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PARAMETRIC INTEGRAL REPRESENTATIONS OF RENORMALIZED FEYNMAN AMPLITUDES*

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ABSTRACT

A parametric integral representation for the amplitudes of renormalized perturbation theory is developed. The result is a closed, well-defined and unique renormalized amplitude to be associated with an arbitrary Feynman graph. By unique we mean that the renormalized amplitude is explicitly independent of the initial choice of independent integration momenta and the routing of external momenta through the graph. Our prescription is applicable to conventionally unrenormalizable as well as renormalizable theories. It is shown that for renormalizable theories, our representation is formally equivalent to the usual recursive subtraction formula for writing renormalized amplitudes and hence can be interpreted in terms of mass and coupling constant renormalization. To investigate the practical advantages of this formalism, a calculation of the fourth order vacuum polarization in Quantum Electrodynamics is carried out.

I. Introduction

Parametric integral representations of Feynman amplitudes have been used for a variety of purposes (1) ever since the beginning of modern quantum field theory. They have been especially useful in the investigation of analyticity properties in perturbation theory and in carrying out calculations in quantum electrodynamics. The purpose of the present work is to develop a parametric integral form for renormalized Feynman amplitudes which is convenient for discussing some of the formal aspects of renormalization theory and which will provide a general framework for carrying out higher order calculations in quantum electrodynamics.

With any subtraction scheme for expressing renormalized amplitudes, there are basically two formal problems. It must be shown that the subtractions lead to a unique finite renormalized amplitude and that the cutoff dependent terms which are subtracted can be related to Lagrangian counter terms and hence to renormalization effects. After presenting a definition of renormalized amplitudes, we will discuss both of these problems and then illustrate the calculational advantages of this formalism by looking at the fourth order vacuum polarization contribution in quantum electrodynamics.

In Section II, we will derive a parametric integral form for an arbitrary unsubtracted, regularized Feynman amplitude. We will employ the notation and several of the results of Nakanishi (2) to express the result in a way which is explicitly independent of the routing of external momenta through the graph and the choice of independent integration momenta. The integrand of the parametric integral will reflect only the structure of the corresponding graph. We will list the properties of the parametric functions in the integrand and give the condition for convergence of the integral in the absence of regularization.

In Section III, a parametric integral representation for renormalized Feynman amplitudes will be established using the result of Section II as a starting point. The necessary subtractions will be made by making use of the well-known formula (3.1) for the remainder of the Taylor series. This will avoid the topological complexities associated with overlapping divergences and lead to a unique nonrecursive expression applicable to arbitrary interactions.

Section IV will be devoted to showing that the parametric integral form of renormalized amplitudes is a well-defined expression in the absence of regularization. The proof involves a careful power counting in the parametric integral and does not rely on Weinberg's proof (3) of Dyson's power counting theorem which involves an unjustified contour rotation.

In Section V, we will show that the expression for renormalized amplitudes developed here is equivalent to a recursive subtraction formula in which the subtraction terms are directly related to Lagrangian counter terms (4) and hence, in the case of renormalizable theories, to renormalization effects.

In Section VI, we will carry through a calculation of the fourth order vacuum polarization contribution in quantum electrodynamics using the formalism developed in Section III. There are several features of this formalism which together simplify the calculation considerably. First of all, since the momentum integrals have been carried out the only momentum in the problem is the external momentum and hence the trace calculations become trivial. By the use of (3.1), the subtractions will be made at the origin of momentum space. This will eliminate the infrared divergent terms which appear in the intermediate stages of the calculation when the subtractions are made on the mass shell. We will be primarily interested in the high energy behavior of the vacuum polarization. In this energy region, each of the graphs of Figure 6 gives contributions proportional to $\log^2(-k^2/m^2)$ and $\log(-k^2/m^2)$. It is well-known from direct calculation (5) and from renormalization group techniques (6) that the $\log^2(-k^2/m^2)$ contributions cancel and that the leading term in fourth order goes as $\log(-k^2/m^2)$. In our approach, this cancellation occurs at an early stage of the calculation without actually carrying out the integrals giving rise to the $\log^2(-k^2/m^2)$ contributions.

The present work is similar in some respects to the approach of Yennie and Kuo (7) which is formulated in momentum space rather than parameter space.

II. Parametric Integral Formulas

We begin by considering an arbitrary proper Feynman graph G containing N directed internal lines and n independent basic circuits. The momentum of each line r will be denoted by $p_r + q_r$, where p_r is an integration momentum and q_r is a constant momentum which will be related to the momenta external to the graph. Due to momentum conservation at each vertex, only n of the p_r will be independent integration (loop) momenta. If there are v vertices, we have

$$n = N - v + 1 , \tag{2.1}$$

the + 1 accounting for over-all momentum conservation.

With each line in the graph will be associated a propagator of the form

$$\frac{i Z_r (p_r + q_r)}{(p_r + q_r)^2 - m_r^2 + i\epsilon} \tag{2.2}$$

where Z_r depends upon the type of propagator. Then, apart from constant factors and vertex γ-matrices, the amplitude will be

$$W^{(G)} = \int \prod_{i=1}^{n} d^4 p_i \prod_{r \epsilon G} \left\{ \frac{i Z_r (p_r + q_r)}{(p_r + q_r)^2 - m_r^2 + i\epsilon} \right\} \tag{2.3}$$

where we have chosen a particular set of the p_r as independent integration momenta. A convenient starting point for changing (2.3) into parametric form is to express the propagator (2.2) in the form (8)

$$\frac{i Z_r (p_r + q_r)}{(p_r + q_r)^2 - m_r^2 + i\epsilon} =$$

$$\int_0^\infty dx_r \, Z_r\left(\frac{1}{ix_r} \nabla_{\ell_r}\right) \exp\left[ix_r \left((p_r + q_r)^2 + (p_r + q_r)\cdot \ell_r - m_r^2 + i\epsilon\right)\right]\Bigg|_{\ell_r = 0} \tag{2.4}$$

The ultra-violet divergences show up in parametric form as singularities of the integrand at the lower limit of the parametric integration. To avoid these divergences, we regularize each propagator by changing the lower limit of the parametric integration from zero to a small positive constant ρ. Substituting this into (2.3) gives a regularized amplitude

$$W_\rho^{(G)} = \int \prod_{i=1}^{n} d^4 p_i \times$$

$$\times \prod_{r \in G} \left\{ \int_\rho^\infty dx_r \, Z_r\left(\frac{1}{ix_r} \nabla_{\ell_r}\right) \exp\left[ix_r \left((p_r + q_r)^2 + (p_r + q_r)\cdot \ell_r - m_r^2 + i\epsilon\right)\right]\Bigg|_{\ell_r = 0} \right\} \tag{2.5}$$

Since the propagators have been regularized, the momentum and parametric integrations in (2.5) can be interchanged and the momentum integrations can be carried out by diagonalizing the quadratic form in the exponential and repeatedly employing the formula

$$\int d^4 p \, e^{iap^2} = \frac{\pi^2}{ia^2} \tag{2.6}$$

The details of this procedure are similar to those carried out by Nakanishi (2) and so we omit them here. The result can be written in a form explicitly independent of the original choice of loop momenta.

$$W_\rho^{(G)} = \left(\frac{\pi^2}{i}\right)^n \int_\rho^\infty dx_G \prod_{r\in G} Z_r\left(\frac{1}{ix_r}\nabla_{\ell_r}\right)\frac{1}{U^2}$$

$$\times \exp\left[i \sum_{r\in G} x_r(q_r^2 + q_r\cdot\ell_r - m_r^2 + i\epsilon) - \frac{i}{U}\sum_C U_C\left(\sum_{r\subset C} \pm x_r(q_r + \frac{\ell_r}{2})\right)^2\right]\Bigg|_{\ell=0} \tag{2.7}$$

where the sum $\sum_C$ is over all possible simple closed circuits in G and

$$\int_\rho^\infty dx_G = \prod_{r\in G}\int_\rho^\infty dx_r\,. \tag{2.8}$$

U and U_C are functions of the integration parameters only.

$$U = \sum x_{\nu_1} x_{\nu_2} \dots x_{\nu_n} \tag{2.9}$$

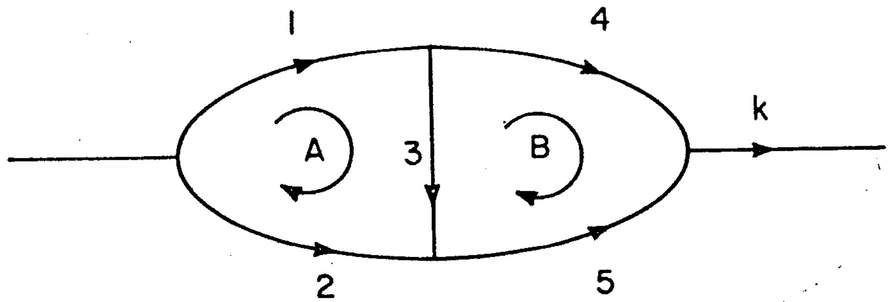
where the summation is over all possible sets $\{\nu_1 \dots \nu_n\}$ such that $p_{\nu_1}, p_{\nu_2}, \dots p_{\nu_n}$ is a possible set of independent integration momenta and

$$U_C = \sum x_{\nu_1} x_{\nu_2} \dots x_{\nu_{n-1}} \tag{2.10}$$

where the summation is over all possible sets $\{\nu_1 \dots \nu_{n-1}\}$ such that none of the corresponding lines belongs to C and such that $p_{\nu_1}, p_{\nu_2} \dots p_{\nu_{n-1}}$ is a possible set of independent integration momenta. The double sign in (2.7) corresponds to the relative direction of lines in circuit C.*

* The initial assignment of directions to internal lines is arbitrary. However, it is convenient to use the direction of fermion propagation for the fermion lines. Then each member of a closed fermion loop or of fermion path through the graph will have the same direction. With this choice, each Z_r will always be of the form $(+\frac{1}{ix_r}\not\nabla_{\ell r} + m_r)$.

As an example, consider the self energy graph of Figure 1.



A simple self energy graph.

Figure 1

where circuit A is composed of lines 1, 2, and 3 and circuit B is composed of lines 3, 4, and 5. Then

$$U = (x_1 + x_2)(x_4 + x_5) + x_3(x_1 + x_2 + x_4 + x_5) \qquad (2.11)$$

$$U_A = x_4 + x_5 \qquad\qquad U_B = x_1 + x_2$$

Expression (2.7) still seems to depend upon how the external momenta are routed through the graph since this determines the values of the various q_r's. To see that this is not the case, we interchange orders of summation in (2.7) and re-express it in the form

$$W_\rho^{(G)} = \left(\frac{\pi^2}{i}\right)^n \int_\rho^\infty dx_G \prod_{r\epsilon G} Z_r\left(\frac{1}{ix_r}\nabla_{\ell_r}\right)\frac{1}{U^2}$$

$$\times \exp\left[i\left(V + \sum_{r\epsilon G} x_r\ell_r\cdot Y_r - \tfrac{1}{4}\sum_{r,s\epsilon G} x_r x_s \ell_r\cdot\ell_s X_{rs}\right) - i\sum_{r\epsilon G} x_r(m_r^2 - i\epsilon)\right]\Bigg|_{\ell=0} \tag{2.12}$$

where

$$V = \sum_{r\epsilon G} x_r q_r^2 - \frac{1}{U}\sum_C U_C\left(\sum_{r\epsilon C} \pm x_r q_r\right)^2, \tag{2.13a}$$

$$Y_{r\mu} = q_{r\mu} - \frac{1}{U}\sum_{C\epsilon C(r)} U_C\left(\sum_{s\epsilon C} \pm x_s q_{s\mu}\right), \tag{2.13b}$$

$$X_{rs} = \frac{1}{U}\sum_{C\epsilon C(r,s)} \pm U_C\,. \tag{2.13c}$$

$C(r)$ is the set of all simple closed circuits in G containing line r and $C(r, s)$ is the set of all simple closed circuits containing both line r and line s. The double sign in (2.13b) and (2.13c) corresponds to the relative direction of lines r and s on circuit C.

