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The Notions of Mass in Gravitational and Particle Physics

A dissertation presented

by

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to The Department of Physics

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in the field of

Physics

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ABSTRACT

It is presently thought that the mass of all of the elementary particles is determined by the Higgs field. This scalar field couples directly into the trace of the energy momentum tensor of the elementary particles. The attraction between two or more masses arises from the exchange of gravitational quantum particles of spin 2, called gravitons. The gravitational field couples directly into the energy momentum tensor. Then there is a close connection between the Higgs field, that originates the mass, and the gravitational field that dictates how the masses interact. Our purpose in this thesis is to discuss this close connection in terms of fundamental definitions of inertial and gravitational masses. On a practical level we explore two properties of mass from the viewpoint of coupling into the Higgs field: (i) The coupling of the both the Higgs and gravity to the energy-pressure tensor allows for the decay of the Higgs particle into two gravitons. We use the self energy part of the Higgs propagator to calculate the electromagnetic, weak, fermionic and gravitational decay rate of the Higgs particle. We show that the former process appears to dominate the other decay modes. Since the gravitons are detectable with virtually zero probability, the number of Higgs particles with observable decay products will be much less than previously expected. (ii) Some new experimental results seem to indicate that the mass of the heavy elementary particles like the Z, W^+, W^- and especially the top quark, depends on the particle environment in which these particles are produced. The presence of a Higgs field due to neighboring particles could be responsible for induced mass shifts. Further measurements of mass shift effects might give an indirect proof of the Higgs particle. Such can be in principle done by re-analyzing some of the production data $e^+e^- \rightarrow ZZ$ (or W^+W^-) already collected at the LEP experiment. About the physical property of the top quark, it is too early to arrive at any conclusion. In the foreseeable future, there will be more extended top quark production statistics from the Tevatron accelerator so that the mass shift hypothesis can be experimentally probed.

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Chapter 1

Introduction

Notions of inertial and gravitational mass have been fundamental in physics since the time of Newton first introduced both concepts. Presently the manner in which notions of gravitational mass enter into contemporary physical theory is via Einstein's general relativistic theory of gravity in which the gravitational field describes the metric geometry of the space time. Inertial mass of elementary particles enters into Standard Model physical theory is via the Higgs field. For non-relativistic (slowly moving massive) particles the inertial and gravitational masses are equal to the high degree of accuracy, but this rule is not absolute. In detail, the source of the Higgs field particle is the trace of the stress (energy-pressure) tensor $T = T^{\mu}_{\mu}$. While there have been a considerable number of experiments checking the theory of gravity, and thereby sources of gravitational mass, so far there has no direct experimental confirmation of that the Higgs field as the source of elementary particle inertial masses. The attempts at verification consists of high energy experimental searches for an observable Higgs particle.

The purpose of this thesis is to explore theoretically the new evidentiary signatures for the Higgs hunting experimentalists probe:

- 1. Since the source of both gravitational and inertial mass come from the stress tensor it follows that the coupling between the Higgs particle and gravitons should be particularly strong. A process such the Higgs to two graviton decay modes appears to dominate the width of the Higgs particle once it is produced. If such proves to be the case, then one would have to detect the Higgs by the "missing four momentum". Recall that Z particle decays into a neutrino anti-neutrino pair also has such "missing four momenta". The Higgs to two graviton decay rate will be computed in detail.
- 2. The Higgs is the source of the inertial mass. If two heavy particle are very closely situated together, then the presence of one of the particles shifts the mass of the other and vice-versa. It is possible (for example) that the mass of a Z or W boson or a top quark can be shifted by the fact that they can be produced in pairs. Mass shift will be computed in details.

So far there is experimental evidence for any scalar field producing inertial masses, although there have been and continue to be extensive high energy experimental programs searching for experimental confirmation of the Higgs particle.

To develop the above concepts, we will explain in Chapt.2 how in a classi-

cal limit it is possible to define a Newtonian theory of gravity starting from an action principle. Based on the principle of equivalence, the non-relativistic gravitational field due to many massive particles is a linear superposition of the fields due to each of the masses taken separately. Applying to the non-relativistic gravitational case, the same superposition principle conventional to the electromagnetic case, it is possible to write the gravitational Lagrangian as quadratic in the gravitational field. (ii) The most general local rotationally invariant quadratic scalar formed from the vector gravitational is $|\mathbf{g}|^2$. The transition from the classic scenario to the quantum non-relativistic one will be done studying the solution of a massive particle in a linear potential. Because the solution is dependent on the mass of the particle we will prove the principle of equivalence does not hold in a quantum mechanical system. In the final part of Chapt.2 we will derive the Einstein's field equation and point out the connection between the space-time curvature and the trace of the matter energy momentum tensor. In Chapt.3 we will explain why the Higgs particle couples to the trace of the stress tensor. We will derive a standard technique to calculate the decay rate of the Higgs into a particle anti-particle pair. For example we will calculate the decay rate for an Higgs particle originating a fermion anti-fermion pair, two spin one bosons, such as the photon, the W^{\pm} and the Z^{0} . We will also explain why the $H \to gg$ is the predominant decay channel in the Standard Model. In the Chapt.4 we will explain how the theory of final state interaction will be a prove to determine or not the presence of a mean Higgs field. Final state interaction is a theory usually employed in nuclear physics. The enhancement or the suppression of certain precesses are explain as an effect of the Coulomb potential between the nucleons and the leptons inside the nucleus. To be able to reproduce the already known Coulomb result starting from an action principle will permit to develop a technique to calculate the final state interaction due by the Higgs exchange, for example between two electro-weak bosons or the gluon exchange between two quarks. We will show also how to obtain a mass shift in the massive particle pairs $(ZZ, W^+W^-, t\bar{t})$ produced near threshold. This will allow to explain the reason why the mass of a particle depends by the presence (very closely) of another.

Chapter 2

General Relativity

2.1 Inertial and Gravitational Mass

In this section we consider the concept of inertial and gravitational mass in classical non-relativistic theory. The inertial mass occurs originally in Newton's second law of motion for a test particle.

$$\mathbf{F} = M_{\text{inertial}} \mathbf{a}.$$
 (2.1)

The gravitational force (known as the weight) is a special force normally written as

$$\mathbf{W} = M_{\text{gravitational}} \mathbf{g} \tag{2.2}$$

where \mathbf{g} is the gravitational acceleration field. The equality of inertial and gravitational mass is normally assumed; i.e.

$$M_{\rm gravitational} = M_{\rm inertial} = M. \tag{2.3}$$

The gravitation field itself is derived from a potential

$$\mathbf{g} = -\mathbf{grad}\Phi\tag{2.4}$$

so that the motion of a particle in a gravitational field may be described by the non-relativistic Lagrangian

$$L(\mathbf{v}, \mathbf{r}) = \frac{1}{2}M|\mathbf{v}|^2 - M\Phi(\mathbf{r}).$$
(2.5)

The particle orbit obeys an equation of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{v}} \right) = \left(\frac{\partial L}{\partial \mathbf{r}} \right) \quad \Rightarrow \quad \dot{\mathbf{v}} = \mathbf{g}$$
(2.6)

which is independent of the mass M. This is an expression of the *principle* of equivalence implicit in Eq.(2.3).

For several masses the interaction between masses and the gravitational field may be described by the Lagrangian

$$L_{\rm int} = -\sum_{a} M_a \Phi(\mathbf{r}_a) = -\int \rho(\mathbf{r}) \Phi(\mathbf{r}) d^3 \mathbf{r}$$
(2.7)

wherein the mass density is defined as

$$\rho(\mathbf{r}) = \sum_{a} M_a \delta(\mathbf{r} - \mathbf{r}_a).$$
(2.8)

To obtain the complete Newtonian field equations for \mathbf{g} , it is necessary to consider the Lagrangian for gravitational field; i.e.

$$L_{\text{field}} = -\frac{1}{8\pi G} \int |\mathbf{g}|^2 d^3 \mathbf{r}.$$
 (2.9)

Eq.(2.9) may be understood as follows: (i) By the principle of equivalence, the gravitational field due to many massive particles must be a linear superposition of the fields due to each of the masses taken separately. The gravitational field Lagrangian must thereby be quadratic in the gravitational field. (ii) The most general local rotationally invariant quadratic scalar formed from the vector gravitational acceleration is $|\mathbf{g}|^2$. (iii) The sign in Eq.(2.9) is chosen for *attractive* gravitational forces. (iv) The coupling strength must be experimentally determined; i.e.

$$G \approx 6.674 \times 10^{-8} \left[\frac{\mathrm{cm}^3}{\mathrm{gm sec}^2} \right].$$
 (2.10)

The classical action principle for the gravitational field equations reads

$$\delta S = \int (\delta L_{\text{field}} + \delta L_{\text{int}}) dt = 0 \qquad (2.11)$$

From Eq.(2.9), we have

$$\delta L_{\text{field}} = -\frac{1}{4\pi G} \int \mathbf{g} \cdot \delta \mathbf{g} d^3 \mathbf{r} = \frac{1}{4\pi G} \int \mathbf{g} \cdot \mathbf{grad} \delta \Phi d^3 \mathbf{r}.$$
 (2.12)

Integrating Eq.(2.12) by parts yields

$$\delta L_{\text{field}} = -\frac{1}{4\pi G} \int (div \ \mathbf{g}) \delta \Phi d^3 \mathbf{r}.$$
 (2.13)

From Eq.(2.7),

$$\delta L_{\rm int} = -\int \rho \delta \Phi d^3 \mathbf{r}. \qquad (2.14)$$

Eqs.(2.11), (2.13) and (2.14) imply

$$\delta S = -\int \int \left\{ \frac{1}{4\pi G} div \ \mathbf{g} + \rho \right\} \delta \Phi d^3 \mathbf{r} dt = 0.$$
 (2.15)

The full Newtonian gravitational field equations are thereby

$$div \mathbf{g} = -4\pi G\rho,$$

$$curl \mathbf{g} = 0$$

$$\mathbf{g} = -\mathbf{grad}\Phi.$$
(2.16)

Gravitational static forces then follow from

$$\Delta \Phi = 4\pi G \rho,$$

$$\Phi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}') d^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|},$$
(2.17)

yielding the Newtonian two body gravitational potentials

$$U_{ab} = -G\left(\frac{M_a M_b}{r_{ab}}\right)$$
$$U = \sum_{a < b} U_{ab}.$$
(2.18)

The important point is that the form of the Newtonian gravitational interaction follows from Galilean invariance, the equality of gravitational and inertial mass, the superposition principle for gravitational forces and the principle of equivalence.

2.2 Non-relativistic Quantum Theory

While the equality of gravitational and inertial mass in Eq.(2.3) holds in both the classical and quantum mechanical versions of non-relativistic gravitational dynamics, the principle of equivalence does not hold true in the quantum mechanical theory. To see what is involved one may consider the problem of a Newtonian gravity "bouncing ball". The Hamiltonian of a bouncing ball has the form

$$H = \frac{p_z^2}{2M} + M\Phi(z),$$

$$\Phi(z) = gz, \text{ if } z > 0,$$

$$\Phi(z) = \infty, \text{ if } z \le 0.$$
(2.19)

Classically a ball falls from a height h, hits the ground with an elastic bounce and rises back to a height h. The classical frequency for this periodic process is

$$\omega_c = \pi \sqrt{\frac{g}{2h}} \tag{2.20}$$

independently of the mass of the ball. The mass independence is an expression of the classical principle of equivalence. Note that

$$\omega_c = \pi g \sqrt{\frac{M}{2E}}.$$
(2.21)

For the quantum mechanical bouncing ball problem, for z > 0 and $\psi(0^+) = 0$, we have

$$H\phi(z) = \left\{ -\frac{\hbar^2}{2M} \left(\frac{d}{dz}\right)^2 + Mgz \right\} \psi(z) = E\Psi(z) = Mgh\psi(z).$$
(2.22)

Employing the change of variable

$$\xi = 1 - \left(\frac{z}{h}\right) = 1 - \left(\frac{Mgz}{E}\right),\tag{2.23}$$

we have

$$\beta^{2} = \left(\frac{\hbar^{2}}{2M^{2}h^{3}g}\right) = \left(\frac{\hbar\omega_{c}}{\pi E}\right)^{2},$$

$$\beta^{2} = \frac{\hbar^{2}g^{2}M}{2E^{3}},$$

$$-\beta^{2}\psi''(\xi) = \xi\psi(\xi),$$

$$\psi(1) = 0 \text{ and } \xi < 1.$$
(2.24)

The solution of Eq.(2.24) in terms of the Airy function is

$$\psi(\xi) = Ai\left(-\frac{\xi}{\beta}\right). \tag{2.25}$$

The energy levels are thereby given by the roots of the Airy function [26]

$$Ai\left(-\frac{1}{\beta_{n}}\right) = 0, \quad n = 1, 2, 3, 4, \dots,$$

$$E_{n} = \left[\frac{\hbar^{2}g^{2}M}{2\beta_{n}^{2}}\right]^{1/3},$$

$$E_{n} \approx \frac{M^{1/3}}{2} \left[3\pi\hbar g\left(n - \frac{1}{4}\right)\right]^{2/3} \quad n \gg 1,$$
(2.26)

which depend explicitly on mass. The principle of equivalence does not work for the quantum mechanical case.

Several experiments investigate on the principle of equivalence [1, 6], among them we consider the one performed by R. Colella, A. W. Overhauser, and S. A. Werner (COW-experiment) [2, 3, 4, 5]. This experiment is essentially a thermal-neutron interferometry. A beam of neutrons is diffracted by a Si glass. The two coherent beams move along different paths (ABCEF and ABDEF) and are then detected by a He^3 counter at F. Fig. 2.1 represents a schematic drawing of the COW experimental neutron interferometer.

The COW-experiment analysis employs the Hamiltonian

$$H(\mathbf{p}, \mathbf{r}) = \left(\frac{|\mathbf{p}|^2}{2M} + M\mathbf{g} \cdot \mathbf{r}\right).$$
(2.27)



Figure 2.1: Schematic drawing of the COW experiment interferometer. Rotating the apparatus along the AD axis allows to study the relative role of the two paths.

2.3 Classical Relativistic Particle Mechanics

A relativistic event is specified by where \mathbf{r} and when t the event takes place. In flat space-time, a moving particle is endowed with an internal clock which reads an invariant proper time between two neighboring events along the particle path

$$-c^2 d\tau^2 = |d\mathbf{r}|^2 - c^2 dt^2.$$
(2.28)

In terms of flat inertial space-time coordinates $x^{\mu} = (x^1, x^2, x^3, x^0) = (\mathbf{r}, ct)$ Eq.(2.28) reads

$$-c^2 d\tau^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu}.$$
 (2.29)

In an non-inertial system of reference in flat space-time or in curved spacetime where there is a true gravitational field, the interval is defined by a quadratic form with a space-time metric

$$-c^2 d\tau^2 = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}.$$
 (2.30)

A massive particle has a four vector velocity along its path defined as

$$v^{\mu} = \frac{dx^{\mu}}{d\tau}$$
 where $g_{\mu\nu}(x)v^{\mu}v^{\nu} = -c^2$. (2.31)

The classical action for a moving point particle has the form

$$S = -Mc^{2} \int d\tau$$

$$S = \int \mathcal{L}(v, x) d\tau,$$

$$\mathcal{L}(v, x) = \frac{1}{2} M \left(g_{\mu\nu}(x) v^{\mu} v^{\nu} - c^{2} \right).$$
(2.32)

The variational principle $\delta S = 0$ yields the Lagrangian equations of motion

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial v^{\mu}} = \frac{\partial \mathcal{L}}{\partial x^{\mu}},$$

$$\frac{d}{d\tau} (g_{\mu\sigma} v^{\sigma}) = \frac{1}{2} \partial_{\mu} g_{\lambda\sigma} v^{\lambda} v^{\sigma},$$

$$g_{\mu\sigma} \frac{dv^{\sigma}}{d\tau} = \frac{1}{2} [\partial_{\mu} g_{\lambda\sigma} - \partial_{\lambda} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\lambda}] v^{\lambda} v^{\sigma}.$$
(2.33)

With the matrix $\{g^{\mu\nu}\}$ defined as the inverse of the matrix $\{g_{\mu\nu}\}$ and the gravitational connection coefficients defined as

$$\Gamma^{\mu}_{\lambda\sigma} = \frac{1}{2} g^{\mu\nu} \left(\partial_{\lambda} g_{\nu\sigma} + \partial_{\sigma} g_{\nu\lambda} - \partial_{\nu} g_{\lambda\sigma} \right), \qquad (2.34)$$

the classical geodesic equations of motion of a classical particle in a gravitational field read

$$\frac{dv^{\mu}}{d\tau} + \Gamma^{\mu}_{\lambda\sigma} v^{\lambda} v^{\sigma} = 0.$$
(2.35)

The four momentum of the particle $p_{\mu} = (\mathbf{p}, -E/c)$ is given by

$$p_{\mu} = \frac{\partial \mathcal{L}}{\partial v^{\mu}} = M g_{\mu\nu}(x) v^{\nu},$$

$$v^{\mu} = \frac{g^{\mu\nu}(x) p_{\nu}}{M}.$$
(2.36)

From Eqs.(2.31) and (2.36) one finds

$$g^{\mu\nu}(x)p_{\mu}p_{\nu} + M^2c^2 = 0.$$
(2.37)

More generally, the Hamiltonian corresponding to the Lagrangian in Eq.(2.32)is defined as

$$\mathcal{H}(p,x) = v^{\mu}p_{\mu} - \mathcal{L}(v,x),$$

$$\mathcal{H}(p,x) = \frac{(g^{\mu\nu}(x)p_{\mu}p_{\nu} + M^{2}c^{2})}{2M}.$$
 (2.38)

The single particle action and classical equations of motion now read

$$S = \int (p_{\mu}dx^{\mu} - \mathcal{H}(p,x)d\tau),$$

$$\delta S = 0$$

$$\frac{dx^{\mu}}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_{\mu}}$$

$$\frac{dp_{\mu}}{d\tau} = -\frac{\partial \mathcal{H}}{\partial x^{\mu}}.$$
(2.39)

The classical definition of the particle mass comes directly from the *constraint* Eq.(2.37), i.e.

$$2M\mathcal{H}(p,x) = g^{\mu\nu}p_{\mu}p_{\nu} + M^2c^2 = 0.$$
(2.40)

The proper time $d\tau$ enters in Eq.(2.39) as a Lagrange multiplier for the constraint Eq.(2.40).

The particle proper time in Eq.(2.30) has two classical solutions

$$cd\tau_{\pm} = \pm \sqrt{-g_{\mu\nu}dx^{\mu}dx^{\nu}}.$$
 (2.41)

A classical particle moves forward in proper time $d\tau_+ > 0$ while a classical anti-particle moves backward in proper time $d\tau_- < 0$. In the presence of an electromagnetic field,

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \qquad (2.42)$$

the Lagrangian and Hamiltonian for a particle with mass M and charge e

have, respectively, the forms

$$\mathcal{L}(v,x) = \frac{1}{2}M\left(g_{\mu\nu}(x)v^{\mu}v^{\nu} - c^{2}\right) + \left(\frac{e}{c}\right)v^{\mu}A_{\mu}(x),$$

$$\mathcal{H}(p,x) = \frac{1}{2M}\left[g^{\mu\nu}(x)\left(p_{\mu} - \frac{e}{c}A_{\mu}(x)\right)\left(p_{\nu} - \frac{e}{c}A_{\nu}(x)\right) + M^{2}c^{2}\right](2.43)$$

Either of the sets of equations of motion

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial v^{\mu}} \right) = \left(\frac{\partial \mathcal{L}}{\partial x^{\mu}} \right),$$
or
$$\frac{dx^{\mu}}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_{\mu}} \quad \text{and} \quad \frac{dp_{\mu}}{d\tau} = -\frac{\partial \mathcal{H}}{\partial x^{\mu}},$$
(2.44)

yields the Lorentz force on a charge equation of motion

$$Ma^{\mu} = M \left[\frac{dv^{\mu}}{d\tau} + \Gamma^{\mu}_{\lambda\sigma} v^{\lambda} v^{\sigma} \right],$$

$$Ma^{\mu} = \frac{e}{c} F^{\mu\nu} v_{\nu}.$$
(2.45)

The Hamilton-Jacobi equation for a classical particle in an electromagnetic field allows for the solution of the Lorentz force on a charge Eq.(2.45) in the following form: (i) There is a velocity field $v^{\mu}(x)$ obeying the rule

$$Mv_{\mu}(x) = \partial_{\mu}S(x) - \frac{e}{c}A_{\mu}(x), \qquad (2.46)$$

wherein the action function S(x) obeys the constraint Hamilton-Jacobi equation

$$\mathcal{H}(p = \partial S(x), x) = 0$$
 i.e. $g^{\mu\nu}(x)v_{\mu}(x)v_{\nu}(x) + c^2 = 0.$ (2.47)

The classical particle orbit may be found from the first order differential equation

$$\frac{dx^{\mu}}{d\tau} = v^{\mu}(x). \tag{2.48}$$

In reality, there are two sets of orbits that one may compute employing the two velocity fields, both obeying the Hamilton-Jacobi equation

$$v_{\pm}^{\mu}(x) = \pm v^{\mu}(x),$$

$$g_{\mu\nu}(x)v_{\pm}^{\mu}(x)v_{\pm}^{\nu}(x) = -c^{2},$$

$$\frac{dx_{\pm}^{\mu}}{d\tau_{\pm}} = v_{\pm}^{\mu}(x).$$
(2.49)

One path is a charged particle moving forward in time and the conjugate path is an anti-particle moving backward in time.

