CRITICAL PHENOMENA AND THE RENORMALIZATION GROUP

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Abstract

The recent theory of critical phenomena and the renormalization group as promoted by Wilson is considered on an introductory level. The main emphasis is on the idea of the fixed point Hamiltonian (asymptotic invariance of the critical Hamiltonian under change of the length scale) and the resulting homogeneity laws.

1. CRITICAL BEHAVIOR

A. Critical Points

The transition[1] from one phase to another like melting or boiling changes the properties of a system discontinuously. Such a phase transition is called a first order transition or discontinuous transition. By varying one or several thermodynamic variables like the temperature, it is frequently possible to follow the coexistence curve so that the two distinct phases become more and more similar until both phases become equal at a certain point. If beyond this point only one homogeneous phase exists and all changes are smooth and continuous, then this point is called a critical point.[2]

Examples of critical points are (a) the termination point of the coexistence curve of a liquid and its vapor (or two phases of different density of a lattice gas like hydrogen in metals) at the critical temperature T_c and pressure p_c, (b) the critical point of separation of mixtures and alloys above (or below) which the components mix without a miscibility gap, (c) the ordering temperature of a homogeneous binary crystal below which one sublattice is primarily occupied by one species.

A second class of systems exhibits domains of magnetic or electric moments of different orientation which vanish at the critical temperature. Examples are (d) ferromagnets with ferromagnetic domains of different orientation whose spontaneous magnetization vanish continuously at the Curie temperature, (e) ferroelectrics with ferroelectric domains whose spontaneous polarization go to zero at the critical temperature, and (f) NH_4 compounds whose electric octupole moments order primilary in one or the other direction below T_c. (g) The alternating spin order of antiferromagnets goes to zero at the Neel point so that two counterphase domains become indistinguishable. Analogously one observes (h) alternating ordering of electric dipole moments in antiferroelectrics and (i) alternating ordering of electric octupole moments in NH_4 compounds.

Thirdly (j) superfluid helium and (k) superconductors are characterized by a condensate associated with a phase below T_c. This condensate vanishes continuously approaching T_c from below so that domains with different phase cannot be distinguished above T_c. This list does not exhaust the types of critical points observed. But it gives an impression of the variety of phenomena which can be described by the theory of critical phenomena.

To unify the description of critical phenomena one has introduced the concept of the order parameter. For the liquid-vapor transition and other transitions characterized by a difference of densities in both phases (sublattices) the order parameter is the difference between the expectation value of the density of the phase (sublattice) from its value at criticality. For orientational transitions the expectation value of the electric (magnetic) moment (or the difference on the sublattices) serves as the order parameter. In superfluid helium and superconductors the expectation of the condensate wave function is the order parameter. The amount of the order parameter is (approximately) the same in all phases but it differs in sign, direction and phase, respectively, in different phases (domains).

The field conjugate to the order parameter is often called the symmetry breaking field, since it breaks the symmetry of the Hamiltonian in the case of orientational phase transitions and the transitions to the superfluid and superconducting state. Without this field the Hamiltonian is invariant under certain rotations of the order parameter or invariant under the change of the phase (gauge transformation). This symmetry breaking field is the magnetic field for ferromagnets, the electric field for ferroelectrics, the chemical potential for the liquid vapor transition, the difference of chemical potentials for mixtures. In several cases the symmetry breaking field is not experimentally accessible as in superfluids and in superconductors. But it is often introduced in theoretical physics for conceptual reasons like the staggered field in antiferromagnets and antiferroelectrics.

This unified description allows us to restrict to one class of system in explaining the main features of critical phenomena, provided we neglect a number of peculiar features of certain systems. Two features we will often neglect are (i) the quantum mechanic (or discrete) nature of the microscopic origin of many phase transitions (superfluid He, superconductors, spin and exchange interaction in magnets, etc). Since critical phenomena become apparent on a macroscopic scale, it is assumed that the commutators can be neglected and the order parameter can be handled like a continuous classical variable. (ii) In many cases we will neglect long range interactions or the long range part of these interactions. Therefore we will neglect dipolar interactions and the coupling of the interaction to lattice distortions which induce long range interactions.

We will mainly use the magnetic language. Thus we will discuss the critical behaviour of a ferromagnet consisting of classical spins on a rigid lattice for which the exchange interaction dominates so that the dipolar interaction can be neglected.

B. Critical Exponents - the Homogeneity Assumption

The first theory to explain the critical behavior of ferromagnets was the molecular field theory by P. Weiss.[3] According to this theory the spontaneous magnetization m is zero above T_c and goes to zero from below like $\sqrt{T_c - T}$. The susceptibility diverges like $|T-T_c|^{-1}$ from below and above T_c and the specific heat shows a finite jump at T_c. Experimentally however one finds $m \propto (T_c-T)^{\beta}$ with $\beta \approx 1/3$ for the spontaneous magnetization, $\chi \propto |T_c-T|^{-\gamma}$ with $\gamma \approx 4/3$ and a singular contribution to the specific heat like $c_{sing} \propto |T_c-T|^{-\alpha}$ with α close to zero. For negative α the specific heat shows a cusp, for positive α it diverges. The exponents α,β and γ are called critical exponents. The deviation of the molecular field exponents from the experimental critical exponents has led to the search of soluble models. Unfortunately most models (approximations) lead back to the molecular field behavior. Two models which give different sets of critical exponents are the spherical model and the two dimensional Ising model. The exponents are listed in table 1. None of these models give exponents which are close to the experimentally observed exponents. The reason is that the molecular field theory completely neglects the critical fluctuations (apart from the homogeneous component) which leads to a γ which is too small; the spherical model overestimates the critical fluctuations which leads to a γ which is too large. The two dimensional Ising model describes the

