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Analytical Solutions of the Schrödinger Equation for the Manning-Rosen plus Hulthén Potential Within SUSY Quantum Mechanics

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Abstract. In this paper, the bound state solution of the modified radial Schrödinger equation is obtained for the Manning-Rosen plus Hulthén potential by implementing the novel improved scheme to surmount the centrifugal term. The energy eigenvalues and corresponding radial wave functions are defined for any $l \neq 0$ angular momentum case via the Nikiforov-Uvarov (NU) and supersymmetric quantum mechanics (SUSYQM) methods. By using these two different methods, equivalent expressions are obtained for the energy eigenvalues, and the expression of radial wave functions transformations to each other is demonstrated. The energy levels are worked out and the corresponding normalized eigenfunctions are represented in terms of the Jacobi polynomials for arbitrary l states. A closed form of the normalization constant of the wave functions is also found. It is shown that, the energy eigenvalues and eigenfunctions are sensitive to n_r radial and l orbital quantum numbers.

1. Introduction

The exact solutions of the stationary Schrödinger equation play an important role in quantum mechanics since they contain all the necessary information regarding the quantum system under consideration. Since the wave function contains all necessary information for full description of a quantum system, therefore, an analytical solution of the Schrödinger, Klein-Fock-Gordon and Dirac equations is of high importance in quantum mechanics [1, 2]. There are few potentials for which the Schrödinger equation can be solved explicitly for all n_r radial and l orbital quantum states. However, analytic solutions are possible only for a few simple quantum systems like the movement in the spherical simmetrical field and the linear harmonic oscillator [1, 3, 4, 5]. In general, many quantum systems can be treated only by approximation methods or numerical solutions.

As known, one of the main objectives in theoretical physics since the early years of quantum mechanics (QM) is to obtain an exact solution of the Schrödinger equation for some special potentials of physical interest.

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It would be interesting and an other side important to solve the Schrödinger equation for the Manning-Rosen plus Hulthén potential for $l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interacting systems.

The central Manning-Rosen potential is defined by [6, 7]

$$V(r) = \frac{1}{kb^2}\left[\frac{\alpha(\alpha-1)e^{-2r/b}}{(1-e^{-r/b})^2} - \frac{A'e^{-r/b}}{(1-e^{-r/b})}\right], k = 2\mu/\hbar^2. \tag{1}$$

where A' and α are two dimensionless parameters, but parameter b has dimensions of length.

The Hulthén potential is defined by [8, 9]

$$V(r) = -\frac{Ze^2\delta e^{-\delta r}}{(1-e^{-\delta r})} \tag{2}$$

where Z and δ are respectively the atomic number and the screening parameter, determining the range for the Hulthén potential. The Hulthén potential is one of the important short-range potentials in physics.

It should be noted that, the radial Schrödinger equation for the Hulthén potential can be solved analytically for only the states with zero angular momentum [8, 9, 10, 11, 12]. For any l states a number of methods have been employed to evaluate bound-state energies numerically [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

The combined potential which is considered in this study is obtained by adding Hulthén potential term to Manning-Rosen potential as:

$$V(r) = \frac{1}{k}\left[\frac{\alpha(\alpha-1)e^{-2r/b}}{b^2(1-e^{-r/b})^2} - \frac{A'e^{-r/b}}{b^2(1-e^{-r/b})}\right] - \frac{Ze^2\delta e^{-\delta r}}{(1-e^{-\delta r})}. \tag{3}$$

Unfortunately, for an arbitrary l-states ($l \neq 0$), the Schrödinger equation does not get an exact solution. So far, many methods were developed and applied, for the solution Schrödinger equation, such as supersymmetry (SUSY), factorization, Laplace transform approach and the path integral methods. An other method known as the Nikiforov-Uvarov (NU) method [24] was proposed for solving the Schrödinger equation analytically. Many works show the power and simplicity of NU method in solving central and noncentral potentials. This method is based on solving the second order linear differential equation by reducing to a generalized equation of hypergeometric type which is a second order homogeneous differential equation with polynomials coefficients of degree not exceeding the corresponding order of differentiation.