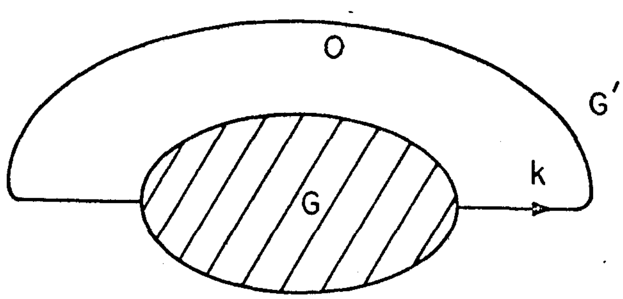
The functions V, $Y_{r\mu}$, and X_{rs} appear in a somewhat different form in the work of Nakanishi (2) and, as he shows, $Y_{r\mu}$ and V can be written in terms of the external momenta in a way which is explicitly route-independent. The reader is referred to the paper of Nakinishi for a proof of this, and the results are simply reproduced here.

We first consider the case when G is a self energy graph. Then V and $Y_{r\mu}$ are given by

$$V = \frac{W}{U} k^2 \tag{2.14a}$$

$$Y_{r\mu} = \left(\frac{1}{U} \sum_{C \epsilon C(0,r)} \pm W_C \right) k_\mu \tag{2.14b}$$

where k_μ is the external momentum and W and W_C are defined as follows. From the self energy graph G, we form a graph G' by connecting the external lines of G and label the new line as 0 with parameter x_0.



Graph formed by joining the external lines of the self energy graph G'.

Figure 2

Then W is just the U function of the graph G' with x_0 set equal to zero and W_C is the U_C function of the graph C. The double sign in (2.14b) corresponds to the relative direction of lines 0 and r on circuit C.* Clearly the forms (2.14a) and (2.14b) are independent of the routing of k_μ through the graph.

* The direction of external lines is defined by the momentum labeling.

As an example, we return to the self energy graph of Figure 2. For this graph,

$$W = x_1x_2(x_3 + x_4 + x_5) + x_4x_5(x_1 + x_2 + x_3) + x_3(x_1x_5 + x_2x_4)$$

$$Y_{1\mu} = \frac{1}{U}\left[(x_2x_3 + x_2x_5 + x_3x_5) + x_2x_4\right] k_\mu \tag{2.15}$$

$$Y_{3\mu} = \frac{1}{U}\left[x_2x_4 - x_1x_5\right] k_\mu$$

and similarly for the other Y_r's.

We next consider the general case when G has ℓ external momenta $k_1, k_2, \ldots, k_\ell$ which we take to be directed inward. Then by momentum conservation,

$$\sum_{i=1}^{\ell} k_\ell = 0. \tag{2.16}$$

For this graph, V is given by

$$V = \frac{1}{U} \sum_{i>j} W^{ij} \, (-k_i \cdot k_j) \tag{2.17}$$

where W^{ij} is the W function for the self energy graph formed from G by setting $k_i = -k_j$ and all others equal to zero. Similarly, $Y_{r\mu}$ is given by

$$Y_{r\mu} = \sum_{i=1}^{\ell-1} Y_{r\mu}^{\ell i} \tag{2.18}$$

where $Y^{\ell i}_{r\mu}$ is the $Y_{r\mu}$ frunction for the self energy graph formed from G by setting $k_\ell = -k_i$ and all others equal to zero.

Equation (2.12) is the representation of an arbitrary unsubtracted, regularized amplitude which we shall use as a starting point for defining a renormalized amplitude. If each field in the theory is a spin zero field and if there are no derivative couplings, then each Z_r is equal to one and (2.12) becomes simply

$$W^{(G)}_\rho = \left(\frac{\pi^2}{i}\right)^n \int_\rho^\infty dx_G \frac{1}{U^2} \exp\left[iV - i \sum_{r\in G} x_r(m_r^2 - i\epsilon)\right] \tag{2.19}$$

This simplified form will be used in the discussions of Sections IV and V.

In developing a parametric form for renormalized amplitudes, it will be necessary to know the properties of the parametric functions appearing in the integrand in (2.12). To this end, we will list several of these properties here. The first six follow immediately from the definitions and the proof of the seventh is given by Nakanishi (2).

1) The functions U, U_C, W^{ij} and W^{ij}_C are homogeneous polynomials in the x_r with the following order:

U : n-th order	W^{ij} : (n + 1)-th order
U_C : (n - 1)-th order	W^{ij}_C : n-th order

2) The functions V, Y_r and X_{rs} are homogeneous with respect to the x_r with the following order:

V : first order $\qquad$ Y_r : 0-th order $\qquad$ X_{rs} : (-1)-th order

3) The functions U, U_C, W^{ij} and W_C^{ij} are all non-negative definite in the region of integration.

$$U,\ U_C,\ W^{ij},\ W_C^{ij} \geq 0 \text{ in } x_r \geq 0,\ r = 1, 2, \ldots, N$$

4) V vanishes only when $x_r = 0$, for all $r \in C'$, where C' is any closed circuit in G. The same holds for U_C for all C' in G except $C' = C$. W^{ij} vanishes only when $x_r = 0$, for all $r \in C'$ where C' is any closed circuit in G or any path through G connecting external lines i and j. The same holds for W_C^{ij} for all C' in G except $C' = C$.

5) Let H be a collection of lines in G containing m independent closed circuits. Then U has an m-th order zero at $x_r = 0$, for all $r \in H$. The same holds for U_C apart from the circuit C. For W^{ij} and W_C^{ij}, the order of the zero in m + 1 if H includes a path through the graph connecting lines i and j, and m if it does not.

6) X_{rs} has a first order pole only when $x_t = 0$, for all $t \in C$, where $C \in C(r, s)$.

7) Let H be a union of m independent circuits in G. Let R be the graph formed from G by shrinking each line of H to a point. Denote the U function for G, (H, R) by U_G, (U_H, U_R). Then

$$U_G = U_H U_R + U'_G \tag{2.20}$$

where U'_G is at least of order m + 1 in x_r, $r \in H$. Similarly for W_G^{ij},

$$W_G^{ij} = U_H W_R^{ij} + W_G^{ij\,'} \tag{2.21}$$

where $W_G^{ij\,'}$ is at least of order m + 1 in x_r, $r \in H$. We will only use subscripts on the parametric functions when it is necessary to avoid confusion.

The limit $\rho \to 0$ in $W_\rho^{(G)}$ may not exist due to the existence of non-integrable poles of the integrand at the lower limit of the parametric integration. These divergences correspond to the "ultra-violet" divergences of the momentum space representation of Feynman amplitudes. At the upper end of the integration, the integrand dies off exponentially due to the negative imaginary part associated with each mass.

To make this more precise, we first look at the momentum representation (2.5) of the amplitude. We consider a particular sub-integration corresponding to some proper subgraph S_i consisting of N_i lines and n_i independent circuits. Let z_r be the power of the momentum in the numerator factor Z_r. The degree of divergence d_i for the subgraph S_i is defined to be the power of the integration momenta internal to S_i in the numerator minus the power of the integration momenta internal to S_i in the denominator:

$$d_i = 4n_i + \sum_{r \in S_i} z_r - 2N_i \qquad (2.22)$$

According to Dyson's power counting theorem, the limit $\rho \to 0$ in $W_\rho^{(G)}$ will exist providing that $d_i < 0$ for all proper subgraphs S_i of G.

The same condition holds for the existence of the limit $\rho \to 0$ in the parametric form (2.12). According to property 5, U will have an n_i-th order zero at $x_r = 0$, for all $r \in S_i$. The ∇_{ℓ_r} operators will bring Y_r and X_{rs} factors into the numerator. Y_r is zeroth order in any subset of the parameters while X_{rs} has poles given by property 6. By inspection of (2.12), the order of the pole introduced by the ∇_{ℓ_r} operators is

$$\frac{1}{2}\sum_{r\epsilon S_i} z_r \quad \text{for} \quad \sum_{r\epsilon S_i} z_r \ \text{even}$$

$$\frac{1}{2}\left(\sum_{r\epsilon S_i} z_r - 1\right) \quad \text{for} \quad \sum_{r\epsilon S_i} z_r \ \text{odd} \tag{2.23}$$

It follows that the numerator will contain sufficient powers of the parameters x_r, $r\epsilon S_i$ to make the corresponding sub-integration converge provided that

$$2N_i > 4n_i + \sum_{r\epsilon S_i} z_r \tag{2.24}$$

This is just the condition $d_i < 0$. A rigorous proof of the power counting theorem can easily be constructed using the parametric form and, in fact, it will be a special case of the proof of finiteness for renormalized amplitudes to be presented in Section IV.

III. Renormalized Amplitudes

Using the power counting theorem as a guide, we will define the renormalized amplitude corresponding to a Feynman graph G by locating those proper subgraphs S_i of G for which $d_i \geq 0$ and performing a sufficient number of subtractions to make the corresponding sub-integration convergent in the limit $\rho \to 0$. These subtractions can conveniently be made by using the well-known formula for the remainder of a Taylor series

$$f(x) - f(0) - \cdots - \frac{f^{(n)}(0)}{n!} x^n = \int_0^1 d\xi \, \frac{(1-\xi)^n}{n!} \left(\frac{\partial}{\partial \xi}\right)^{n+1} f(\xi x) \tag{3.1}$$

Combining this method of performing subtractions with the parametric integral form of Feynman amplitudes will yield our parametric integral form for renormalized amplitudes.

We consider an arbitrary Feynman graph G and begin by performing the subtractions corresponding to a particular subgraph S_i for which $d_i \geq 0$. Working in the momentum representation, we choose a set of integration momenta for G so that exactly n_i of them are internal to S_i. Let the external momenta of S_i be $k_1, k_2, \ldots, k_{\ell_i}$. They will depend upon the integration momenta of G not internal to S_i and the external momenta of G. The unsubtracted regularized amplitude is

$$W_\rho^{(G)} \left[W_\rho^{(S_i)} (k_1 \ldots k_{\ell_i}) \right] \tag{3.2}$$

where the functional dependence of $W_\rho^{(G)}$ upon $W_\rho^{(S_i)}$ is denoted by the square brackets. We subtract from $W_\rho^{(S_i)}$ all terms up to order d_i in its Taylor expansion about the point $k_1 = k_2 = \cdots = k_{\ell_i} = 0$. Using (3.1), this gives

$$W_\rho^{(G)}\left[\int_0^1 d\xi_i \frac{(1-\xi_i)^{d_i}}{d_i!} \left(\frac{\partial}{\partial \xi_i}\right)^{d_i+1} W_\rho^{(S_i)}\left(\xi_i k_1, \ldots, \xi_i k_{\ell_i}\right)\right] \tag{3.3}$$

The steps leading from (2.5) to (2.12) can then be carried out keeping track of the ξ_i parameter. This leads to an expression which can be formed from (2.12) by inserting the ξ_i parameter into the parametric functions, U, V, Y_r and X_{rs} in a simple way and applying the operator

$$\int_0^1 d\xi_i \frac{(1-\xi_i)^{d_i}}{d_i!} \left(\frac{\partial}{\partial \xi_i}\right)^{d_i+1} . \tag{3.4}$$

to the integrand. Rather than bore the reader with this bookkeeping or even its result, we shall simply present the more general result of the fully renormalized amplitude.

The renormalized amplitude is defined by starting with (2.12) and performing the above operations for a large enough class of subgraphs to insure the convergence of (2.12) in the limit $\rho \to 0$. Let $\mathscr{S}$ denote the set of proper subgraphs S_i of G which

(a) are superficially divergent, $d_i \geq 0$

(b) cannot be formed from another superficially divergent graph by simply opening one line.