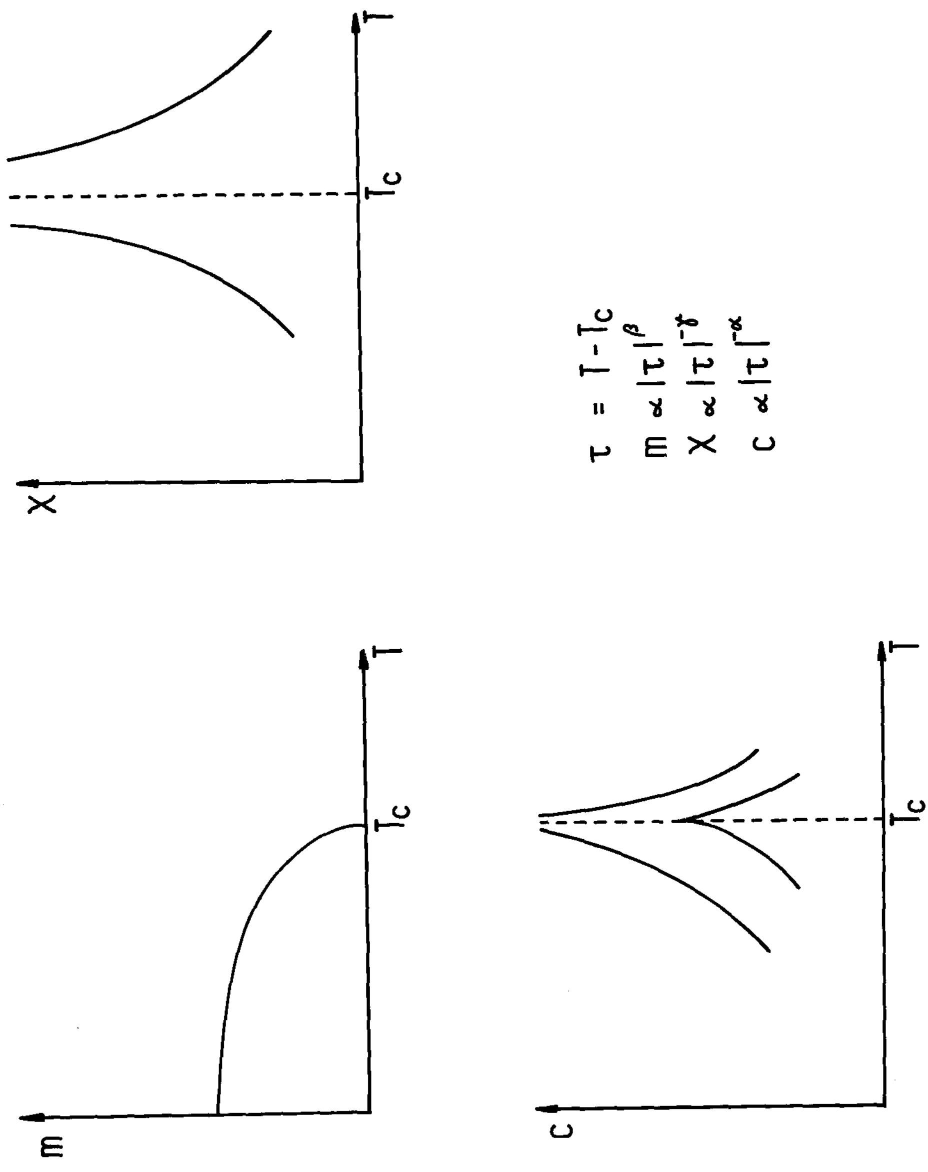

Fig. 1. The schematic behavior of the spontaneous magnetization, susceptibility and specific heat near T_c.

Power law	Exponent	Molecular Field Approxim.	Spherical Model d = 3	Ising Model d = 2	Experiments d = 3	High temperature Expansions d = 3		
						n=1	n=2	n=3
$m \propto \lvert\tau\rvert^{\beta}$	β	1/2	1/2	1/8	$\approx 1/3$	.31		
$\chi \propto \lvert\tau\rvert^{-\gamma}$	γ	1	2	7/4	$\approx 4/3$	1.25	1.32	1.38
$c_{sing} \propto \lvert\tau\rvert^{-\alpha}$	α	0 (disc.)	-1 (kink)	0 (log)	≈ 0	.13	.00	-.10

Table 1. Critical Exponents of various models.

fluctuations properly. However the dimensionality of the system plays an important role in critical phenomena so that the two dimensional Ising model does not yield a reasonable approximation for the three dimensional Ising model.

Apart from some other two dimensional models (F-model, KDP-model, eight-vertex-model), there are no exactly soluble models available. Therefore one has tried a different approach to determine critical exponents by means of series expansions. One expands for example the susceptibility or the specific heat of a model like the Ising model in powers of the inverse[4] temperature $\beta = (k_B T)^{-1}$, assumes that the quantity considered shows a power law behavior close to T_c and analyzes the series accordingly. This yields estimates for the critical exponents listed in the last columns of table 1 for three models: The Ising model (a model of spins S with two states $S = \pm 1$), the XY-model (a model of planar spins S, that is spins with two components $S_x = \cos\varphi$, $S_y = \sin\varphi$) and the classical Heisenberg model (a model of three dimensional (classical) vectors S with $S^2 = 1$). The spins are located at the sites of a lattice and interact via an (isotropic) short range (in most cases nearest neighbor) interaction . Low temperature expansions are only available for the Ising model. Therefore β is quoted only for the Ising model. One can estimate the low temperature exponents for the specific heat and the susceptibility of the Ising model. They are slightly different from the high temperature exponents. Since it is hard to estimate the accuracy of the exponents determined, it is hard to decide whether high and low temperature exponents are equal within the error bars. One finds that the exponents determined from the expansions are quite close to the experimentally observed ones. But unfortunately one does not learn from these expansions why the systems exhibit these broken power laws near T_c. It is the aim of this paper to review on an introductory level the ideas which provide an understanding of the critical behavior.

A first step to link different aspects was the homogeneity assumption by Widom[5]. We bring a modified version of it. (Widom's assumption included the possibility of logarithmic singularities which will not be considered in this section). Widom assumes that the free energy can be separated as a function of the magnetic field h and the temperature difference $\tau = T - T_c$ into a regular and a singular part

$$F = F_{reg}(\tau) + F_{sing}(\tau, h) \quad , \qquad (1.1)$$

where the singular part is responsible for the critical behavior. He assumes that the singular part is a homogeneous function of the variables τ and h, that is

$$F_{sing}(\tau,h) = |\tau|^{2-\alpha} f_{\pm}\left(\frac{h}{|\tau|^{\Delta}}\right) \tag{1.2}$$

where the $\pm$ denotes that the function is different for positive and negative τ . Δ is called the gap exponent. Homogeneity means that multiplying τ by a factor c and h by a factor c^{Δ} multiplies the function by a factor $c^{2-\alpha}$.