Both types of solutions (particle and anti-particle) must in reality exist. To see this most clearly, consider the particle Hamiltonian in a given system of coordinates

$$\{x^{\mu}\} = (x^{1}, x^{2}, x^{3}, x^{0}) = (r^{1}, r^{2}, r^{3}, ct) = (\mathbf{r}, ct),$$

$$\{p_{\mu}\} = (p_{1}, p_{2}, p_{3}, p_{0}) = (p_{1}, p_{2}, p_{3}, -E/c) = (\mathbf{p}, -E/c).$$
(2.50)

The constraint equation

$$2M\mathcal{H}(p,x) = g^{\mu\nu}(x) \left(p_{\mu} - \frac{e}{c}A_{\mu}(x)\right) \left(p_{\nu} - \frac{e}{c}A_{\nu}(x)\right) + M^{2}c^{2} = 0 \quad (2.51)$$

is quadratic equation in the energy E and thereby has two solutions

$$E = H_{\pm}(\mathbf{p}, \mathbf{r}, t) \tag{2.52}$$

and the two sets of solutions (particle and anti-particle) follow from the two Hamiltonians

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H_{\pm}(\mathbf{p}, \mathbf{r}, t)}{\partial \mathbf{p}},$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H_{\pm}(\mathbf{p}, \mathbf{r}, t)}{\partial \mathbf{r}}.$$
(2.53)

Two examples will suffice: (i) For a charged particle in *flat* space-time in a vector potential $A_{\mu} = (\mathbf{A}, -\Phi)$ the two Hamiltonians are

$$H_{\pm}(\mathbf{p},\mathbf{r},t) = e\Phi(\mathbf{r},t) \pm c\sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r},t)\right)^2 + M^2c^2} .$$
(2.54)

With e = -|e| as the electronic charge, $H_+(\mathbf{p}, \mathbf{r}, t)$ is the electron Hamiltonian and $H_-(\mathbf{p}, \mathbf{r}, t)$ is the positron Hamiltonian. (ii) For an uncharged particle moving in the gravitational field of the Sun,

$$r_{s} \equiv \frac{2GM_{Sun}}{c^{2}} ,$$

$$c^{2}d\tau^{2} = \left(1 - \frac{r_{s}}{r}\right)c^{2}dt^{2} - \frac{dr^{2}}{1 - (r_{s}/r)} - r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2}), \quad (2.55)$$

the particle Hamiltonian is

$$H(p_r, p_{\theta}, p_{\phi}, r, \theta, \phi) = c\left(1 - \frac{r_s}{r}\right) \times$$

$$\sqrt{p_r^2 + \frac{1}{r^2 [1 - (r_s/r)]} \left(p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right) + M^2 c^2} .$$
 (2.56)

The orbit of a neutral massive particle moving in the gravitational field of the Sun may be found from

$$\dot{r} = \frac{\partial H}{\partial p_r}, \qquad \dot{\theta} = \frac{\partial H}{\partial p_{\theta}}, \qquad \dot{\phi} = \frac{\partial H}{\partial p_{\phi}},$$
$$\dot{p}_r = -\frac{\partial H}{\partial r}, \qquad \dot{p}_{\theta} = -\frac{\partial H}{\partial \theta}, \qquad \dot{p}_{\phi} = -\frac{\partial H}{\partial \phi}. \qquad (2.57)$$

The relativistic orbit in space depends on the mass M. In the limit $M \to 0$ we have the photon Hamiltonian

$$H_{\gamma}(p_r, p_{\theta}, p_{\phi}, r, \theta, \phi) = c \left(1 - \frac{r_s}{r}\right) \sqrt{p_r^2 + \frac{1}{r^2 [1 - (r_s/r)]} \left(p_{\theta}^2 + \frac{p_{\phi}^2}{\sin^2 \theta}\right)} .$$
(2.58)

The Hamiltonian Eq.(2.58) describes the photon orbit in the neighborhood of the Sun's gravitational field, i.e. the bending of light around the Sun.

2.4 Quantum Relativistic Particle Mechanics

The presence of a gravitational field changes the curvature of space-time. The classical motion of a free particle no longer corresponds to a space-time straight line. The classical curve corresponds to a *geodesic*. To illustrate the quantum motion of a spin-zero particle let us consider the Klein Gordon equation and a scalar field $\phi(x)$. It reads

$$\left(-\Box_g + \kappa^2\right)\phi(x) = 0. \tag{2.59}$$

In Eq.(2.59) $\kappa = Mc/\hbar$. The presence of the gravitational field appears directly in the definition of the d' Alambertian operator

$$\Box_g \phi(x) = \frac{1}{\sqrt{-g(x)}} \partial_\lambda \left(\sqrt{-g(x)} g^{\lambda\sigma}(x) \partial_\sigma \phi(x) \right), \qquad (2.60)$$

and $g \equiv det ||g_{\mu\nu}||$. The spin-zero Green's function in space-time obeys an equation of the form

$$\left(-\Box_g + \kappa^2\right) G(x, y) = \delta^4(x, y),$$

$$\mathcal{H}\left(p = -i\hbar \frac{\partial}{\partial x}, x\right) G(x, y) = \left(\frac{\hbar^2}{2M}\right) \delta^4(x, y).$$
(2.61)

where the delta function is defined with respect to the invariant space-time volume $d\Omega_x = \sqrt{-g} d^4 x$; i.e.

$$f(x) = \int \delta^4(x, y) f(y) d\Omega_y.$$
(2.62)

The propagator G(x, y) represents the amplitude to go in space-time from the point x to the point y; In detail, for $x \in \Omega$

$$\begin{split} \phi(x) &= \int_{\Omega} \delta^4(x, y) \phi(y) d\Omega_y, \\ \phi(x) &= \int_{\Omega} \left(-\Box_{g(x)} + \kappa^2 \right) G(x, y) \phi(y) d\Omega_y, \end{split}$$

$$\phi(x) = \int_{\Omega} \left\{ \left(-\Box_{g(y)} + \kappa^2 \right) G(x, y) \right\} \phi(y) d\Omega_y, \qquad (2.63)$$

which, together with $(-\Box_{g(y)} + \kappa^2)\phi(y) = 0$ yields

$$\phi(x) = \int_{\Omega} \left\{ \left(-\Box_{g(y)} G(x, y) \right) \phi(y) + G(x, y) \left(\Box_{g(y)} \phi(y) \right) \right\} d\Omega_y, \qquad (2.64)$$

where Ω is a space-time region with a boundary "surface" volume $\Sigma = \partial \Omega$. Converting Eq.(2.64) into an integral over the boundary

$$\phi(x) = \oint_{\Sigma} \left\{ G(x,y) \frac{\partial \phi(y)}{\partial y^{\mu}} - \phi(y) \frac{\partial G(x,y)}{\partial y^{\mu}} \right\} d^{3} \Sigma_{y}^{\mu}.$$
 (2.65)

From the values of the field ϕ and the field derivatives $\partial_{\mu}\phi$ on the boundary $\Sigma = \partial \Omega$ one may find from the Green's function the value of the field in the space-time region Ω .

If the Feynman boundary conditions are used for G(x, y), then $\phi(x \in \Omega)$ is determined in part by boundary conditions in the future on Σ_+ and partly by boundary conditions in the past on Σ_- . Thus, anti-particles in Ω arrive from the future Σ_+ and particles in Ω arrive from the past Σ_- . The Green's function with the proper boundary conditions to describe both particles and anti-particles is written as

$$G(x,y) = \left(\frac{1}{-\Box_g + \kappa^2 - i0^+}\right) \delta^4(x,y),$$

$$G(x,y) = \left(i \int_0^\infty e^{-is(\kappa^2 - \Box_g)} ds\right) \delta^4(x,y).$$
(2.66)

Introducing the proper time $\tau = (2Ms/\hbar)$ yields

$$G(x,y) = \left(\frac{i\hbar}{2M} \int_0^\infty e^{-i\tau \mathcal{H}(p=-i\hbar\partial,x)/\hbar} d\tau\right) \delta^4(x,y),$$

$$G(x,y) = \left(\frac{i\hbar}{2M}\right) \int_0^\infty \langle x| e^{-i\tau \mathcal{H}(p,x)/\hbar} |y\rangle d\tau.$$
(2.67)

The formal Dirac notation is employed, which obeys

$$\int d\Omega_x |x\rangle \langle x| = 1,$$

$$\langle x|y\rangle = \delta^4(x,y).$$
(2.68)

The propagator is defined in Eq.(2.67) as an integral over all possible values of the proper time. For a quantum particle, the amplitude for going from xto y is given by the sum of the amplitudes along different paths with different proper times.

Eq.(2.67) cannot in general be solved. Here we will review an expression for the propagator in a flat space-time metric. Eq.(2.60) then reads

$$\Box \phi(x) = \eta^{\lambda \sigma} \partial_{\lambda} \partial_{\sigma} \phi(x) \tag{2.69}$$

where the tensor $\eta^{\mu\nu}$ is diagonal and its elements are (1, 1, 1, -1). In flat space-time there is translational invariance. The propagator has the form

$$G(x,y) = G(x-y).$$
 (2.70)

Using the completeness relation in $|k\rangle$ space

$$\int |k\rangle \langle k| \frac{d^4k}{(2\pi)^4} = 1 \tag{2.71}$$

for

$$\langle x|k\rangle = e^{ik \cdot x},\tag{2.72}$$

the propagator in Eq.(2.67) now reads

$$G(x-y) = \left(\frac{i\hbar}{32\pi^4 M}\right) \int d^4k \int_0^\infty d\tau \exp\left[-\frac{i\hbar(k^2+\kappa^2)\tau}{2M} + ik(x-y)\right].$$
(2.73)

The k-integrals are Gaussian. With z = y - x, the propagator is given by

$$G(z) = \left(\frac{M}{8\pi^2\hbar}\right) \int_0^\infty \frac{d\tau}{\tau^2} \exp\left[-\frac{iMc^2\tau}{2\hbar} + \frac{iMz^2}{2\hbar\tau}\right]$$
$$= \left(\frac{M}{8\pi^2\hbar}\right) \int_0^\infty \frac{d\tau}{\tau^2} \exp\left[-\left(\frac{iMz^2}{2\hbar\tau}\right)\left(\frac{c^2\tau^2}{z^2} - 1\right)\right]. \quad (2.74)$$

Employing the change of variables $t = \tau c / \sqrt{z^2}$ Eq. (2.74) reads

$$G(z) = \left(\frac{Mc}{8\pi^2\hbar\sqrt{z^2}}\right) \int_0^\infty \frac{dt}{t^2} \exp\left[-\left(\frac{iMc\sqrt{z^2}}{2\hbar}\right)\left(t - \frac{1}{t}\right)\right].$$
 (2.75)

Noting that the function $\exp[w(t-1/t)/2]$ is the generating function of the first order Bessel's functions

$$e^{w(t-1/t)/2} = \sum_{n=-\infty}^{+\infty} t^n J_n(w), \qquad (2.76)$$

the propagator 2.75 assumes the form

$$G(z) = \left(\frac{Mc}{8\pi^2\hbar\sqrt{z^2}}\right)\sum_{n=-\infty}^{+\infty}\int_0^\infty dt t^{n-2}J_n\left(-\frac{iMc\sqrt{z^2}}{\hbar}\right).$$
 (2.77)

The last equation is not very useful to understand the behavior of the propagator near $z^2 \approx 0$. Thus we prefer to use an other, equivalent representation

$$G(z) = 2i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot z}}{k^2 - K^2} = 2i\theta(z^0)G^+(z) - 2i\theta(-z^0)G^-(z), \quad (2.78)$$

where $\theta(z^0)$ is the Heaviside function. The functions $G^{\pm}(z)$ are defined

$$G^{+}(z) = \frac{1}{2} \left(\Delta(z) - i\Delta'(z) \right) G^{-}(z) = \frac{1}{2} \left(\Delta(z) + i\Delta'(z) \right).$$
(2.79)

The two functions are odd $(\Delta(x))$ and even $(\Delta'(z))$, but both satisfy

$$(\Box + K^2)\Delta(z) = 0 \tag{2.80}$$

$$(\Box + K^2)\Delta'(z) = 0. (2.81)$$

Let's focus on the function $\Delta(x)$

$$\Delta(z) = -\frac{i}{2(2\pi)^3} \int \frac{d^3 \mathbf{k}}{k^0} \left(e^{-ik \cdot z} - e^{ik \cdot z} \right)$$

= $-\frac{i}{(2\pi)^3} \int d^4 k \epsilon(k^0) \delta(k^2 - K^2) e^{-ik \cdot z}$ (2.82)

¹Only in this paragraph we use the metric with a signature equal to + - - -.

$$\epsilon(k^0) = \theta(k^0) - \theta(-k^0). \tag{2.83}$$

This function can be expressed as

$$\Delta(z) = \frac{1}{4\pi r} \frac{\partial}{\partial r} F(r, z^{0}), \text{ where } r = |\mathbf{z}|$$

$$F(r, z^{0}) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{k^{2} + K^{2}}} \cos(kr) \sin\left(\sqrt{k^{2} + K^{2}} z^{0}\right)$$

$$= J_{0} \left(K\sqrt{z_{0}^{2} - r^{2}}\right) \quad \text{for } z^{0} > r$$

$$= 0 \quad \text{for } -r < z^{0} < r$$

$$= -J_{0} \left(K\sqrt{z_{0}^{2} - r^{2}}\right) \quad \text{for } z^{0} < -r. \qquad (2.84)$$

We can write explicitly $\Delta(z)$ as

$$\Delta(z) = -\frac{1}{2\pi} \epsilon(z^0) \left(\delta(z^2) - \frac{K^2}{2} \theta(z^2) \frac{J_1(K\sqrt{z^2})}{K\sqrt{z^2}} \right), \qquad (2.85)$$

where the Bessel function $J_1(w)$ can be obtained from

$$J_0(w) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\beta \sin(w \cosh \beta)$$

$$J_1(w) = -\frac{\partial}{\partial w} J_0(w).$$
(2.86)

Near the light cone $(z^2 \approx 0)$ Eq. (2.85) reads

$$\Delta(z) = -\frac{1}{2\pi} \epsilon(z^0) \left(\delta(z^2) - \frac{K^2}{2} \theta(z^2) + \dots \right), \qquad (2.87)$$

this last equation shows that $\Delta(z)$ [9, 10, 11] has a delta function singularity

as well as a finite (jump) discontinuity on the light cone. Note that the delta function singularity is independent of the mass (that is central to the result of Higgs case in Chap.4 whereas the $\theta(z^2)$ depends upon the mass. Let's consider the other function $\Delta'(z)$. In the same fashion we can write

$$\begin{aligned} \Delta'(z) &= \frac{1}{4\pi r} \frac{\partial}{\partial r} F(r, z^0) \\ F'(r, z^0) &= -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{k^2 + K^2}} \cos(kr) \cos\left(\sqrt{k^2 + K^2} z^0\right) \\ &= N_0 \left(K\sqrt{z_0^2 - r^2}\right) \quad \text{for } |z^0| > r \\ &= -iH'_0 \left(iK\sqrt{z_0^2 - r^2}\right) \quad \text{for } r > |z^0|, \end{aligned}$$
(2.88)

where $N_n(w)$ are the Neuman functions

$$N_0(w) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} d\beta \cos(w \cosh\beta)$$
 (2.89)

and $H_n(w)$ are the Hankel functions. Near the light cone $(z^2 \approx 0)$ Eq. (2.87) reads

$$\Delta'(z) = -\frac{1}{2\pi^2} \left(P \frac{1}{z^2} - \frac{K^2}{2} \ln\left(\frac{\gamma}{2}\sqrt{K^2|z^2|}\right) + \frac{K^2}{4} + \dots \right), \qquad (2.90)$$

where P denotes the principal part. We must note that the function $\Delta'(z)$ does not vanish outside the light cone.

The difference between the motion of a classical and a quantum particle

can be described in terms of the action in the exponential of Eq.(2.73); i.e.

$$S(x-y) = \frac{M}{2} \left(\frac{(x-y)^2}{\tau^2} - c^2 \right) \tau.$$
 (2.91)

For a classical particle

$$-c^{2}\tau^{2} = (x-y)^{2},$$

$$S_{classical}(x-y) = -Mc^{2}\tau = -Mc\sqrt{-(x-y)^{2}}.$$
(2.92)

For a quantum particle Eq.(2.92) does not hold true. There are quantum fluctuations about the classical path so that $c\tau \neq \sqrt{-(x-y)^2}$ with amplitude $\exp(S(x-y)/\hbar)$. The strength of this amplitude depends on the mass M so that the principle of equivalence no longer holds true. For example, the amplitude for a particle going backward in time to switch to going forward in time (quantum particle anti-particle pair production) is very strongly dependent upon the mass.

2.5 Einstein's Field Equation

The set of equations describing a system, that from now on we consider as *matter*, and a gravitational field are generally referred to as Einstein's equations. These express the relation between the gravitational Ricci tensor and the matter energy momentum tensor.

To obtain these equations we write the total action as the sum of the gravi-
tational and matter parts.

$$S = S_g + S_{matter}.$$
 (2.93)

The action of a gravitational field, S_g , must be expressed as a scalar integral, and the integrand function must contain the derivatives of $g_{\mu\nu}$ not higher than the first order. The usual choice is to write the gravitational action as

$$S_g = \frac{c^3}{16\pi G} \int d\Omega_x R, \qquad (2.94)$$

where R is the curvature of the space-time. The curvature is normally derived contracting the Ricci tensor,

$$R = g^{\mu\nu} R_{\mu\nu}$$

= $g^{\mu\nu} \left(\frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\mu\nu} \Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\sigma}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} \right).$ (2.95)

Although the curvature contains derivatives of the second order in $g_{\mu\nu}$, S_g can be expressed as

$$S_g = \frac{c^3}{16\pi G} \int d\Omega_x R$$

= $\frac{c^3}{16\pi G} \int d\Omega_x F + \frac{c^3}{16\pi G} \delta \int d^4x \frac{\partial(\sqrt{-g}w^{\sigma})}{\partial x^{\sigma}}$ (2.96)

where the function F contains only the first derivatives of $g_{\mu\nu}$.