Note that for Quantum Electrodynamics, condition (b) is automatically satisfied by superficially divergent graphs, however, this is not so in general. With each member S_i of $\mathscr{S}$, we associate a parameter ξ_i. If $d_G \geq 0$, we let $G = S_0$ and associate ξ_0 with it. Then the renormalized amplitude is

$$W_R^{(G)} = \lim_{\rho\to 0} W_{R\rho}^{(G)} = \left(\frac{\pi^2}{i}\right)^n \int_0^\infty dx_G \prod_{S_i\in\mathscr{S}} \int_0^1 d\xi_i \prod_{S_j\in\mathscr{S}} \frac{(1-\xi_j)^{d_j}}{d_j!} \left(\frac{\partial}{\partial\xi_j}\right)^{d_j+1}$$

$$\times \prod_{r\in G} Z_r\left(\frac{1}{ix_r}\nabla_{\ell_r}\right)\frac{1}{\overline{U}^2} \exp\left[i\left(\overline{V}+\sum_{r\in G}\hat{x}_r\ell_r\overline{Y}_r - \tfrac{1}{4}\sum_{r,s\in G}\hat{x}_r\hat{x}_s\ell_r\cdot\ell_s\overline{X}_{rs}\right) - i\sum_{r\in G}x_r(m_r^2-i\epsilon)\right]\Bigg|_{\ell=0} \tag{3.5}$$

where $\overline{U}$, $\overline{V}$, $\overline{Y}_r$, $\overline{X}_{rs}$ and $\hat{x}_r$ are defined in the following way:

Def: $\overline{U}$ is formed by multiplying each term in U which is of order $n_i + m$ in x_r, $r\in S_i$ by ξ_i^{2m}. This is done for each $S_i\in\mathscr{S}$.

Def: $\overline{W}^{ij}$ is defined in the same way and then

$$V = \frac{1}{\overline{U}} \sum_{i>j} \overline{W}^{ij}(-k_i\cdot k_j) \tag{3.6}$$

Def: $\overline{U}_C$ and $\overline{W}_C^{ij}$ are also defined in the same way as $\overline{U}$ and then

$$\overline{X}_{rs} = \frac{1}{\overline{U}} \sum_{C\in C(r,s)} \pm\, \overline{U}_C \tag{3.7a}$$

and

$$\overline{Y}_{r\mu} = \sum_{i=1}^{\ell-1}\left(\frac{1}{\overline{U}}\sum_{C\in C(0,r)} \pm\, \overline{W}_C^{i\ell}\right)k_{i\mu} \tag{3.7b}$$

Def: For each line r, denote the subset of $\mathscr{S}$ whose members contain r by $\mathscr{S}_r$. Then

$$\hat{x}_r = \prod_{S_i\in\mathscr{S}_r} \xi_i x_r \tag{3.8}$$

We wish to make several observations concerning the definition (3.5) of the renormalized amplitude. First of all, we note that it is a non-recursive expression, applicable to arbitrary interactions which is explicitly independent of the choice of independent integration momenta and the routing of external momenta through the graph. It will be shown in the next section that (3.5) is an absolutely convergent integral. The members of $\mathscr{S}$ can be either disjoint, nested or overlapping, however, the ordering of the ξ integrations, both among themselves and relative to the x integrations, is irrelevant. The subtractions in (3.5) have been made at the origin of momentum space. There is no fundamental reason for doing this but such a choice for the subtraction point yields a simple form for the renormalized amplitude. This choice will, of course, necessitate finite renormalizations to insure that propagators have poles on the physical mass shell but will also simplify calculations somewhat since it eliminates infra-red divergence problems. These things will be discussed in Sections V and VI. Finally, we might mention that as far as the parameter ξ_0 corresponding to the entire graph is concerned, the effect of rules for forming $\overline{U}$, $\overline{V}$, $\overline{Y}_r$, $\overline{X}_{rs}$ and $\hat{x}_r$ is simply to multiply each external momentum by ξ_0. This is certainly expected.

When each Z_r is equal to one, (3.5) simplifies a great deal just as (2.12) simplified to give (2.19). It becomes

$$W_R^{(G)} = \left(\frac{\pi^2}{i}\right)^n \int_0^\infty dx_G \prod_{S_i \in \mathscr{S}} \int_0^1 d\xi_i \prod_{S_j \in \mathscr{S}} \frac{(1-\xi_j)^{d_j}}{d_j!} \left(\frac{\partial}{\partial \xi_j}\right)^{d_j+1}$$
$$\times \frac{1}{\overline{U}^2} \exp\left[i\left(\overline{V} - \sum_{r \in G} x_r(m_r^2 - i\epsilon)\right)\right] \tag{3.9}$$

The condition $d_i \geq 0$ which the members of $\mathscr{S}$ must satisfy is now

$$\nu_i = \frac{d_i}{2} = 2n_i - N_i \geq 0 \tag{3.10}$$

since $z_r = 0$ for all $r \in G$. Since each ξ_i parameter appears only in the form ξ_i^2, the formula

$$\int_0^1 dx \frac{(1-x)^{2n}}{2n!} \left(\frac{\partial}{\partial x}\right)^{2n+1} f(x^2) = \int_0^1 dy \frac{(1-y)^n}{n!} \left(\frac{\partial}{\partial y}\right)^{n+1} f(y) \tag{3.11}$$

can be applied for each $S_i \in \mathscr{S}$ and using (3.10) we get

$$W_R^{(G)} = \left(\frac{\pi^2}{i}\right)^{n_G} \int_0^\infty dx_G \prod_{S_i \in \mathscr{S}} \int_0^1 d\xi_i \prod_{S_j \in \mathscr{S}} \frac{(1-\xi_j)^{\nu_j}}{\nu_j!} \left(\frac{\partial}{\partial \xi_j}\right)^{\nu_j+1}$$

$$\times \frac{1}{\hat{U}^2} \exp\left[i\hat{V} - i \sum_{r \in G} x_r (m_r^2 - i\epsilon)\right] . \tag{3.12}$$

$\hat{U}$ is defined by multiplying each term in U which is of order $n_i + m$ in x_r, $r \in S_i$ by ξ_i^m. $\hat{W}^{ij}$ is defined in the same way and then $\hat{V}$ is given by

$$\hat{V} = \frac{1}{\hat{U}} \sum_{i>j}^{\ell} \hat{W}^{ij} (-k_i \cdot k_j) . \tag{3.13}$$

The ξ operations were constructed to produce subtractions at the origin of momentum space. To see how this works in the parametric form, we consider the simple case of a graph G for which each Z_r is equal to one and for which $\mathscr{S}$ contains only one member S with $\nu_S = 0$. Then the regularized renormalized amplitude is

$$W_{R\rho}^{(G)} = \left(\frac{\pi^2}{i}\right)^{n_G} \int_\rho^\infty dx_G \int_0^1 d\xi \, \frac{\partial}{\partial \xi} \, \frac{1}{\hat{U}_G^2} \exp\left[i\hat{V}_G - 1 \sum_{r\epsilon G} x_r(m_r^2 - i\epsilon)\right]$$

$$= \left(\frac{\pi^2}{i}\right)^{n_G} \int_\rho^\infty dx_G \left\{ \frac{1}{\hat{U}_G^2(\xi=1)} \exp\left[i\hat{V}_G(\xi=1) - i \sum_{r\epsilon G} x_r(m_r^2 - i\epsilon)\right] \right. \quad (3.14)$$

$$\left. - \frac{1}{\hat{U}_G^2(\xi=1)} \exp\left[i\hat{V}_G(\xi=0) - i \sum_{r\epsilon G} x_r(m_r^2 - i\epsilon)\right] \right\}$$

The G subscript has been included on the parametric functions. Clearly

$$\hat{U}_G(\xi=1) = U_G \quad (3.15a)$$

and

$$\hat{V}_G(\xi=1) = V_G . \quad (3.15b)$$

From the definitions of $\hat{U}_G$ and $\hat{V}_G$ and Eqs. (2.20) and (2.21), we have

$$\hat{U}_G(\xi=0) = U_S U_R \quad (3.16a)$$

and

$$\hat{V}_G(\xi=0) = V_R \quad (3.16b)$$

where R is the graph formed from G by shrinking each line of S to a point. Using (3.15) and (3.16), expression (3.14) becomes

$$W_{R\rho}^{(G)} = W_{\rho}^{(G)} - \left\{ \left(\frac{\pi^2}{i}\right)^{n_S} \int_{\rho}^{\infty} dx_S \frac{1}{U_S^2} \exp\left[-i \sum_{r \epsilon S} x_r (m_r^2 - i\epsilon) \right] \right\}$$

$$\times \left\{ \left(\frac{\pi^2}{i}\right)^{n_R} \int_{\rho}^{\infty} dx_R \frac{1}{U_R^2} \exp\left[i V_R - i \sum_{r \epsilon R} x_r (m_r^2 - i\epsilon) \right] \right\} \tag{3.17}$$

$$= W_{\rho}^{(G)} - W_{\rho}^{(S)}(0)\, W_{\rho}^{(R)}$$

as expected.

IV. Finiteness of the Renormalized Amplitude

The renormalized amplitude is defined in Section III by using the subtraction operator (3.4). In order to prove that it is a well-defined expression, we will carry out the derivatives appearing in the subtraction operators and then investigate the remaining integral over the x and ξ parameters. In order to keep things as simple as possible, we will restrict ourselves to the case of spin zero propagators and no derivative couplings for which the renormalized amplitude is given by (3.12).

The result of doing the ξ-derivatives in (3.12) is an expression of the form

$$\int_0^\infty dx_G \int_0^1 \prod_{S_i \in \mathscr{S}} d\xi_i \sum_\sigma S_\sigma(k) \frac{R_\sigma(x,\xi)}{\hat{U}^2 + p_\sigma} \exp\left[i\hat{V} - i \sum_{r\in G} x_r(m_r^2 - i\epsilon)\right] \tag{4.1}$$

where the summation is over the terms generated by carrying out the derivative operators. $S_\sigma(k)$ depends only upon the invariants formed from the external momenta and $R_\sigma(x,\xi)$ is a produce of x_r's and ξ_i's. The integrand of (4.1) decreases exponentially at infinity and the only possible poles occur at the zeros of $\hat{U}$. From its definition, we know that

$$\hat{U} \geq 0 \tag{4.2}$$

and that it vanishes only when some subset of the x_r's and ξ_i's is set equal to zero. Let H be a collection of lines in the graph G under consideration with N_H members and let $\mathscr{S}'$ be a subset of $\mathscr{S}$ with S' members. Let $n(\mathscr{S}', H)$, $(m_\sigma(\mathscr{S}', H))$ denote the order of the zero of $\hat{U}$, $(R_\sigma(x,\xi))$ when

$x_r = 0$, $r \in H$ and $\xi_i = 0$, $S_i \in \mathscr{S}'$. Then in order to show that (4.1) is well-defined, it is sufficient to show that for any H, any $\mathscr{S}'$ and any term in the sum over σ,

$$N_H + S' + m_\sigma(\mathscr{S}', H) > n(\mathscr{S}', H)(p_\sigma + 2). \tag{4.3}$$

We first consider the zeros of $\hat{U}$. When $\mathscr{S}'$ is empty, $n(\mathscr{S}', H)$ is given simply by n_H. This follows from the properties of the parametric functions listed in Section II. When $\mathscr{S}'$ is not empty, the situation becomes substantially more complicated and the general result is given by the following theorem.

Theorem 1: Let $\mathscr{S}' = \{S_1, S_2, \ldots S_{S'}\}$. These graphs may overlap in various ways and we construct from them a sequence of nested sets of lines. We define S(i) to be the set of lines in G which belong to at least i members of $\{S_1, S_2, \ldots S_{S'}\}$. Then $S(S') \subset S(S'-1) \subset \ldots \subset S(1) \subset S(0) \equiv G$. We define R(i) to be the set of lines formed from S(i) by shrinking all the lines of S(i+1) to a point and $n_{H \cap R(i)}$ to be the number of independent closed loops formed by the lines of H in R(i). Then

$$n(\mathscr{S}', H) = n_{H\cap R(S')} + n_{H \cap R(S'-1)} + \cdots n_{H\cap R(1)} + n_{H \cap R(0)} + \ell_{\mathscr{S}'} \tag{4.4}$$

where

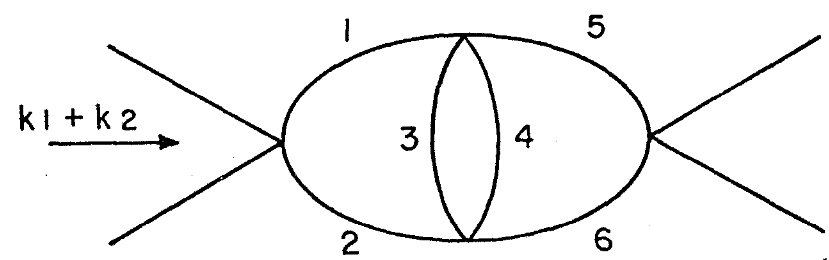
$$\ell_{\mathscr{S}'} = \sum_{i=1}^{S'} n_{S(i)} - \sum_{j=1}^{S'} n_{S_i} \,. \tag{4.5}$$

This theorem will be proved in Appendix A. The desired result (4.3) will then follow from the next theorem concerning the zeros of $R_\sigma(x, \xi)$ which will also be proved in Appendix A.