It is very important to notice that modified radial Schrödinger equation for the Manning-Rosen potential is fully studied in Ref. [25].

Taking into account the facts which are mentioned, in this article we obtain the energy eigenvalues and corresponding eigenfunctions for arbitrary l states by solving the modified radial Schrödinger equation for the Manning-Rosen plus Hulthén potential using Nikiforov-Uvarov (NU) method [24] and the shape invariance concept that was introduced by Gendenshtein [26, 27].

The rest of the present work is organized as follows. Bound-state Solution of the radial Schrödinger equation for Manning-Rosen plus Hulthén potential by NU method within ordinary quantum mechanics is provided in Section 2. In Section 3 we present the Solution of Schrödinger equation for Manning-Rosen plus Hulthén potential within SUSY QM and the numerical results and concluding remarks for energy levels and the corresponding normalized eigenfunctions are presented in Section 4.

2. Bound state Solution of the Radial Schrödinger equation for Manning-Rosen Plus Hulthén potential within ordinary quantum mechanics.

The radial Schrödinger equation for Manning-Rosen plus Hulthén potential is

$$R''(r) + \frac{2}{r}R'(r) + \frac{2\mu}{\hbar^2}\Big[E + Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}} - \\ -\frac{l(l+1)\hbar^2}{2\mu r^2} - \frac{\alpha(\alpha-1)e^{-2r/b}}{b^2(1-e^{-r/b})^2} + \frac{A'e^{-r/b}}{b^2(1-e^{-r/b})}\Big]R(r) = 0, \quad (4)$$

respectively.

The effective Manning-Rosen plus Hulthén potential is defined in this form:

$$V_{\text{eff}}(r) = -Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}} + \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{\alpha(\alpha-1)e^{-2r/b}}{b^2(1-e^{-r/b})^2} - \frac{A'e^{-r/b}}{b^2(1-e^{-r/b})} \quad (5)$$

In order to solve Eq.(4) for $l \neq 0$, we must make an approximation for the centrifugal term. When $\delta r << 1$, $(r/b << 1)$ we use an improved approximation scheme [25, 28, 29] to deal with the centrifugal term,

$$\frac{1}{r^2} \approx \delta^2\left[C_0 + \frac{e^{-\delta r}}{(1-e^{-\delta r})^2}\right], \quad (6)$$

where the parameter $C_0 = \frac{1}{12}$ (Ref. [30]) is a dimensionless constant. It should be noted that this approximation, is only valid for small δr and it breaks down in the high screening region. After using this approximation radial Schrödinger equation is solvable analytically.

We assume $R(r) = \frac{1}{r}\chi(r)$ and $1/b = \delta$ in Eq.(4) then the modified radial Schrödinger equation becomes

$$\chi''(r) - l(l+1)\delta^2\left(C_0 + \frac{e^{-\delta r}}{(1-e^{-\delta r})^2}\right)\chi(r) + \\ \frac{2\mu}{\hbar^2}\left[E + Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}} - \frac{\delta^2\alpha(\alpha-1)e^{-2r\delta}}{k^2(1-e^{-r\delta})^2} + \frac{A'\delta^2e^{-r\delta}}{k^2(1-e^{-r\delta})}\right]\chi(r) = 0. \quad (7)$$

According to the (6) approximation scheme the modified effective potential $\widetilde{V}_{\text{eff}}(r)$ becomes

$$\widetilde{V}_{\text{eff}}(r) = -Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}} + \frac{\delta^2\alpha(\alpha-1)e^{-2\delta r}}{(1-e^{-\delta r})^2} - \frac{A'\delta^2e^{-r/b}}{(1-e^{-\delta r})} + \\ + \frac{l(l+1)\hbar^2}{2\mu}\delta^2\left(C_0 + \frac{e^{-\delta r}}{(1-e^{-\delta r})^2}\right) \quad (8)$$