Let us discuss some consequences. We obtain the specific heat by differentiating[6] F twice with respect to α. This gives the singular part of the specific heat at constant vanishing field h

$$c_{sing} \propto |\tau|^{-\alpha} f_{\pm}(0) \tag{1.3}$$

which was the reason for calling the exponent in eq. (1.2) $2-\alpha$. The magnetization is obtained from eq. (1.2) by differentiating with respect to h

$$m = -|\tau|^{2-\alpha-\Delta} f'_{\pm}\left(\frac{h}{|\tau|^{\Delta}}\right) \quad . \tag{1.4}$$

At h = 0 this leads to

$$m = -|\tau|^{\beta} f'_{\pm}(0) \tag{1.5}$$

with

$$\beta = 2-\alpha-\Delta \quad . \tag{1.6}$$

Differentiating twice with respect to h we obtain

$$\chi = -|\tau|^{2-\alpha-\Delta} f''_{\pm}\left(\frac{h}{|\tau|^{\Delta}}\right) \tag{1.7}$$

which yields

$$\gamma = \alpha + 2\Delta - 2 \quad . \tag{1.8}$$

From eqs. (1.6) and (1.8) we find a relation between the exponents α, β, γ

$$\alpha + 2\beta + \gamma = 2 \quad . \tag{1.9}$$

A look at table 1 shows that this relation is fulfilled for all listed sets of exponents. We note that from eqs. (1.6) and (1.8) we obtain

$$\Delta = \beta + \gamma \quad . \tag{1.10}$$

Then eq. (1.4) can be easily cast in the form

$$\frac{m}{|\tau|^{\beta}} = -f'_{\pm}\left(\frac{h}{|\tau|^{\Delta}}\right) \quad . \tag{1.11}$$

Solving with respect to $h/|\tau|^{\Delta}$ yields

$$\frac{h}{|\tau|^{\Delta}} = g_{\pm}\left(\frac{m}{|\tau|^{\Delta}}\right) \tag{1.12}$$

with some function g, which can be written

$$\frac{h}{m|\tau|^{\gamma}} = w_{\pm}\left(\frac{m}{|\tau|^{\beta}}\right) \tag{1.13}$$

with $w(x) = g(x)/x$. Thus $h/|\tau|^{\Delta}$ should be a function of $m/\ |\tau|^{\beta}$ only. In Fig.2 data[7] of the magnetization m of $CrBr_3$ as a function of the two variables, temperature and field, are plotted in the variables $m/\ |\tau|^{\beta}$ and $h/(m\ |\tau|^{\gamma})$. If the homogeneity assumption would not hold, the data points would be scattered in the whole plot. Since the data follow the homogeneity assumption,they lie on two lines corresponding on the behaviour above and below T_c.

In the following section we will show how the homogeneity relation can be derived.

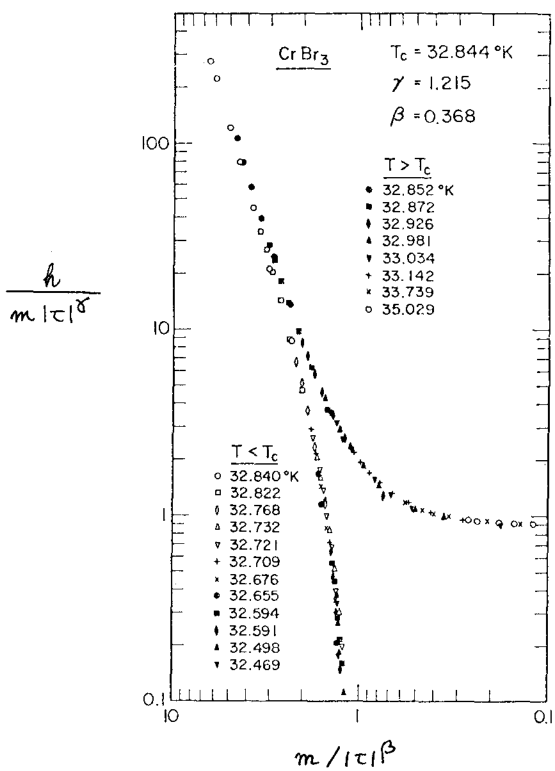

Fig. 2. This plot of $h/(m|\tau|^{\gamma})$ against $m/|\tau|^{\beta}$ confirms the scaling hypothesis for $CrBr_3$. The two branches are for $T>T_c$ and $T<T_c$. After J.T. Ho and J.D. Litster, J. Appl. Phys. 40, 1270 (1969)

2. RENORMALIZATION GROUP EQUATION

A. Motivation

A hint on how the critical state can be characterized can be obtained from the correlation functions. Let us consider the auto correlation function of the spins $S_o(r)$. From what one knows from exactly solvable systems this correlation function decays at criticality with a power law for large distances

$$\langle S_o(o)\, S_o(r)\rangle_{crit} = \frac{c}{r^{d-2+\eta}} \qquad (2.1)$$

where η is a new critical exponent and d the dimensionality of the system. η describes the deviation from the Ornstein-Zernicke-behavior of the correlation function. Let us now consider the same ferromagnet under a different length scale. To accomplish this we divide the sample into cubic cells of length b lattice spacings in each direction. Then the magnetization of a cell

$$s = \sum_{cell} S_o(r') \qquad (2.2)$$

obeys asymptotically

$$\langle s(o)\, s(r)\rangle = \frac{c\, b^{2d}}{r^{d-2+\eta}} \qquad (2.3)$$

since each cell contains b^d spins. Now we change the length scale by a factor b and the scale for the magnetization by a factor $b^{(d+2-\eta)/2}$

$$r = bR \;, \quad s(r) = b^{(d+2-\eta)/2}\, S_1(r) \qquad (2.4)$$

Then we obtain the asymptotic behavior of our new spin variables

$$\langle S_1(o)\, S_1(R)\rangle = \frac{c}{R^{d-2+\eta}} \quad . \qquad (2.5)$$

Therefore the correlation function is invariant under the change of the scale (2.4). This invariance of the correlation function suggests that the effective interaction at criticality is invariant with respect to the change of the length scale. We call the procedure which changes the scale of the hamiltonian (effective interaction) renormalization group (RG) procedure and the corresponding transformation is called RG transformation. In the remainder of this section we outline some requirements and properties of RG transformation and derive one of the RG equations.