Theorem 2: For any term in the sum over σ,

$$m_\sigma(\mathscr{S}', H) > \left[n_{H \cap R(S')} + n_{H \cap R(S'-1)} + \cdots + n_{H \cap R(0)} + \ell_{\mathscr{S}'}\right](p_\sigma + 2) - N_H - S' \quad (4.6)$$

The general proofs of these theorems are somewhat tedious and it would probably be helpful to first look at a simple example. We consider the graph of Figure 3 which arises in the ϕ^4 theory.



Vertex graph in the ϕ^4 theory with overlapping vertex subgraphs.

Figure 3

The set $\mathscr{S}$ contains four members; the entire graph S_0, the graph S_1 consisting of lines 1, 2, 3 and 4, the graph S_2 consisting of lines 3, 4, 5 and 6, and the graph S_3 consisting of lines 3 and 4. For each of these, $\nu_i = 0$ and the renormalized amplitude is given by

$$\left(\frac{\pi^2}{i}\right)^3 \int_0^\infty d_{x_1} \ldots d_{x_6} \int_0^1 d\xi_0 \ldots d\xi_3 \prod_{i=0}^{3} \frac{\partial}{\partial \xi_k} \frac{1}{\hat{U}^2} \times$$

$$\times \exp\left[i \frac{\hat{W}}{\hat{U}} (k_1 + k_2)^2 - i(m^2 - i\epsilon)(x_1 + \ldots + x_6)\right] \tag{4.7}$$

where

$$\hat{U} = \left[\xi_3 \;\; \xi_1 x_3 x_4 (x_1 + x_2) + \xi_2 x_3 x_4 (x_5 + x_6)\right] + (x_3 + x_4)(x_1 + x_2)(x_5 + x_6) \tag{4.8a}$$

$$\hat{W} = \xi_0 \xi_3 x_3 x_4 \left[\xi_1^2 x_1 x_2 + \xi_1 \xi_2 (x_1 x_6 + x_2 x_5) + \xi_2^2 x_5 x_6\right]$$

$$+ \xi_0 (x_3 + x_4) \left[\xi_1 x_1 x_2 (x_5 + x_6) + \xi_2 x_5 x_6 (x_1 + x_2)\right] \tag{4.8b}$$

Suppose that H consists of lines 5 and 6 and $\mathscr{S}'$ consists of one member S_1. Then $\ell_{\mathscr{S}'} = 0$, $R(0) = G/S_1$ and

$$n(\mathscr{S}', H) = n_{H \cap R(0)} = 1. \tag{4.9}$$

In addition, $N_H = 2$ and $S' = 1$ and by inspection, one can see that the condition of Theorem 2,

$$m_\sigma(\mathscr{S}', H) > (P_\sigma + 2) - 3, \tag{4.10}$$

holds for each tern in the sum over σ.

Finally, we wish to briefly discuss the limit $\epsilon \to +0$ in (3.12). To do this, we transform it into the Feynman denominator form. This is done by inserting the factor

$$\int_0^\infty \frac{d\lambda}{\lambda}\, \delta\left(1 - \frac{1}{\lambda} \sum_{r\epsilon G} x_r\right) = 1$$

into (3.12), rescaling $x_r \to \lambda x_r$ for all $r \epsilon G$ and doing the λ integration. For the case $2n_G - N_G < 0$, the result is

$$W_R^{(G)} \sim \int_0^\infty dx_G\, \delta\left(1 - \sum_{r\epsilon G} x_r\right) \prod_{S_i \epsilon \mathscr{S}} \int_0^1 d\xi_i \prod_{S_j \epsilon \mathscr{S}} \frac{(1-\xi_j)^{\nu_j}}{\nu_j!} \left(\frac{\partial}{\partial \xi_j}\right)^{\nu_j+1}$$

$$\times \frac{1}{\hat{U}^2} \frac{1}{\left[\hat{V} - \sum_{r\epsilon G} x_r(m_r^2 - i\epsilon)\right]^{N_G - 2n_G}} \qquad (4.11)$$

In the limit $\epsilon \to +0$, $W_R^{(G)}$ will have singularities determined by the zeros of $\hat{V} - \sum_{r\epsilon G} x_r m_r^2$. These singularities correspond to the existence of absorptive parts due to the opening up of inelastic channels. The usual treatment (2) (6) of these singularities for unrenormalized amplitudes, leading to the Landau conditions (10), can be carried over directly to the renormalized amplitude (4.11). For a careful treatment of the $\epsilon \to 0$ limit using the language of distribution theory, we refer the reader to the paper of Hepp (9).

The proof that the renormalized amplitude is well-defined can easily be generalized to the case of an arbitrary Feynman amplitude given by (3.5). One again carries out the ξ derivatives and examines each term generated by these operations. From the form of the functions U, V, Y_r and X_{rs}, it is clear that the effect of the derivative operators is again to make the integral "less divergent" and the proof, although somewhat more complicated, goes through just as above.

V. Mass and Coupling Constant Renormalization

We have given a prescription for associating a well-defined renormalized amplitude with an arbitrary Feynman graph. The usual method of defining renormalized amplitudes is by means of a recursive subtraction formula of the type used by Salam (11) and more recently by Bogoluibov and Parasiuk (1) and Hepp (9). In this section we intend to show the formal equivalence of our subtraction scheme with a recursive subtraction formula in which the subtraction terms can be related to Lagrangian counter-terms and hence to field, coupling constant and mass renormalizations. These renormalization effects are usually dealt with via the Green's functions of the theory, however, the Lagrangian counter-term approach seems to be simpler for our purposes. We will restrict ourselves to the ϕ^4 theory in this discussion however it applies to any renormalizable theory. We have not, as yet, been able to completely prove that our prescription is equivalent to a recursive subtraction formula for an arbitrary unrenormalizable theory although we feel that this is the case.

We first consider briefly some of the features of the ϕ^4 theory. The Lagrangian density for this theory written in terms of unrenormalized quantities is

$$\mathscr{L}(x) = \frac{1}{2}\left(\partial_\mu \phi_0 \partial^\mu \phi_0 - m_0^2 \phi_0^2\right) - \frac{1}{4!}\lambda_0 \phi^4 \tag{5.1}$$

We introduce a new mass, coupling constant and field as follows:

$$m_0^2 = m^2 + \delta m^2 \tag{5.2a}$$

$$\phi_0 = Z_2^{\frac{1}{2}} \phi \tag{5.2b}$$

$$\lambda_0 = \frac{Z_1}{Z_2^2}\lambda \tag{5.2c}$$

Then letting

$$Z_2 = 1 + B \qquad Z_1 = 1 - L\,, \tag{5.3}$$

$\mathscr{L}(x)$ can be written as

$$\mathscr{L}(x) = \tfrac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \tfrac{1}{4}\lambda\phi^4 - \tfrac{1}{2}Z_2\delta m^2\phi^2 + \tfrac{1}{2}B\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + L\frac{1}{4!}\lambda\phi^4 \tag{5.4}$$

We next carry through the canonical formalism and construct the interaction Hamiltonian in the interaction representation. The result is

$$\mathscr{H}_I(x) = \frac{1}{4!}\lambda\phi_I^4 - L\frac{1}{4!}\lambda\phi_I^4 + \frac{1}{2}\delta m^2 Z_2\phi_I^2 - \frac{1}{2}B\left(\partial_\mu\phi_I\partial^\mu\phi_I - m^2\phi_I^2\right) + \frac{1}{2}\frac{B^2}{1+B}\dot{\phi}_I^2, \tag{5.5}$$

or letting

$$A = -\delta m^2 Z_2 - Bm^2, \tag{5.6}$$

$$\mathscr{H}_I(x) = \frac{1}{4!}\lambda\phi_I^4 - L\frac{1}{4!}\lambda\phi_I^4 - \frac{1}{2}A\phi_I^2 - \frac{1}{2}B\partial_\mu\phi_I\partial^\mu\phi_I + \frac{1}{2}\frac{B^2}{1+B}\dot{\phi}_I^2. \tag{5.7}$$

In perturbation theory, this interaction Hamiltonian gives rise to the usual four-line vertex coming from the first term as well as several new types of vertices. It is well known (12) that there are only two types of divergences in the graphs arising from the first term in (5.7). These divergences correspond to vertex subgraphs (four external lines) for which

$\nu = 0$ and self-energy graphs (two external lines) for which $\nu = 1$. The vertex subgraphs can overlap as in the graph of Figure 3. The second term in $\mathcal{H}(x)$ gives rise to four-line vertices which serve as counter-terms for the vertex subgraphs while the next two terms which produce two-line vertices are the counter-terms necessary to remove the divergences corresponding to self-energy subgraphs. The last term in $\mathcal{H}(x)$ serves to cancel the non-covariant contributions coming from the next to last term due to the fact that the derivative operator does not commute with the time-ordering operator.

The effect of these counter-terms in $\mathcal{H}_I(x)$ is that all graphs arising from them can be forgotten provided that for any graph G arising from the first term in $\mathcal{H}_I(x)$, the corresponding unsubtracted amplitude $W^{(G)}$ is replaced by a renormalized amplitude $W_R^{(G)}$ which we are about to define. Let $\{S_1 \ldots S_m\}$ be any set of mutually disjoint vertex and self-energy subgraphs of a given graph G. We denote the functional dependence of $W^{(G)}$ upon $W^{(S_1)} \ldots W^{(S_m)}$ by

$$W^{(G)} = W^{(G/S_1 \ldots S_m)}\left[W^{(S_1)} \ldots W^{(S_m)}\right] \tag{5.8}$$

Everything is assumed to be regularized in some consistent manner in this discussion. We then define a quantity $\overline{W}^{(G)}$ recursively by

$$\overline{W}^{(G)} = W^{(G)} + \sum_{\{S_1 \ldots S_m\}} W^{(G/S_1 \ldots S_m)}\left[-t\overline{W}^{(S_1)} \ldots -t\overline{W}^{(S_m)}\right] \tag{5.9}$$

where the summation is over all non-empty sets of mutually disjoint vertex and self-energy subgraphs. The regularized amplitudes $\overline{W}^{(S_i)}$ depend upon the invariants formed from the external momenta. The effect of the operator t is to project out terms up to order ν_i in the Taylor expansion of these amplitudes about the origin in their invariants. For a vertex subgraph, which depends upon six invariants,

$$t\bar{\Lambda}^{(S_i)} = \bar{\Lambda}^{(S_i)}(\text{all invs.} = 0) = -i\lambda L^{(S_i)} \tag{5.10a}$$

and for a self-energy subgraph

$$t\bar{\Sigma}^{(S_i)}(k^2) = \bar{\Sigma}^{(S_i)}(0) + \bar{\Sigma}'^{(S_i)}(0)k^2 = A^{(S_i)} + B^{(S_i)}k^2 . \tag{5.10b}$$

Then $W_R^{(G)}$ is given by

$$W_R^{(G)} = (1-t)\,\bar{W}^{(G)} \tag{5.11a}$$

if G is a self-energy or vertex graph and

$$W_R^{(G)} = \bar{W}^{(G)} \tag{5.11b}$$

otherwise. The constants in (5.10a) and (5.10b) are related to those in the interaction Hamiltonian (5.7) by

$$A = \sum_G A^{(G)} \qquad B = \sum_G B^{(G)} \tag{5.12}$$

where the summation is over all proper self-energy graphs and

$$L = \sum_G L^{(G)} \tag{5.13}$$

where the summation is over all proper vertex graphs.