In order to transform Eq.(7), the equation of the generalized hypergeometric-type which is in the form [24]

$$\chi''(s) + \frac{\tilde{\tau}}{\sigma}\chi'(s) + \frac{\tilde{\sigma}}{\sigma^2}\chi(s) = 0, \quad (9)$$

we use the following ansatz in order to make the differential equation more compact,

$$-\varepsilon^2 = \frac{2\mu}{\hbar^2\delta^2}E, \;\; E < 0, \beta^2 = \frac{2\mu Ze^2}{\hbar^2\delta}, s = e^{-\delta r}. \quad (10)$$

Hence, we obtain

$$\chi''(s) + \frac{\chi'(s)}{s} - \frac{1}{s^2(1-s)^2}\Big[-\varepsilon^2(1-s)^2 - l(l-1)(C_0(1-s)^2 + s) + (\beta^2 + A')s(1-s) - \alpha(\alpha-1)s^2\Big]\chi(s) = 0, \tag{11}$$

Now, we can successfully apply NU method for defining eigenvalues of energy. Finally for energy eigenvalues we find:

$$E_{nl} = \frac{-h^2\delta^2}{2\mu} \cdot \left[\frac{l(l+1) + 1/2 + n_r(n_r+1) + \sqrt{\frac{1}{4} + l(l+1) + \alpha(\alpha-1)}(2n_r+1) - \beta^2 - A'}{2n_r + 1 + 2\sqrt{\frac{1}{4} + \alpha(\alpha-1) + l(l+1)}}\right]^2 + \frac{h^2\delta^2}{2\mu} l(l+1)C_0, \tag{12}$$

For the radial eigenfunction we find:

$$\chi_{n_r}(s) = C_{n_r} s^{\sqrt{c}}(1-s)^K \frac{\Gamma(n_r + 2\sqrt{c} + 1)}{n_r!\Gamma(2\sqrt{c}+1)} \underset{21}{F}\left(-n_r, 2\sqrt{c} + 2K + n_r, 1 + 2\sqrt{c}; s\right). \tag{13}$$

The normalization constant C_{n_r} can be found from normalization condition

$$\int_0^\infty |R(r)|^2 r^2 dr = \int_0^\infty |\chi(r)|^2 dr = b\int_0^1 \frac{1}{s}|\chi(s)|^2 ds = 1, \tag{14}$$

by using the following integral formula [32]:

$$\int_0^1 (1-s)^{2(\delta+1)} s^{2\lambda-1}\left\{\underset{21}{F}(-n_r, 2(\delta+\lambda+1) + n_r, 2\lambda+1; s)\right\}^2 dz = \frac{(n_r + \delta + 1)n_r!\Gamma(n_r + 2\delta + 2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n_r + \delta + \lambda + 1)\Gamma(n_r + 2\lambda + 1)\Gamma(2(\delta+\lambda+1) + n_r)}, \tag{15}$$

for $\delta > \frac{-3}{2}$ and $\lambda > 0$. After simple calculations, we obtain normalization constant as

$$C_{n_r} = \sqrt{\frac{n_r!2\sqrt{c}(n_r + K + \sqrt{c})\Gamma(2(K+\sqrt{c}) + n_r)}{b(n_r + K)\Gamma(n_r + 2\sqrt{c} + 1)\Gamma(n_r + 2K)}}. \tag{16}$$

3. The Solution of Schrödinger equation for Manning-Rosen Plus Hulthén potential within SUSY Quantum Mechanics

According to Supersymmetric quantum mechanics, the eigenfunction of ground state $\chi_0(r)$ in Eq.(7) is a form as below [33, 34]

$$\chi_0(r) = Nexp\left(-\frac{\sqrt{2\mu}}{\hbar}\int W(r)dr\right). \tag{17}$$

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where N and $W(r)$ are normalized constant superpotential, respectively. The connection between the supersymmetric partner potentials $V_1(r)$ and $V_2(r)$ of the superpotential $W(r)$ is as follows [33, 34]:

$$V_1(r) = W^2(r) - \frac{\hbar}{\sqrt{2\mu}}W'(r) + E\ , \quad V_2(r) = W^2(r) + \frac{\hbar}{\sqrt{2\mu}}W'(r) + E. \tag{18}$$

