied and that:

$$\kappa = -q\sum_i \frac{k_i}{2a_i} - h\,q\frac{k_1k_2k_3}{2a_1a_2a_3} - h\sum_I\sum_{j\neq i}\frac{Q_j^{(I)}k_I}{8(a_I - a_j)}\,, \tag{A.27}$$

$$m_0 = -q\sum_i \frac{k_i}{2a_i} - hq\frac{k_1k_2k_3}{2a_1a_2a_3} + q\sum_I\sum_{j\neq i}\frac{Q_j^{(I)}k_I}{8a_I(a_I - a_j)}\,. \tag{A.28}$$

B.1.3 The Minkowski-space limit

For $h = 0$ the BPS and non-BPS solutions must be the same and in Section 5.3.5 it was argued that this should be achieved by the coordinate change (5.65) [1]. For the angular

[1] Here for almost-BPS we are using the conventions with self-dual field strengths and anti-self-dual base metric

momentum vector this means

$$\begin{aligned} \mu_-(d\psi_- - q\,\cos\theta\,d\phi_-) \;+\; \omega_{\phi\,-}d\phi_- \;&=\; \mu_+(d\psi_+ + q\,\cos\theta\,d\phi_+) \;+\; \omega_{\phi\,+}d\phi_+ \\ &=\; \tilde{\mu}_+(-q\,d\phi_- + \cos\theta\,d\psi_-) \;+\; \frac{1}{q}\,\tilde{\omega}_{\phi\,+}d\psi_-/,\,, \end{aligned} \tag{A.29}$$

where the quantities with the tildes have the replacement $k_I^+ \to \widehat{k}_I$. This implies that we must have (for $h = 0$):

$$\mu_- \;=\; \tilde{\mu}_+\cos\theta \;+\; \frac{1}{q}\,\tilde{\omega}_{\phi\,+}\,, \qquad \omega_{\phi\,-} \;=\; \tilde{\omega}_{\phi\,+}\,\cos\theta \;-\; q\;\tilde{\mu}_+\sin^2\theta\,. \tag{A.30}$$

We have verified this by explicit computation and it works provided that one also makes the substitutions:

$$m_i^+ \;\to\; \frac{a_i}{q}\,m_i^-\,, \quad m_0^+ \;\to\; \frac{1}{q}\,m_{div}^-\,, \quad m_\infty^+ \;\to\; \frac{1}{q}\,\kappa^-\,, \quad m_{div}^+ \;\to\; \frac{1}{q}\,m_\infty^-\,, \quad \kappa^+ \;\to\; m_0^-\,. \tag{A.31}$$

Thus the transformation (5.65) does indeed map the complete BPS description to the almost-BPS description. Note also that we have included the terms $m_{div}^\pm$ so as to complete this dictionary.

B.2 The asymptotic structure and charges

The general scaling geometries discussed in Sections 5.3 and 5.4 look, at larger scales, like a black-ring wrapping the Taub-NUT fiber and, as a result, look like a black hole in the four-dimensional, non-compact space time. Here we complete this description by examining the asymptotic structure. For the general configuration of supertubes described in Section 5.3 we give the remainder of the solution and compare it with black-ring solution and we calculate the asymptotic charges. It should be remembered that the complete solution isn't simply an isolated black hole but is a two centered configuration with a $D6$-brane at the center of the space and so there will be contributions to the asymptotic charges from both centers and the interactions between them. In this calculations we use the convention with $\epsilon = -1$.

B.2.1 The BPS configuration

In the scaling limit were the distances between the supertubes are very small compared to their distance from the center of the space we expect the configuration to look like a black ring of radius R. It follows immediately from (5.27)–(5.29) that the asymptotic electric charges are:

$$Q_1 \ = \ Q_2^{(1)} + Q_3^{(1)}\,, \qquad Q_2 \ = \ Q_1^{(2)} + Q_3^{(2)}\,, \qquad Q_3 \ = \ Q_1^{(3)} + Q_2^{(3)}\,. \tag{A.32}$$

The functions describing the solution of a black ring in Taub-NUT (see [105] for details) precisely match the ones for a system of three clustered supertubes. In particular, it is useful to introduce the effective angular momentum parameter:

$$m_R \ \equiv \ m_1 + m_2 + m_3\,, \tag{A.33}$$

and then note that (5.52) and (5.67) yield the radius relation of the effective black ring:

$$m_R V_R \;=\; \frac{1}{2}(k_1 + k_2 + k_3)\,. \tag{A.34}$$

In the scaling limit, to first order, we find $\omega_{10} \approx \omega_{20} \approx \omega_{30} \approx \omega_{R0}$ and $\omega_{21} \approx \omega_{32} \approx \omega_{31}$. The latter, when combined with (5.71), means that the "supertube interaction term" in (A.17) vanishes to leading order. This is the familiar merger condition on local angular momentum contributions [71]. The remaining term in (A.17) is the sum of the angular momentum contributions for each supertube about the center of space thus give the correct effective ω for the black ring.