This connection between subtractions for Feynman graphs and Lagrangian counter-terms is well-known (13) and we do not intend to discuss it further. It is our purpose to show that $W_R^{(G)}$ as defined by (3.12) is equivalent to the definition given by (5.9) and (5.11) in this chapter. Before proceeding to this, we should point out that since we have made all subtractions at the origin of momentum space, it is necessary to perform additional finite subtractions for self-energy parts to insure that the renormalized propagator will have a pole on the physical mass shell. These finite mass renormalizations are discussed in detail by Yennie and Kuo (7) and a method of performing the subtractions directly on the mass shell is given by the present author in reference (14). This question shall not concern us further.

The proof that (3.12) is equivalent to the definition of $W_R^{(G)}$ given in this chapter is straightforward. The operator

$$\int_0^1 d\xi_i \, \frac{(1-\xi_i)^{\nu_i}}{\nu_i!} \left(\frac{\partial}{\partial \xi_i}\right)^{\nu_i+1} \tag{5.14}$$

appearing in (3.12) is, through relation (3.1), equivalent to the operator $(1-t)$ used in this chapter. Thus

$$\int_0^1 d\xi_i \, \frac{(1-\xi_i)^{\nu_i}}{\nu_i!} \left(\frac{\partial}{\partial\xi_i}\right)^{\nu_i+1} f(\xi_i) = f(1) - \sum_{n=0}^{\nu_i} \frac{\xi_i^n}{n!} f^{(n)}(0) = (1-t_i) f(\xi_i) \tag{5.15}$$

where the 1 in $1-t_i$ is really an operator which sets $\xi_i = 1$. We first consider the subset $\mathcal{S}_0$ of $\mathcal{S}$ whose members do not properly contain members of $\mathcal{S}$

themselves. We can use the representation (5.15) for the ξ operations corresponding to the members of $\mathscr{S}_0$ and if it can be shown that $t_i t_j$ vanishes in (3.12) whenever $S_i \in \mathscr{S}_0$ and $S_j \in \mathscr{S}_0$ overlap, then (3.12) takes the following form.

$$W_R^{(G)} = \left(\frac{\pi^2}{i}\right)^{n_G} \int_0^\infty dx_G \prod_{S_i \in \mathscr{S}-\mathscr{S}_0} \int_0^1 d\xi_i \prod_{S_j \in \mathscr{S}-\mathscr{S}_0} \frac{(1-\xi_j)^{\nu_j}}{\nu_j!} \left(\frac{\partial}{\partial \xi_j}\right)^{\nu_j+1}$$

$$\times \left\{1 + \sum_{\{S_1 \ldots S_m\} \in \mathscr{S}_0} (-t_1) \ldots (-t_m)\right\} \frac{1}{\hat{U}^2} \exp\left[i\hat{V} - i \sum_{r \in G} x_r (m_r^2 - i\epsilon)\right] . \tag{5.16}$$

where the summation is over all sets of mutually disjoint members of $\mathscr{S}_0$. When a term in this summation does not contain t_i where $S_i \in \mathscr{S}_0$, then ξ_i is to be set equal to 1.

We next consider those members of $\mathscr{S}$ which properly contain only the subgraphs in $\mathscr{S}_0$. We again use (5.15) for the ξ operations corresponding to these graphs and we will show that a product of t operators corresponding to two overlapping members of this set gives zero when in (5.16) due to ξ operations corresponding to the union of these two graphs. This procedure can be continued working from the inside out and the result is that $W_R^{(G)}$ becomes

$$W_R^{(G)} = \left(\frac{\pi^2}{i}\right)^{n_G} \int_0^\infty dx_G \left\{1 + \sum_{\{S_1 \ldots S_m\} \in \mathscr{S}} (-t_i) \ldots (-t_m)\right\}$$

$$\times \frac{1}{\hat{U}^2} \exp\left[i\hat{V} - i \sum_{r \in G} x_r (m_r^2 - i\epsilon)\right] . \tag{5.17}$$

The summation is over all non-empty sets $\{S_1 \ldots S_m\}$ of non-overlapping members of $\mathscr{S}$. Note that members of $\{S_1 \ldots S_m\}$ may be nested.

In order to show that (5.17) is equivalent to the definition of $W_R^{(G)}$ given by (5.9) and (5.11), we re-arrange the summation in (5.17) by defining an operator Q_G recursively by

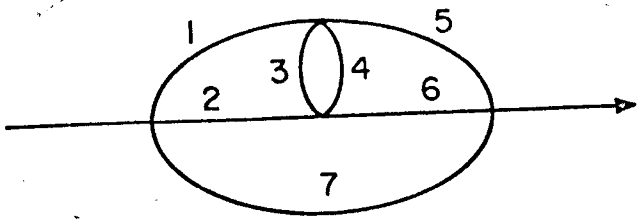
$$Q_G = \left\{1 + \sum_{\{S_1 \ldots S_m\}} (-t_i Q_{S_i}) \ldots (-t_m Q_{S_m})\right\}, \tag{5.18}$$

the sum being over all non-empty sets of mutually disjoint members of $\mathscr{S}$ which are properly contained in G. Then

$$W_R^{(G)} = \left(\frac{\pi^2}{i}\right)^{n_G} \int_0^\infty dx_G (1 - t_0) Q_G \frac{1}{\hat{U}^2} \exp\left[i\hat{V} - i \sum_{r \in G} x_r (m_r^2 - i\epsilon)\right] \tag{5.19}$$

where $t_0 = 0$ for $\nu_0 = \nu_G < 0$. The reader can easily convince himself by inspection that (5.19) is identical to (5.17). The equivalence of (5.19) with the definition of $W_R^{(G)}$ via (5.9) and (5.11) should be clear since these two expressions have the same form. Regularization is not necessary in (5.19) since the subtraction operations are performed directly on the integrand.

It remains to prove that the overlaps do indeed vanish as we have stated above. We have shown in reference (14) that if two members S_a and S_b of $\mathscr{S}$ overlap, then they must both be vertex subgraphs and $S_a \cup S_b$ must be either a self-energy or vertex subgraph. If $S_a \cup S_b \in \mathscr{S}$, we let $S_c = S_a \cup S_b$. If $S_a \cup S_b$ is a vertex subgraph which is not an element of $\mathscr{S}$, we let S_c be the self-energy subgraph formed from $S_a \cup S_b$ by adjoining one line. Thus the vertex subgraph of Figure 4 composed of lines 1 to 6 is not a member of $\mathscr{S}$ but the self-energy graph formed by adjoining line 7 is a member of $\mathscr{S}$.



A graph with a superficially divergent subgraph which is not a member of $\mathscr{S}$.

Figure 4

In either of the above cases, $S_c \in \mathscr{S}$ and hence there will be a ξ operation corresponding to this graph. The operation $t_a t_b$ simply sets ξ_a and ξ_b equal to zero. Since $\hat{U}$ does not have a zero when $\xi_a = \xi_b = 0$, the $t_a t_b$ operation can be commuted with the ξ derivatives and what we must show is that

$$\left(\frac{\partial}{\partial \xi_c}\right)^{\nu_c+1} \frac{1}{\hat{U}^2(\xi_a=\xi_b=0)} \exp\left[i\hat{V}(\xi_a=\xi_b=0) - \sum_{r\in G} x_r(m_r^2 - i\epsilon)\right] = 0 \tag{5.20}$$

Consider any term in $\hat{U}$ or $\hat{W}^{ij}$ which contain a factor ξ_c. This term must be at least of order $n_{S_c}+1$ in x_r, $r \in S_c$ and hence at least of order $n_{S_a \cup S_b}+1$ in x_r, $r \in S_a \cup S_b$. Suppose that it is of order $n_{S_a} + m_a$, $(n_{S_b} + m_b)$ in x_r, $r \in S_a$, (S_b). Then since any term is at least of order $n_{S_a \cap S_b}$ in x_r, $r \in S_a \cap S_b$,

$$n_{S_a \cup S_b} < n_{S_a} + m_a + n_{S_b} + m_b - n_{S_a \cap S_b} . \tag{5.21}$$

But clearly

$$n_{S_a \cup S_b} = n_{S_a} + n_{S_b} - n_{S_a \cap S_b} \tag{5.22}$$

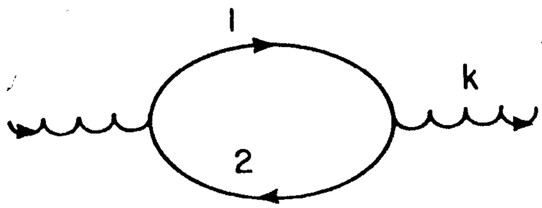
for two overlapping vertex graphs and hence

$$m_a + m_b > 0 \tag{5.23}$$

This means that any term in U or W^{ij} which contains a factor ξ_c must also contain either a factor ξ_a or ξ_b. Thus all dependence on ξ_c vanishes when ξ_a and ξ_b are set equal to zero. This gives the result (5.20).

VI. Fourth Order Vacuum Polarization

The starting point for any calculation in quantum electrodynamics is expression (3.5) with vertex γ-matrices, constant factors and traces over closed fermion loops inserted. To illustrate the technique, we first look at the second order vacuum polarization.



Second order vacuum polarization graph.

Figure 5

The unsubtracted amplitude is

$$\Pi^{(2)}_{\mu\nu\, 0}(k) = \frac{\alpha}{4\pi^3 i} \int d^4p \; T_r \left\{ \gamma_\mu \frac{i}{\not p + \not k - m} \gamma_\nu \frac{i}{\not p - m} \right\} \tag{6.1}$$

where α is the fine structure constant. According to (3.5), the corresponding renormalized amplitude is

$$\Pi^{(2)}_{\mu\nu}(k) = \frac{\alpha}{4\pi^3 i} \left(\frac{\pi^2}{i}\right) \int_0^\infty dx_1 dx_2 \int_0^1 d\xi \; \frac{(1-\xi)^2}{2!} \left(\frac{\xi}{\partial \xi}\right)^3 \frac{1}{(x_1+x_2)^2}$$

$$\times\, T_r \left\{ \gamma_\mu \left(\frac{x_1}{x_1+x_2} \xi \not k \qquad \right) \gamma_\nu \left(\frac{-x_2}{x_1+x_2} \xi \not k + m \right) + \frac{1}{2} \frac{1}{x_1+x_2} \gamma_\mu \gamma^\sigma \gamma_\nu \gamma_\sigma \right\} \tag{6.2}$$

$$\times\, \exp\left[i \frac{x_1 x_2}{x_1+x_2} \xi^2 k^2 \qquad {}_1 + x_2) m^2 \right]$$

where m^2 includes a small negative imaginary part. Since $\Pi_{\mu\nu}(k)$ is gauge invariant, it must be of the form

$$\Pi_{\mu\nu}(k) = (g_{\mu\nu} k^2 - k_\mu k_\nu)\, \Pi(k) \, . \tag{6.3}$$

Thus doing the trace in (6.2) and extracting the coefficient of $-k_\mu k_\nu$ gives $\Pi^{(2)}(k)$.

$$\Pi^{(2)}(k) = -\frac{2\alpha}{\pi} \int_0^\infty dx_1 dx_2 \int_0^1 d\xi \, \frac{(1-\xi)^2}{2!} \left(\frac{\partial}{\partial \xi}\right)^3 \xi^2 \, \frac{x_1 x_2}{(x_1 + x_2)^4} \times \exp\left[i \frac{x_1 x_2}{x_1 + x_2} \xi^2 k^2 - i m^2 (x_1 + x_2) \right] \tag{6.4}$$