The particular solution of the Riccati equation (18) searches the following form:

$$W(r) = -\frac{\hbar}{\sqrt{2\mu}}\left(A + \frac{Be^{-\delta r}}{1-e^{-\delta r}}\right), \tag{19}$$

where A and B unknown constants. Since $V_1(r) = \tilde{V}_{eff}(r)$, having inserted the relations Eq.(8) and Eq.(19) into the expression Eq.(18), and from comparison of compatible quantities in the left and right sides of the equation, we find the following relations for A and B constants :

$$A^2 = -\frac{2\mu E}{\hbar^2} + \delta^2 C_0 l(l+1) = \varepsilon^2\delta^2 + \delta^2 C_0 l(l+1), \tag{20}$$

$$2AB - \delta B = \delta^2 l(l+1) - \beta^2\delta^2 - A'\delta^2 - \delta^2\alpha(\alpha-1), \tag{21}$$

$$B^2 - \delta B = \delta^2 l(l+1) - \delta^2\alpha(\alpha-1). \tag{22}$$

Considering extremity conditions to wave functions, we obtain $B > 0$ and $A < 0$.
Solving Eq.(22) yields

$$B = \frac{\delta \pm \sqrt{\delta^2 + 4\delta^2 l(l+1) + \alpha(\alpha+1)}}{2} = \frac{\delta \pm 2\delta\sqrt{\frac{1}{4} + l(l+1) + \alpha(\alpha+1)}}{2}, \tag{23}$$

and considering $B > 0$ from Eqs.(21) and (22), we find

$$2AB - B^2 = \delta^2 l(l+1) - \beta^2\delta^2 - A'\delta^2, \tag{24}$$

or

$$A = \frac{B}{2} - \frac{\delta^2(\beta^2 + A')}{2B}, \tag{25}$$

From Eq.(20) and (25), we find

$$-\frac{2\mu E_0}{\hbar^2\delta^2} = \frac{1}{\delta^2}\left(\frac{B}{2} - \frac{\delta^2(\beta^2 + A')}{2B}\right)^2 - C_0\lambda. \tag{26}$$

Finally, for energy eigenvalue, we obtain

$$E_0 = \frac{\hbar^2\lambda C_0\delta^2}{2\mu} - \frac{\hbar^2}{2\mu}\left(\frac{B}{2} - \frac{\delta^2(\beta^2 + A')}{2B}\right)^2, \tag{27}$$

When $r \to \infty$, the chosen superpotential $W(r)$ is $W(r) \to -\frac{\hbar A}{\sqrt{2\mu}}$.
Having inserted the Eq.(19) into Eq.(18), then we can find supersymmetric partner potentials $V_1(r)$ and $V_2(r)$ in the form

$$\begin{aligned} V_1(r) &= W^2(r) - \frac{\hbar}{\sqrt{2\mu}} W'(r) + E = \\ &= \frac{\hbar^2}{2\mu}\left[A^2 + \frac{(2AB - \delta B)e^{-\delta r}}{1 - e^{-\delta r}} + \frac{(B^2 - \delta B)e^{-2\delta r}}{(1 - e^{-\delta r})^2}\right]. \end{aligned} \tag{28}$$

and

$$\begin{aligned} V_2(r) &= W^2(r) + \frac{\hbar}{\sqrt{2\mu}} W'(r) + E = \\ &= \frac{\hbar^2}{2\mu}\left[A^2 + \frac{(2AB + \delta B)e^{-\delta r}}{1 - e^{-\delta r}} + \frac{(B^2 + \delta B)e^{-2\delta r}}{(1 - e^{-\delta r})^2}\right], \end{aligned} \tag{29}$$