Using the Gauss theorem, the last integral can be evaluated on the hypersurface where, according to the least action principle, the variation of the field is just equal to zero.

$$\frac{c^3}{16\pi G}\delta \int d^4x \frac{\partial(\sqrt{-g}w^{\sigma})}{\partial x^{\sigma}} = \frac{c^3}{16\pi G}\delta \int_{\partial\Sigma} d\Sigma_{\sigma}\sqrt{-g}w^{\sigma} = 0$$
(2.97)

The variation of the gravitational action is hence

$$\delta S_g = \frac{c^3}{16\pi G} \delta \int d\Omega_x F$$

= $\frac{c^3}{16\pi G} \delta \int d\Omega_x R - \frac{c^3}{16\pi G} \delta \int d^4x \frac{\partial(\sqrt{-g}w^{\sigma})}{\partial x^{\sigma}}$
= $\frac{c^3}{16\pi G} \delta \int d\Omega_x R.$ (2.98)

The Einstein equations are derived setting equal to zero the variation of the total action S respect to $g^{\mu\nu}$.

$$\delta S = \delta (S_g + S_{matter})$$

= $\frac{c^3}{16\pi G} \delta \int d\Omega_x R + \delta S_{matter} = 0.$ (2.99)

The variation of the matter action with respect to the metric tensor $g^{\mu\nu}$ allows us to derive the energy-momentum tensor $T_{\mu\nu}$, defined as a constant times the functional derivative of the S_{matter} .

$$T_{\mu\nu} = -2c \frac{\delta S_{matter}}{\delta g^{\mu\nu}} \tag{2.100}$$

or simply

$$\delta S_{matter} = -\frac{1}{2c} \int d\Omega_x T_{\mu\nu} \delta g^{\mu\nu} = \frac{1}{2c} \int d\Omega_x T^{\mu\nu} \delta g_{\mu\nu}.$$
 (2.101)

The variation of the gravitational action reads

$$\delta S_g = \frac{c^3}{16\pi G} \delta \int d\Omega_x R$$

= $\frac{c^3}{16\pi G} \int d^4 x \delta(\sqrt{-g} g^{\mu\nu} R_{\mu\nu})$
= $\frac{c^3}{16\pi G} \int d\Omega_x \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu}.$ (2.102)

Using this result and Eq (2.101) we write.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} = G_{\mu\nu}, \qquad (2.103)$$

wherein $G_{\mu\nu}$ is the Einstein tensor. Because $T_{\mu\nu}$ is a symmetric 4 × 4 tensor and R is defined in Eq. (2.95) as the product of the partial derivatives of the metric, the Einstein equations are a set of 10 coupled elliptic-hyperbolic nonlinear partial differential equations. Their solutions are generally not trivial, but they can be solved in a *weak* gravitational field,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$
 (2.104)

where $|h_{\mu\nu}| \ll 1$, and the equations become linear. The general solution of the Einstein equations is the metric of the space-time.

The relation between the curvature and the energy-momentum tensor trace

is explicit once we multiply both sides of Eq. (2.103) by $g^{\mu\nu}$

$$R = -\frac{8\pi G}{c^4}T.$$
 (2.105)

To summarize this section we say that the Einstein equations describe how, in a gravitational field, a particle curves in space and how the gravitational field stretches or squeezes (deforms) matter, depending upon whether we consider R = R(T) or T = T(R).

2.6 The Tolman Mass and the Gravitational Pressure Tensor

The Einstein equations may be written as Eq. (2.103) or as

$$T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = \frac{c^4}{8\pi G}R_{\mu\nu}, \qquad (2.106)$$

wherein the explicit expression for T reads

$$T = g^{\mu\nu}T_{\mu\nu} = \frac{c^4}{8\pi G}R^{\mu}_{\ \mu} = T^0_{\ 0} + T^1_{\ 1} + T^2_{\ 2} + T^3_{\ 3}$$
$$= -\mathcal{E} + 3P. \qquad (2.107)$$

Where these two quantities correspond to the pressure P and the energy density \mathcal{E} .

Using this information we can write an exact expression for the tensor $R^{\mu}_{\ \nu}$,

that for the particular case of $\mu=\nu=0$ reads

$$R^{0}_{\ 0} = \frac{8\pi G}{c^{4}} \left(T^{0}_{\ 0} - \frac{1}{2}T\right)$$
(2.108)

$$= -\frac{4\pi G}{c^4} \left(\mathcal{E} + 3p\right).$$
 (2.109)

When the gravitational field is weak the metric tensor is diagonal and Eq. (2.30) is written as

$$c^2 d\tau^2 \approx \left(1 + \frac{\phi}{c^2}\right) dt^2 - \left(1 - \frac{\phi}{c^2}\right) d\mathbf{r}^2.$$
 (2.110)

where $\phi \ll c^2$. In this approximation Eq. (2.109) reads

$$R^0_{\ 0} = -\frac{1}{c^2} \Delta \phi. \tag{2.111}$$

According to Eq. (2.16), $\Delta \Phi = 4\pi G \rho_{gr}$, therefore we find

$$R_{0}^{0} = -\frac{4\pi G}{c^{4}} \left(\mathcal{E} + 3P\right)$$

$$\approx -\frac{1}{c^{2}} \Delta \Phi = -\frac{4\pi}{c^{2}} G \rho_{gr}.$$
(2.112)

Solving the last equation for ρ_{gr} defines the gravitational mass density

$$\rho_{gr} \approx \frac{1}{c^2} \left(\mathcal{E} + 3p \right). \tag{2.113}$$

The mass obtained from integrating ρ_{gr} with respect to space is called the Tolman Mass

$$M_{Tolman} \approx \frac{1}{c^2} \int d^3 \mathbf{r} \left(\mathcal{E} + 3p \right).$$
 (2.114)

Because $3P < \mathcal{E}$ the inertial and gravitational masses satisfy this relation

$$M_I \le M_{Tolman} \le 2M_I. \tag{2.115}$$

For any given mass distribution the force per unit volume in a gravitational field is

$$\mathbf{w} = \rho_{gr} \mathbf{g} = -\frac{1}{4\pi G} (div\mathbf{g}) \mathbf{g}$$
$$w_i = -\frac{1}{4\pi G} (\partial_j g_j) g_i. \tag{2.116}$$

Using the fact that

$$\mathbf{curlg} = \mathbf{0}.\tag{2.117}$$

we write Eq. (2.116) as

$$w_i = -\frac{1}{4\pi G} \left(\partial_j (g_j g_i) - g_i \partial_i g_j \right), \qquad (2.118)$$

or simply $\mathbf{w} = -div \ \overrightarrow{P}_{gr}$, where \overrightarrow{P}_{gr} is the gravitational pressure tensor defined in dyadic notation as

$$\vec{P}_{gr} = \frac{1}{8\pi G} \left(2\mathbf{gg} - \mathbf{g}^2 \,\vec{1} \right). \tag{2.119}$$

Taking the trace of this tensor we obtain the energy density for the gravitational field

$$Tr \vec{P}_{gr} = \frac{1}{8\pi G} \left(2\mathbf{g}^2 - 3\mathbf{g}^2 \right) = -\frac{1}{8\pi G} \mathbf{g}^2 = u_{gr}.$$
(2.120)

For the case of a static mass distribution, the equilibrium condition reads

$$div\left(\vec{\vec{P}} + \vec{\vec{P}}_{gr}\right) = 0, \qquad (2.121)$$

hence writing the integral

$$\int d^{3}\mathbf{r} div\left(\vec{\vec{P}} + \vec{\vec{P}}_{gr}\right) = 0 \quad \text{integrating by parts} -\int d^{3}\mathbf{r}\left(\vec{\vec{P}} + \vec{\vec{P}}_{gr}\right) = 0.$$
(2.122)

and taking its trace allows us to find out how the energy density of the gravitational field is related to the pressure

$$tr \int d^3r \left(\vec{\vec{P}} + \vec{\vec{P}}_{gr} \right) = 0$$

$$\int d^3r \left(3P + u_{gr} \right) = 0 \qquad (2.123)$$

The difference between M_I and M_{gr} seems to be zero up to many significant digits, the reason is because for an everyday object like a chair or a table the ratio $3p/\mathcal{E}$ can have a value as big as 10^{-15} . An immediate consequence is that any experiment with an accuracy less than 10^{-15} will confirm the principle of equivalence. A completely different picture can be found in many stellar objects such as a neutron star or a black hole where the pressure does play an important role in the definition of the gravitational mass.

2.7 Pressure-Energy Tensor

In a macroscopic body the configuration of the molecules corresponds to a status of thermal equilibrium. Once the body is deformed the positions of the molecules change and therefore there are forces, defined as internal stresses, that tend to bring the body back in its equilibrium position. The exerted force on an internal element of surface area ΔA_k is defined as

$$F^i = P^{ik} \Delta A_k, \tag{2.124}$$

wherein P^{ik} is the classical three-dimensional pressure tensor. Considering a four dimensional space-time approach we change from P^{ik} to $T^{\mu\nu}$ and write Eq. (2.124) as

$$P^{\mu} = \frac{1}{c} \int T^{\mu\nu} d^{3} \Sigma_{\nu}, \qquad (2.125)$$

where the P^{μ} is the four-momentum of the system. The Noether theorem states that for every continuous symmetry of the action there exists a conserved current. In a relativistic theory the invariance under space-time translations corresponds to the conservation of the energy momentum tensor. In the presence of a gravitational field the conservation of energy and momentum reads

$$T^{\mu\nu}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\lambda\nu} T^{\lambda\nu} + \Gamma^{\nu}_{\lambda\nu} T^{\mu\lambda} = 0, \qquad (2.126)$$

while in a flat space-time metric Eq. (2.126) reads

$$T^{\mu\nu}_{,\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0.$$
 (2.127)

As examples we will derive the energy-momentum tensor in two cases, the first one for an electromagnetic field and the second one for a free particle. The action for an electromagnetic system is

$$S_{EM} = -\frac{1}{16\pi c} \int d\Omega_x F_{\mu\nu} F^{\mu\nu}$$

= $-\frac{1}{16\pi c} \int d\Omega_x F_{\mu\nu} F_{\sigma\tau} g^{\mu\sigma} g^{\tau\nu},$ (2.128)

therefore the variation with respect to the metric gives

$$\delta S_{EM} = -\frac{1}{16\pi c} \delta \int d\Omega_x F_{\mu\nu} F_{\sigma\tau} g^{\mu\sigma} g^{\tau\nu}$$

$$= -\frac{1}{16\pi c} \int d^4 x F_{\mu\nu} F_{\sigma\tau} \delta \left(\sqrt{-g} g^{\mu\sigma} g^{\tau\nu} \right)$$

$$= -\frac{1}{16\pi c} \int d\Omega_x F_{\mu\nu} F_{\sigma\tau} \left(\delta g^{\mu\sigma} g^{\tau\nu} + g^{\mu\sigma} \delta g^{\tau\nu} - \frac{1}{2} g^{\mu\sigma} g^{\tau\nu} g_{\alpha\beta} \delta g^{\alpha\beta} \right)$$

$$= -\frac{1}{8\pi c} \int d\Omega_x \left(F_{\alpha\nu} F_{\beta}^{\ \nu} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right) \delta g^{\alpha\beta}. \qquad (2.129)$$

This expression corresponds to Eq. (2.101) and therefore we can now write the energy-momentum tensor as

$$T_{\alpha\beta} = \frac{1}{4\pi} \left(F_{\alpha\nu} F_{\beta}^{\ \nu} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right). \tag{2.130}$$

The relativistic form of the action, with an explicit expression of the metric tensor, permits one to define the energy momentum tensor directly in a symmetric way.

We now consider the case of a particle moving in a gravitational field. The action is

$$S_{matter} = -Mc^2 \int d\tau. \qquad (2.131)$$

To vary the action with respect to the metric we start varying Eq. (2.30)

$$-2c^{2}d\tau\delta d\tau = \delta g_{\mu\nu}dx^{\mu}dx^{\nu}$$
$$-c^{2}\delta d\tau = \frac{1}{2}\delta g_{\mu\nu}v^{\mu}v^{\nu}d\tau. \qquad (2.132)$$

The action can be written as

$$S_{matter} = -Mc^2 \int d\tau \int d\Omega_y \delta^4(x, y).$$
 (2.133)

and Eq. (2.100) reads

$$\frac{1}{2c} \int d\Omega_x T^{\mu\nu} \delta g_{\mu\nu} = \frac{1}{2} M \int d\tau \int d\Omega_y \delta^4(x,y) \delta g_{\mu\nu} v^\mu v^\nu d\tau.$$
(2.134)

Taking the functional derivative of this equation with respect to the variation in the metric and multiplying both sides by 2c we find the energy-momentum tensor to be

$$T^{\mu\nu} = Mc \int d\tau \delta^4(x, x(\tau)) v^{\mu} v^{\nu}, \qquad (2.135)$$

this equation, in the case of many non-interacting particles, defines the energy

momentum tensor for a system of *dust*.

2.8 Electromagnetic and Gravitational Waves

The existence and the variation of electromagnetic and gravitational fields are inferred through the detection and study of electromagnetic and gravitational waves. In the theory of electromagnetism the interaction between a current $j_{\mu} = (\mathbf{j}, -c\rho)$ and the four-potential $A_{\nu} = (\mathbf{A}, -\phi)$ is described in the action term

$$S_{int} = -\frac{1}{c^2} \int d^4x j^{\mu} A_{\mu}, \qquad (2.136)$$

where space-time is considered to be flat $(\sqrt{-g} = 1)$. The presence of the electromagnetic field occurs with a term in the action given by Eq.(2.128), therefore the total action reads

$$S = -\frac{1}{c^2} \int d^4x j^{\mu} A_{\mu} - \frac{1}{16\pi c} \int d^4x F^{\mu\nu} F_{\mu\nu}.$$
 (2.137)

According to the least action principle the "equation of motion" for j_{μ} are determined by varying the action with respect to the four-potential A_{ν} ,

$$\delta S = -\frac{1}{c^2} \int d^4 x j^{\mu} \delta A_{\mu} - \frac{1}{16\pi c} \int d^4 x \delta \left(F^{\mu\nu} F_{\mu\nu}\right)$$

$$= -\frac{1}{c^2} \int d^4 x j^{\mu} \delta A_{\mu} + \frac{1}{4\pi c} \int d^4 x \frac{\partial F^{\mu\nu}}{\partial x^{\mu}} \delta A_{\nu}$$

$$= -\frac{1}{c^2} \int d^4 x \left(j^{\nu} - \frac{c}{4\pi} \frac{\partial F^{\mu\nu}}{\partial x^{\mu}}\right) \delta A_{\nu} = 0. \qquad (2.138)$$

Hence this equation relates the current j^{ν} and the electromagnetic tensor $F^{\mu\nu}$ according to

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = -\frac{4\pi}{c}j^{\mu}.$$
(2.139)

Recalling that the first two Maxwell's equations are derived from the definition of the electric field $\mathbf{E} = -\dot{\mathbf{A}}/c - \mathbf{grad}\phi$ and of the magnetic field $\mathbf{B} = \mathbf{curl}\mathbf{A}$.

$$\mathbf{curl}\mathbf{E} = \mathbf{curl}\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \mathbf{grad}\phi\right)$$
$$= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
(2.140)

and

$$div\mathbf{B} = div\left(\mathbf{curlA}\right) = 0. \tag{2.141}$$

The other two Maxwell equation are obtained writing Eq. (2.139) for $\nu = 0$

$$\frac{\partial F^{\mu 0}}{\partial x^{\mu}} = \frac{4\pi}{c} j^{0}$$

$$div \mathbf{E} = 4\pi\rho, \qquad (2.142)$$

or for $\nu=1,2,3$

$$\frac{\partial F^{\mu\nu}}{\partial x^{\mu}} = \frac{4\pi}{c} j^{\nu}$$

curlB = $\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j}$ (2.143)

In the case of an empty space ($\rho = 0$ and $\mathbf{j} = 0$) Eq. (2.142), (2.143) reduce to

$$div\mathbf{E} = 0 \tag{2.144}$$

$$\mathbf{curlB} = -\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{2.145}$$

Writing the very last equation in a gauge where the scalar potential is zero $(\phi = 0)$ and $div \mathbf{A} = 0$ leads to

$$\mathbf{curlB} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = -\Delta \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t} = 0$$
$$-\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t}\right) \mathbf{A} = 0$$
(2.146)

This is the equation of an electromagnetic wave. Taking either the **curl** or the div of Eq. (2.146) we find that the electric field **E** and the magnetic field **B** satisfy the very same equation. In the case of a plane Electromagnetic wave for any given direction of propagation n the three vectors **E**, **B**, **n** satisfy this relation

$$\mathbf{E} = \mathbf{n} \times \mathbf{B},\tag{2.147}$$

therefore the waves are transverse. The Poynting vector for a plane wave reads

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$
$$= \frac{c}{4\pi} E^2 \mathbf{n} = \frac{c}{4\pi} B^2 \mathbf{n}, \qquad (2.148)$$

hence the energy flux along the direction of propagation is proportional to the energy density of the wave

$$W_{EM} = \frac{1}{8\pi} (E^2 + B^2) \tag{2.149}$$

and then $S = cW\mathbf{n}$. To summarize, electromagnetic waves propagate in vacuum at the speed of light, they transport energy according to the definition of the Poynting vector. Moreover the photon, although it is a spin 1 particle, comes only in two helicity states, the reason is $F^{\mu\nu}$ is a 4×4 anti-symmetric tensor with 6 degrees of freedom, but there are 4 constraints given by Maxwell's equations hence the total number of degrees of freedom is just 2 = 6 - 4.

The theory of General Relativity predicts that a non-static gravitational field produces gravitational waves [12, 13]. Because the coupling constant G is very *small* compared to the strong and electro-weak ones, it is very difficult to have a strong enough source of gravitational waves in a laboratory based on earth. Such sources of these waves are to be found through all those phenomena, happening inside stars or galaxies that involve a very fast change in the mass distribution. As an example we can cite the supernova explosion which occurred in 1987, the Supernova 1987A where the collapse of a massive star produced a large amount of gravitational radiation. During a supernova explosion there is almost immediately the emission of a neutrinos flux that precedes the electromagnetic and gravitation radiation, just because the neutrinos interacts very weakly and hence propagate as soon as the star starts to

collapse before the explosion. In 1987, in Italy, there were two running experiments denominated UNO (underground neutrino observatory) and Geograv, the first was an experiment about the detection of neutrinos while the second was a gravitational antenna. UNO detected a huge neutrino flux just before the supernova explosion, while Geograv detected a signal contemporaneous to the supernova explosion [7, 8]. The probability that these two events were casual and not correlated was calculated to be about 3%. The hunt for the gravitational waves started with Weber with his detector made by a resonant alluminium bar [14, 15, 16] and nowadays there are several experiments around the globe employing a laser interferometer [17, 19, 20, 21, 23] or a super conductive alluminum bar [24]. Their discovery, or better their study, if we consider reliable the Weber and Geograv's results, will give another confirmation of the theory of General Relativity and moreover of the quantum nature of the gravitational radiation. Recently is available a *Tesi* di Laurea, from University of Perugia, where the author, Silvia Chiacchiera, explores different possibilities of detection of gravitational waves [25]. To see the connection between gravitational waves and General Relativity we use the assumption made in Eq. (2.104), remembering that $|h_{\mu\nu}| \ll 1$ does not fix an unique choice of a reference system. In a weak gravitational field the Ricci tensor reads

$$R_{\mu\nu} = \frac{1}{2} \left(\partial^{\sigma} \partial_{\mu} h_{\sigma\nu} + \partial^{\sigma} \partial_{\nu} h_{\sigma\mu} - \Box h_{\mu\nu} - \partial_{\mu} \partial_{\nu} h \right) + O(h^2), \qquad (2.150)$$

where to lower an index we just use the Minkowski's metric $\eta_{\mu\nu}$ and to raise

one we use $\eta^{\mu\nu}$. The curvature of the gravitational field is obtained from the Ricci tensor according to

$$\eta^{\mu\nu}R_{\mu\nu} = R = \partial^{\mu}\partial^{\nu}h_{\mu\nu} - \Box h + O(h^2).$$
(2.151)

The Einstein equations, linear in h, read

$$2R_{\mu\nu} - \eta_{\mu\nu}R = \frac{1}{2}\partial^{\sigma}\partial_{\mu}h_{\sigma\nu} + \partial^{\sigma}\partial_{\nu}h_{\sigma\mu} - \Box h_{\mu\nu}$$
$$-\partial_{\mu}\partial_{\nu}h - \eta_{\mu\nu}\left(\partial^{\sigma}\partial^{\mu}h_{\sigma\mu} - \Box h\right)$$
$$= \frac{16\pi G}{c^4}T_{\mu\nu}.$$
(2.152)

An appropriate choice of a gauge condition allows to substantially simplify these equations. For example in harmonic coordinate system

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \tag{2.153}$$

the Einstein equations read

$$\Box h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \Box h = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$
 (2.154)

In writing these equations we have use the fact that in the weak field approximation Eq. (2.153) reads

$$\partial_{\mu}h^{\mu\lambda} - \frac{1}{2}\partial^{\lambda}h = 0, \qquad (2.155)$$

that defines the *Lorentz gauge*. Usually one introduces a tensor $\overline{h}_{\mu\nu}$ defined as

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \qquad (2.156)$$

because in this way Eq. (2.152) assumes a form similar to the wave equation for the electromagnetic field, Eq. (2.139)

$$\Box \overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$
(2.157)

In this case considering an empty space means $T_{\mu\nu} = 0$ and Eq. (2.156) reduces to

$$\Box \overline{h}_{\mu\nu} = 0, \qquad (2.158)$$

that describes the propagation of a gravitational wave in the vacuum. A generic solution to Eq. (2.158) is

$$\overline{h}_{\mu\nu} = \epsilon_{\mu\nu} e^{ik^{\alpha}x_{\alpha}} + \epsilon^*_{\mu\nu} e^{-ik^{\alpha}x_{\alpha}}, \qquad (2.159)$$

where $\epsilon_{\mu\nu}$ is the polarization tensor and k^{α} is the four vector wave number. These satisfy the relations

$$k^{\alpha}k_{\alpha} = 0 \tag{2.160}$$

and

$$k^{\mu}\epsilon_{\mu\nu} = 0. \tag{2.161}$$

According to Eq. (2.160) the graviton is a massless particle with a dispersion

relation equal to the photon one.

$$\omega^2 = c^2 \mathbf{k}^2. \tag{2.162}$$

The approximation defined in Eq. (2.148) requires that the metric must be globally invariant. A change of coordinates

$$x^{\mu} \to x^{'\mu} = x^{\mu} + \xi^{\mu}(x),$$
 (2.163)

must leave the metric unchanged. The perturbation $h_{\mu\nu}$ transforms according to

$$h_{\mu} \to h_{\prime\mu} = h_{\mu} - \partial_{\mu}\xi_{\nu}(x) - \partial_{\nu}\xi_{\mu}(x), \qquad (2.164)$$

wherein $|\partial_{\mu}\xi_{\nu}(x)| \ll |h_{\mu\nu}|$. The condition expressed in the Eq. (2.155) fixes an unique choice of a reference system only is the four functions ξ^{μ} satisfy the condition

$$\Box \xi^{\mu} = 0. \tag{2.165}$$

To summarize, the gravitational waves propagate in the vacuum at the speed of light and carry out gravitational energy as was theorized by Einstein in 1916 and verified by Hulse and Taylor in 1974 studying the declining orbit period of the binary pulsar system called PSR 1913 + 16. The graviton is a spin 2 particle, but like the photon it comes in only two helicity states. The reason is because $h_{\mu\nu}$ is a 4 × 4 symmetric tensor with 10 degrees of freedom, but there are 8 constraints given by the Lorentz gauge (4) and by Eq. (2.165) (4), hence the total number of degrees of freedom is just 2 = 10 - 4 - 4.