Inserting the identity

$$1 = \int_0^\infty \frac{d\lambda}{\lambda} \, \delta\left(1 - \frac{1}{\lambda} \sum_{r \epsilon G} x_r\right) \tag{6.5}$$

scaling $x_r \to \lambda x_r$ and doing the ξ integral gives

$$\Pi^{(2)}(k) = -\frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \int_0^\infty \frac{d\lambda}{\lambda} \left\{ \exp\left[i\lambda x(1-x) k^2 - i\lambda m^2 \right] - \exp\left[-i\lambda m^2 \right] \right\} \, . \tag{6.6}$$

Using the identity

$$\int_0^\infty \frac{d\lambda}{\lambda} \left(e^{ia\lambda} - e^{ib\lambda} \right) = \log\left(\frac{b}{a}\right) \tag{6.7}$$

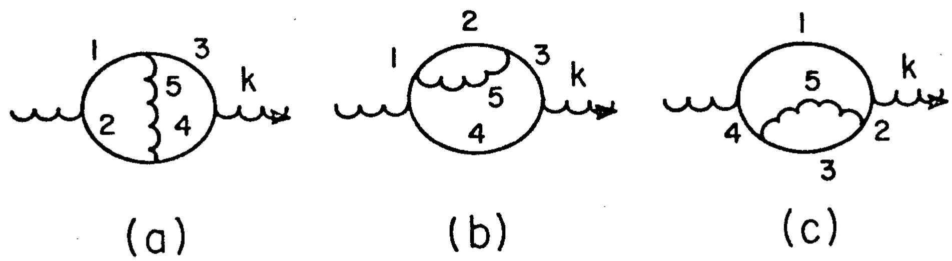
we have

$$\Pi^{(2)}(k) = \frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \log\left(1 - x(1-x) \frac{k^2}{m^2} \right) \tag{6.8}$$

which for $-k^2/m^2 >> 1$ becomes

$$\frac{\alpha}{3\pi}\log\left(\frac{-k^2}{m^2}\right) - \frac{5}{9}\,\frac{\alpha}{\pi} \tag{6.9}$$

The fourth order vacuum polarization $\Pi_{\mu\nu}^{(4)}$ consists of contributions from the three diagrams of Figure 6.



Fourth order vacuum polarization graphs.

Figure 6

The contributions from graphs 6b and 6c are identical and hence

$$\Pi_{\mu\nu}^{(4)}(k) = (g_{\mu\nu}k^2 - k_\mu k_\nu)\Pi^{(4)}(k) = \Pi_{\mu\nu}^{(a)}(k) + 2\,\Pi_{\mu\nu}^{(b)}(k) \tag{6.10}$$

where $\Pi_{\mu\nu}^{(a)}(k)$ and $\Pi_{\mu\nu}^{(b)}(k)$ are the amplitudes corresponding to graphs 6a and 6b, respectively. For the graph of Figure 6a, the set $\mathscr{S}$ in (3.5) consists of the entire graph and the two overlapping vertex graphs composed of lines 1, 4, 5 and 2, 3, 5. We associate the parameter ξ_0 with the entire graph, the parameter ξ_1 with the 1, 4, 5 vertex and the parameter ξ_2 with the 2, 3, 5 vertex. Then defining parametric functions $\overline{U}_a$, $\overline{W}_a$, $\overline{Y}_{ra}$ and $\overline{X}_{rsa}$ according to the rules of Section III and introducing the notation

$$\int_0^1 d\xi_i \frac{(1-\xi_i)^n}{n!}\left(\frac{\partial}{\partial \xi_i}\right)^{n+1} = R_i^{(n+1)} \tag{6.11}$$

we have

$$\begin{aligned} \Pi^{(a)}_{\mu\nu}(k) = i\left(\frac{\alpha}{4\pi}\right)^2 \int_0^\infty dx_1,\dots,dx_5\, R_0^{(3)} R_1^{(1)} R_2^{(1)} \\ \mathrm{Tr}\left\{\gamma_\mu\left(\frac{1}{ix_1}\nabla\!\!\!/_{\ell_1}+m\right)\gamma_\sigma\left(\frac{1}{ix_2}\nabla\!\!\!/_{\ell_2}+m\right)\gamma_\nu\left(\frac{1}{ix_3}\nabla\!\!\!/_{\ell_3}+m\right)\gamma_\sigma\left(\frac{1}{ix_4}\nabla\!\!\!/_{\ell_4}+m\right)\right\}\frac{1}{U_a^2} \\ \times \exp\left[i\left(\frac{W_a}{U_a}k^2+\sum_{r=1}^4 x_r\ell_r\cdot Y_{ra}-\frac{1}{4}\sum_{r,s=1}^4 x_r x_s \ell_r\cdot\ell_s X_{rsa}\right)-im^2\sum_{r=1}^4 x_r-\epsilon x_5\right]\Bigg|_{\ell_r=0} \end{aligned} \tag{6.12}$$

To form $\Pi^{(b)}_{\mu\nu}(k)$, we first apply the rules of Section III which effect subtractions at the origin of momentum space. The result will be a contribution similar to (6.12). In order to insure that the second order electron propagator composed of lines 1, 2, 3 and 5 has a simple pole at $k\!\!\!/=m$, we must subtract from this contribution a term in which the second order renormalized electron self energy $\Sigma^{(2)}(k)$ is replaced by $\Sigma^{(2)}(k\!\!\!/=m)$. Associating the parameter ξ_0 with the entire graph and the parameter ξ_1 with the self energy subgraph composed of lines 2 and 5, we have

$$\Pi^{(b)}_{\mu\nu}(k) = i\left(\frac{\alpha}{4\pi}\right)^2 \int_0^\infty dx_1, \ldots, dx_5 \, R_0^{(3)} R_1^{(2)}$$

$$\times \operatorname{Tr}\left\{\gamma_\mu\left(\frac{1}{ix_1}\nabla_{\ell_1}+m\right)\gamma_\sigma\left(\frac{1}{ix_2}\nabla_{\ell_2}+m\right)\gamma_\sigma\left(\frac{1}{ix_3}\nabla_{\ell_3}+m\right)\gamma_\nu\left(\frac{1}{ix_4}\nabla_{\ell_4}+m\right)\right\}\frac{1}{U_2^2} \tag{6.13}$$

$$\times \exp\left[i\left(\frac{\overline{W}_b}{\overline{U}_b}k^2 + \sum_{r=1}^{4} x_r\ell_r\cdot\overline{Y}_{rb} - \frac{1}{4}\sum_{r,s=1}^{4}\hat{x}_r\hat{x}_s\ell_r\cdot\ell_s\overline{X}_{rsb}\right) - im^2\sum_{r=1}^{4}x_r - \epsilon x_5\right]\Bigg|_{\ell_r=0}$$

$$- \Sigma^{(2)}(m)\,\Pi^{(R)}_{\mu\nu}(k)$$

where R is the graph formed from that of Figure 6b by shrinking lines 2 and 5 to a point.

To calculate $\Pi^{(4)}(k)$, we carry out the ℓ_r derivatives in (6.12) and (6.13), do the traces and keep only the coefficient of $-k_\mu k_\nu$ in each. Defining these coefficients to be $\Pi^{(a)}(k)$ and $\Pi^{(b)}(k)$ respectively,

$$\Pi^{(4)}(k) = \Pi^{(a)}(k) + 2\Pi^{(b)}(k). \tag{6.14}$$

The ∇_{ℓ_r} operators in (6.12) and (6.13) generate terms which have either zero, one or two factors of $\overline{X}_{rs}$. The last of these clearly do not give rise to $k_\mu k_\nu$ terms. The trace calculations are very simple since there is only one momentum k_μ involved. The symmetry of the graph of Figure 6a can be used to write $\Pi^{(a)}(k)$ in a simple form in which the overlapping divergence has been removed. The results are

$$\Pi^{(a)}(k) = 2\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_1 \ldots dx_5\, \delta(1-x_1-\ldots-x_5) \int_0^1 d\xi\, \frac{\partial}{\partial \xi}\, \frac{1}{\hat{U}_a^2}$$

$$\times \left\{ \left[\xi\, \frac{x_5}{\hat{U}_a} (A_1 A_2 - \hat{A}_1 \hat{A}_4) - \frac{x_1+x_4}{\hat{U}_a} A_1 A_4 \right] \log\left(1 - \frac{k^2}{m^2}\, \frac{\hat{W}_a}{\hat{U}_a (x_1+\ldots+x_4)}\right) \right. \quad (6.15)$$

$$\left. + \frac{1}{(x_1+\cdots+x_4)} \left[\xi A_2 (A_1 + A_4) - \hat{A}_1 \hat{A}_4\right] \left(1 - \frac{m^2}{k^2}\, \frac{(x_1+\ldots+x_4)\hat{U}_a}{\hat{W}_a}\right)^{-1} \right\}$$

where

$$\hat{U}_a = (x_1+x_4)(x_2+x_3+x_5) + \xi\, x_5 (x_2+x_3)$$

$$\hat{W}_a = x_1 x_4 (x_2+x_3+x_5) + \xi\, x_2 x_3 (x_1+x_4+\xi\, x_5) + \xi\, x_5 (x_1 x_3 + x_2 x_4) \quad (6.16a)$$

$$A_1 = \frac{1}{\hat{U}_a}\left[x_4(x_2+x_3+x_5) + \xi\, x_3 x_5\right]$$

$$\hat{A}_4 = -\frac{1}{\hat{U}_a}\left[x_1(x_2+x_3+x_5) + \xi\, x_2 x_5\right] \quad (6.16b)$$

$$A_2 = \frac{1}{U_a}\left[x_1 x_3 + x_3 x_4 + x_3 x_5 + x_4 x_5\right]$$

and where U_a, W_a, A_1 and A_4 are formed by setting $\xi = 1$ in $\hat{U}_a$, $\hat{W}_a$, $\hat{A}_1$ and $\hat{A}_4$ respectively and

$$2\Pi^{(b)}(k) = 2\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_1 \ldots dx_5\ \delta(1-x_1-\cdots-x_5) \int_0^1 d\xi\ \frac{\partial}{\partial \xi}$$

$$\times \left\{ \left[\frac{x_4^2 x_5 (x_2+x_r)^2}{\hat{U}_b^5} - 3\hat{B}_4 \frac{x_4 x_5 (x_2+x_5)}{\hat{U}_b^4} \right] \log\left(1 - \frac{k^2}{m^2} \frac{\hat{W}_b}{\hat{U}_b (x_1 + \cdots + x_4)} \right) \right.$$

$$+ \int_0^1 d\xi_0 \frac{\partial}{\partial \xi_0} \hat{B}_4 \left[\xi_0 k^2 \frac{x_4^3 x_5 (x_2+x_5)^2}{\hat{U}_b^5} - m^2 x_4 \frac{4x_2 + 3x_5}{\hat{U}_b^3} \right] \tag{6.17}$$

$$\left. \times \left(\frac{\hat{W}_b}{\hat{U}_b} \xi_0 k^2 - m^2 (x_1 + \cdots + x_1) \right)^{-1} \right\} - 2\,\Sigma^{(2)}(m)\,\Pi^{(R)}(k)$$

where

$$\begin{aligned} \hat{U}_b &= (x_1 + x_3 + x_4)(x_2 + x_5) + \xi\, x_2 x_5 \\ \hat{W}_b &= x_4 (x_1 + x_3)(x_2 + x_5) + \xi\, x_2 x_4 x_5 \\ \hat{B}_4 &= \frac{1}{U_b} \left[(x_1 + x_3)(x_2 + x_5) + \xi\, x_2 x_5 \right]. \end{aligned} \tag{6.18}$$

We now restrict our attention to the asymptotic region $-k^2 >> m^2$ and keep only those terms in $\Pi^{(a)}(k)$ and $\Pi^{(b)}(k)$ which behave like $\log(-k^2/m^2)$ or $\log^2(-k^2/m^2)$ in this region. $\Pi^{(a)}(k)$ becomes

$$\Pi^{(a)}(k) = \Pi_1^{(a)}(k) + \Pi_2^{(a)}(k) + \Pi_3^{(a)}(k) \tag{6.19}$$

where $\Pi_1^{(a)}(k)$ comes from those terms multiplying the log in (6.15) which do not require the internal subtraction (those containing a factor of ξ or ξ^2).