By using superpotential $W(r)$ from Eq.(19) we can find $\chi_0(r)$ radial eigenfunction in this form:

$$\begin{aligned} \chi_0(r) = N_0 exp\left(-\frac{\sqrt{2\mu}}{\hbar}\int W(r)dr\right) = N_0 exp\left[\int\left(A + \frac{Be^{-\delta r}}{1 - e^{-\delta r}}\right)dr\right] = \\ N_0 e^{Ar} exp\left[\frac{B}{\delta}\int\frac{d(1 - e^{-\delta r})}{1 - e^{-\delta r}}\right] = N_0 e^{Ar}(1 - e^{-\delta r})^{\frac{B}{\delta}} \end{aligned} \tag{30}$$

here $r \to 0$; $\chi_0(r) \to 0, B > 0$ and $r \to \infty$; $\chi_0(r) \to 0, A < 0$

Two partner potentials $V_1(r)$ and $V_2(r)$ which differ from each other with additive constants and have the same functional form are called invariant potentials [26, 27]. Thus, for the partner potentials $V_1(r)$ and $V_2(r)$ given with Eq.(28) and Eq.(29), the invariant forms are:

$$\begin{aligned} R(B_1) = V_2(B, r) - V_1(B_1, r) = \frac{\hbar^2}{2\mu}\left[A^2 - A_1^2\right] = \\ = \frac{\hbar^2}{2\mu}\left[\left(\frac{B}{2} - \frac{\delta^2(\beta^2 + A')}{2B}\right)^2 - \left(\frac{B + \delta}{2} - \frac{\delta^2(\beta^2 + A')}{2(B + \delta)}\right)^2\right]. \end{aligned} \tag{31}$$

So we have

$$\begin{aligned} R(B_i) = V_2[B + (i-1)\delta, r] - V_1[B + i\delta, r] = \\ -\frac{\hbar^2}{2\mu}\left[\left(\frac{B + i\delta}{2} - \frac{\delta^2(\beta^2 + A')}{2(B + i\delta)}\right)^2 - \left(\frac{B + (i-1)\delta}{2} - \frac{\delta^2(\beta^2 + A')}{2(B + (i-1)\delta)}\right)^2\right]. \end{aligned} \tag{32}$$

where the reminder $R(B_1)$ is independent of r.

If we continue this procedure and make the substitution $B_{n_r} = B_{n_r - 1} + \delta = B + n_r\delta$ at every step until $B_n \geq 0$, the whole discrete spectrum of Hamiltonian $H_-(B)$:

$$E_{n_r l} = E_0 + \sum_{i=0}^{n} R(B_i) =$$

$$= \frac{\hbar^2\lambda}{2\mu}\delta^2 C_0 - \frac{\hbar^2}{2\mu}\left(\frac{B}{2} - \frac{\delta^2(\beta^2+A')}{2B}\right)^2 - \frac{\hbar^2}{2\mu}\Bigg[\left(\frac{B+\delta}{2}\frac{\delta^2(\beta^2+A')}{2(B+\delta)}\right)^2 -$$
$$-\left(\frac{B}{2} - \frac{\delta^2(\beta^2+A')}{2B}\right)^2 + \left(\frac{B+2\delta}{2} - \frac{\delta^2(\beta^2+A')}{2(B+2\delta)}\right)^2 + ... +$$
$$+\left(\frac{B+(n_r-1)\delta}{2} - \frac{\delta^2(\beta^2+A')}{2(B+(n_r-1)\delta)}\right)^2 - \Bigg(\frac{B+(n_r-2)\delta}{2} -$$
$$-\frac{\delta^2(\beta^2+A')}{2(B+(n_r-2)\delta)}\Bigg)^2 - \left(\frac{B+(n_r-2)\delta}{2} - \frac{\delta^2(\beta^2+A')}{2(B+(n_r-2)\delta)}\right)^2 +$$
$$+\left(\frac{B+n_r\delta}{2} - \frac{\delta^2(\beta^2+A')}{2(B+n_r\delta)}\right)^2 -$$
$$-\left(\frac{B+(n_r-1)\delta}{2} - \frac{\delta^2(\beta^2+A')}{2(B+(n_r-1)\delta)}\right)^2\Bigg] =$$
$$= \frac{\hbar^2\lambda}{2\mu}\delta^2 C_0 - \frac{\hbar^2}{2\mu}\left(\frac{B+n_r\delta}{2} - \frac{\delta^2(\beta^2+A')}{2(B+n_r\delta)}\right)^2, \quad (33)$$