2.9 Non-Commutative Geometry

It is useful to write Maxwell's and Einstein equations for electromagnetic and gravitational waves in a gauge invariant form.

2.9.1 Gauge Invariance for Photons and Gravitons

For the Maxwell case we may write in complex form

$$\mathbf{F} = \mathbf{E} + i\mathbf{B}$$

div
$$\mathbf{F} = 0$$

 $i\frac{\partial \mathbf{F}}{\partial t} = c \operatorname{\mathbf{curl}} \mathbf{F}$ (2.166)

One can write Eq.(2.166) in the column vector form

$$\psi = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix},$$

$$\frac{\partial \psi}{\partial t} = -c(\mathbf{S} \cdot \mathbf{grad})\psi, \qquad (2.167)$$

wherein the spin operator components $\mathbf{S} = (S_x, S_y, S_z)$ are represented by

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \text{ and } S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
(2.168)

The spin operator compenets obey

$$[S_i, S_j] = i\epsilon_{ijk}S_k \text{ and } \mathbf{S} \cdot \mathbf{S} \equiv S_x^2 + S_y^2 + S_z^2 = S(S+1) = 2, \quad (2.169)$$

which is expected for a spin S = 1 photon. Introducing the momentum operator operator $\mathbf{p} = -i\hbar \mathbf{grad}$ allows us to write

$$i\hbar \frac{\partial \psi(\mathbf{r},t)}{\partial t} = \mathcal{H}\psi(\mathbf{r},t)$$
 (2.170)

with the single photon Hamiltonian

$$\mathcal{H} = c\mathbf{S} \cdot \mathbf{p}. \tag{2.171}$$

The form of Eqs.(2.170) and Eqs.(2.171) makes manifest that the wave function for a single photon is merely Maxwell's equations in a thinly disguised form. The main point is that the vacuum Maxwell's equations can be written in a gauge invariant form for the complex field tensor

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + i \,^* F_{\mu\nu}$$
$$\partial_{\mu} \mathcal{F}^{\mu\nu} = 0, \qquad (2.172)$$

wherein ${}^*F_{\mu\nu}$ is the dual tensor to $F_{\mu\nu}$.

Let us now derive the gauge invariant form of the graviton equation. One starts from the Weyl conformal tensor $C_{\mu\nu\alpha\beta}$ [18] defined in terms of the curvature tensor $R_{\mu\nu\alpha\beta}$ via

$$R_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + \frac{1}{2} \left(g_{\mu\alpha} R_{\nu\beta} + g_{\nu\beta} R_{\mu\alpha} - g_{\mu\beta} R_{\nu\alpha} - g_{\nu\alpha} R_{\mu\beta} \right) - \frac{1}{6} \left(g_{\mu\alpha} g_{\nu\beta} - g_{\nu\alpha} g_{\mu\beta} \right) R. \quad (2.173)$$

The complex gauge invariant gravitational wave amplitude is

$$\mathcal{G}^{\mu\nu}{}_{\alpha\beta} = C^{\mu\nu}{}_{\alpha\beta} + i \, {}^*C^{\mu\nu}{}_{\alpha\beta} \tag{2.174}$$

wherein ${}^{*}C^{\mu\nu}{}_{\alpha\beta}$ is the dual tensor to $C^{\mu\nu}{}_{\alpha\beta}$ From the Einstein equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \text{ and } R = -\left(\frac{8\pi G}{c^4}\right) T, \qquad (2.175)$$

it follows that

$$R_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta}$$
 (vacuum). (2.176)

From the Bianchi identities follows the vacuum equations of motion for the graviton

$$D_{\mu}\mathcal{G}^{\mu\nu}{}_{\alpha\beta} = D_{\nu}\mathcal{G}^{\mu\nu}{}_{\alpha\beta} = 0. \qquad (2.177)$$

Eqs.(2.174) and (2.177) are the gravitational wave analogs to the electromagnetic Eqs.(2.172). Given a vacuum space-like unit normal Killing vector $N^{\mu}N_{\mu} = -1$, one may define gravitational-electric and gravitationalmagnetic fields according to

$$\mathcal{G}_{\mu\nu\alpha\beta}N^{\nu}N^{\beta} = \mathcal{G}_{\mu0\alpha0} = E_{\mu\alpha} + iB_{\mu\alpha}, \qquad (2.178)$$

in direct analogy to the electromagnetic fields

$$\psi_{\mu} \equiv \mathcal{F}_{\mu\nu} N^{\nu} = \mathcal{F}_{\mu0} = E_{\mu} + iB_{\mu}, \qquad (2.179)$$

where Eq.(2.167) has been invoked. It is now evident that the gauge invariant graviton wave function is a spatial tensor

$$\psi_{\mu\nu} \equiv \mathcal{G}_{\mu0\nu0} = E_{\mu\nu} + iB_{\mu\nu}.$$
 (2.180)

While the photon is described by the complex three vector ψ_{μ} , the graviton is described by the symmetric three tensor $\psi_{\mu\nu}$. Since the Weyl conformal tensor is traceless $C^{\mu}_{\ \alpha\mu\beta} = 0$, the gravitational wave tensor is also traceless;

$$g^{\mu\nu}\psi_{\mu\nu} = 0. (2.181)$$

A three by three symmetric traceless tensor has (2S + 1) = 5 independent amplitudes corresponding to the graviton spin S = 2. Let us consider this in more detail.

The equation of motion for the tensor graviton wave function follows from the vacuum Einstein equations via Eqs.(2.177) and (2.180); It is

$$i\hbar \frac{\partial \psi_{ab}}{\partial t} = \frac{c\mathbf{p}}{S} \cdot \left(\delta_{bb'} \mathbf{s}_{aa'} + \delta_{aa'} \mathbf{s}_{bb'}\right) \psi_{a'b'}, \qquad (2.182)$$

where $\mathbf{s}_{cc'}$ are the matrix elements of spin one operators as in Eq.(2.168).

Equivalently, the graviton wave function obeys

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H}\psi,$$

$$\mathcal{H} = c \frac{\mathbf{p} \cdot \mathbf{S}}{S}$$

$$\mathbf{S} \cdot \mathbf{S} \equiv S(S+1) = 5 \text{ (graviton)}. \tag{2.183}$$

The spin two operators follow from adding two spin one operators. In general, one may add two spin operators and find the possibilities S = 0 or S = 1or S = 2. Since the wave function is spin symmetric, $\psi_{ab} = \psi_{ba}$ it is not possible to have S = 1. Since the wave function is traceless $\sum_{a} \psi_{aa} = 0$ it is not possible to have S = 0. The graviton is thus shown to be S = 2 object as in Eq.(2.183).

2.9.2 Momentum Space wave functions

In momentum space a massless particle such as a graviton (S = 2) or a photon (S = 1) obeys a Schrödinger equation of the form

$$i\hbar \frac{\partial \psi(\mathbf{p}, t)}{\partial t} = \mathcal{H}\psi(\mathbf{p}, t) = c \frac{\mathbf{S} \cdot \mathbf{p}}{S} \psi(\mathbf{p}, t).$$
 (2.184)

Since these particles are massless only the two values $\Lambda = \pm S$ of the Helicity

$$\Lambda = \frac{\mathbf{p} \cdot \mathbf{S}}{|\mathbf{p}|} \tag{2.185}$$

appear in physical states

$$\mathcal{H} = \epsilon \left(\frac{\Lambda}{S}\right) \quad \text{where} \quad \epsilon = c|\mathbf{p}|.$$
 (2.186)

The fact that helicity is used instead of spin implies interesting properties concerning the position of a massless particle.

To see what is involved consider the *total* angular momentum of a massless particle. It is

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} + \hbar \mathbf{S}$$
 where $\mathbf{r} = i\hbar \frac{\partial}{\partial \mathbf{p}}$. (2.187)

Only the helicity components of the spin are observable so we may write

$$\mathbf{S} = \frac{\mathbf{p}}{p} \Lambda \left(\mathbf{1} - \frac{\mathbf{p}\mathbf{p}}{\mathbf{p}^2} \right) \cdot \mathbf{S}, \qquad (2.188)$$

or equivalently by the sequence of relations

$$\mathbf{u}(\mathbf{p}) = \hbar \frac{\mathbf{S} \times \mathbf{p}}{p^2}$$
$$\mathbf{r} = \mathbf{R} + \mathbf{u}(\mathbf{p})$$
$$\mathbf{J} = \mathbf{R} \times \mathbf{p} + \hbar \left(\frac{\mathbf{p}}{p}\right) \Lambda.$$
(2.189)

When only the helicity components of the spin contribute to the total angular momentum, then the position becomes an operator with components which do not commute. In detail,

$$\frac{i}{\hbar} \left[R_a, R_b \right] = \epsilon_{abc} \left(\frac{\hbar p_c \Lambda}{p^3} \right).$$
(2.190)

For example, consider a photon or graviton with helicity $\Lambda = S$ moving along the the z-axis. The coordinates of the particle $\mathbf{R} = (X, Y, Z)$ in the plane normal to the axis of propagation obey a non-commutative geometry relation

$$i[X,Y] = \frac{\hbar^2}{p^2} S \equiv L^2.$$
 (2.191)

For example the pythagorean theorem in the (X, Y) plane is quantized according to

$$X^{2} + Y^{2} = (2n+1)L^{2} = (2n+1)S\lambda^{2}$$
 wherein $\lambda = \frac{\hbar}{p}$. (2.192)

The quantization of position functions according to the non-commutative geometry is not without interest for quantum optics when the positions of photons are made manifest by the "pixel" positions of the detectors. For the case of gravitational wave detection the pixel positions of uncertainty

$$\Delta X \Delta Y > (L^2/2) \tag{2.193}$$

with a graviton length scale of (perhaps) $L \sim 10^6$ meters.

Chapter 3

The Higgs Model

In this chapter we show the analogy between the mass grown mechanism of a photon inside a super-conductor and the mass grown mechanism of the electro-weak bosons in a Higgs field. We define the action describing the interactions of the Higgs particle. We derive the Yukawa potential between massive particle distributions, where the Higgs particle is the mediator. We compute and confront several decay rates for the Higgs particle. These calculations are very important if we want to understand the chances of detecting the Higgs particle at the next high energy accelerators.

3.1

The Standard Model (SM) is a successful model describing the interactions of elementary particles. As every model, it is not perfect and presents some aspects which do not totally agree with the experimental results, these may be the evidences for *new Physics*. One of the weak point of the SM is that the mass of the elementary particles cannot be determined *a priori*. Over the past decades several different models have been proposed for the mass problem, but among them the most well received one was the Higgs model by Peter Higgs. He used some of the concepts already developed in the theory of super-conductivity to explain how a mass is grown on all the particles through a spontaneous symmetry breaking of the local gauge invariance [28, 29]. The connection between these two theories can be understood considering the diamagnetic properties of a super-conductor. Given a super-conductor in a magnetic field, the intensity of the field, inside the material, decreases exponentially, according to this law

$$Prob. \propto e^{-x/\Lambda_L},\tag{3.1}$$

wherein the typical length scale of this phenomenon is defined as the London penetration depth Λ_L . The magnetic field cannot deeply penetrate inside the superconductor because the mass-less photon acquires a mass through its interaction with the condensate made by super-conductive electrons. The elementary particle physics approach to the mass-growth problem reads that a mass-less particle acquires its mass through the interaction with the Higgs condensate. To understand how these two Physics fields are connected we will start deriving the mass of the photon inside a super-conductor and the London penetration. We will employ only the Newton's first law of mechanics, $\mathbf{F} = m\mathbf{a}$, and the Maxwell's equations. In a gauge where $\mathbf{grad}\phi = 0$ and $div \mathbf{A} = 0$ the electric field is equal to:

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.$$
(3.2)

Inside a super-conductor only a fraction of the total number of electrons is responsible for carrying the super-conductive currents. These electrons pair up in couple called a Cooper pair. The density of the Cooper's pair is a critical parameter of the super-conductive state and it is indicated by $n_c = |\Psi|^2$, where

$$\Psi \propto \langle \psi_{\uparrow} \psi_{\downarrow} \rangle \tag{3.3}$$

and ψ is the wave-function of a single electron. A Cooper pair can be considered as one particle boson with spin 0, charge $e^* = -2e$ and mass $m = 2m_e$. Because the super-conductor is a dia-magnet, the Lorentz force on a Cooper pair is just given by the electric field **E**. The equation of motions then reads

$$\mathbf{F} = m^* \frac{d\mathbf{v}}{dt} = e^* \mathbf{E} = -\frac{e^*}{c} \frac{\partial \mathbf{A}}{\partial t}.$$
(3.4)

The expression for the super-conductive current density can be derived directly from Eq. (3.4)

$$\mathbf{J} = n_c e^* \mathbf{v} = -\frac{n_c e^{*2} \mathbf{A}}{m^* c}.$$
(3.5)

To obtain the equation of motion for a photon inside a superconductor we note that the fourth Maxwell's equation and Eq. (3.5) are just functions of

$$\mathbf{curlB} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}$$
$$\mathbf{curl}(\mathbf{curlA}) = -\Delta \mathbf{A} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{4\pi n_c e^{*2}\mathbf{A}}{m^* c^2}$$
$$\left(\Box + \frac{4\pi n_c e^{*2}}{m^* c^2}\right)\mathbf{A} = 0$$
(3.6)

This last equation is known as Klein-Gordon (K-G) equation, it describes a massive spin 1 particle and differs from the electromagnetic wave equation by a term that can be identified as a $(mass \times c/\hbar)^2$. The mass M_{γ} is here obtained as a function of fundamental physical constants (like the electron mass and charge, the Plank's constant and the speed of light) and the London penetration depth.

$$M_{\gamma} = \frac{\hbar}{\Lambda_L c}, \qquad (3.7)$$

$$\Lambda_L = \sqrt{\frac{m^* c^2}{4\pi e^{*2} n_c}}.$$
(3.8)

The main result of this section is that inside a superconductor a mass-less gauge boson (the photon γ) acquires a mass due to its interaction with the super-conductive electrons. In the next section we will generalize the concept of symmetry gauge symmetry and we will discuss the Higgs mechanism. To obtain the masses for the three electro-weak gauge bosons we write in complete analogy the density of super conductive electrons as $v = | < \phi(x) >$

Α

| over the vacuum and the electro-weak currents as¹

$$J_{W^{\pm}}^{\mu} = -g_{W}^{2}v^{2}W_{\pm}^{\mu} \tag{3.9}$$

$$J_Z^{\mu} = -g_Z^2 v^2 Z^{\mu} \tag{3.10}$$

$$M_{W^{\pm}} = g_W v / 2 \tag{3.11}$$

$$M_{Z^{\pm}} = g_Z v/2 \tag{3.12}$$

In the next session we will derive how the Higgs particle couples with other elementary particles.

3.2 The Higgs Particle

In the Standard Model the Higgs field is responsible for creating masses for all the elementary particles in the universe. The mediator of this interaction is a scalar neutral particle called the Higgs boson. The search and discovery of the Higgs boson is the principal goal of the modern experimental High Energy Physics as it was in the last thirty years. Although the Standard Model does not predict a value for the Higgs particle mass, present opinion is that its mass is confined, with a 95% confidence, in the interval between $114 \ GeV/c^2$ and $195 \ GeV/c^2$. As is the case for most "elementary particles", the Higgs boson is unstable and decays in several different channels. There is an extend bibliography about all the decay modes of the Higgs particle. To detect these decay products will be one of the main goals of the Large

¹In natural units $\hbar = c = 1$

Hadron Collider (LHC), under construction at CERN, Geneve Switzerland. The discovery of the Higgs particle would complete the Standard Model, and thereby confirm that there is no new physics at presently attainable energies. On the other hand if the Higgs particle remains undetected, then there will be a valid reason to look with a more objective eye to many alternative theories which have been proposed in the last forty years and unfairly ignored by the majority of the physics community.