For these terms, the log can be expanded and

$$\Pi_1^{(a)}(k) = 2\left(\frac{\alpha}{\pi}\right)^2 \log\left(\frac{-k^2}{m^2}\right) \int_0^\infty dx_1 \ldots dx_5 \, \frac{1}{U_a^3} \left\{ x_5(A_1A_2 - A_1A_4) \right.$$

$$\left. + (x_1 + x_4) \frac{1}{U_a^2} \left[x_5(x_2 + x_3 + x_5)(x_1x_3 + x_2x_4) + x_2x_3x_5^2 \right] \right\}. \tag{6.20}$$

$\Pi_2^{(a)}(k)$ comes from that term multiplying the log which does require the internal subtraction. For this term, the log cannot be expanded since this would introduce a logarithmic divergence in x_1, x_4 and ξ .

$$\Pi_2^{(a)}(k) = 2\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_1 \ldots dx_5 \, \delta(1 - x_1 - \ldots - x_5) \int_0^1 d\xi \, \frac{\partial}{\partial \xi} \frac{1}{\hat{U}_a^5}$$

$$\times (x_1 + x_4) x_1 x_4 (x_2 + x_3 + x_5)^2 \log\left(1 - \frac{k^2}{m^2} \frac{\hat{W}_a}{\hat{U}_a(x_1 + \ldots + x_4)}\right) \tag{6.21}$$

Since the log serves as a cutoff for a logarithmically divergent integral, we expect $\Pi_2^{(a)}(k)$ to give a $\log^2(-k^2/m^2)$ contribution. The remaining terms in (6.15) are convergent without the internal subtraction, however, there is one non-vanishing subtraction term ($\xi = 0$ term). It is only for this term, $\Pi_3^{(a)}(k)$, that the limit $-k^2/m^2 \to \infty$ is not finite.

$$\Pi_3^{(a)}(k) = -2\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_1 \ldots dx_5 \, \delta(1 - x_1 - \ldots - x_5) \frac{x_1 x_4}{(x_1 + x_4)^4 (x_2 + x_3 + x_5)^2 (x_1 + \ldots + x_4)}$$

$$\times \left(1 - \frac{m^2}{k^2} \frac{(x_1 + \ldots + x_4)(x_1 + x_4)}{x_1 x_4}\right)^{-1} \tag{6.22}$$

This integral diverges logarithmically when $-k^2/m^2 \to \infty$ and hence it will give a $\log(-k^2/m^2)$ contribution.

Similarly,

$$2\Pi^{(b)}(k) = \Pi_1^{(b)}(k) + \Pi_2^{(b)}(k) + \Pi_3^{(b)}(k) \tag{6.23}$$

where $\Pi_1^{(b)}(k)$ comes from the term multiplying the log in (6.17) for which the subtraction term ($\xi = 0$ term) vanishes. The log can be expanded and

$$\Pi_1^{(b)}(k) = -6\left(\frac{\alpha}{\pi}\right)^2 \log\left(\frac{-k^2}{m^2}\right) \int_0^\infty dx_1 \ldots dx_5\, \delta(1-x_1-\ldots-x_5)\, \frac{x_2 x_4 x_5^2 (x_2+x_5)}{U_b^5} . \tag{6.24}$$

$\Pi_2^{(b)}$ comes from those terms multiplying the log in (6.17) which require the internal subtraction. Expanding the log would produce a logarithmic divergence for these terms in x_1, x_2, x_4 and ξ and so we expect a $\log^2(-k^2/m^2)$ contribution.

$$\begin{aligned}\Pi_2^{(b)}(k) = & -2\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_1 \ldots dx_5\, \delta(1-x_1-\ldots-x_5) \int_0^1 d\xi\, \frac{\partial}{\partial \xi}\, \frac{1}{\hat{U}_b^5} \\ & \times x_4 x_5 (x_2+x_5)^2 (3x_1+3x_3-x_4) \log\left(1 - \frac{k^2}{m^2}\, \frac{\hat{W}_b}{\hat{U}_b(x_1+\ldots+x_4)}\right)\end{aligned} \tag{6.25}$$

$\Pi_3^{(b)}$ comes from the term in (6.17) with k^2 in the numerator which requires the internal subtraction.

$$\Pi_3^{(b)}(k) = 2\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_1 \ldots dx_5\, \delta(1 - x_1 - \cdots - x_5) \int_0^1 d\xi \, \frac{\partial}{\partial \xi}$$

$$\times \frac{x_4^3 x_5 (x_1 + x_3)(x_2 + x_5)^3}{\hat{U}_b^6} \left(\frac{\hat{W}_b}{\hat{U}_b} - \frac{m^2}{k^2}(x_1 + \cdots + x_4)\right)^{-1} \qquad (6.26)$$

When $-k^2/m^2 \to \infty$, this integral diverges logarithmically in x_1, x_3, x_4 and ξ. Hence we expect it to give a $\log(-k^2/m^2)$ contribution. The finite mass counterterm $\Sigma^{(2)}(m)\, \Pi^{(R)}(k)$ in (6.17) remains finite at $-k^2/m^2 \to \infty$.

There are six contributions to the asymptotic form of $\Pi^{(4)}(k)$. Four of them, $\Pi_1^{(a)}$, $\Pi_3^{(a)}$, $\Pi_1^{(b)}$ and $\Pi_3^{(b)}$, give rise to $\log(-k^2/m^2)$ contributions while $\Pi_2^{(a)}$ and $\Pi_2^{(b)}$ both give rise to $\log^2(-k^2/m^2)$ and $\log(-k^2/m^2)$ contributions. It will be shown in Appendix B that the $\log^2(-k^2/m^2)$ terms cancel and that $\Pi_2^{(a)} + \Pi_2^{(b)}$ gives only a $\log(-k^2/m^2)$ contribution. The integrals giving rise to $\log(-k^2/m^2)$ contributions are straightforward. The calculations are presented briefly in Appendix B. The results are

$$\Pi_1^{(a)} = 5/8\, (\alpha/\pi)^2 \log(-k^2/m^2) \qquad (6.27a)$$

$$\Pi_3^{(a)} = -1/3\, (\alpha/\pi)^2 \log(-k^2/m^2) \qquad (6.27b)$$

$$\Pi_1^{(b)} = -1/8\, (\alpha/\pi)^2 \log(-k^2/m^2) \qquad (6.27c)$$

$$\Pi_3^{(b)} = -1/12\, (\alpha/\pi)^2 \log(-k^2/m^2) \qquad (6.27d)$$

$$\Pi_2^{(a)} + \Pi_2^{(b)} = 1/6\, (\alpha/\pi)^2 \log(-k^2/m^2) \qquad (6.27e)$$

The integrations have also been checked numerically (15). Adding these contributions together we have the complete renormalized fourth order vacuum polarization contribution in the asymptotic region $-k^2 >> m^2$.

$$\Pi^{(4)}(k) = 1/4\ (\alpha/\pi)^2 \log(-k^2/m^2) \tag{6.28}$$

This result for $\Pi^{(4)}(k)$ is identical to that appearing elsewhere in the literature (5)(6).

VII. Discussion

The formalism developed here has been shown to be useful from both a formal and practical point of view. It gives a concise way of expressing renormalized amplitudes for arbitrary graphs and the proof of convergence of these integrals is a great deal simpler than the corresponding proof when the renormalized amplitude is given by a recursive subtraction formula. We do not claim great mathematical rigor but the present discussion could be transcribed into a mathematically more precise language without too much difficulty.

From a practical point of view, we feel that this formalism could be very useful especially when combined with numerical integration techniques. For example, the fourth order vacuum polarization is a sum of two parametric integrals, (6.15) and (6.17). These integrals can be done numerically quite easily to give the result (6.28). Several such calculations are being looked into at the present time. A further point of interest is the question of the gauge invariance of the vacuum polarization. A proof of this for the renormalized fourth order amplitude using the parametric formalism is being looked into.

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Appendix A

Two Theorems Concerning the Convergence of the Renormalized Amplitudes

In this appendix we will prove the two theorems which were stated in Section IV.

Proof of Theorem 1: We first consider the nested sequence of sets $S(S') \subset S(S') \cup [H \cap S(S'-1)] \subset S(S'-1) \subset \cdots \subset S(1) \subset S(1) \cup H \subset G$. For any nested sequence of sets $K_1, K_2 \cdots K_m$, it is certainly always possible to choose independent integration momenta such that n_{K_i} of the momenta internal to K_i are independent and hence there is always at least one term in $\hat{U}$ which is of order n_{K_i} in x_r, $r \in K_i$ for $i=1,2,\ldots,m$. For the above sequence it then follows that this term in $\hat{U}$ will be of order

$$n_{H \cap R(i)} = n_{S(i+1) \cup [H \cap S(i)]} - n_{S(i+1)} \tag{A.1}$$

in x_r, $r \in H \cap R(i)$. It will thus be of order

$$n_{H \cap R(S')} + n_{H \cap R(S'-1)} + \cdots n_{H \cap R(1)} + n_{H \cap R(0)}$$

in x_r, $r \in H$. Suppose that this term is of order $n_{S_i} + m_i$ in x_r, $r \in S_i$ for $i=1,2,\ldots,S'$. It will then be of order $m_1 + m_2 + \ldots + m_{S'}$ in $\xi_1, \xi_2, \ldots, \xi_{S'}$. Certainly one restriction on the above numbers is

$$\sum_{i=1}^{S'} (n_{S_i} + m_i) = \sum_{j=1}^{S'} n_{S(j)} . \tag{A.2}$$

Then defining

$$\ell_{\mathscr{S}'} = \sum_{i=1}^{S'} n_{S(i)} - \sum_{i=1}^{S'} n_{S_i} , \tag{A.3}$$

it follows that the term in $\hat{U}$ under consideration is of order

$$n_{H\cap R(S')} + n_{H\cap R(S'-1)} + \cdots + n_{H\cap R(1)} = n_{H\cap R(0)} + \ell_{\mathscr{S}'} \tag{A.4}$$

in $\xi_1, \xi_2, \ldots, \xi_{S'}$ and x_r, $r \in H$. A similar analysis shows that any term in $\hat{U}$ is at least of this order and hence we have the result (4.4) of Theorem 1.

Proof of Theorem 2: We first prove the useful fact that for any non-empty set K of lines in G, the expression $R_\sigma(x, \xi)$ appearing in (4.1) is at least of order

$$n_K(p_\sigma + 2) - N_K + 1 \tag{A.5}$$

in x_r, $r \in K$. In this expression, N_K is the number of lines in K, and n_K is the number of independent loops formed by these lines. If $\nu_K < 0$, the proof is trivial since each time a derivative operator in (4.1) acts in such a way as to increase the power $\hat{U}$ in the denominator, it also introduces a term into the numerator which is at least of order n_K in x_r, $r \in K$. It follows that $R_\sigma(x, \xi)$ is at least of order $n_K p_\sigma \geq n_K p_\sigma + \nu_K + 1 = n_K(p_\sigma + 2) - N_K + 1$ in x_r, $r \in K$. Next suppose that $\nu_K \geq 0$. For any set K we can associate a member of $\mathscr{S}$ as follows. We first throw out as many lines of K as possible without decreasing the number of independent loops. We assume the resulting set is connected since if it is not we can apply the following considerations to each connected part individually. To this set we add all lines connecting any two of its vertices. Suppose that there are ℓ such lines. Then for the resulting graph S(K),

$$n_{S(K)} = n_K + \ell \tag{A.6}$$

$$N_{S(K)} \leq N_K + \ell \tag{A.7}$$

Thus $\nu_{S(K)} \geq \nu_K + \ell \geq 0$ and hence S(K) must be a member of $\mathscr{S}$. The ξ operation corresponding to S(K) will insure that $R_\sigma(x, \xi)$ is at least of order $n_{S(K)}p_\sigma + \nu_{S(K)} + 1$ in x_r, $r \in S(K)$. Thus it is at least of order $n_{S(K)}p_\sigma + \nu_{S(K)} + 1 - \ell$ in x_r, $r \in K$. But $\nu_{S(K)} - \ell \geq \nu_K = 2n_K - N_K$ and $n_{S(K)} \geq n_K$, and therefore $R_\sigma(x, \xi)$ is at least of order $n_K(p_\sigma + 2) - N_K + 1$ in x_r, $r \in K$.