and we obtain

$$E_{n_r l} = \frac{\hbar^2\lambda}{2\mu}\delta^2 C_0 - \frac{\hbar^2}{2\mu}\left(\frac{B+n_r\delta}{2} - \frac{\delta^2(\beta^2+A')}{2(B+n_r\delta)}\right)^2, \quad (34)$$

Finally, for energy eigenvalues we found

$$E_{nl} = \frac{-h^2\delta^2}{2\mu}\cdot$$
$$\left[\frac{l(l+1)+1/2+n(n+1)+\sqrt{\frac{1}{4}+\lambda+\alpha(\alpha-1)}(2n+1)-\beta^2-A'}{2n+1+2\sqrt{\frac{1}{4}+\alpha(\alpha-1)+\lambda}}\right]^2 +$$
$$\frac{h^2\delta^2}{2\mu}l(l+1)C_0, \quad (35)$$

The obtained result of radial Schrödinger equation by using the Eq.(17) of the ground state eigenfunction is exactly same with the result obtained by using NU method.

4. Numerical Results and Conclusion

Solutions of the modified radial Schrödinger equation for the Manning-Rosen plus Hulthén potential are obtained within ordinary quantum mechanics by applying the Nikiforov-Uvarov method and within SUSY QM by applying the shape invariance concept that was introduced by Gendenshtein method in which we have used the improved approximation scheme to the centrifugal potential for arbitrary l states. Both ordinary and SUSY quantum mechanical energy eigenvalues and corresponding eigenfunctions obtained for arbitrary l quantum numbers. Solutions of the modified radial Schrödinger equation for the Manning-Rosen plus Hulthén potential are obtained within ordinary quantum mechanics by applying the Nikiforov-Uvarov method and within SUSY QM by applying the shape invariance concept that was introduced by Gendenshtein method in which we have used the improved approximation scheme to the centrifugal potential for arbitrary l states. Both ordinary and SUSY quantum mechanical energy

eigenvalues and corresponding eigenfunctions obtained for arbitrary l quantum numbers. In the Table I, we present numerical results for the energy eigenvalues of the Hulthén, Manning-Rosen and Manning-Rosen plus Hulthén potentials as a function of screening parameter for various state in atomic units which is obtained by using ordinary (NU method) and SUSY QM(shape invariance method) methods. It can be seen from Table I that, the bound states more stable in the case Manning-Rosen plus Hulthén potential the relative Manning-Rosen and Hulthén potentials cases. This opens a new possibilities for determining of the properties of the interactions in atomic system. As a conclusion of the results presented in these tables, the numerical analyses obtained of the analytically solution is very sensitive to the n_r radial and l orbital quantum numbers. The results are sufficiently accurate for practical purpose.

Our detailed analysis show that, if we take $C_0 = 0$, $\alpha = 0$, $A' = 0$ and $l \neq 0$ in Eqs. (12) and (35) leads to the same result obtained in the work [35] and $l = 0$, $\alpha = 0$, and $A' = 0$ to the same result as in [10].

It is shown, that energy eigenvalues and corresponding eigenfunctions are identical for both ordinary and SUSY QM. These results in the special cases are compared to the results [25, 36] and the numerical results for the energy eigenvalues are obtained by using MATHEMATICA package programm.

We firstly, present an alternative method to obtain the energy eigenvalues and corresponding eigenfunctions of the Manning-Rosen plus Hulthén potential within the framework SUSY quantum mechanics for any l states. Here after, an analytical study of the Schrödinger equation have been performed for Manning-Rosen plus Hulthén potential using the improved approximation scheme to the centrifugal term for arbitrary l-states.