3.3 Stress as the Source of the Higgs Field

Theoretical calculations about the decay rates of the Higgs particle into diverse channels, provides an understanding of the possibility of detection and the possible discovery the Higgs boson. To derive all these products we must start from a microscopic action. Although the Higgs particle is responsible for creating the mass of all the elementary particles, an effective action can be derived starting from the detailed model action. Suppose that the Higgs field obeys

$$\phi = \langle 0 | \phi | 0 \rangle + \sigma = v + \sigma,$$

$$\hbar cv \approx 246 \text{ GeV}. \qquad (3.13)$$

The source of the Higgs σ -field may be written as

$$T(x) = c \left[\phi(x) \left(\frac{\delta S}{\delta \phi(x)} \right) \right]_{\phi(x)=v}.$$
 (3.14)

In terms of the elementary particle masses that enter into the Lagrangian density, source becomes

$$T(x) = \sum_{a} M_a \frac{\partial \mathcal{L}(x)}{\partial M_a}$$
(3.15)

so that

$$S_{int} = \frac{1}{vc} \int T(x)\sigma(x)d^4x.$$
(3.16)

Eq.(3.15) establishes the Higgs source as the trace of the stress tensor, $T = T^{\mu}{}_{\mu}$. Finally, the Fermi weak interaction coupling strength G_F is determined by the mean Higgs field according to

$$\sqrt{2}\left(\frac{\hbar G_F}{c^3}\right) = \frac{1}{v^2}.\tag{3.17}$$

To obtain the trace of the stress tensor we use the formula

$$T(x) = M \frac{\partial \mathcal{L}(x, M)}{\partial M}$$
(3.18)

A few examples will suffice:

$$T_{\text{Fermi}}(x) = M_F \frac{\partial \mathcal{L}_{\text{Fermi}}(x)}{\partial M_F} = -M_F c^2 \bar{\psi}(x) \psi(x), \text{ for a massive fermion}$$

$$T_W(x) = M_W \frac{\partial \mathcal{L}_W(x)}{\partial M_W} = -\left(\frac{2M_W^2 c^2}{\hbar}\right) W^{\dagger}(x) W(x), \text{ for the } W^{\pm} \text{ bosons},$$

$$T_Z(x) = M_Z \frac{\partial \mathcal{L}_Z(x)}{\partial M_Z} = -\left(\frac{M_Z^2 c^2}{\hbar}\right) \bar{Z}(x) Z(x), \text{ for the } Z^0 \text{ boson.} \quad (3.19)$$

For a free *classical* particle with mass m, the action and Lagrangian density is given by

$$S_{\text{classical}} = -mc^2 \int d\tau = \frac{1}{c} \int \mathcal{L}_{\text{classical}}(x) d^4 x$$
$$\mathcal{L}_{\text{classical}}(x) = -mc^3 \int \delta^4(x - x(\tau)) d\tau. \qquad (3.20)$$

From Eqs.(3.15) and (3.20) we have

$$T_{\text{classical}} = m \frac{\partial \mathcal{L}_{\text{classical}}(x)}{\partial m} = -mc^3 \int \delta^4(x - x(\tau)) d\tau.$$
(3.21)

Our reasoning is as follows: (i) One may obtain a Yukawa potential between any two massive particles via an effective action to be discussed below; i.e.

$$U_Y = -\frac{\sqrt{2}G_F M_1 M_2}{4\pi R} e^{-KR}$$
(3.22)

(ii) The "classical" coupling between (say) fermions and the Higgs scalar is generally written as

$$S_{F} = -\frac{1}{c} \int f\phi(x)\bar{\psi}(x)\psi(x)d^{4}x$$

$$M_{F}c^{2} = fv$$

$$S_{F} = -M_{F}c^{3} \int \left(1 + \frac{\sigma(x)}{v}\right)\bar{\psi}(x)\psi(x)d^{4}x.$$
(3.23)

in agreement with Eqs.(3.16) and Eqs.(3.19). We stress the importance of the connection between the Yukawa potential U_Y and the Higgs particle because U_Y was first introduced in order to describe the interaction between two nucleons exchanging a scalar meson. Below, we wish to study how two particles, one at the origin and the other at \mathbf{r} , interact through the Higgs field. We derive an expression for the effective action up to the second order in v^{-1} employing perturbation theory in the form

$$exp\left[\frac{i}{\hbar}S_{eff}\right] = \left\langle exp\left[\frac{i}{\hbar}S_{int}\right] \right\rangle_{+}$$

$$\frac{i}{\hbar}S_{eff} \approx -\left\langle \frac{1}{2\hbar^{2}}S_{int}^{2} \right\rangle_{+}$$

$$S_{eff} = \frac{i\sqrt{2}G_{F}}{2c^{5}}\int\int T(x) < \sigma(x)\sigma(y) >_{+} T(y)d^{4}xd^{4}y$$

$$S_{eff} = \frac{\sqrt{2}G_{F}}{2c^{5}}\int\int T(x)D(x-y)T(y)d^{4}xd^{4}y.$$
(3.24)

In particular the effective action describes how the sources in T(x) interact through a scalar Higgs field described by the propagator D(x-y). The static limit of Eq.(3.24) yields the Yukawa potential. We assume here the static limit of classical point particle sources as in Eq.(3.21) which read

$$T_a(x) = -M_a c^2 \delta^3 (\mathbf{x} - \mathbf{r}_a)$$

$$T_b(y) = -M_b c^2 \delta^3 (\mathbf{y} - \mathbf{r}_b).$$
(3.25)

According to this definition Eq. (3.24) reads

$$S_{eff} = \frac{\sqrt{2}G_F}{2c^5} \int \int T(x)D(x-y)T(y)d^4xd^4y$$

$$S_{ab} = \frac{\sqrt{2}G_F M_a M_b}{c} \int \int \delta^3(\mathbf{x} - \mathbf{r}_a)D(x-y)\delta^3(\mathbf{y} - \mathbf{r}_b)d^4xd^4y$$

$$S_{ab} = \sqrt{2}G_F M_a M_b c \int \int D(\mathbf{r}_{ab} - c(t_a - t_b))dt_a dt_b.$$
(3.26)

wherein $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$. Because the above integral depends on the difference between the two times it comes on handy to compute it after we have made the following change of variables

$$t = \frac{t_a + t_b}{2}$$

$$t_{ab} = t_a - t_b.$$
(3.27)

The Fourier transform of the Higgs boson propagator reads

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik \cdot (x-y)}}{k^2 + K^2 - i0^+} \quad \text{where} \quad \hbar K = M_H c. \tag{3.28}$$

We can write the Eq.(3.26) as

$$S_{ab} = \sqrt{2}G_F M_a M_b \int dt \int \frac{d^4k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}_{ab}}}{k^2 + K^2 - i0^+} \delta(ck^0)$$

$$S_{ab} = \sqrt{2}G_F M_a M_b \int dt \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{e^{i\mathbf{k}\cdot\mathbf{r}_{ab}}}{|\mathbf{k}|^2 + K^2 - i0^+}\right]$$

$$S_{ab} = \int \left[\frac{\sqrt{2}G_F M_a M_b}{4\pi r_{ab}} e^{-Kr_{ab}}\right] dt, \qquad (3.29)$$

wherein the integral in t_{ab} yielded $2\pi\delta(ck^0)$. Normally a static potential is obtained from the action as

$$\Re e(S_{ab}) = -\int U_{ab}(r_{ab})dt.$$
(3.30)

Therefore from equation Eq. (3.76) we derive that the static Yukawa potential simply reads

$$U_{ab}(r_{ab}) = -\left[\frac{\sqrt{2}G_F M_a M_b}{4\pi r_{ab}}e^{-Kr_{ab}}\right].$$
 (3.31)

We have here shown that the Higgs particle couples to the trace of the stress tensor. To consolidate our hypothesis about the properties of the Higgs boson, we seek to compute its decay rate into two gravitons [31, 32], or a fermion anti-fermion pair or two photons and so forth. In order to do the computation we will start again from the very same action Eq. (3.70), but we will look at it from an other prospective. In this case we want to see how to write the total Higgs propagator in momentum space, including the "loop contribution" at the lowest order perturbation theory in v^{-1} . We will then obtain all the decay modes of the Higgs boson taking the imaginary part of the self-energy part of the full propagator. The appropriate choice of the trace of the energy momentum tensor will determine which particular decay mode is calculated. The generic decay rate, $H \to X\overline{X}$, is obtained from the expansion of the effective action and considering only the quadratic term in the source T(x). Practically we want to find out how a scalar particle propagates in space time from x to y. We employ a new effective action,

$$exp\left[\frac{i}{\hbar}S_{eff}\right] = \left\langle exp\left[\frac{i}{\hbar}S_{int}\right] \right\rangle_{+}$$
$$\frac{i}{\hbar}S_{eff} \approx -\left\langle \frac{1}{2\hbar^{2}}S_{int}^{2} \right\rangle_{+}$$
$$S_{eff} \approx \frac{i}{2\hbar c^{2}v^{2}} \int \int d^{4}x d^{4}y < T(x)T(y) >_{+} \sigma(x)\sigma(y)$$
$$S_{eff} = \frac{i\sqrt{2}G_F}{c^5} \int \int d^4x d^4y < T(x)T(y) >_+ \sigma(x)\sigma(y).$$
(3.32)

To understand the physical meaning of Eq. (3.30) we write the total action for the Higgs boson as

$$S_H = -\frac{\hbar}{2} \int d^4x \left(\partial^\mu \sigma(x) \partial_\mu \sigma(x) + K^2 \sigma(x)^2 \right), + \frac{i\sqrt{2}G_F}{c^5} \int \int d^4x d^4y < T(x)T(y) >_+ \sigma(x)\sigma(y), \quad (3.33)$$

where the constant κ is defined as $\kappa = M_H c/\hbar$. The exact propagator for a scalar particle reads

$$\mathcal{D}_H(k^2) = \frac{1}{k^2 + K^2 - \Pi(k^2)}.$$
(3.34)

The self-energy part follows from Eq.(3.32) according to

$$\widetilde{\Pi}(x-y) = \frac{2i\sqrt{2}G_F}{\hbar c^5} < T(x)T(y) >_+
= \frac{1}{(2\pi)^4} \int d^4k \Pi(k) e^{ik(x-y)}.$$
(3.35)

and the decay rate is given by

$$\Gamma = \frac{c}{\kappa} \Im m \Pi (-\kappa^2 - i0^+). \tag{3.36}$$

In the next sections we will derive the gravitational, fermionic and electroweak decay rates.

3.4 Particle Propagators

In the next sections we need to employ the propagator of a fermion, a spin 1 boson (either massive or massless) and a spin 2 massless boson. These function are defined as

$$S_{F}(x-y) = i < \psi(x)\psi(y) >_{+}$$

= $\int \frac{d^{4}k}{(2\pi)^{4}} \frac{e^{ik \cdot (x-y)}}{k^{2} + K_{F}^{2} - i0^{+}} (\gamma \cdot k - K_{F}),$ (3.37)
 $K_{F} = \frac{M_{F}c}{\hbar}$

for a fermion,

$$D_{\gamma}^{\mu\nu}(x-y) = \left(\frac{i}{\hbar c}\right) < A^{\mu}(x)A^{\nu}(y) >_{+} = \int \frac{d^{4}k}{(2\pi)^{4}} \frac{4\pi e^{ik\cdot(x-y)}}{k^{2} - i0^{+}} \eta^{\mu\nu} \qquad (3.38)$$

for a photon [33],

$$D_{W^{\pm}}^{\mu\nu}(x-y) = \left(\frac{i}{\hbar c}\right) < W^{\dagger}(x)W(y) >_{+} \\ = \int \frac{d^{4}k}{(2\pi)^{4}} \frac{e^{ik \cdot (x-y)}}{k^{2} + K_{W}^{2} - i0^{+}} \left(\eta^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{K_{W}^{2}}\right), \quad (3.39) \\ K_{W} = \frac{M_{W}c}{\hbar}$$

for a W^{\pm} electro-weak boson,

$$D_{Z}^{\mu\nu}(x-y) = \left(\frac{i}{\hbar c}\right) < Z(x)Z(y) >_{+} \\ = \int \frac{d^{4}k}{(2\pi)^{4}} \frac{e^{ik \cdot (x-y)}}{k^{2} + K_{Z}^{2} - i0^{+}} \left(\eta^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{K_{Z}^{2}}\right), \quad (3.40)$$

$$K_Z = \frac{M_Z c}{\hbar}$$

for a ${\cal Z}$ electro-weak boson and

$$D_{g}^{\mu\nu\alpha\beta}(x-y) = \left(\frac{ic^{3}}{\hbar G}\right) < h^{\mu\nu}(x)h^{\alpha\beta}(y) >_{+}$$
$$= G^{\mu\nu\alpha\beta} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{4\pi e^{ik\cdot(x-y)}}{k^{2} - i0^{+}}, \qquad (3.41)$$
$$G^{\mu\nu\alpha\beta} = 4(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta})$$

for a graviton.

3.5 An Useful Integral

To calculate the decay rate of the Higgs particle we must always solve this integral

$$I(q^2, K) = \int d^4Q \delta(Q^2 + K^2) \delta((Q - q)^2 + K^2), \qquad (3.42)$$

where q^{μ} is a four vector such as $q^2 = \mathbf{q}^2 - q_0^2 = -M_H^2 c^2/\hbar^2$ and $K = Mc/\hbar$, being M the mass of the particle originating the loop in the Higgs particle propagator. We calculate $I(q^2, K)$ in all the details.

$$I(q^{2}, K) = \int d^{4}Q \delta((Q + q/2)^{2} + K^{2}) \delta((Q - q/2)^{2} + K^{2})$$

$$= \int d^{4}Q \frac{ds}{2\pi} \frac{dt}{2\pi} e^{is((Q+q/2)^{2} + K^{2})} e^{it((Q-q/2)^{2} + K^{2})}$$

$$= \int d^{4}Q \frac{ds}{2\pi} \frac{dt}{2\pi} e^{i(s+t)(Q^{2}+q^{2}/4+K^{2})} e^{i(s-t)(Q\cdot q)}$$

$$= \frac{1}{2} \int d^{4}Q \delta(Q^{2} + q^{2}/4 + K^{2}) \delta(Q \cdot q) \qquad (3.43)$$

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The product $Q \cdot q$ is equal to $\mathbf{Q} \cdot \mathbf{q} - Q^0 q^0$, therefore the integral in dQ^0 reads

$$\int dQ^{0}\delta(Q^{2} + q^{2}/4 + K^{2})\delta(Q \cdot q) = \int dQ^{0}\delta(Q^{2} + q^{2}/4 + K^{2})\delta(\mathbf{Q} \cdot \mathbf{q} - Q^{0}q^{0})$$
$$= \frac{1}{q^{0}}\delta\left[\mathbf{Q}^{2} - \left(\frac{\mathbf{Q} \cdot \mathbf{q}}{q^{0}}\right)^{2} - g^{2}\right], \qquad (3.44)$$

where

$$g^2 = -q^2/4 - K^2 \tag{3.45}$$

and $I(q^2, K)$ is

$$I(q^{2}, K) = \frac{1}{2q^{0}} \int d^{3}Q \frac{1}{q^{0}} \delta \left[\mathbf{Q}^{2} - \left(\frac{\mathbf{Q} \cdot \mathbf{q}}{q^{0}}\right)^{2} - g^{2} \right]$$

$$= \frac{\pi}{q^{0}} \int_{-1}^{+1} dz \left[1 - \left(\frac{|\mathbf{q}|z}{q^{0}}\right)^{2} \right]^{-3/2} \int_{0}^{+\infty} y^{2} dy \delta(y^{2} - g^{2})$$

$$= \frac{\pi g}{2q^{0}} \int_{-1}^{+1} dz \left[1 - \left(\frac{|\mathbf{q}|z}{q^{0}}\right)^{2} \right]^{-3/2}$$

$$= \frac{\pi g}{2q^{0}} \left\{ \frac{z}{\sqrt{1 - \left(\frac{|\mathbf{q}|z}{q^{0}}\right)^{2}}} \right\}_{-1}^{+1} = \frac{\pi g}{\sqrt{-q^{2}}}$$

$$= \frac{\pi}{2} \sqrt{1 + \frac{4K^{2}}{q^{2}}} = \frac{\pi}{2} \sqrt{1 - 4\left(\frac{M}{M_{H}}\right)^{2}}.$$
(3.46)

Having defined the propagators and calculates the integral $I(q^2, K)$ we can derive the decay rates of the Higgs particle.

3.6 Weak and Electromagnetic decay modes

The Higgs boson couples directly to all the massive elementary particles and indirectly (for example as in $H \to \bar{X}X \to \gamma\gamma$) to massless particles. There exists an wide literature about all these decay modes that will be deeply investigate at the LHC experiments. In particular the Higgs particle can decay into a quark anti-quark pair $H \to \bar{q}q$, up to the bottom quark, and lepton anti-lepton pair $H \to \bar{f}f$. Moreover, if the Higgs mass were big enough, it is also possible for the Higgs to decay into electro-weak bosons, $H \to ZZ, H \to W^+W^-$ or into top quark pairs, $H \to \bar{t}t$.

3.6.1 Electromagnetic decay mode

In this part we will calculate the $H \rightarrow \gamma \gamma$ decay rate. Although there is not direct coupling between the Higgs particle and the photon we can write down the interaction considering that the polarization of the fermionic vacuum creates the electromagnetic field and then we will consider the interaction between the neutral scalar boson and the fermions and between the fermions and the electromagnetic field. Schwinger [30] calculated the action describing the interaction between a scalar meson and a photon. Using directly his result² we write

$$S_{int} = \frac{\alpha}{12\pi^2 v} \int d^4 x \sigma(x) (\mathbf{B}^2 - \mathbf{E}^2)$$

= $\frac{\alpha}{24\pi^2 v} \int d^4 x \sigma(x) F_{\mu\nu}(x) F^{\mu\nu}(x).$ (3.47)

²Schwinger uses this definition of the fine structure constant $\alpha = e^2/(4\pi\hbar c)$, while we simply use $\alpha = e^2/(\hbar c)$

This expression is correct for a lepton originating the electro-magnetic field, in the case of quarks Eq. 3.47 must be multiplied by

$$N_c \times q_q^2$$

where N_c is the number of colors and q_q^2 is the ratio of the quarks and the electron charges squared (either 4/9 or 1/9). The effective action is obtained with the same technique used in the other cases and it is

$$S_{eff} = \frac{i}{2\hbar} \left(\frac{\alpha}{24\pi^2 v}\right)^2 \int d^4x d^4y \sigma(x) \sigma(y) < F_{\mu\nu}(x) F^{\mu\nu}(x) F_{\alpha\beta}(y) F^{\alpha\beta}(y) >_+ .$$
(3.48)

The self-energy part of the propagator is equal to

$$\tilde{\Pi}(x-y) = \frac{2i\sqrt{2}G_F}{\hbar c^3} \left(\frac{\alpha}{24\pi^2}\right)^2 < F_{\mu\nu}(x)F^{\mu\nu}(x)F_{\alpha\beta}(y)F^{\alpha\beta}(y) >_+ . \quad (3.49)$$

Focusing on the time ordered product, we can write it as

$$< F_{\mu\nu}(x)F^{\mu\nu}(x)F_{\alpha\beta}(y)F^{\alpha\beta}(y)>_{+}=$$

 $2 < F_{\mu\nu}(x)F_{\alpha\beta}(y)>_{+}< F^{\mu\nu}(x)F^{\alpha\beta}(y)>_{+}.$ (3.50)

Let us calculate explicitly the first one

$$\langle F_{\mu\nu}(x)F_{\alpha\beta}(y) \rangle_{+} = \langle (\partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x))(\partial_{\alpha}A_{\beta}(y) - \partial_{\beta}A_{\alpha}(y)) \rangle_{+}$$

$$= \langle \partial_{\mu}A_{\nu}(x)\partial_{\alpha}A_{\beta}(y) \rangle_{+} - \langle \partial_{\mu}A_{\nu}(x)\partial_{\beta}A_{\alpha}(y) \rangle_{+} - \langle \partial_{\nu}A_{\mu}(x)\partial_{\alpha}A_{\beta}(y) \rangle_{+}$$

$$+ \langle (\partial_{\nu}A_{\mu}(x)\partial_{\beta}A_{\alpha}(y)) \rangle_{+}$$

$$= \frac{\hbar c}{i} [\eta_{\nu\beta} \partial^x_{\mu} \partial^y_{\alpha} - \eta_{\nu\alpha} \partial^x_{\mu} \partial^y_{\beta} - \eta_{\mu\beta} \partial^x_{\nu} \partial^y_{\alpha} + \eta_{\mu\alpha} \partial^x_{\nu} \partial^y_{\beta}] D_{\gamma}(x-y)$$

$$= \frac{\hbar c}{i} \int \frac{d^4 k}{(2\pi)^4} \frac{[\eta_{\nu\beta} k_{\mu} k_{\alpha} - \eta_{\nu\alpha} k_{\mu} k_{\beta} - \eta_{\mu\beta} k_{\nu} k_{\alpha} + \eta_{\mu\alpha} k_{\nu} k_{\beta}] e^{ik \cdot (x-y)}}{k^2 - i0^+}$$
(3.51)

therefore Eq. (3.50) reads

$$< F_{\mu\nu}(x)F^{\mu\nu}(x)F_{\alpha\beta}(y)F^{\alpha\beta}(y) >_{+}$$

$$= 2\left(\frac{\hbar c}{i}\right)^{2}\int \frac{d^{4}k}{(2\pi)^{4}} \frac{4\pi[\eta_{\nu\beta}k_{\mu}k_{\alpha}-\eta_{\nu\alpha}k_{\mu}k_{\beta}-\eta_{\mu\beta}k_{\nu}k_{\alpha}+\eta_{\mu\alpha}k_{\nu}k_{\beta}]e^{ik\cdot(x-y)}}{k^{2}-i0^{+}} \times$$

$$\int \frac{d^{4}Q}{(2\pi)^{4}} \frac{4\pi[\eta^{\nu\beta}Q^{\mu}Q^{\alpha}-\eta^{\nu\alpha}Q^{\mu}Q^{\beta}-\eta^{\mu\beta}Q^{\nu}Q^{\alpha}+\eta^{\mu\alpha}Q^{\nu}Q^{\beta}]e^{iQ\cdot(x-y)}}{Q^{2}-i0^{+}}$$

$$= -256\pi^{2}\hbar^{2}c^{2}\int \frac{d^{4}k}{(2\pi)^{4}} \frac{d^{4}Q}{(2\pi)^{4}} \frac{(k\cdot Q)^{2}+Q^{2}k^{2}/2}{(k^{2}-i0^{+})(Q^{2}-i0^{+})}e^{i(k+Q)\cdot(x-y)}$$

$$(3.52)$$

The self-energy part of the propagator is equal to

$$\tilde{\Pi}(x-y) = -\frac{i512\pi^2\sqrt{2\hbar}G_F}{c} \left(\frac{\alpha}{24\pi^2}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4} \times \frac{((k\cdot Q)^2 + Q^2k^2/2)e^{i(k+Q)\cdot(x-y)}}{(k^2 - i0^+)(Q^2 - i0^+)}$$
(3.53)