It follows that $R_\sigma(x, \xi)$ will be at least of order $n_{[H \cap S(i)] \cup S(i+1)}(p_\sigma + 2) - N_{[H \cap S(i)] \cup S(i+1)} + \delta_{io}$ in x_r, $r \in [H \cap S(i)] \cup S(i+1)$. The above condition has been relaxed for all i except i=0 to account for the fact that each might be empty. We assume, however, that $H \cup S(1)$ is not empty. Similarly, $R_\sigma(x, \xi)$ must be at least of order $n_{S(i)}(p_\sigma + 2) - N_{S(i)} + 1$ in x_r, $r \in S(i)$. Suppose that it is of order $n_{S(i)}(p_\sigma + 2) - N_{S(i)} + 1 + L(i)$ in x_r, $r \in S(i)$. Subtracting corresponding terms and using the definition of R(i), we see that $R_\sigma(x, \xi)$ must be at least of order

$$\left[n_{H \cap R(S')} + n_{H \cap R(S'-1)} + \dots + n_{H \cap R(0)}\right](p_\sigma + 2) - N_H - S' - \sum_{i=1}^{S'} L(i) + 1$$

in x_r, $r \in H$. Let $R_\sigma(x, \xi)$ be of order $n_{S_i}(p_\sigma + 2) - N_{S_i} + 1 + L i$ in x_r, $r \in S_i$. Then clearly

$$\sum_{i=1}^{S'} \left[n_{S_i}(p_\sigma + 2) + N_{S_i} + 1 + L i \right] = \sum_{j=1}^{S'} \left[n_{S(j)}(p_\sigma + 2) + N_{S(j)} + 1 + L(j) \right]. \quad \text{(A.8)}$$

From this and the definition (A.3) of $\ell_{\mathscr{S}'}$, it follows that $R_\sigma(x, \xi)$ is at least of order

$$\left[n_{H\cap R(S')} + \dots + n_{H\cap R(0)} + \ell_{\mathscr{S}'} \right] (p_\sigma + 2) - N_H - S - \sum_{i=1}^{S'} L i + 1$$

in x_r, $r \in H$. Finally we note that $R_\sigma(x, \xi)$ must be of order $L i$ in ξ_i for $i = 1, \dots, S'$. This gives (4.6) and hence Theorem 2 is proven.

Appendix B

In this appendix we will calculate $\Pi_1^{(b)}$ and $\Pi_3^{(a)}$ and show that the $\log^2(-k^2/m^2)$ terms in $\Pi_2^{(a)}$ and $\Pi_2^{(b)}$ cancel. The other integrals are done in exactly the same way.

We first calculate $\Pi_1^{(b)}(k)$. The integral in (6.24) can be made more symmetric and we have

$$\Pi_1^{(b)}(k) = -\frac{3}{2}\left(\frac{\alpha}{\pi}\right)^2 \log\left(\frac{-k^2}{m^2}\right) \int_0^1 dx_1 \ldots dx_5 \, \delta(1 - x_1 - \cdots - x_5)$$
$$\times \frac{x_2 x_5 (x_2 + x_5)^2 (x_1 + x_4)}{[(x_1 + x_3 + x_4)(x_2 + x_5) + x_2 x_5]^5} \tag{B.1}$$

A convenient substitution of variables is

$$\begin{aligned} x_1 &= u y & x_2 &= v z \\ x_4 &= (1-u) y & x_5 &= (1-v) z \\ dx_1 dx_4 &= y\, dy\, du & dx_2 dx_5 &= z\, dz\, dv\, . \end{aligned} \tag{B.2}$$

Then

$$\Pi_1^{(b)}(k) = -\frac{3}{2}\left(\frac{\alpha}{\pi}\right)^2 \log\left(\frac{-k^2}{m^2}\right) \int_0^1 dx_3\, dy\, dz\, dv \;\; \delta(1 - x_3 - y - z)$$
$$\times \frac{v(1-v) y^2}{[y + x_3 + z v(1-v)]^5} \tag{B.3}$$

The change of variables

$$\begin{aligned} y &= w x \\ x_3 &= (1-w) x \\ dy\, dx_3 &= x\, dx\, dw \end{aligned} \tag{B.4}$$

leads to

$$\Pi_1^{(b)}(k) = -\tfrac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \log\left(\frac{-k^2}{m^2}\right) \int_0^1 dx\,dv\, \frac{v(1-v)\,x^3}{[x+(1-x)\,v\,(1-v)]^5} . \tag{B.5}$$

This integral is elementary and yields

$$\Pi_1^{(b)}(k) = -\frac{1}{8}(\alpha/\pi)^2 \log(-k^2/m^2) . \tag{B.6}$$

In the integral in the expression (6.22) for $\Pi_3^{(a)}$ a variable change of the type (B.2) is also useful. It leads to

$$\Pi_3^{(a)}(k) = -2\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dy\,dz\,dx_5\,du\,\delta(1-y-z-x_5)$$
$$\times \frac{u^2(1-u)^2 z}{(z+x_5)(y+z)} \left(y\,u\,(1-u) - (y+z)\frac{m^2}{k^2}\right)^{-1} . \tag{B.7}$$

We next let

$$y = wx$$
$$z = (1-w)\,x \tag{B.8}$$
$$dy\,dz = x\,dx\,dw$$

and arrive at

$$\Pi_3^{(a)}(k) = -2\left(\frac{\alpha}{\pi}\right)^2 \log\left(\frac{-k^2}{m^2}\right)\int_0^1 dx\,du\,dw\, \frac{u^2(1-u)^2(1-w)}{[1-wx]}\left(w\,u\,(1-u) - \frac{m^2}{k^2}\right)^{-1}$$
$$= -2\;\frac{\alpha}{\pi}^{\;2} \int_0^1 du\,u\,(1-u)\,\log\left[1-\frac{k^2}{m^2}\,u(1-u)\right] \rightarrow -\frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \ln\left(\frac{-k^2}{m^2}\right) \tag{B.10}$$

as $\left(\frac{-k^2}{m^2}\right) \rightarrow \infty$.

Finally, we will show that $\Pi_2^{(a)} + \Pi_2^{(b)}$ behaves as $\log(-k^2/m^2)$ and not $\log^2(-k^2/m^2)$ in the asymptotic region. We first point out that since we are only interested in terms which increase as $(-k^2/m^2) \to \infty$, the terms containing ξ in $\hat{W}_a$ in (6.21) and in $\hat{W}_b$ in (6.25) can be dropped. We show this for (6.25) by rewriting the log appearing there in the form

$$\log\left(1 - \frac{k^2}{m^2}\,\frac{x_4(x_1+x_3)(x_2+x_5)}{\hat{U}_b(x_1+\ldots+x_4)}\right) + \log\left(1 + \frac{\xi\, x_2 x_4 x_5}{x_4(x_1+x_3)(x_2+x_5) - \frac{m^2}{k^2}\,\hat{U}_b(x_1+\ldots+x_4)}\right) \tag{B.11}$$

The limit $(-k^2/m^2) \to \infty$ can be taken in the second term and the integral (6.25) will still be convergent. Thus only the first term in (B.11) which is formed by dropping terms containing ξ in $\hat{W}_b$ contributes in the asymptotic region. A similar result holds for (6.21).

We further simplify (6.25) by making a change of variables. We let

$$x_1 \to u x_1, \quad x_3 \to (1-u)\,x_1, \quad dx_1\,dx_3 \to x_1\,dx_1\,dy \tag{B.12}$$

with the limits on the u integration being zero and one. The result is

$$\Pi_2^{(b)}(k) = -2\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx_1\,dx_2\,dx_4\,dx_5\;\delta(1-x_1-x_2-x_4-x_5)$$
$$\times \int_0^1 d\xi\,\frac{\partial}{\partial\xi}\,\frac{x_1x_4x_5(x_2+x_5)^2(3x_1-x_4)}{\hat{U}^5}\,\log\left(1-\frac{k^2}{m^2}\,\frac{x_1x_4(x_2+x_5)}{\hat{U}(x_1+x_2+x_4)}\right). \tag{B.13}$$

where

$$\hat{U} = (x_1 + x_4)(x_2 + x_5) + \xi \, x_2 x_5 \, . \tag{B.14}$$

In (6.21) the change of variables

$$x_2 \to u x_2, \quad x_3 \to (1-u)\, x_2, \quad dx_2\, dx_3 \to x_2\, dx_2\, dy \tag{B.15}$$

leads to

$$\begin{aligned} \Pi_2^{(a)}(k) &= 2\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx_1\, dx_2\, dx_4\, dx_5\ \delta\,(1 - x_1 - x_2 - x_4 - x_5) \\ &\times \int_0^1 d\xi\, \frac{\partial}{\partial \xi}\, \frac{x_1 x_4 x_2 (x_2 + x_5)^2 (x_1 + x_4)}{\hat{U}^5}\, \log\left(1 - \frac{k^2}{m^2}\, \frac{x_1 x_4 (x_2 + x_5)}{\hat{U}(x_1 + x_2 + x_4)}\right). \end{aligned} \tag{B.16}$$

From (B.13) and (B.16), we have

$$\begin{aligned} \Pi_2^{(a)}(k) + \Pi_2^{(b)}(k) &= 2\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx_1\, dx_2\, dx_4\, dx_5\ \delta\,(1 - x_1 - x_2 - x_4 - x_5) \\ &\times \int_0^1 d\xi\, \frac{\partial}{\partial \xi}\, \frac{x_1 x_4 (x_2 + x_5)^2 \left[(x_1 + x_4)(x_2 - x_5) - 2\, x_5 (x_1 - x_4)\right]}{\hat{U}^5} \\ &\times \log\left(1 - \frac{k^2}{m^2}\, \frac{x_1 x_4 (x_2 + x_5)}{\hat{U}(x_1 + x_2 + x_4)}\right) . \end{aligned} \tag{B.17}$$

The second term in the square brackets clearly gives no contribution since x_1 and x_4 appear symmetrically everywhere else. The remainder can be written in the form

$$2\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx_1\, dx_2\, dx_4\, dx_5\, \delta(1-x_1-x_2-x_4-x_5)$$

$$\times \int_0^1 d\xi\, \frac{\partial}{\partial \xi}\, \frac{x_1 x_2 x_4 (x_2+x_5)^2 (x_1+x_4)}{\hat{U}^5} \log\left(\frac{\dfrac{x_1 x_4 (x_2+x_5)}{\hat{U}(x_1+x_2+x_4)} - \dfrac{m^2}{k^2}}{\dfrac{x_1 x_4 (x_2+x_5)}{\hat{U}(x_1+x_5+x_4)} - \dfrac{m^2}{k^2}}\right). \tag{B.18}$$

The subtraction is no longer necessary to make this integral convergent in x_2 and x_5 and we can examine the $\xi = 1$ and the $\xi = 0$ terms separately. The first is seen to be convergent when $(-k^2/m^2) \to \infty$ while the second becomes logarithmically divergent in x_1 and x_4 when $(-k^2/m^2) \to \infty$. Thus for $-k^2 >> m^2$,

$$\Pi_2^{(a)}(k) + \Pi_2^{(b)}(k) \approx -2\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx_1\, dx_2\, dx_4\, dx_5\, \delta(1-x_1-x_2-x_4-x_5)$$

$$\times \frac{x_1 x_2 x_4}{(x_1+x_4)^4 (x_2+x_5)^3} \log\left(\frac{\dfrac{x_1 x_4}{(x_1+x_4)(x_1+x_4+x_2)} - \dfrac{m^2}{k^2}}{\dfrac{x_1 x_4}{(x_1+x_4)(x_1+x_4+x_5)} - \dfrac{m^2}{k^2}}\right). \tag{B.19}$$

This integral can be evaluated by the type of variable substitution used above and the result is (6.27e).

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