We can compactify our solution on the Taub-NUT circle to obtain a microstate geometry for a four-dimensional black-hole. By standard Kaluza-Klein reduction arguments and in the same spirit as in [65] we can write the metric as:

$$ds^2 = \frac{I_4}{(Z_1 Z_2 Z_3)^{2/3} V^2}\Big(d\psi + A - \frac{\mu V^2}{I_4}(dt+\omega)\Big)^2 + \frac{V(Z_1 Z_2 Z_3)^{1/3}}{\sqrt{I_4}} ds_4^2\,, \tag{A.35}$$

where

$$ds_4^2 = -I_4^{-1/2}(dt+\omega)^2 + I_4^{1/2}(dr^2 + r^2 d\Omega_2^2)\,, \tag{A.36}$$

is the 4-dimensional Lorentzian metric and one defines:

$$I_4 \;\equiv\; Z_1 Z_2 Z_3 V - \mu^2 V^2\,. \tag{A.37}$$

To identify the asymptotic charges and ADM mass it is most convenient to pass to a rest-frame at infinity by re-defining

$$\tilde{\psi} \;=\; \psi - \frac{\mu_0 h^2}{I_0} t\,, \tag{A.38}$$

where $I_0 = h - h^2\mu_0^2$ and μ_0 are the constant values of I_4 and μ at infinity. For the BPS solution one has $\mu_0 = m_\infty = -\frac{m_R}{R} = \frac{J_R}{16R}$.

The asympotic charges can be read from the expansion of the metric and the warp factors Z_I. After simplifying the resulting expressions and by making use of the radius relation. Thus for the electric charges we find:

$$\bar{Q}_I \;=\; Q_I\,, \tag{A.39}$$

and the ADM mass is given by:

$$M \;=\; \frac{1}{8 I_0^{3/2}} \left(4\,q + 16 m_\infty^2 h^2 R + h\,(Q_1 + Q_2 + Q_3)\right)\,. \tag{A.40}$$

The Kaluza-Klein charge coming from the momentum around the ψ-fiber is:

$$P \;=\; -\,\frac{h^2 m_\infty}{4 I_0^2}\Big(4 h m_\infty^2 R V_R + 4h\,(R+1) + 4h^2 m_\infty R + h\big(Q_1 + Q_2 + Q_3\big)\Big)\,, \tag{A.41}$$

and the angular momentum around the ϕ direction is:

$$J_3 \;=\; \frac{q h m_\infty R}{I_0^{3/2}}(h m_\infty^2 - 1) \;=\; -\,\frac{m_\infty q R}{\sqrt{I_0}}\,. \tag{A.42}$$

The black ring horizon area is given by:

$$A_H \; = \; 2\pi^2 q\sqrt{J_4}\,, \tag{A.43}$$

where J_4 is the $E_{7(7)}$ quartic invariant:

$$J_4 \; = \; -\sum_{I=1}^{3} Q_I^2 k_I^2 + 2\sum_{I<J} k_I k_J Q_I Q_J + 8k_1k_2k_3 J_R\,. \tag{A.44}$$

B.2.2 The non-BPS configuration

In the scaling limit, where the distances between the supertubes are very small compared to their distance from the center of the space, the solution should effectively behave as a black ring of radius R. Once more it is easy to see that the functions describing the solution match those of the black ring if one identifies:

$$Q^{(I)} = \sum_{i\neq I} Q_i^{(I)}\,, \qquad m_R = m_1 + m_2 + m_3\,. \tag{A.45}$$