The Fourier transform of Eq. (3.53) reads

$$\Pi(q) = -\frac{i512\pi^2 \sqrt{2\hbar} G_F}{c} \left(\frac{\alpha}{24\pi^2}\right)^2 \int d^4x \frac{d^4k}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4} \times \frac{((k \cdot Q)^2 + Q^2 k^2/2) e^{i(k+Q-q) \cdot (x-y)}}{(k^2 - i0^+)(Q^2 - i0^+)} = -\frac{i8\sqrt{2\hbar} G_F}{c} \left(\frac{\alpha}{3\pi}\right)^2 \int d^4k \frac{d^4Q}{(2\pi)^4} \times \frac{((k \cdot Q)^2 + Q^2 k^2/2) \delta(k + Q - q)}{(k^2 - i0^+)(Q^2 - i0^+)}.$$
(3.54)

The imaginary part of the self energy reads

$$\Im m\Pi(q) = \frac{i\sqrt{2}\hbar G_F}{2c} \left(\frac{\alpha}{3\pi^3}\right)^2 \times \int d^4k d^4Q ((k \cdot Q)^2 + Q^2 k^2/2) \delta(k + Q - q) \delta(k^2) \delta(Q^2) \\ = \frac{\sqrt{2}\hbar G_F q^4}{8c} \left(\frac{\alpha}{3\pi^2}\right)^2 I(q^2, 0) = \frac{\sqrt{2}\hbar G_F q^4}{c} \left(\frac{\alpha}{12\pi}\right)^2. \quad (3.55)$$

The possible spin 0 configurations of two on shell photons are (1, -1), (-1, 1). The photons of the self energy loop are virtual, off shell particle, therefore a spin 0 status can be obtained as (1, -1), (0, 0), (-1, 1). To write the correct value for decay rate we must first multiply Eq. (3.55) by a factor of 2/3 and then by \hbar/M_H . Finally the decay rate reads

$$\Gamma(H \to \gamma \gamma) = \frac{2}{3} \left(\frac{\alpha}{12\pi}\right)^2 \frac{\sqrt{2}G_F M_H^2}{\hbar c} \frac{M_H c^2}{\hbar}.$$
(3.56)

This decay rate, being proportional to the fine structure constant α squared, is much less than the other ones. Anyway it is interesting to show the technique used.

3.6.2 Fermionic decay mode

The Higgs particle interacts with all the massive lepton families e, μ, τ and with all the quark generations, (d, u), (s, c), and (b, t). In particular the decay, $H \rightarrow t\bar{t}$, happens if the Higgs mass were in a region already excluded by past experimental results. The trace of energy-momentum tensor is

$$T(x) = -mc^2 \overline{\psi}(x)\psi(x). \qquad (3.57)$$

Among all the possible fermionic decays of the Higgs particle we have to note that, the coupling being proportional to the mass of the fermion, the $H \rightarrow b\bar{b}$ will be the one with the biggest decay rate. Moreover the Higgs decay into a quark anti-quark couple comes with an extra factor of 3 due to the numbers of color. We can directly write Eq. (3.33) using the trace of the stress tensor defined in Eq. (3.57)

$$\begin{split} \tilde{\Pi}(x-y) &= \frac{i2\sqrt{2}G_F m^2}{\hbar c} < \overline{\psi}(x)\psi(x)\overline{\psi}(y)\psi(y) >_+ \\ &= -\frac{i2\sqrt{2}G_F m^2}{\hbar c} < \psi(y)\overline{\psi}(x)\psi(x)\overline{\psi}(y) >_+ \\ &= \frac{i2\sqrt{2}G_F m^2}{\hbar c} Tr\{S_F(y-x)S_F(x-y)\} \\ &= \frac{i2\sqrt{2}G_F m^2}{\hbar c} Tr\int \frac{d^4k}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4} \frac{e^{i(k+Q)\cdot(x-y)}(\gamma^{\mu}k_{\mu}+K)}{(k^2+K^2-i0^+)} \frac{(\gamma^{\nu}Q_{\nu}-K)}{(Q^2+K^2-i0^+)} \\ &= -\frac{i8\sqrt{2}G_F m^2}{\hbar c}\int \frac{d^4k}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4} \frac{e^{i(k+Q)\cdot(x-y)}(k\cdot Q+K^2)}{(k^2+K^2-i0^+)(Q^2+K^2-i0^+)} \\ &= -\frac{i4\sqrt{2}G_F m^2}{\hbar c}\int \frac{d^4k}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4} e^{i(k+Q)\cdot(x-y)} \times \\ &= \frac{((k+Q)^2+4K^2-(k^2+K^2)-(Q^2+K^2))}{(k^2+K^2-i0^+)(Q^2+K^2-i0^+)}. \end{split}$$
(3.58)

The Fourier transform of the self-energy gives

$$\Pi(q^2) \ = \ \int d^4x e^{-iq\cdot x} \tilde{\Pi}(x)$$

$$= \frac{i4\sqrt{2}G_F m^2}{\hbar c} \int \frac{d^4 Q}{(2\pi)^4} \times \frac{(q^2 + 4K^2 - ((q-Q)^2 + K^2) - (Q^2 + K^2))}{((q-Q)^2 + K^2 - i0^+)(Q^2 + K^2 - i0^+)}.$$
 (3.59)

Taking the imaginary part of this equation allows to simply write

$$\Im m \Pi(q^2) = -\frac{\sqrt{2}G_F m^2 q^2}{4\pi^2 \hbar c} \left(1 + \frac{4K^2}{q^2}\right) \int d^4 Q \delta((q-Q)^2 + K^2) \delta(Q^2 + K^2)$$

$$= -\frac{\sqrt{2}G_F m^2 q^2}{4\pi^2 \hbar c} \left(1 + \frac{4K^2}{q^2}\right) I(q^2, K^2)$$

$$= -\frac{\sqrt{2}G_F m^2 q^2}{8\pi \hbar c} \left(1 + \frac{4K^2}{q^2}\right)^{3/2}.$$
(3.60)

The decay rate is again obtained from Eq. (3.34) and is equal to

$$\Gamma(H \to f\bar{f}) = \left(N_c \frac{\sqrt{2}G_F m^2}{8\pi\hbar c}\right) \left(\frac{M_H}{\hbar c}\right) \left(1 - \frac{4m^2}{M_H^2}\right)^{3/2},\qquad(3.61)$$

where N_c is the number of colors, 3 for a quark and 1 for a lepton.

3.6.3 Weak decay modes

The calculations of decay rate of the Higgs boson into charged and neutral weak bosons are very similar. We calculate them in parallel. The action describing the interaction reads

$$S_{int,W} = -\frac{2K_W^2}{vc} \int d^4x \sigma W^{\dagger}_{\mu} W^{\mu}$$

$$S_{int,Z} = -\frac{K_Z^2}{vc} \int d^4x \sigma Z_{\mu} Z^{\mu}.$$
(3.62)

The effective actions for the self-energy parts read

$$S_{eff,W} = \frac{2i}{\hbar} \frac{K_W^4}{v^2 c^2} \int d^4 x d^4 y N(\sigma(x)\sigma(y)) < W^{\dagger}_{\mu}(x) W^{\mu}(x) W^{\dagger}_{\nu}(y) W^{\nu}(y) > S_{eff,Z} = \frac{i}{2\hbar} \frac{K_Z^4}{v^2 c^2} \int d^4 x d^4 y N(\sigma(x)\sigma(y)) < Z(x) Z^{\mu}(x) Z_{\nu}(y) Z^{\nu}(y) > (3.63)$$

The polarizations are then given by

$$\Pi_{W}(x-y) = \frac{8iK_{W}^{4}}{\hbar^{2}c^{2}v^{2}} \langle W_{\mu}^{\dagger}(x)W^{\nu}(y) \rangle_{+} \langle W_{\nu}^{\dagger}(y)W^{\mu}(x) \rangle_{+}$$

$$= -\frac{8iK_{W}^{4}}{v^{2}}D_{\mu\nu,W}(x-y)D_{W}^{\mu\nu}(x-y)$$

$$\Pi_{Z}(x-y) = \frac{4iK_{Z}^{4}}{\hbar^{2}c^{2}v^{2}} \langle Z_{\mu}(x)Z_{\nu}(y) \rangle_{+} \langle Z^{\mu}(x)Z^{\nu}(y) \rangle_{+}$$

$$= -\frac{4iK_{Z}^{4}}{v^{2}}D_{\mu\nu,Z}(x-y)D_{Z}^{\mu\nu}(x-y). \qquad (3.64)$$

At first we define this function³

$$\mathcal{F}(k,Q,K) = \left(\eta_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{K^2}\right) \left(\eta^{\mu\nu} + \frac{Q^{\mu}Q^{\nu}}{K^2}\right),\qquad(3.65)$$

and then we write the imaginary part of the Fourier transform of the self energy parts

$$\Im m \Pi_W(q) = \frac{K_W^4}{2\pi^2 v^2} \int d^4 k d^4 Q \mathcal{F}(k, Q, K_W) \\ \times \delta(k^2 + K_W^2) \delta(Q^2 + K_W^2) \delta(q - k - Q) \\ = \frac{K_W^4}{8\pi^2 v^2} I(q, K_W) \left(12 + \frac{4q^2}{K_W^2} + \frac{q^4}{K_W^4}\right)$$

³We drop here the pedeces Z and W.

$$= \frac{K_W^4}{16\pi v^2} \sqrt{1 + \frac{4K_W^2}{q^2} \left(12 + \frac{4q^2}{K_W^2} + \frac{q^4}{K_W^4}\right)}$$

$$\Im m \Pi_Z(q) = \frac{K_Z^4}{4\pi^2 v^2} \int d^4 k d^4 Q \mathcal{F}(k, Q, K_Z)$$

$$\times \delta(k^2 + K_Z^2) \delta(Q^2 + K_Z^2) \delta(q - k - Q)$$

$$= \frac{K_Z^4}{32\pi v^2} \sqrt{1 + \frac{4K_Z^2}{q^2}} \left(12 + \frac{4q^2}{K_Z^2} + \frac{q^4}{K_Z^4}\right).$$
 (3.66)

These expressions are simplified defining $x_W = -K_W^2/q^2$ and $x_Z = -K_Z^2/q^2$

$$\Im m \Pi_W(q) = \frac{\hbar q^4}{16\pi v^2} \sqrt{1 - 4x_W^2} (12x_W^4 - 4x_W^2 + 1)$$

$$\Im m \Pi_Z(q) = \frac{\hbar q^4}{32\pi v^2} \sqrt{1 - 4x_Z^2} (12x_Z^4 - 4x_Z^2 + 1).$$
(3.67)

At this point we can write the decay rate as

$$\Gamma_{H \to W^+W^-} = \frac{\hbar}{M_H} \Im m \Pi_W (q^2 - \frac{M_H^2 c^2}{\hbar^2})
= \frac{M_H^3 c^4}{16\pi \hbar^2 v^2} \sqrt{1 - 4x_W^2} (12x_W^4 - 4x_W^2 + 1)
= \left(\frac{\sqrt{2}G_F M_H^2}{16\pi \hbar c}\right) \frac{M_H c^2}{\hbar} \sqrt{1 - 4x_W^2} (12x_W^4 - 4x_W^2 + 1)
\Gamma_{H \to Z^0 Z^0} = \left(\frac{\sqrt{2}G_F M_H^2}{32\pi \hbar c}\right) \frac{M_H c^2}{\hbar} \sqrt{1 - 4x_Z^2} (12x_Z^4 - 4x_Z^2 + 1). \quad (3.68)$$

3.7 Gravitational decay modes

The decay of the Higgs particle into graviton pairs gg is either neglected or wrongly calculated. We believe that because of gravity, the mass and the Higgs field are deeply connected. The value of $\Gamma(H \to gg)$ will predominate all the decay modes of the Higgs boson, bieng the most important one. The gravitons will escape detection with a probability of virtually unity. Thus any Higgs producing two gravitons will be overlooked unless the experimentalists keep their eyes wide open for missing four momentum as is the case for decays into neutrinos. From an experimental prospective, the decay rate into "missing" graviton four momenta can explain why it might not be possible to detect as many Higgs events as already theorized in many books and articles. Nevertheless it may still be possible to indirectly detect the Higgs particle at LHC by looking for the missing gravitational four momenta. Although at first this eventuality could not seem to be very interesting, we should always remember that it is not the first time that information about the physics of an elementary particle is obtained in this way. For example the Z boson is well known to decay ~ 20% of the time according to $Z \rightarrow \nu \bar{\nu} + \gamma_{\text{soft}}$. To calculate the Higgs decay rate into two gravitons, we need a suitable expression for the trace of the gravitational pressure-energy tensor. In the weak field limit and at the second order exapansion in the metric perturbation $h_{\mu\nu(x)}$, one can write

$$\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)} \right\rangle = \frac{8\pi c^4}{G}T^{(2)}_{\mu\nu} = \frac{8\pi c^4}{G}t^{(2)}_{\mu\nu} \tag{3.69}$$

and then employs the pseudo tensor $t^{\mu\nu}(x)$ instead of $T_{\mu\nu}[27]$. The time ordered vacuum expectation value is then written as

$$< t(x)t(y) >_{+} = \left(\frac{c^{4}}{32\pi G}\right)^{2} < \left(\partial^{\mu}h^{\alpha\beta}\partial_{\mu}h_{\alpha\beta}\right)_{x} \left(\partial^{\nu}h^{\gamma\delta}\partial_{\nu}h_{\delta\gamma}\right)_{y} >_{+} .$$
(3.70)

Using Wick's theorem, the explicit expansion of the time order reads

$$<\partial^{\mu}h^{\alpha\beta}(x)\partial_{\mu}h_{\alpha\beta}(x)\partial^{\nu}h^{\gamma\delta}(y)\partial_{\nu}h_{\delta\gamma}(y)>_{+}=$$

$$2<\partial^{\mu}h^{\alpha\beta}(x)\partial^{\nu}h^{\gamma\delta}(y)>_{+}<\partial_{\mu}h_{\alpha\beta}(x)\partial_{\nu}h_{\delta\gamma}(y)>_{+}.$$
(3.71)

Apart from a constant, and the partial derivatives this last expression describes a graviton propagating in a close loop from x to y and y to x, confront with Eq. 3.42. Eq. (3.70) reads

$$< t(x)t(y) >_{+} = -2\left(\frac{\hbar c}{32\pi}\right)^{2} (\partial_{\mu}\partial_{\nu}D_{\alpha\beta\gamma\delta}(x-y))(\partial^{\mu}\partial^{\nu}D^{\alpha\beta\gamma\delta}(x-y))$$

$$= -2\left(\frac{\hbar c}{32\pi}\right)^{2}G_{\alpha\beta\gamma\delta}\int \frac{d^{4}k}{(2\pi)^{4}}\frac{4\pi k_{\mu}k_{\nu}e^{ik\cdot(x-y)}}{k^{2}-i0^{+}} \times$$

$$G_{\alpha\beta\gamma\delta}\int \frac{d^{4}Q}{(2\pi)^{4}}\frac{4\pi Q^{\mu}Q^{\nu}e^{iQ\cdot(x-y)}}{Q^{2}-i0^{+}}.$$

$$(3.72)$$

The product $G^{\alpha\beta\gamma\delta}G_{\alpha\beta\gamma\delta}$ gives

$$G^{\alpha\beta\gamma\delta}G_{\alpha\beta\gamma\delta} = 4(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}) \times 4(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

= 16[(16 + 4 - 4) + (4 + 16 - 4) - (4 + 4 - 16)]
= 640 (3.73)

and therefore Eq.3.72

$$\langle t(x)t(y) \rangle_{+} = -20\hbar^{2}c^{2}\int \frac{d^{4}k}{(2\pi)^{4}} \frac{k_{\mu}k_{\nu}e^{ik\cdot(x-y)}}{k^{2}-i0^{+}} \times \int \frac{d^{4}Q}{(2\pi)^{4}} \frac{Q^{\mu}Q^{\nu}e^{iQ\cdot(x-y)}}{Q^{2}-i0^{+}}.$$
 (3.74)

The Fourier transform of the self energy reads

$$\Pi(q^2) = -\frac{20i}{v^2} \int d^4x \frac{d^4k}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4} \frac{(k \cdot Q)^2 e^{i(k+Q-q) \cdot x}}{(k^2 - i0^+)(Q^2 - i0^+)}$$
$$= -\frac{20i}{v^2} \int d^4k \frac{d^4Q}{(2\pi)^4} \frac{(k \cdot Q)^2 \delta^4(q - k - Q)}{(k^2 - i0^+)(Q^2 - i0^+)}$$
(3.75)

To calculate the decay rate we need to take the imaginary part of Eq. (3.75).

$$\Im m\Pi(q^2) = -\Re e \frac{20}{v^2} \int d^4k \frac{d^4Q}{(2\pi)^4} \frac{(k \cdot Q)^2 \delta^4(q - k - Q)}{(k^2 - i0^+)(Q^2 - i0^+)}$$

$$= \frac{5}{4\pi^2 v^2} \int d^4k d^4Q (k \cdot Q)^2 \delta^4(q - k - Q) \delta(k^2) \delta(Q^2)$$

$$= \frac{5q^4}{16\pi^2 v^2} I(q^2, 0) = \frac{5q^4}{32\pi v^2}$$
(3.76)

To obtain the $H \rightarrow gg$ decay rate we must remember that the outgoing gravitons have only two possible values for the spin polarizations (+2, -2), while the gravitons used in the calculation are off mass-shell particles with five possible values for the spin polarizations (+2, +1, 0, -1, -2). Thus to write down the correct result we must multiply Eq. (3.76) by a factor of 2/5 because of the presence of three ghosts in the theory. The decay rate is then

$$\Gamma(H \to gg) = \frac{\hbar}{M_H} \Im m \Pi(q^2 = -M_H^2 c^2/\hbar^2)$$
$$= \left(\frac{\sqrt{2}G_F M_H^2}{16\pi\hbar c}\right) \frac{M_H c^2}{\hbar}$$
(3.77)

For the well accepted value of the Higgs mass, 114.4 $GeV/c^2 < M_H <$ 195 GeV/c^2 , the decay rate is included in the interval [0.96 : 4.88]GeV. These values make $H \rightarrow gg$ the predominant decay mode compared to the other Standard Model Higgs decays. The branching ratio between the gravitational and the other decays is close to 2.44 in the case of the Higgs mass close to the upper limit of $195 < GeV/c^2$ and it is about equal to 10^2 in the case of an Higgs mass close to the lower limit of $114 < GeV/c^2$. We predict that the LHC experiments will not detect as many Higgs particles as theorized because most of them will "disappear" decaying into two undetectable gravitons.