B. Properties of the RG transformation[8-12]

We denote the hamiltonian function $\mathcal{H}$ and the free energy $\mathcal{F}$.

We introduce

$$H = \beta \mathcal{H} \quad , \quad F = \frac{\beta \mathcal{F}}{V} \quad , \quad \beta = \frac{1}{k_B T} \tag{2.6}$$

where V is the volume of the system. For simplicity's sake we call H and F hamiltonian and free energy, resp.

$$-F = \frac{1}{V} \ln \mathrm{trace} \exp(-H) \quad . \tag{2.7}$$

The RG transformation consists of
(i) a change of the length scale by a factor $b=e^{\ell}$ in all linear dimensions (we leave the partition function Z=trace exp (-H) invariant). Since the volume shrinks by a factor $e^{-d\ell}$ we obtain

$$F_o = e^{-d\ell} F_\ell \tag{2.8}$$

(ii) a transformation and/or elimination of the spin variables S which leaves the free energy invariant. The transformation shall not generate long-range interactions. The new hamiltonian H_ℓ has to be comparable with the original hamiltonian H_o (same Hilbert or function space). This demands an extension of the system to the original volume for finite systems. The RG transformation transforms H_o into H_ℓ

$$H_\ell = R_\ell(H_o) \tag{2.9}$$

$$F(H_o) = e^{-d\ell} F(H_\ell) \quad . \tag{2.10}$$

C. RG equation with smooth momentum cut-off

There are various ways to construct RG equations which transform hamiltonians:
(i) Wilson's recurrence relation (approximation)[13]. Numerical solution for d=3 see Refs[14]. Expansion in $\varepsilon=4-d$ see Refs. 15.
(ii) Wilson's differential RG equation with smooth momentum cut-off[10], generalization Ref.[16].
(iii) Differential RG equation with sharp momentum cut-off (generates long-range interactions): Wegner and Houghton[17]
(iv) Aharony's method[18]
(v) Two-dimensional Ising models: Niemeijer and van Leeuwen [19], Nauenberg and Nienhuis[20].

We do not discuss the other varieties of the RG, which transform the correlation functions. (Compare the review by Zinn-Justin[21]).

Now we derive the RG equation with smooth momentum cut-off. We represent the hamiltonian as a functional of the Fourier components.

$$S_q = \int d^d r \; e^{-iqr} \; s(r) \tag{2.11}$$

of the variable S(r)

$$\begin{aligned} H = H\{S_q\} = & \; V u_0 + u_1 S_0 + \tfrac{1}{2}\int u_2(q)\, S_q S_{-q}\, d^d q \\ & + \tfrac{1}{3!}\int u_3(q_1,q_2)\, S_{q_1} S_{q_2} S_{-q_1-q_2}\, d^d q_1\, d^d q_2 \\ & + \tfrac{1}{4!}\int u_4(q_1,q_2,q_3) S_{q_1} S_{q_2} S_{q_3} S_{-q_1-q_2-q_3}\, d^d q_1\, d^d q_2\, d^d q_3 \; + \ldots \end{aligned} \tag{2.12}$$

We perform an infinitesimal change of the length scale

$$r \rightarrow r(1-\delta) \qquad \delta \text{ infinitesimal} \tag{2.13}$$

$$q \rightarrow q(1+\delta) \tag{2.14}$$

$$S_q^0 = S^\delta_{q+\delta q} = S_q^\delta + \delta q \, \nabla_q \, S_q^\delta \tag{2.15}$$

$$V^0 = V^\delta (1 + d\cdot\delta) = V^\delta + \delta\cdot d\cdot V^\delta \; . \tag{2.16}$$

Therefore the hamiltonian H transforms into

$$H\{S_q\} \rightarrow H\{S_q\} + \delta\Big(\int d^d q \, \big(q \, \nabla S_q \frac{\delta H}{\delta S_q} + \frac{d}{2}\big) + d\cdot V\cdot \frac{\partial H}{\partial V}\Big). \tag{2.17}$$

The transformation (2.11) is unitary apart from a volume dependent constant. This volume dependence produces the term d/2 in eq. (2.17). From this eq. we obtain the generator G_{dil} of the dilatation

$$G_{dil}\, H = \int d^d q \, \big(q\,\nabla S_q \frac{\delta H}{\delta S_q} + \frac{d}{2}\big) + d\cdot V\cdot \frac{\partial H}{\partial V} \; . \tag{2.18}$$

Secondly we allow a transformation of the variables

$$S_q \rightarrow S_q + \delta\cdot \varphi_q\{S\} \tag{2.19}$$

which transforms the hamiltonian according to

$$H \rightarrow H + \delta \sum \varphi_q \frac{\partial H}{\partial S_q} \tag{2.20}$$

where ψ_q is a functional of the spin variables. The volume element in phase space transforms according to

$$\prod dS_q \to \prod d(S_q + \delta \cdot \psi_q) = \prod dS_q \cdot (1 + \delta \sum \frac{\partial \psi_q}{\partial S_q}) \qquad (2.21)$$

which with

$$\sum \frac{\partial \psi_q}{\partial S_q} = \int d^d q \frac{\delta \psi_q}{\delta S_q} \qquad (2.22)$$

yields

$$\int \prod dS_q \exp(-H) = \int \prod dS_q \exp\left(-H - \delta \int \left(\psi_q \frac{\delta H}{\delta S_q} - \frac{\delta \psi_q}{\delta S_q}\right) d^d q\right) \qquad (2.23a)$$

so that the hamiltonian transforms into

$$H \to H + \delta \int \left(\psi_q \frac{\delta H}{\delta S_q} - \frac{\delta \psi_q}{\delta S_q}\right) d^d q \quad . \qquad (2.23b)$$

The generator $G_{tra}(\psi)$ of this transformation is

$$G_{tra}(\psi) H = \int \left(\psi_q \frac{\delta H}{\delta S_q} - \frac{\delta \psi_q}{\delta S_q}\right) d^d q \quad . \qquad (2.24)$$

The transformation is performed so that the free energy is invariant (eq. (2.23a)).