Denoting the terms corresponding to the supertube and the black ring solution with the subscripts "*st*" and "*br*", we then have:

$$\kappa_{br} = \tilde{\kappa}_{st} = \kappa_{st} + h\sum_I \sum_{j\neq i} \frac{Q_j^{(I)} k_I}{8(a_I - a_j)}\;,\,, \tag{A.46}$$

$$m_{0,br} = \tilde{m}_{0,st} = m_{0,st} - q\sum_I \sum_{j\neq i} \frac{Q_j^{(I)} k_I}{8a_I(a_I - a_j)}\,, \tag{A.47}$$

$$\mu_{br}^{(1)} = \mu_{I,st}^{(1)}\,, \qquad \omega_{br}^{(1)} = \omega_{I,st}^{(1)}\,, \tag{A.48}$$

$$\mu_{br}^{(2)} = \mu_{Ij,st}^{(3)}\,, \qquad \omega_{br}^{(2)} = \tilde{\omega}_{Ij,st}^{(3)} = \omega_{Ij,st}^{(3)} - \frac{1}{2(a_I - a_j)}d\phi\,, \tag{A.49}$$

$$\mu_{br}^{(3)} = \mu_{st}^{(2)}\,, \qquad \omega_{br}^{(3)} = \omega_{st}^{(2)}\,, \qquad \mu_{br}^{(4)} = \tilde{\mu}_{Ij,st}^{(5)} = \mu_{Ij,st}^{(5)} - \frac{1}{2(a_j - a_I)a_I r}, \tag{A.50}$$

$$\omega_{br}^{(4)} = \tilde{\omega}_{Ij,st}^{(5)} = \omega_{Ij,st}^{(5)} - \frac{\cos\theta}{2(a_j - a_I)a_I}\,, \qquad \mu_{br}^{(5)} = \mu_{st}^{(4)}\,, \qquad \omega_{br}^{(5)} = \omega_{st}^{(4)}\,, \tag{A.51}$$

$$\mu_{br}^{(i)} = \mu_{st}^{(i)}\,, \qquad \omega_{br}^{(i)} = \omega_{st}^{(i)}\,, \quad i = 6,7,8,9\,. \tag{A.52}$$

The modified terms $\tilde{\kappa}_{st}$, $\tilde{m}_{0,st}$, $\tilde{\omega}_{Ij,st}^{(3)}$, $\tilde{\mu}_{Ij,st}^{(5)}$, $\tilde{\omega}_{Ij,st}^{(5)}$ still describe exactly the same three-supertube system because the modifications involve the re-shuffling of harmonic functions into the $\mu^{(i)}V$ with compensating shifts in M (along with corresponding changes in ω's).

The non-BPS radius relation for the black ring is:

$$\hat{m}_R V_R = 16\sum_I \hat{k}_I + 16hq\frac{\hat{k}_1\hat{k}_2\hat{k}_3}{V_R R^3}\,, \tag{A.53}$$

which is to be identified with (5.68), where

$$\hat{m}_i = m_i\left(h + \frac{q}{|a_i|}\right)^{-1}\,, \tag{A.54}$$

is the effective angular-momentum parameter.

The electric charges of the non-BPS configuration are

$$\bar{Q}_I \;=\; Q_I + \frac{4q}{R^2}\frac{C^{IJK}}{2}k_J k_K\,, \tag{A.55}$$

while the ADM mass is given by

$$M \ = \ \frac{1}{I_0^{3/2}} \left(\frac{q}{2} + \frac{h}{8} \left(\bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3 \right) - mh \ (k_1 + k_2 + k_3) \right) . \tag{A.56}$$

The Kaluza Klein charge is now

$$P = \frac{1}{I_0^2} \Big(h^2 (h + m^2) \ (k_1 + k_2 + k_3) - \tfrac{1}{4} \, mh^2 \ \left(\bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3 \right) - m^3 q \Big) \, , \tag{A.57}$$

and the angular momentum is:

$$J_3 = \frac{q}{2\sqrt{I_0}} \Big(\, (k_1 + k_2 + k_3) + \frac{1}{R} \, \vec{k} \cdot \vec{Q} + \frac{1}{R^2} (h + \frac{2q}{R}) k_1 k_2 k_3 \Big) \, . \tag{A.58}$$

For the horizon area we find:

$$\hat{A}_H = 2\pi^2 q \sqrt{\hat{J}_4} \, , \tag{A.59}$$

where

$$\hat{J}_4 = - \sum_{I=1}^{3} Q_I^2 \hat{k}_I^2 + 2 \sum_{I<J} \hat{k}_I \hat{k}_J Q_I Q_J - 8 \hat{k}_1 \hat{k}_2 \hat{k}_3 16 \hat{m}_R \, , \tag{A.60}$$

which, upon setting $16\hat{m}_R = -J_R$, matches the horizon area of a BPS black ring with dipole and angular momentum parameters equal to the effective ones.