Chapter 4

Proximity Effects

4.1 Final State Interaction

In nuclear physics, it is well known that the electromagnetic (Coulomb) interaction between final state products can drastically effect particle reaction rates. Near thresholds, for example, nuclear alpha decay is strongly suppressed while nuclear beta decay is enhanced by final state Coulomb interactions. Here we discuss high energy physics enhancement and/or suppression of reactions wherein the potentials must include weak and strong as well as electromagnetic interactions. Potentials due to the exchange of gluons and the exchange of a hypothetical Higgs particle are explicitly considered. The Coulomb interaction

$$U_{Coul} = Z_1 Z_2 \left(\frac{\hbar c\alpha}{r}\right) \tag{4.1}$$

between the final products of nuclear reactions can have a large effect on particle reaction rates and cross sections. If the final state Coulomb potential is repulsive, then the reaction is suppressed. Such is the case for (say) nuclear alpha decay or inverse nuclear beta decay. If the Coulomb final state interaction is attractive, then the reaction is enhanced. Such is the case for nuclear beta decay. The effects of the final state Coulomb potential is (i) particularly large near threshold and (ii) requires methods far beyond standard low order perturbation theory for a proper calculation.

Although the application of final state interaction theory to problems of nuclear physics is by now fairly routine, the theory is *not* yet quite standard practice in high energy physics wherein perturbation theory perhaps *too often reigns supreme*. Yet the potentials of the weak and strong interactions, if *not* the gravitational potential

$$U_{Newton} = -G\left(\frac{M_1M_2}{r}\right),\tag{4.2}$$

surely play a final state interaction role similar to the Coulomb interaction in nuclear physics. In particular, we wish to discuss these final state interactions due to both weak and strong forces. There has been considerable earlier work[36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48] with applications to the W^+W^- and heavy flavor $q\bar{q}$ production.

The strong force potential, presumed due to gluon exchange, has the form

$$U_{Glue} = \left(\frac{\hbar c \alpha_s}{r}\right) \mathbf{T}_1 \cdot \mathbf{T}_2, \qquad (4.3)$$

in which the matrices $\{\mathbf{T}\}\$ are the color SU(3) group generators. The gluon exchange potential Eq. (4.3) is written down in close analogy to the photon exchange potential Eq. (4.1); It reads

$$U_{\bar{q}q} = V_{q\bar{q}} = -\frac{4}{3} \left(\frac{\hbar c \alpha_s}{r} \right) \quad (\text{quark anti - quark}),$$
$$U_{\bar{q}\bar{q}} = V_{qq} = -\frac{2}{3} \left(\frac{\hbar c \alpha_s}{r} \right) \quad (\text{quark quark}). \tag{4.4}$$

However, Eqs.(4.3) and (4.4) hold true only in the $r \to 0$ limit. For large r, the presumed confinement (linear) portion of the potential is presently only partially understood. The details of the full quark potentials are summarized in 4.7.

The weak Higgs exchange potential has the form

$$U_{Higgs} = -\left(\frac{\sqrt{2}G_F}{4\pi}\right) \left(\frac{M_1 M_2}{r}\right) e^{-(M_H c/\hbar)r}$$
(4.5)

in close analogy to the graviton exchange potential Eq. (4.2). Here, the Fermi interaction strength G_F plays a role analogous to the Newtonian gravitational coupling G while the mass M_H of the Higgs particle plays the role of an inverse screening length. That the graviton exchange potential should bear a strong resemblance to the Higgs exchange potential (apart from screening) is due to the fact that gravitational mass is the source and sink of the gravitational field while inertial mass is the source and sink of the Higgs field. The principle of equivalence between gravitational and inertial mass dictates that the Higgs particle (if it exists) is intimately connected with gravity.

To compute the final state interaction effects of the effective exchange potentials which may enhance or may suppress the reaction, it is convenient to employ the quasi-classical relativistic Hamilton-Jacobi equation. If the potential is repulsive and the reaction suppressed, then the effect lies mainly in the classically disallowed region (quantum tunneling). If the potential is attractive and the reaction is enhanced, then the effect arises due to the strong overlap of the attracted particle wave functions. This point is illustrated in Sec.4.2 wherein the amplification of beta decay and the suppression of inverse beta decay will be reviewed. In Sec.4.3 the attractive gluon exchange potential will be discussed with regard to enhancement factors for the production of quark anti-quark pairs, i.e. quark jets. Final state interactions induced by the Higgs field are discussed in Sec.4.4 for $Z\bar{Z}$ and W^+W^- production. The Higgs effects become more important as the mass increases. In principle these effects may be of use in experimental probes which seek to verify that the Higgs field exists. This point is briefly discussed in the concluding Sec.4.6.

4.2 The Coulomb Potential

Consider the inverse beta decay of a nucleus written as the reaction

$$\bar{\nu}_e + (Z+1, A) \to (Z, A) + e^+.$$
 (4.6)

The finally produced positron interacts with the final nucleus via the repulsive Coulomb potential

$$U_{+}(r) = \frac{\hbar c \alpha Z}{r}.$$
(4.7)

Since the nucleus is much more massive than is the positron, it is normally sufficient to treat the Coulomb interaction potential as if it were *external*. The positron energy equation then reads

$$(E - U_{+}(r))^{2} = m^{2}c^{4} + c^{2}|\mathbf{p}|^{2}.$$
(4.8)

Relativistic Hamilton-Jacobi dynamics asserts that the momentum is the gradient of the positron action

$$\mathbf{p} = \mathbf{grad}W(\mathbf{r}, E). \tag{4.9}$$

The radial solution of Eqs.(4.8) and (4.9) reads

$$c^{2}p(r,E)^{2} = c^{2} \left[\frac{\partial W(r,E)}{\partial r}\right]^{2} = (E - U_{+}(r))^{2} - m^{2}c^{4},$$

$$c^{2}p(r,E)^{2} = \left[E - \frac{\hbar c\alpha Z}{r} + mc^{2}\right] \left[E - \frac{\hbar c\alpha Z}{r} - mc^{2}\right].$$
 (4.10)

The classically allowed $(p^2 > 0)$ and classically disallowed $(p^2 < 0)$ regions in the radial coordinate r are defined by

$$0 < r < a \text{ or } r > b \implies (allowed),$$

 $a < r < b \implies (disallowed), \qquad (4.11)$

wherein

$$a = \frac{\hbar c \alpha Z}{E + mc^2}$$
 and $b = \frac{\hbar c \alpha Z}{E - mc^2}$. (4.12)

The reaction suppression is described by the barrier factor B for the regime in which classical motion is forbidden; In detail

$$B = \frac{2}{\hbar} \Im m |W(b, E) - W(a, E)| = \frac{2}{\hbar} \int_{a}^{b} |\Im m[p(r, E)]| dr,$$

$$B = \frac{2}{\hbar c} \int_{a}^{b} \sqrt{\left|E - \frac{\hbar c \alpha Z}{r} + mc^{2}\right| \left|E - \frac{\hbar c \alpha Z}{r} - mc^{2}\right|} dr,$$

$$B(E, Z\alpha) = 2\pi Z\alpha \left[\frac{E}{\sqrt{E^{2} - m^{2}c^{4}}} - 1\right] = 2\pi Z\alpha \left[\left(\frac{c}{v}\right) - 1\right], \quad (4.13)$$

where v is the positron velocity. In the non-relativistic limit $v \ll c$, the Coulomb barrier factor $B \approx (2\pi Z \alpha c/v)$ is well known. Eq. (4.13) represents the relativistic theory in which the barrier factor vanishes in the high energy limit $(v \rightarrow c)$.

The physical picture in the relativistic theory is worthy of note. The "tunneling" through the barrier is in reality electronic "pair creation" under the barrier for (a < r < b). When the pair is created the positron half of the pair rushes off to infinity $(b < r < \infty)$. The electron half of the pair falls into the center (0 < r < a) converting one of the nuclear protons into a neutron and emitting an electron neutrino. The total inverse beta decay reaction may then be represented as

$$(\text{vacuum}) \rightarrow e^- + e^+,$$

$$\bar{\nu}_e + e^- + (Z+1, A) \rightarrow (Z, A), \qquad (4.14)$$

for which Eq. (4.6) is the *total* reaction. The full suppression factor cross section ratio induced by the Coulomb repulsion between the positron and

the final state nucleus is given by

$$S(E,Z) = \frac{\sigma \left[\bar{\nu}_e + (Z+1,A) \to (Z,A) + e^+ \right]}{\sigma^{(0)} \left[\bar{\nu}_e + (Z+1,A) \to (Z,A) + e^+ \right]},$$

$$S(E,Z) = \frac{B(E,Z\alpha)}{exp(B(E,Z\alpha)) - 1}.$$
(4.15)

Eq. (4.15) concludes our discussion for the case of inverse beta decay.

For the case of beta decay

$$(Z-1,A) \to (Z,A) + e^- + \bar{\nu}_e,$$
 (4.16)

the Coulomb potential between the outgoing electron and the nucleus is attractive

$$U_{-}(r) = -\frac{\hbar c \alpha Z}{r} . \qquad (4.17)$$

The Hamilton-Jacobi equation for the attractive Coulomb energy reads

$$(E - U_{-}(r))^{2} = m^{2}c^{4} + c^{2}|\mathbf{p}|^{2}$$
 wherein $\mathbf{p} = \mathbf{grad}W(E, \mathbf{r}).$ (4.18)

Since there is a particle anti-particle "duality" corresponding to positive and negative energy solutions in any relativistic theory, if an electron sees an attractive potential then the positron will see a repulsive potential. Relativistic dynamics with Poincaré symmetry automatically includes both particle and antiparticle dynamics. Employing this duality of solutions one finds that the beta decay for the electron is again described by Eq. (4.13) but this time with an amplification factor. The full ratio of decay rates corresponds to



Figure 4.1: For an outgoing beta decay electron or inverse beta decay positron with energy $E = \{mc^2/\sqrt{1 - (v/c)^2}\}$ there will be, respectively, an attraction or repulsion from the central nuclear final state charge Ze. Shown are the curves for the electron rate amplification A(Z, E) and the positron rate suppression S(Z, E) implicit in the conventional Coulomb final state corrections.

[34, 35]

$$A(E,Z) = \frac{\Gamma[(Z-1,A) \to (Z,A) + e^{-} + \bar{\nu}_{e}]}{\Gamma^{(0)}[(Z-1,A) \to (Z,A) + e^{-} + \bar{\nu}_{e}]},$$

$$A(E,Z) = \frac{B(E,Z\alpha)}{1 - exp(-B(E,Z\alpha))}.$$
(4.19)

The suppression factor for an outgoing positron and the amplification factor for an outgoing electron are plotted in Figure 4.1. For the inverse beta decay of Eq. (4.15), the positron emerges with velocity

$$v = \frac{c\sqrt{E^2 - m^2 c^4}}{E}$$
(4.20)

and the cross section is suppressed by the coulomb interaction factor S. For the beta decay case in Eq. (4.19), the electron can still emerge with the velocity in Eq. (4.20) but the decay rate is enhanced with an amplification factor A.

4.3 The Gluon Exchange Potential

Consider the production of a quark and an anti-quark with momenta p and \bar{p} . The pair interacts with an attractive gluon exchange potential $U_{\bar{q}q}(r)$. On a short distance scale one expects a Coulomb-like potential with a strong interaction charge which dominates the actual Coulomb potential; i.e.

$$U_{\bar{q}q}(r) = -\frac{4}{3} \left(\frac{g^2}{4\pi r} \right) = -\frac{4}{3} \left(\frac{\hbar c \alpha_s}{r} \right) \quad \text{as} \quad r \to 0.$$
 (4.21)

On a larger distance scale, the potential is discussed in 4.7.

The total mass \sqrt{s} of the final state pair is determined by

$$c^{2}s = -P^{2} = -(p + \bar{p})^{2} = 2(c^{2}m^{2} - \bar{p} \cdot p).$$
(4.22)

In the center of mass reference frame of the pair ($\mathbf{P} = \mathbf{p} + \bar{\mathbf{p}} = 0$), kinematics dictates

$$-c^{2}\bar{p}\cdot p = \bar{E}E - c^{2}\bar{\mathbf{p}}\cdot\mathbf{p} = c^{4}m^{2} + 2c^{2}|\mathbf{p}|^{2}; \qquad (4.23)$$

In detail, the relative momentum of the quark anti-quark pair is given by

$$|\mathbf{p}| = c\sqrt{(s/4) - m^2} \ . \tag{4.24}$$

The enhancement factor for the quark anti-quark jet production then follows a form closely analogous to the Coulomb case in Eqs.(4.17) and (4.19). The production amplification is

$$A_{\bar{q}q}(s) = \frac{\Gamma_{\bar{q}q}(s)}{\Gamma_{\bar{q}q}^{(0)}(s)},$$

$$B_{\bar{q}q}(s) = \frac{4\pi\alpha_s}{3} \left[\sqrt{\frac{s}{s-4m^2}} - 1 \right],$$

$$A_{\bar{q}q}(s) = \frac{B_{\bar{q}q}(s)}{1-exp(-B_{\bar{q}q}(s))},$$
(4.25)

which has been plotted in Figure 4.2. The amplification is particularly strong near the threshold value $s_0 = 4m^2$.





Figure 4.2: The gluon exchange potential amplification of quark anti-quark jet production is plotted as a function of the invariant mass squared. The amplification begins at threshold. A reasonable but approximate value for the strong coupling strength α_s has been employed.

4.4 The Higgs Exchange Potential

The calculation of Higgs exchange amplification factor from the potential in Eq. (4.5) is a bit more delicate due to the screening effect of the Higgs mass M_H . As shown in what follows, it turns out that the Higgs mass drops out of the result since the amplification factor is determined by the wave function of the two produced particle at zero distance for a fixed time. In effect, this represents a "zero space time interval" for the exchange and it is well known that the nature of the light cone singularity in the mass propagator is mass independent. The Higgs boson exchange Feynman diagram producing the exchange potential is shown in Figure 4.3.

The action associated with this exchange is given by

$$S_{Higgs} = \frac{\sqrt{2}G_F}{2c^5} \int \int T(x)D(x-y)T(y)d^4xd^4y,$$
 (4.26)

wherein T(x) is the trace of the stress tensor and D(x-y) is the Higgs boson propagator

$$D(x-y) = \hbar^2 \int \left[\frac{e^{ip \cdot (x-y)/\hbar}}{p^2 + (M_H c)^2 - i0^+} \right] \frac{d^4 p}{(2\pi\hbar)^4}.$$
 (4.27)

A more physical space-time representation of the Higgs boson propagation follows from the Schwinger proper time representation

$$D(x-y) = \frac{M_H}{8\pi^2\hbar} \int_0^\infty e^{[iM_H/2\hbar]\{-c^2\tau + [(x-y)^2/\tau]\}} \left(\frac{d\tau}{\tau^2}\right).$$
 (4.28)

For two particles moving at uniform velocities the trace of the stress tensor quasi-classical sources reads $T_{a,b}(x) = -M_{a,b}c^3 \int \delta(x - v_{a,b}\tau)d\tau$. Eq. (4.26) now yields the action

$$S_{ab} = \left(\frac{\sqrt{2}G_F M_a M_b}{c}\right) c^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(v_a \tau_a - v_b \tau_b) d\tau_a d\tau_b .$$
(4.29)

If Eq. (4.28) is substituted into Eq. (4.29), then the resulting Gaussian integrals over $d\tau_a$ and $d\tau_b$ can be performed yielding

$$S_{ab} = \left(\frac{\sqrt{2}G_F m_a m_b}{c}\right) \int_0^\infty \tilde{F}(v_a, v_b, \tau) \left(\frac{d\tau}{\tau}\right), \qquad (4.30)$$

wherein

$$\tilde{F}(v_a, v_b, \tau) = \left(\frac{c^2}{4\pi\sqrt{(v_a \cdot v_b)^2 - c^4}}\right) e^{-iM_H c^2 \tau/2\hbar}.$$
(4.31)

The Higgs mass M_H drops out of the final expression for the imaginary part of the action,

$$\Im m \ S_{ab} = -\left(\frac{\sqrt{2}G_F M_a M_b}{8c}\right) \left(\frac{M_a M_b c^2}{\sqrt{(p_a \cdot p_b)^2 - (M_a M_b c^2)^2}}\right),\tag{4.32}$$

wherein the momenta $p_a = M_a v_a$ and $p_b = M_b v_b$ have been introduced.

Suppose the production of a particle anti-particle pair each of mass M. Associated with such a mass is a weak coupling strength

$$\alpha_F(M) = \left(\frac{\sqrt{2}G_F M^2}{4\pi\hbar c}\right) \tag{4.33}$$

such that

$$B_{pair}(s) = -\frac{2}{\hbar} \Im m S_{pair} = 2\pi \alpha_F(M) \left(\frac{M^2}{\sqrt{s(s-4M^2)}}\right).$$
(4.34)

The resulting Higgs induced amplification factor is determined by

$$A_{pair}(s) = \frac{B_{pair}(s)}{1 - \exp(-B_{pair}(s))} .$$
 (4.35)

In this regard one may consider the reactions

$$e^{+} + e^{-} \rightarrow W^{+} + W^{-},$$

$$e^{+} + e^{-} \rightarrow Z + \overline{Z}.$$
(4.36)

The amplification coupling strengths for the above reactions are, respectively,

$$2\pi \alpha_F(M_W) \approx 0.0532,$$

 $2\pi \alpha_F(M_Z) \approx 0.0687.$ (4.37)

For these massive particles the Higgs boson exchange induced amplification is somewhat larger than the photon exchange amplification which contributes in the W^+W^- production case.

In Figure 4.4, we exhibit the amplification factor for $Z\bar{Z}$ production due to the exchange potential of the Higgs boson; It is

$$A_{(\rm Z-pair)}(s) = \frac{\Gamma(e^+ + e^- \to Z + \bar{Z})}{\Gamma^{(0)}(e^+ + e^- \to Z + \bar{Z})},$$



Figure 4.3: The exchange of a Higgs boson between two particles gives rise to the attractive potential $U_{ab} = -(\sqrt{2}/4\pi)(G_F M_a M_b/r) \exp(-M_H r/\hbar c)$. The action S_{ab} of the exchange is examined in detail.



Figure 4.4: Shown is the amplification factor $A_{(Z-pair)}(s)$ of the Z pair production reaction $e^+ + e^- \rightarrow Z + \overline{Z}$ due to a Higgs boson exchange.

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Figure 4.5: Shown is the amplification factor $A_{(W-pair)}(s)$ of the W^+W^- pair production reaction $e^+ + e^- \rightarrow W^+ + W^-$ due to *both* Higgs boson exchange *and* photon exchange. Both Higgs exchange and photon exchange contribute to the amplification factor yielding a somewhat larger effect than for the case of $e^+ + e^- \rightarrow Z + \overline{Z}$.

$$A_{(Z-pair)}(s) = \frac{B_{(Z-pair)}(s)}{1 - \exp[-B_{(Z-pair)}(s)]} .$$

$$B_{(Z-pair)}(s) = 2\pi \alpha_F(M_Z) \left(\frac{M_Z^2}{\sqrt{s(s - 4M_Z^2)}}\right), \quad (4.38)$$

For the case of W^+W^- production, the enhancement is due to both photon exchange (which surely exists) and Higgs boson exchange (which may exist). The complete answer for W^+W^- amplified production reads

$$A_{\rm (W-pair)}(s) = \frac{\Gamma(e^+ + e^- \to W^+ + W^-)}{\Gamma^{(0)}(e^+ + e^- \to W^+ + W^-)} ,$$

$$A_{(W-pair)}(s) = \frac{B_{(W-pair)}(s)}{1 - \exp[-B_{(W-pair)}(s)]},$$

$$B_{(W-pair)}(s) = 2\pi\alpha_F(M_W) \left(\frac{M_W^2}{\sqrt{s(s - 4M_W^2)}}\right) + \pi\alpha \left(\frac{\sqrt{s} - \sqrt{s - 4M_W^2}}{\sqrt{(s - 4M_W^2)}}\right),$$
 (4.39)

which is plotted in Figure 4.5. The amplification factor for W^+W^- production is more pronounced than the amplification factor for $Z\bar{Z}$ production since photon exchange contributes to the former process but not to the latter.