From eqs. (2.18) and (2.24) we obtain the renormalization group equation

$$\frac{\partial H}{\partial \ell} = G(\psi) H = \left(G_{dil} + G_{tra}(\psi)\right) H \qquad (2.25)$$

in differential form. Wilson[10] has choosen a special dependence for ψ

$$\psi_q = \frac{d}{2} S_q + (c + 2q^2)\left(S_q - \frac{\delta H}{\delta S_q}\right) \qquad (2.26)$$

where the constant c has to be adjusted properly. This choice guarantees that the Fourier components S_q with large q are eliminated and survive only in $u_2(q)$ which for large q approaches unity.

3. SCALING AND THE LINEARIZED RG EQUATION

A. Fixed point, classification of operators

In the Wilson theory of critical phenomena the following two assumptions are made:

(i) It is assumed that a fixed point hamiltonian H^* exists

$$G H^* = 0 \qquad (3.1)$$

This is a hamiltonian which maps into itself.

(ii) It is assumed that for a critical hamiltonian

$$\lim_{\ell\to\infty} H_\ell = H^* \quad . \tag{3.2}$$

The RG equation

$$\frac{\partial H}{\partial \ell} = G_{\text{Wilson}} H = \int d^dq \left(q \nabla_q S_q + \frac{d}{2} S_q + (c+2q^2) S_q\right) \frac{\delta H}{\delta S_q}$$
$$+ \int d^dq\, (c+2q^2)\left(-\frac{\delta H}{\delta S_q}\frac{\delta H}{\delta S_{-q}} + \frac{\delta^2 H}{\delta S_q\, \delta S_{-q}}\right) \tag{3.3}$$

yields in linear order in ΔH

$$G_{\text{lin}}(H^* + \Delta H) = \int d^dq \left(q \nabla_q S_q + \frac{d}{2} S_q + (c+2q^2) S_q\right) \frac{\delta \Delta H}{\delta S_q}$$
$$+ \int d^dq\, (c+2q^2)\left(-2\frac{\delta H^*}{\delta S_{-q}} + \frac{\delta}{\delta S_{-q}}\right)\frac{\delta \Delta H}{\delta S_q} \equiv L \Delta H \quad . \tag{3.4}$$

We define eigenoperators O_i by the eigenvalue equation

$$L O_i = y_i O_i \quad . \tag{3.5}$$

We assume in the following that the eigenoperators form a complete set of operators so that any hamiltonian H_o can be expanded

$$H_o = H^* + \sum \mu_i O_i \quad . \tag{3.6}$$

Then we obtain in linear order in μ

$$H_\ell = H^* + \sum \mu_i e^{y_i \ell} O_i \quad . \tag{3.7}$$

Corresponding to the eigenvalues y one distinguishes

$y > 0$ relevant operator,

$y \doteq 0$ marginal operator,

$y < 0$ irrelevant operator. (3.8)

From equation (3.7) we find immediately that at the critical point the fields (in high energy physics sources) μ_i of all relevant operators have to vanish.

Depending on the nonlinear terms marginal operators may act as relevant, irrelevant, and substantially marginal operators (as in the eight-vertex-model), resp.

There is a special operator, the constant $V(u_o - u_o^*)$, eq. (2.12) which formally has

$$O_o = 1, \qquad y_o = d \tag{3.9}$$

However the addition of a constant to the hamiltonian does not change its critical behavior. Therefore $\mu_o = 0$ is not necessary for criticality. This is the origin of the regular part of the free energy.

The type of the critical behavior depends on the number of symmetry conserving relevant operators. (Symmetry conserving means that the symmetry of the hamiltonian is conserved, it does not exclude a spontaneously broken symmetry of the system). Let us expand

$$\mathcal{H} = \sum \mu_i^o \, O_i \tag{3.10}$$

$$H^* = \sum \mu_i^* \, O_i \tag{3.11}$$

then we obtain

$$\mu_i = \beta \mu_i^o - \mu_i^* \quad . \tag{3.12}$$

For a normal critical point one has one relevant symmetry conserving operator (apart from O_o) O_E which determines the critical temperature

$$\mu_E \equiv \tau = \beta \mu_E^o - \mu_E^* = (\beta - \beta_c)\mu_E^o \; ; \; \beta_c = \frac{\mu_E^*}{\mu_E^o} \quad . \tag{3.13}$$

Crudely speaking O_E is proportional to the hamiltonian minus its expectation value at the critical point. At a tricritical point one has two relevant symmetry conserving operators (apart from O_o) and consequently two conditions for criticality.

Redundant operators:

We state a few results on redundant operators (see ref. 16). The hamiltonian H^* is not uniquely defined, it depends on the functional ψ. Varying this functional one can show that any hamiltonian $H^* + \delta\, G_{tra}(\varphi)\, H^*$ can be a fixed point. (δ infinitesimal). We call these hamiltonians equivalent to H^* and the operators $G_{tra}(\varphi)\, H^*$ redundant operators. One shows that L applied to a redundant operator yields again a redundant operator. Since equivalent hamiltonians can be obtained from H^* by means of the transformation (2.23) both hamiltonians have equal free energy. Therefore the redundant operators O_i do not contribute to the critical behavior. Therefore $\mu_i = 0$ need not be fulfilled for redundant operators at the critical point.