B.3 Details of the quartic

The bubble equations can be written in a more symbolic form as:

$$\begin{aligned} g(\beta-\alpha)+b(\gamma-\beta)+d_1\beta\gamma+f_1 &= 0\,, && \text{(A.61)}\\ c(\alpha-\gamma)+g(\alpha-\beta)+d_2\alpha^2+f_2 &= 0\,, && \text{(A.62)}\\ b(\beta-\gamma)+c(\gamma-\alpha)+d_3\beta\gamma+f_3 &= 0\,, && \text{(A.63)} \end{aligned}$$

with the following definitions:

$$g=\frac{1}{|a_1-a_2|}\,,\qquad b=\frac{1}{|a_1-a_3|}\,,\qquad c=\frac{1}{|a_2-a_3|}\,,\qquad d_i=\frac{V_{a_i}}{4d^2}\,,\tag{A.64}$$

$$V_{a_i}=\left(h+\frac{q}{a_i}\right)\,,\qquad f_1=f_3=-4-Y\,,\qquad f_2=-4+Y\,.\tag{A.65}$$

Summing the bubble equations we get the relatively simple condition:

$$\beta\gamma(d_1+d_3)+d_2\alpha^2+f=0\,,\tag{A.66}$$

where $f=f_1+f_2+f_3$. For later convenience we define the following:

$$A = c\left(1-\frac{b}{g}\right)-b\,,\qquad B=\frac{cd_1}{g}-d_3\,,\qquad \Gamma=\frac{cf_1}{g}-f_3\,,\tag{A.67}$$

$$\Psi = 1-\frac{b}{g}\,,\qquad \Phi=\frac{2\,b}{g}-\frac{2\,b^2}{g^2}+\frac{2\,d_1f_1}{g^2}+\frac{d_1}{d_2}+\frac{d_3}{d_2}\,.\tag{A.68}$$

The foregoing system of equations can be explicitly solved by reducing it to a quartic polynomial for the charge β:

$$s_4\beta^4 + s_3\beta^3 + s_2\beta^2 + s_1\beta + s_0 \;=\; 0\,, \tag{A.69}$$

from which α and γ can be determined using:

$$\gamma \;=\; \frac{\Gamma + A\beta}{A - B\beta}\,, \qquad \alpha \;=\; \beta + \frac{b}{g}(\gamma-\beta) + \frac{d_1}{g}\beta\gamma + \frac{f_1}{g} = 0\,. \tag{A.70}$$

The coefficients of the quartic (A.69) are given by:

$$s_0 \;=\; \left(\Gamma\frac{b}{g} + A\frac{f_1}{g}\right)^2 + A^2\frac{f}{d_2}\,, \qquad s_4 = \left(B\Psi - A\frac{d_1}{g}\right)^2\,, \tag{A.71}$$

$$s_1 \;=\; 2A\Gamma\frac{b^2}{g^2} + \Gamma A\Phi + 2\Gamma^2\frac{bd_1}{g^2} + 2A^2\frac{f_1}{g}\Psi + 2\left(A^2 - \Gamma B\right)\frac{f_1 b}{g^2} - 2AB\left(\frac{f_1^2}{g^2} + \frac{f}{d_2}\right), \tag{A.72}$$

$$s_2 \;=\; A^2\Psi^2 + A^2\frac{b^2}{g^2} + \Gamma^2\frac{d_1^2}{g^2} + \left(A^2 - \Gamma B\right)\Phi + 2\Gamma A\frac{d_1}{g}\Psi + 4\Gamma A\frac{bd_1}{g^2} - 4AB\frac{f_1}{g}\Psi$$

$$-2AB\frac{f_1 b}{g^2} + B^2\left(\frac{f_1^2}{g^2} + \frac{f}{d_2}\right)\,, \tag{A.73}$$

$$s_3 \;=\; -2AB\Psi^2 + 2\Gamma A\frac{d_1^2}{g^2} - AB\Phi + 2\frac{d_1}{g}\left(A^2 - B\Gamma\right)\Psi + 2A^2\frac{d_1 b}{g^2} + 2B^2\frac{f_1}{g}\Psi\,. \tag{A.74}$$

For $q = 0$ one has $s_3 = s_1 = 0$ and the quartic, (A.69), collapses to a quadratic in β^2.