4.5 Environmental Mass Shifts

The experimental evidence that the mass of electro-weak bosons seem to depend by the number of Ws or Zs produced in one event does not have so far a theoretical explanation. In this section we will prove how it is possible to obtain the mass shift starting from the action described by Eq.(4.26). So far we have consider only the imaginary part of the action. We now wish to determine the mass shift from the real part of the action. The argument is as follows: (i) The real part of the action for a particle whose life time is $\tau_0 = \Gamma^{-1}$, obeys

$$\Re e(\Delta S) = -\Delta M c^2 \int d\tau = \Delta M c^2 \tau_0,$$

$$\Delta M = -\left[\frac{\Gamma \Re e(\Delta S)}{c^2}\right].$$
 (4.40)

The dispersion relation for the full action

$$\Delta S(k^2) = -\frac{2}{\pi} \int_0^{+\infty} \frac{\Im m \Delta S(\mu^2 - i0^+)}{k^2 + \mu^2 - i0^+} d\mu.$$
(4.41)

We have already proved that the $\Im mS$ does not depend by the mass of the Higgs and therefore we can write $\Im mS = W$, where W is a constant. According to this Eq. (4.41) reads

$$\Delta S(k^2) = -\frac{W}{\pi} \int_0^{+\infty} \frac{1}{k^2 + \mu^2} d\mu^2$$

$$\Delta S(k^2) = -\frac{W}{\pi} P \int_0^{+\infty} \frac{1}{k^2 + \mu^2} d\mu^2$$

$$\Delta S(k^2) = -\frac{W}{\pi} \left(\int_0^{k^2 - \epsilon} \frac{1}{\mu^2 + k^2} d\mu^2 + \int_{k^2 + \epsilon}^{+\infty} \frac{1}{\mu^2 + k^2} d\mu^2 \right) \quad (4.42)$$

Introducing a logarithmic high mass cut-off, we can express the real part of the action to a sufficient degree of accuracy

$$\Re e(\Delta S) = \frac{W}{\pi} \ln\left(\frac{M_H c^2}{\hbar \Gamma_H}\right). \tag{4.43}$$

Eqs.(4.32), (4.40) and (4.43) imply the mass shift when two massive particles are produced together is

$$\Delta M = -\left(\frac{\hbar\Gamma}{c^2}\right) \frac{\alpha_F(M^2)M^2}{\sqrt{s(s-4M^2)}} \ln\left(\frac{M_H c^2}{\hbar\Gamma_H}\right),$$

$$\alpha_F(M^2) = \frac{\sqrt{2}G_F M^2}{4\pi\hbar c}.$$
(4.44)

A complete and organic explanation of this theory of proximity effects is clearly exposed in a new paper [49].

4.6 Conclusions

The threshold amplification and/or suppression factors familiar from the theory of final state interactions have been applied in this work in a higher energy regime. In particular we have considered final state interactions involving the Higgs boson under the supposition that it exists. Even below the threshold for the physically real Higgs particle production, the Higgs field can act as a messenger field entering into enhanced production rates for pairs of heavy particles such as $Z\bar{Z}$, W^+W^- or $t\bar{t}$ pairs[44]. The sharp peaks shown in the plots of enhancement factors will be considerably "rounded" due to (i) particle lifetime effects, (ii) radiative corrections and (iii) energy resolution factors from the energy distributions in incoming beams. Nevertheless, even if a sharp peak no longer appears, the physically "smoothed" threshold region will be affected. Since the production amplification is above the threshold mass squared, i.e. $s > s_0 \equiv 4M^2$, it follows that the threshold transition region will occur at a mass slightly higher than the threshold to be expected if the amplification were ignored. For example, experimental reaction threshold mass shifts of order

$$e^+ + e^- \to Z + \bar{Z} \quad \Rightarrow \quad \Delta M_Z \approx M_Z \alpha_F(M_Z),$$

 $e^+ + e^- \to W^+ + W^- \quad \Rightarrow \quad \Delta M_W \approx M_W[\alpha_F(M_W) + 0.5\alpha], (4.45)$



Figure 4.6: The $e^+ + e^- \rightarrow ZZ$ cross section calculated and enabled and the LEP experiment data points are shown in the picture.

would not be unreasonable and might constitute an unexpected probe of the Higgs field existence.

The experimental evidence of the final state interaction has been done calculating the cross sections for the electro weak gauge boson pair production and comparing the experimental data from the LEP experiment with the enhanced cross section. This should provide a check on the final state higgs field interaction. Moreover in the near future it will be possible to apply the same technique to the study of the top quark pair production and determine if there is as important mass difference when the quark top is produced in


Figure 4.7: The $e^+ + e^- \rightarrow W^+W^-$ cross section has been calculated with enhance factor and the LEP experiment data points are shown in the figure.

pairs or not. The reaction $e^+e^- \rightarrow ZZ$ was widely studied at LEP. In the particular case the final state correction depends only on the Higgs potential. The reaction $e^+e^- \rightarrow W^+W^-$ was also measured at LEP and in this case both Coulomb and Higgs potentials have been taken into account.



Figure 4.8: Shown is the amplification factor for the $t\bar{t}$ pair production.

4.7 Quark Potentials

The one gluon exchange potential between a quark and anti-quark has been approximated as

$$V_{Glue}(r) = \left(\frac{\hbar c \alpha_s}{r}\right) (\mathbf{T}_1 \cdot \mathbf{T}_2) = \int \left(\frac{4\pi \hbar c \alpha_s}{|\mathbf{k}|^2}\right) e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d^3\mathbf{k}}{(2\pi)^3} (\mathbf{T}_1 \cdot \mathbf{T}_2). \quad (4.46)$$

In reality, the strong interaction coupling strength itself depends on $|\mathbf{k}|^2$ so that the Coulomb-like potential is modified to read

$$\tilde{V}_{Glue}(r) = 4\pi\hbar c \int \left(\frac{\alpha_s(|\mathbf{k}|^2)}{|\mathbf{k}|^2}\right) e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d^3\mathbf{k}}{(2\pi)^3} (\mathbf{T}_1 \cdot \mathbf{T}_2).$$
(4.47)

More simply,

$$\tilde{V}_{Glue}(r) = \left[\frac{\hbar c(\mathbf{T}_1 \cdot \mathbf{T}_2)}{r}\right] \chi(r),$$

$$\chi(r) = \frac{2}{\pi} \int_0^\infty \alpha_s(k^2) \sin(kr) \frac{dk}{k}.$$
(4.48)

If $\alpha_s(k^2)$ were a constant, then Eqs.(4.48) would reduce to Eq. (4.3). However the Coulomb-like law from gluon exchange breaks down at large distances.

To see what happens as $r \to \infty$, one may presume a finite limit in the form

$$\lim_{k^2 \to 0^+} \{\hbar c k^2 \alpha_s(k^2)\} = 2\sigma, \tag{4.49}$$

and differentiate Eq. (4.48) twice with respect to r; i.e.

$$\chi''(r) = -\frac{2}{\pi} \int_0^\infty k \alpha_s(k^2) \sin(kr) dk,$$

$$\lim_{r \to \infty} \chi''(r) = -\frac{2\sigma}{\hbar c} . \tag{4.50}$$

What is called a "QCD motivated potential" results from the assertion that $\chi''(r) = -(2\sigma/\hbar c)$ for all of the important distance scales. If this is indeed the case, then

$$\tilde{V}_{Glue}(r) = \mathbf{T}_1 \cdot \mathbf{T}_2 \left\{ \frac{\hbar c \alpha_s}{r} - \sigma r \right\}, \qquad (4.51)$$

wherein the long range linear part of the potential describes the intrinsic tension σ in a QCD string. In detail, for the quark anti-quark potential

$$U_{\bar{q}q}(r) = -\frac{4}{3} \left(\frac{\hbar c \alpha_s}{r} \right) + \tau_{\bar{q}q} r \quad \text{where} \quad \tau_{\bar{q}q} = \frac{4\sigma}{3} , \qquad (4.52)$$

and for the quark-quark potential

$$U_{qq}(r) = -\frac{2}{3} \left(\frac{\hbar c \alpha_s}{r} \right) + \tau_{qq} r \quad \text{where} \quad \tau_{qq} = \frac{2\sigma}{3} . \tag{4.53}$$

Chapter 5

Conclusion

5.1 Mass and Stress

In this thesis we have proved how the principle of equivalence holds true only in a classical system. We have derived the energies eigenvalues for the non-relativistic quantum case of a linear potential (quantum bouncing ball) and shown that the mass is a parameter of the problem that does not drop out. We derived the expression for the free particle propagator. It can be written as as series of mass dependent Bessel, Hankel, and Neumann's functions and a mass-less term. This term has a primary importance because it explain why the mass of the Higgs particle does not appear in the massenhancement factor calculated in the fourth chapter. In the third chapter we have determinated the connection between the trace of the stress tensor and the Higgs field. We were able to prove that from the action

$$S_{int} = \frac{1}{vc} \int T(x)\sigma(x)d^4x$$
(5.1)

it is possible to obtain the correct Yukawa potential between two mass distributions and we calculated the decay rates of the Higgs particle into massive leptons, quarks and electroweak bosons and into mass-less gravitons and photons. We calculate that $H \rightarrow gg$ decay will be the predominant decay mode. It is important to note that this decay is a function of the Fermi constant G_F and not of the gravitational one G therefore it is many order of magnitudes bigger than previously thought [50]. This decay rate must to be taken in serious consideration to correctly understand and evaluate the number of Higgs particles that will be produced at LHC.

5.2 Mass and Higgs

Using some of the ideas developed in the third chapter we obtain in the fourth the enhancement and suppression factor for the beta and inverse beta decay. Then we applied the same technique to obtain also the enhancement factor due to the Higgs and gluon field. We determined the amplification factor in the $t\bar{t}$ production and we have shown how the Higgs mean field modifies the cross section in the reaction $e^+e^- \rightarrow W+W^-$ and $e^+e^- \rightarrow ZZ$. From the real part of the action we were able to determinate the mass shift that seems to happen when two heavy particles are produced next to each other. It is important to note that the mass-shift effect is due to the Higgs

field and does not depend upon the detection of the Higgs particle. It is very important to have a large statistics about near to threshold production of W^+W^- , Z^0Z^0 and $\bar{t}t$. Within the measure a mass difference between single and pair production will indirectly determine the Higgs particle. If no mass difference will be measured then the Higgs particle will should be removed from the Standard Model and the research for new Physics should be intensified.

In this thesis we exposed different properties of the Higgs particle physics and proposed an original way to look at the Higgs field detection. We expect that the experimental results will bring the necessary data to completely understand the Standard Model.

Note Added in Proof

This note is my response to the comments of the thesis review committee.

Introduction

To fulfill the requirements of the thesis review committee and the current Physics Department Chairman, Professor Sridhar, I have been asked to address the following issues:

- 1. Chapter 2: Explain what is your original contribution in Chapter 2 and provide a more appropriate and up-to-date set of references.
- 2. Chapter 3: Describe why your calculations presented in this chapter are original. Show in detail the differences between your calculations and the others done by Srivastava-Widom and Srivastava's former student. Describe how the Delbourgo paper results are (or are not) related to your calculation.
- 3. Chapter 4: Compare your result with the known literature and experimental results.

Chapter 2

The subject of the thesis is to describe different ideas regarding the concept of mass in Particle Physics. This is partially achieved giving a broad introduction about the interaction between particles and the gravitational field in both relativistic and non-relativistic contexts and in classical and quantum ones. It is important to remark how the principle of equivalence does not hold true in the non-relativistic quantum case.

In the second part of the chapter the Einstein's equations are introduced to explain why the inertial and gravitational mass are not equivalent. They also define how matter interacts with the gravitational field.

The final part of the chapter draws a parallel between electromagnetic and gravitational waves using non-commutative geometry.

The original contribution of the author of the thesis are:

- The definition of the real coupling between the gravitational waves and matter. The gravitational field couples with the trace of the energy momentum stress tensor. This can help to understand and correctly interpret some experimental results such as Weber's¹.
- Gauge invariance for Photons and Gravitons. The electromagnetic result has already been published². The Gravitational one has not yet been published, but it gives a fundamental scale limit about the size of the arms of a gravitational wave interferometer.

¹J. Weber, "Gravitational-Wave-Detector Events," Phys. Rev. Lett. 20, 1307 (1968)

²G. Castellani, S. Sivasubramanian, N. Fabiano, A. Widom, J. Swain, Y.N. Srivastava, G. VitielloNon-Commutative Geometry and Measurements of Polarized Two Photon Coincidence Counts, Annals Phys. 311 (2004) 191-203

Part of the department committee criticisms regarded the lack of proper references. The following publications complete the standard derivations:

A. Einstein, The Meaning of Relativity, Princeton University Press, Princeton, (1953).

R.C. Tolman, Relativity Thermodynamics and Cosmology, Oxford University Press, Oxford, (1934).

J. Weber, General Relativity and Gravitational Waves, Interscience Publishers, New York, (1961).

S. Weinberg, The Quantum Theory of Fields II, Cambridge University Press, Cambridge, (1996).

F. W. Byron, R. W. Fuller, Mathematics of Classical and Quantum Physics, Dover Publications, (1992).

S. Sivasubramanian, Y. N. Srivastava, A. Vitiello and A. Widom, Phys. Lett. A 311, 97, (2003).

S. Sivasubramanian, G. Castellani, N. Fabiano, A. Widom, J. Swain, Y.N.
Srivastava, G. Vitiello, "Quantum Limits on pixel resolution from non-commutative photon coordinates", Journal of Modern Optics, vol. 51, 1529-1534, (2004).
J. Schwinger, "Particles, Sources and Fields", Percous Books, Reading, Vol. 1, Sec. 1-3, (1998).

Chapter 3

This chapter contains an independent derivation of the result previously obtained by Srivastava and Widom [arXiv:hep-ph/0003311]. The same result has been obtained in three different ways: the way of this thesis, Srivastava and Widom's one and a former Srivastava student's one.

To derive my result one needs to start from the definition of the Higgs propagator.

$$\mathcal{D}_H(k^2) = \frac{1}{k^2 + K^2 - \Pi(k^2)}.$$
(5.2)

The graviton self-energy part of the Higgs particle propagator is related to the decay rate of the Higgs particle into gravitons5.1:

$$\Gamma = \frac{c}{\kappa} \Im m \Pi (-\kappa^2 - i0^+).$$
(5.3)

In Chapter 3 the full derivation is carried out from the definition of the



Figure 5.1: Higgs propagator with a self-energy graviton contribution.

effective and interaction action to the value of the decay rate.

Srivastava and Widom, instead, started from an effective action such as:

$$S_{eff} = -\left(\frac{2}{G < \phi}\right) \int d\Omega \chi \mathcal{L}_g, \qquad (5.4)$$

where $d\Omega = \sqrt{-g}d^4x$, $\langle \phi \rangle = M_F \hbar c$, \mathcal{L}_g is the gravitational Lagrangian, χ is related to the Higgs particle total field as $\phi = \langle \phi \rangle + \chi$ and G is the gravitational constant. Srivastava and Widom derived the $H \to gg$ decay rate calculating the value of the $\langle gg|S_{eff}|H \rangle$ matrix element. Moreover they used a Lagrangian of a harmonic oscillator, $L = (\hbar \omega/2)(a^{\dagger}a^{\dagger} + aa)$, that, in the gravitational field case, may create or may destroy two gravitons. The third approach, developed by Professor Srivastava' s former student, starts from the definition of the graviton field:

$$h_{ij}(x) = \sqrt{32G\pi} \sum_{\lambda} \int d\tilde{\mathbf{k}} \epsilon_{ij}^{\lambda}(\mathbf{k}) a^{\lambda}(\mathbf{k}) e^{ikx} + \epsilon_{ij}^{*\lambda}(\mathbf{k}) a^{\dagger\lambda}(\mathbf{k}) e^{-ikx}.$$
 (5.5)

Then she derived the $H \rightarrow gg$ decay rate using:

- a Lagrangian similar to Srivastava-Widom's one.
- The S-Matrix expansion method³.

On the other hand Professor R. Delbourgo and Doctor D. Liu calculated the decay rate of the $H \rightarrow gg$ using a different process. In this case they started from the loop-level rather than three level diagram. Not surprisingly they arrived at a different conclusion [R. Delbourgo and Dongsheng Liu hepph/0004156]. The difference between the Feynman diagrams used in this

³This method is well described in many Quantum Field Theory books. For example, at University of Perugia, Quantum Field Theory, by Mandl and Shaws, is widely used.

derivation and Srivastava-Widom's one can be better understood with picture 5.2:



Figure 5.2: Different $H \rightarrow gg$ diagrams leading to different decay rates.

Chapter 4

The committee's review pointed out that our Higgs calculations were not correct. They based their assumption on a paper where the Higgs mass was either zero or ∞ . On the other hand our Higgs phenomenology is based on the SM Higgs. Experiments over the past two decades have put stringent limits on the Higgs mass. Our calculations use a low mass Higgs in accordance with the experimental data⁴. The limit of an infinite Higgs mass is outside the scope of experiments and also our theoretical computations.

The goal of Chapter 4 is to demonstrate how it is possible to find the presence

 $^{^{4}{\}rm LEP}$ Electroweak Working Group and LEP Collaborations http://lepewwg.web.cern.ch/LEPEWWG/.

of Higgs field looking at the mass shift of massive particles when these are produced in pairs or single. Near threshold pair production of electroweak bosons like the W^{\pm} and the Z^0 or the top quark t can be used to detect the Higgs field⁵. The definition of the mass difference between massive particles produced in pairs or not was given in Eq. 4.44 of the my thesis. This equation can be rewritten as:

$$\Delta M_X \approx -\Gamma_X \left(\frac{M_X^2}{2\pi v^2}\right) \times \left(\frac{M_X^2}{M_{XX}^2}\right) \sqrt{\frac{1}{M_{XX}^2 - 4M_X^2}} \times \ln\left[\frac{M_X}{\Gamma_X}\right].$$
(5.6)

Where M_X and Γ_X are the mass and width of either W, Z or t and M_{XX} is the effective mass of the W^+W^- , Z^0Z^0 and $t\bar{t}$ pair and v is the Higgs vacuum expectation value.

Different decay modes of the electroweak bosons and the top quark make the accurate reconstruction of the particle masses a task of different difficulty. In detail it is a standard technique to precisely determine the mass of the Z^0 when it decays in e^+e^- or $\mu^+\mu^-$. A plot of the invariant masses versus the relative speed of the Z^0Z^0 allows to detect the mass shift. Fig. 5.3 shows how the mass shift is relevant only in the region close to the Z^0Z^0 effective mass threshold. The mass of the Z has been determined at LEP1⁶ and LEP2 ⁷. The average value for the mass is $M_Z = 91.1875 \pm 0.0021 GeV/c^2$. Fig.

 $^{^5\}mathrm{arXiv:hep-ph}/0511233$ contains an exhaustive analysis of the electroweak bosons and top quark mass shifts.

⁶R. Barate et al. (ALEPH Collaboration), Euro. Phys. J. C14 (2000) 1.

P. Abreu et al. (DELPHI Collaboration), Euro. Phys. J. C16 (2000) 371.

M. Acciarri et al. (L3 Collaboration), Euro. Phys. J. C16 (2000) 1.

⁷H. Li et al. (ALEPH Collaboration), ALEPH-2001-006 (2001).

R. Barate et al. (ALEPH Collaboration), Phys. Lett. B469 (1999) 287.

J. Abdallah et al. (DELPHI Collaboration), Euro. Phys. J. C30 (2003) 447.



Figure 5.3: The Z mass shift plotted against the ZZ effective mass in the ZZ threshold region.

5.4 shows the expected mass shift over the LEP2 center of mass (CM) energies. The weighted average value of the mass shift over the energy range is ~ $140 MeV/c^2$. The expected value of the Z^0 mass shift nearby threshold $(E_{CM} = 182.7 GeV)$ is almost $1 GeV/c^2$, this implies that $M_Z \sim 90.3 GeV/c^2$. The LEP experiments did not have enough near threshold statistic⁸ to precisely measure the mass of the Z^0Z^0 pair. This measure will be obtained with the new LHC accelerator at CERN from Spring 2009 to the next ten, twenty years.

⁸Note the cross section value near threshold at:

J. Alcaraz et al. A Combination of preliminary electroweak measurements and constraints on the standard model, CERN-PH-EP-2006-042, 2006, figure 9.4 page 63.



Figure 5.4: The Z mass shifts plotted against the LEP CM energy.

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