The eigenvalues y for redundant operators are not uniquely defined. They depend on the choice of ψ for the RG equation. The eigenvalues of the other operators are uniquely defined. An example of a redundant operator (see Hubbard and Schofield[22]) which corresponds to a shift of S(r) by a constant: $\varphi_q = \delta(q)$

$$G_{tra}(\varphi)\,H = \frac{\delta H}{\delta S_0} \tag{3.14}$$

For the hamiltonian

$$H = \frac{1}{2}\int (r_0+q^2) S_q S_{-q}\, d^d q + \frac{1}{4!} u_0 \int S_{q_1} S_{q_2} S_{q_3} S_{-q_1-q_2-q_3}\, d^{3d} q \tag{3.15}$$

one obtains

$$G_{tra}(\varphi)\,H = r_0 S_0 + \frac{u_0}{3!}\int S_{q_1} S_{q_2} S_{-q_1-q_2}\, d^{2d} q = \int \left(r_0 S(r) + \frac{u_0}{3!} S^3(r)\right) d^d r . \tag{3.16}$$

B. Scaling of the free energy

Within a simplified picture (Kadanoff's cell model[23]) we consider only two operators O_E and the magnetization O_h

$$H_0 = H^* + \tau O_E + h\, O_h \tag{3.17}$$

which yields

$$H_\ell = H^* + \tau e^{y_E \ell} O_E + h\, e^{y_h \ell} O_h \tag{3.18}$$

$$F(\tau, h) = e^{-d\ell} F(\tau e^{y_E \ell}, h\, e^{y_h \ell}) . \tag{3.19}$$

We choose τ by

$$|\tau|\, e^{y_E \ell} = 1 \tag{3.20}$$

and obtain Widom's scaling law (1.2)

$$F(\tau, h) = |\tau|^{d/y_E}\, F\left(\pm 1, \frac{h}{|\tau|^{y_h/y_E}}\right) \tag{3.21}$$

with

$$d/y_E = 2-\alpha \qquad y_h/y_E = \Delta . \tag{3.22}$$

Normally one has an infinite number of perturbations O_i in equation (3.17). To study their effect on the scaling law we add at least one further operator pars pro toto

$$H_0 = H^* + \tau O_E + h O_h + \mu_i O_i \tag{3.23}$$

and obtain

$$H_\ell = H^* + \tau e^{y_E \ell} O_E + h e^{y_h \ell} O_h + \mu_i e^{y_i \ell} O_i \tag{3.24}$$

$$F(\tau, h, \mu_i) = |\tau|^{d/y_E} F\left(\pm 1, \frac{h}{|\tau|^{y_h/y_E}}, \frac{\mu_i}{|\tau|^{y_i/y_E}}\right) . \tag{3.25}$$

We are interested in the critical behavior that is in the limit $\tau \to 0$

$$\lim_{\tau \to 0} \frac{\mu_i}{|\tau|^{y_i/y_E}} \to \begin{cases} 0 & y_i < 0 \ \text{ or } \ \mu_i = 0 \\ \pm\infty & y_i > 0 \ \text{ and } \ \mu_i \neq 0 \end{cases} . \tag{3.26}$$

If O_i is relevant ($y_i > 0$) then μ_i has explicitly to be taken into account. For irrelevant operators the term $\mu_i/|\tau|^{y_i/y_E}$ can be neglected if F can be expanded in powers of μ_i. Note that the right hand side of eq. (3.26) contains the free energy well apart from the critical point. The irrelevant operator yields a correction to scaling

$$F = |\tau|^{d/y_E} F\left(\pm 1, \frac{h}{|\tau|^{y_h/y_E}}, 0\right) + |\tau|^{(d-y_i)/y_E} F'\left(\pm 1, \frac{h}{|\tau|^{y_h/y_E}}, 0\right) + \cdots \tag{3.27}$$

as observed in superfluid He (Ahlers[24]). If F cannot be expanded in powers of μ_i, then Fisher's idea of the anomalous dimension of the vacuum might apply[25].

C. Correlations

Until now we considered only translational invariant perturbations. Let us consider the eigenvalue equation for localized operators $\tilde{O}_i$ (Wilson and Kogut[10])

$$L \tilde{O}_i = - x_i \tilde{O}_i . \tag{3.28}$$

From the representation

$$\tilde{O}_i = \tilde{O}_i\{S_q\} \tag{3.29}$$

we define

$$\tilde{O}_i(r) = \tilde{O}_i\{S_q e^{iqr}\} . \tag{3.30}$$

It follows that

$$L\tilde{O}_i(r) = -x_i\tilde{O}_i(r) - r\nabla_r\tilde{O}_i(r) \quad . \tag{3.31}$$

In linear approximation we obtain from

$$H_0 = H^* + \lambda_i\tilde{O}_i(r) \tag{3.32}$$

$$H_\ell = H^* + \lambda_i e^{-x_i\ell}\,\tilde{O}_i(re^{-\ell}) \quad . \tag{3.33}$$

Let us define

$$O_i(q) = \int\tilde{O}_i(r)\,e^{-iqr}\,d^dr \quad . \tag{3.34}$$

Then

$$H_0 = H^* + \lambda_i\,O_i(q) \tag{3.35}$$

yields

$$H_\ell = H^* + \lambda_i\,e^{(d-x_i)\ell}\;O_i(q\,e^\ell) \quad . \tag{3.36}$$

By comparison for q=0 with (3.5) we see

$$y_i = d - x_i \tag{3.37}$$

From

$$H_0 = H^* + \Sigma\mu_i O_i + \lambda_1 O_1(q) + \lambda_2 O_2(-q) \tag{3.38}$$

we obtain in linear approximation

$$H_\ell = H^* + \sum\mu_i e^{y_i\ell}O_i + \lambda_1 e^{y_1\ell}O_1(qe^\ell) + \lambda_2 e^{y_2\ell}O_2(-qe^\ell) . \tag{3.39}$$

Differentiating the free energy of the Hamiltonian (3.38) yields the correlation function

$$G(q,H_0) = \langle O_1(q)O_2(-q)\rangle_0 = \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}F(H_0) \tag{3.40}$$

Therefore we obtain

$$\langle O_1(q)O_2(-q)\rangle_0 = e^{(y_1+y_2-d)\ell}\langle O_1(qe^\ell)\,O_2(-qe^\ell)\rangle_\ell \tag{3.41}$$

$$G(q,\tau,h) = e^{(y_1+y_2-d)\ell}\;G(qe^\ell,\tau e^{y_E\ell},h\,e^{y_h\ell}) \;. \tag{3.42}$$