The energy eigenvalues of the bound states and corresponding eigenfunctions are analytically found via both of NU and SUSY quantum mechanics. The same expressions were obtained for the energy eigenvalues, and the expression of radial wave functions transformed each other is shown by using these methods. A closed form of the normalization constant of the wave functions is also found.

It is worth to mention that the Manning-Rosen and Hulthén potential is one of the important exponential potential, and it is a subject of interest in many fields of physics and chemistry. The main results of this paper are the explicit and closed form expressions for the energy eigenvalues and the normalized wave functions. The method presented in this paper is a systematic one and in many cases it is one of the most concrete works in this area.

Consequently, studying of analytical solution of the modified radial Schrödinger equation for the Manning-Rosen plus Hulthén potential within the framework ordinary and SUSY QM could provide valuable information on the QM dynamics at atomic and molecule physics and opens new window.

We can conclude that our analytical results of this study are expected to enable new possibilities for pure theoretical and experimental physicist, because the results are exact and more general.

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state	δ	Manning-Rosen potential	Hulthén potential	Manning-Rosen plus Hulthén potential
2p	0.025	-0.221994	-0.225573	-0.447567
	0.050	-0.202355	-0.202292	-0.404646
	0.075	-0.183505	-0.180156	-0.363662
	0.10	-0.165446	-0.159167	-0.324613
	0.150	-0.131699	-0.120625	-0.252324
	0.200	-0.101113	-0.0866667	-0.18778
	0.250	-0.0736887	-0.0572917	-0.13098
	0.300	-0.0494253	-0.0325	-0.0819253
	0.350	-0.02833231	-0.0122917	-0.0406147
3p	0.025	-0.216964	-0.0874653	-0.304429
	0.050	-0.192805	-0.0665278	-0.259333
	0.075	-0.169949	-0.0482986	-0.218247
	0.100	-0.148394	-0.0327778	-0.181172
	0.150	-0.109192	-0.00986111	-0.119053
3d	0.025	-0.216964	-0.0874653	-0.304429
	0.050	-0.192805	-0.0665278	-0.259333
	0.075	-0.169949	-0.0482986	-0.218247
	0.100	-0.148394	-0.0327778	-0.181172
	0.150	-0.109192	-0.00986111	-0.119053
4p	0.025	-0.223478	-0.0399479	-0.263426
	0.050	-0.205195	-0.0222917	-0.227486
	0.075	-0.187573	-0.00953125	-0.197105
	0.100	-0.170614	-0.00166667	-0.172281
4d	0.025	-0.223478	-0.0399479	-0.263426
	0.050	-0.205195	-0.0222917	-0.227486
	0.075	-0.187573	-0.00953125	-0.197105
	0.100	-0.170614	-0.00166667	-0.172281

state	δ	Manning-Rosen potential	Hulthén potential	Manning-Rosen plus Hulthén potential
4f	0.025	-0.223478	-0.0399479	-0.263426
	0.050	-0.205195	-0.0222917	-0.227486
	0.075	-0.187573	-0.00953125	-0.197105
	0.100	-0.170614	-0.00166667	-0.172281
5p	0.025	-0.242051	-0.0188542	-0.260905
	0.050	-0.241574	-0.00541667	-0.246991
5d	0.025	-0.242051	-0.0188542	-0.260905
	0.050	-0.241574	-0.00541667	-0.246991
5f	0.025	-0.242051	-0.0188542	-0.260905
	0.050	-0.241574	-0.00541667	-0.246991
5g	0.025	-0.242051	-0.0188542	-0.260905
	0.050	-0.241574	-0.00541667	-0.246991
6p	0.025	-274104	-0.00835069	-0.282454
6d	0.025	-274104	-0.00835069	-0.282454
6g	0.025	-274104	-0.00835069	-0.282454

Table 1. Energy eigenvalues of the Manning-Rosen, Hultheń and Manning-Rosen plus Hulthén potentials as a function of the screening parameter for 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states in atomic units ($\hbar = m = e = 1$) and for $Z = 1$.