As an example we consider the spin-spin-correlation $(y_1=y_2=y_h)$

(i) $\tau=h=0$

$$G_c(q) = e^{(2y_h-d)\ell}\, G_c(qe^{\ell}) \qquad (3.43)$$

With $qe^{\ell}=1$ we obtain

$$G_c(q) = q^{d-2y_h}\, G_c(1) \qquad (3.44)$$

and identify the exponent

$$d-2y_h = -2+\eta \qquad (3.45)$$

(ii) $\tau \neq 0$, $h=0$

$$G(q,\tau) = e^{(2y_h-d)\ell}\, G(qe^{\ell}, \tau e^{y_E \ell}) \qquad (3.46)$$

With $|\tau| e^{y_E \ell} = 1$ we find

$$G(q,\tau) = |\tau|^{-\frac{2y_h-d}{y_E}}\, G(q|\tau|^{-1/y_E}, \pm 1) = |\tau|^{-\gamma} g_0(q\xi) . \qquad (3.47)$$

Apart from a constant factor ξ is called the correlation length and scales with an exponent ν. We obtain

$$\gamma = \frac{2y_h-d}{y_E} \quad , \quad \xi \propto |\tau|^{-\nu} \quad , \quad \nu = \frac{1}{y_E} \quad , \quad \gamma = \nu(2-\eta) . \qquad (3.48)$$

4. NONLINEAR CONTRIBUTIONS

A. Scaling fields[9]

Apart from certain exceptions which will be discussed below the nonlinearities of the RG equation can be absorbed in scaling fields g_i which depend nonlinearly on the fields μ_j, so that g can be formally expanded in powers of μ and

$$F\{g_i\} = e^{-d\ell}\, F\{g_i\, e^{y_i \ell}\} \qquad (4.1)$$

From equation (3.3) we find

$$\frac{d\mu_i}{d\ell} = y_i \mu_i + \frac{1}{2}\sum_{jk} a'_{ijk}\, \mu_j \mu_k \qquad (4.2)$$

with

$$-2\int d^d q\,(c+2q^2)\frac{\delta O_j}{\delta S_q}\frac{\delta O_k}{\delta S_{-q}} = \sum a'_{ijk}\,O_i \qquad (4.3)$$

To obtain equation (4.1) we require

$$\frac{dg_i}{d\ell} = y_i g_i \qquad (4.4)$$

and expand

$$\mu_i = g_i + \frac{1}{2}\sum b_{ijk}\,g_j g_k + O(g^3) \qquad (4.5)$$

which yields

$$\frac{d\mu_i}{d\ell} = y_i g_i + \frac{1}{2}\sum(y_i b_{ijk} + a'_{ijk})g_j g_k = y_i g_i + \frac{1}{2}\sum(y_j+y_k)b_{ijk}g_j g_k + O(g^3) \qquad (4.6)$$

$$(y_j + y_k - y_i)\,b_{ijk} = a'_{ijk} \qquad (4.7)$$

which can be solved provided $y_i \neq y_j+y_k$. Similarly the terms of nth order in g can be calculated if y_i differs from any sum of n exponents y.

B. Logarithmic Corrections[9]

If $y_i = y_j+y_k$, then logarithmic factors arise. We give an example in which we neglect all terms in the equations (4.2) which do not contribute to the logarithm. Suppose

$$2y_E = y_0 = d \qquad (4.8)$$

$$\frac{d\mu_0}{d\ell} = d\cdot\mu_0 + \frac{1}{2}\,a'_{0EE}\,\mu_E^2 \qquad (4.9)$$

$$\frac{d\mu_E}{d\ell} = y_E\,\mu_E \qquad (4.10)$$

then we obtain ($\tau = \mu_E$)

$$\mu_E(\ell) = \mu_E(0)\,e^{d/2\,\ell} \qquad (4.11)$$

$$\frac{d\mu_0}{d\ell} = d\cdot\mu_0 + \frac{1}{2}\,a'_{0EE}\,\mu_E^2(0)\,e^{d\ell} \qquad (4.12)$$

$$\mu_0(\ell) = \mu_0(0)\,e^{d\ell} + \frac{1}{2}\,\ell\,a'_{0EE}\,\mu_E^2(0)\,e^{d\ell} \qquad (4.13)$$

$$F(0,\mu_E) = e^{-d\ell}\mu_0(\ell) + e^{-d\ell}F(0,\mu_E e^{d/2\,\ell}) \\ = \frac{1}{2}\ell\, a'_{0EE}\mu_E^2 + \mu_E^2 F(0,\pm 1) \ . \tag{4.14}$$

Again we choose $|\mu_E| e^{d/2\,\ell} = 1$ and obtain

$$F(0,\mu_E) = -\frac{a'_{0EE}}{d}\mu_E^2 \ln|\mu_E| + \mu_E^2 F(0,\pm 1) \ . \tag{4.15}$$

C. Broken powers of Logarithms[31]

If $y_u = 0$, then logarithms to some broken powers arise. Again we start from simplified equations to demonstrate the singularity

$$\frac{d\mu_u}{d\ell} = \frac{1}{2} a'_{uuu}\mu_u^2 \tag{4.16}$$

$$\frac{d\mu_i}{d\ell} = y_i\mu_i + a'_{iui}\mu_i\mu_u \tag{4.17}$$

and obtain

$$\mu_u = s(\ell+\ell_0)^{-1} \qquad s = -\frac{2}{a'_{uuu}} \tag{4.18}$$

$$\mu_i(\ell) = \left(\frac{\ell+\ell_0}{\ell_0}\right)^{p_i} e^{y_i\ell}\mu_i(0) \ , \ p_i = s\, a'_{iui} \ . \tag{4.19}$$

Such singularities appear in four dimensions at a critical point and in three dimensions at a tricritical point[31-33].

D. Correlation Functions

The operators $O_i(q)$ become extremely small, if $q \gg q_0$ (q_0 momentum cut-off). Therefore the correlations G become extremely small as soon as $q \gg q_0$ and equation (3.42) will not apply for $q e^{\ell} \gg q_0$. The perturbations $\lambda_1 O_1(q) + \lambda_2 O_2(-q)$ will generate contributions $O_0, O_E, \ldots$ because of the nonlinear terms of the RG equation. Similarly homogeneous perturbations and nonhomogeneous perturbations generate contributions nonlinear in μ and γ. To discuss these effects we make the following simplifying assumption:

(i) We assume that the linear approximation is good for $q \lesssim q_0$.
(ii) In a narrow region around q_0 the nonlinear contributions dominate.
(iii) For $q \gtrsim q_0$ we neglect the inhomogeneous perturbations.

Then from

$$\mathcal{H}_0 = \mathcal{H}^* + \tau O_\varepsilon + \lambda_1 O_h(q) + \lambda_2 O_h(-q) \tag{4.20}$$

we obtain according to (i) with $e^{\ell} = q_0/q$

$$\mathcal{H}_\ell = \mathcal{H}^* + \tau\left(\frac{q_0}{q}\right)^{y_\varepsilon} O_\varepsilon + \lambda_1\left(\frac{q_0}{q}\right)^{y_h} O_h(q_0) + \lambda_2\left(\frac{q_0}{q}\right)^{y_h} O_h(-q_0) \tag{4.21}$$

where actually $q\,e^{\ell}$ should be slightly smaller than q_o. According to our assumption (ii), we obtain with a slight change of ℓ (which we do not indicate explicitly in the next equation)

$$\tilde{\mathcal{H}}_\ell = \mathcal{H}^* + \tau\left(\frac{q_0}{q}\right)^{y_\varepsilon} O_\varepsilon + \lambda_1\lambda_2\left(\frac{q_0}{q}\right)^{2y_h}\left(A + B O_\varepsilon + C\tau\left(\frac{q_0}{q}\right)^{y_\varepsilon} + \cdots\right). \tag{4.22}$$

We have already neglected the inhomogeneous perturbations in this equation according to (iii). Now we have a homogeneous interaction and we can apply the inverse RG transformation

$$\tilde{\mathcal{H}}_0 = \mathcal{H}^* + \tau O_\varepsilon + \lambda_1\lambda_2\left(\frac{q_0}{q}\right)^{2y_h - d}\left(A + B\left(\frac{q_0}{q}\right)^{d-y_\varepsilon} O_\varepsilon + C\tau\left(\frac{q_0}{q}\right)^{y_\varepsilon} + \cdots\right). \tag{4.23}$$

Since the free energy is conserved under the total of these transformations we obtain

$$\langle O_h(q)\, O_h(-q)\rangle = \frac{\partial^2 F(\tilde{\mathcal{H}}_0)}{\partial\lambda_1\,\partial\lambda_2} = \left(\frac{q_0}{q}\right)^{2y_h - d}\left(A + B\left(\frac{q_0}{q}\right)^{d-y_\varepsilon}\langle O_\varepsilon\rangle + C\tau\left(\frac{q_0}{q}\right)^{y_\varepsilon} + \cdots\right) \tag{4.24}$$

a result suggested by Fisher and Langer[26] and which has also been derived by means of the Callan-Symanzik-equation[27]. We emphasize that the conditions (i) and (ii) are not necessary to derive eq. (4.23). It is only necessary[30] that the operators $O_i(q)$ can be neglected for $q \gg q_o$. Then, however, the derivation of (4.23) becomes more complicated. We note that for a linear RG equation (3.42) holds exactly which means that $O_i(q)$ does not become negligible for $q \gg q_o$. Therefore a linear RG does not eliminate the Fouriers components for large q. We see that the elimination of short wave length fluctuations and the linearity of a RG equation exclude each other.

5. FINAL REMARKS

We have outlined the basic ideas of the RG procedure as initiated by Wilson. These ideas can be applied to actual calculations.

We display in table 2 the critical exponents α and γ for three-dimensional systems as obtained by various methods. In section 2C we mentioned already a number of them. We can distinguish three types of calculations:

(i) Approximate calculations. Wilson's recurrence relation[13] can be used to calculate numerically the critical exponents. They are shown in the table (r.r. numerical).

(ii) The critical exponents can be expanded around dimensionality 4 (ε-expansion, $\varepsilon = 4-d$). Unfortunately, however, the series seem to be asymptotically. As a thumb-rule one finds that the exponents in order ε^2 yield a good approximation. In 4 dimensions one obtains molecular-field behaviour (with logarithmic corrections) which can be described by a free fixed-point. It is possible to expand around this fixed point since the coupling constant g for the four-spin interaction (which is marginal for $d = 4$) can be expanded in powers of ε .

(iii) The critical exponents can be expanded in powers of $1/n$. For $n = \infty$ one obtains the critical exponents for the spherical model[34]

$$\gamma = \frac{2}{d-2} \quad , \quad \alpha = \frac{d-4}{d-2} \quad , \quad 2 \leq d < 4 \quad . \tag{5.1}$$

One can perform a systematic expansion[35] around this limit which yields for $d = 3$

$$\gamma = 2 - \frac{24}{\pi^2 n} + O\left(\frac{1}{n^2}\right) , \quad \alpha = -1 + \frac{32}{\pi^2 n} + O\left(\frac{1}{n^2}\right) . \tag{5.2}$$

These numbers are not yet good approximations although they tend into the correct direction. One has to wait for terms in order $1/n^2$.

Table 2. Critical exponents as obtained by various methods for d=3

	n=1	n=2	n=3	Ref.
α high temp.exp.	.13	.00	-.10	28, compare 1
α r. r. numerical	.17	.07	-.04	13,14
α in O (ε)	.17	.10	.05	15
α in O (ε^2)	.08	-.02	-.10	15
α in O(ε^3)	.20	+.08	.01	15
α experiment	.16	-.02	-.14	24. 29
γ high temp. exp.	1.25	1.32	1.38	28, compare 1
γ r. r. numerical	1.22	1.29	1.36	13, 14
γ in O (ε)	1.17	1.20	1.23	15
γ in O (ε^2)	1.24	1.30	1.35	15
γ in O (ε^3)	1.19	1.26	1.32	15

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