

# On certain classes of algebraic curvature tensors\*

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## ABSTRACT

In this text we deal with Rakić duality principle. We search for a connection between Osserman and Rakić curvature tensor. We prove that 3-dimensional Rakić is Osserman. We investigate Rakić duality using Fiedler's skew-symmetric decomposition, and prove that Osserman curvature tensor with constant Fiedler's signs is Rakić.

## 1. Introduction

Let us begin with the basic notation and terminology which are used throughout this work. Let  $R$  be an algebraic curvature tensor on a vector space  $\mathcal{V}$  equipped with an indefinite metric  $g$  of the signature  $(\nu, n - \nu)$ . The sign  $\varepsilon_X = g(X, X)$  denotes the norm of  $X \in \mathcal{V}$ , and it determines various types of vectors. We say that  $X \in \mathcal{V}$  is timelike (if  $\varepsilon_X < 0$ ), spacelike ( $\varepsilon_X > 0$ ), null ( $\varepsilon_X = 0$ ), nonnull ( $\varepsilon_X \neq 0$ ), or unit ( $\varepsilon_X \in \{-1, 1\}$ ). The curvature operator  $\mathcal{R}$  is connected with  $R$  via equation  $R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W)$ . If  $(E_1, E_2, \dots, E_n)$  is an orthonormal basis of  $\mathcal{V}$ , then we use short notations  $\varepsilon_i = \varepsilon_{E_i}$  and  $R_{ijkl} = R(E_i, E_j, E_k, E_l)$ . For the initial definitions and deeper explanations of this topic, the reader can consult Gilkey's books [9, 10].

The Jacobi operator  $\mathcal{J}_X : \mathcal{V} \rightarrow \mathcal{V}$  is a natural operator defined by  $\mathcal{J}_X(Z) = \mathcal{R}(Z, X)(X)$  for all  $Z \in \mathcal{V}$ . In the case of nonnull  $X \in \mathcal{V}$ ,  $\mathcal{J}_X$  preserves nondegenerate hyperspace  $\{X\}^\perp = \{Y \in \mathcal{V} : X \perp Y\}$ , and we have the reduced Jacobi operator  $\tilde{\mathcal{J}}_X : \{X\}^\perp \rightarrow \{X\}^\perp$ , given by  $\tilde{\mathcal{J}}_X = \mathcal{J}_X|_{\{X\}^\perp}$ .

We say that  $R$  is an Osserman curvature tensor if the characteristic polynomial of  $\mathcal{J}_X$  is constant on both pseudo-spheres, in particular on positive ( $\varepsilon_X = 1$ ) and negative ( $\varepsilon_X = -1$ ) one. In a pseudo-Riemannian setting,

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Jordan normal form plays a crucial role, since characteristic polynomial does not determine the eigen-structure of a symmetric linear operator. We say that  $R$  is a Jordan Osserman curvature tensor if the Jordan normal form of  $\mathcal{J}_X$  is constant on both pseudo-spheres. An Osserman curvature tensor, whose Jacobi operator  $\mathcal{J}_X$  is diagonalizable for all nonnull  $X$ , we call diagonalizable Osserman.

In the Riemannian setting ( $\nu = 0$ ), it is known that a local two-point homogeneous space (flat or locally rank one symmetric space) has a constant characteristic polynomial on the unit sphere bundle. Osserman wondered if the converse held [15], and this question has been called the Osserman conjecture by subsequent authors. During the solution of some particular cases of the conjecture, the implication

$$\mathcal{J}_X(Y) = \lambda Y \Rightarrow \mathcal{J}_Y(X) = \lambda X \quad (1)$$

appeared naturally, and if it holds, it can significantly simplify some calculations. The first results in this topic are given by Chi [7], who proved the conjecture in the cases of dimensions  $n \neq 4k$ ,  $k > 1$ . In his work he used the fact that (1) holds for extremal eigenvalues  $\lambda$  of the Jacobi operator. Rakić [16] proved the correctness of (1) for every eigenvalue  $\lambda$  of the Jacobi operator, and this statement has been called Rakić duality principle [9]. After that, Rakić duality is reproved by Gilkey [9], and it become a beneficial tool for the solution of the conjecture. Moreover, the best results in this topic gave Nikolayevsky [12, 13, 14], who used Rakić duality [13] to prove Osserman conjecture in all dimensions, except some possibilities in dimension  $n = 16$ .

The variant of the Osserman conjecture has been appeared in a pseudo-Riemannian setting. For example, in the Lorentzian setting ( $\nu = 1$ ), an Osserman manifold necessarily has a constant sectional curvature [5]. Observation of Osserman manifolds in the signature  $(2, 2)$  become very popular, and it is worth noting results from [6], which are based on the discussion of possible Jordan normal forms of the Jacobi operator.

This is why we start to investigate the duality principle for Osserman curvature tensor in a pseudo-Riemannian setting. In a pseudo-Riemannian setting, the implication (1) looks inaccurate, and therefore we corrected it in the following way [1, 4].

**Definition 1 (Rakić duality)** *We say that a curvature tensor  $R$  satisfies Rakić duality for the value  $\lambda$ , if for all mutually orthogonal units  $X, Y \in \mathcal{V}$  holds*

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \Rightarrow \mathcal{J}_Y(X) = \varepsilon_Y \lambda X. \quad (2)$$

*We say that  $R$  is Rakić if it satisfies Rakić duality for all  $\lambda \in \mathbb{R}$ .*

The Rakić duality for Osserman curvature tensor works nicely for every known example, which motivated us to post the following conjecture.

**Conjecture 1** *Osserman pseudo-Riemannian curvature tensor is Rakić.*

Unfortunately we failed to prove this conjecture in general. In our previous work we gave the affirmative answer only for the conditions of small index ( $\nu \leq 1$ ) [1, 4], low dimension ( $n \leq 4$ ) [1, 4, 3], and some possibilities with small numbers of eigenvalues of the reduced Jacobi operator [2]. Seeing that Rakić property is natural for Osserman curvature tensor (at least in the Riemannian setting), one can ask if the converse held.

**Conjecture 2** *Rakić pseudo-Riemannian curvature tensor is Osserman.*

## 2. Three-dimensional case

In this section we deal with three-dimensional Rakić pseudo-Riemannian curvature tensor. Let us start with the following universal lemma.

**Lemma 1** *If  $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$  and  $\mathcal{J}_Y(X) = \varepsilon_Y \lambda X$ , then for all  $\alpha, \beta \in \mathbb{R}$  holds  $\mathcal{J}_{\alpha X + \beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y) = \varepsilon_{\alpha X + \beta Y} \lambda (\varepsilon_Y \beta X - \varepsilon_X \alpha Y)$ .*

**Proof.** This lemma is a consequence of the straightforward calculations.

$$\begin{aligned}
 \mathcal{J}_{\alpha X + \beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y) &= \mathcal{R}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y, \alpha X + \beta Y)(\alpha X + \beta Y) \\
 &= \mathcal{R}(\varepsilon_Y \beta X, \beta Y)(\alpha X + \beta Y) + \mathcal{R}(-\varepsilon_X \alpha Y, \alpha X)(\alpha X + \beta Y) \\
 &= -\varepsilon_Y \alpha \beta^2 \mathcal{J}_X(Y) + \varepsilon_Y \beta^3 \mathcal{J}_Y(X) - \varepsilon_X \alpha^3 \mathcal{J}_X(Y) + \varepsilon_X \alpha^2 \beta \mathcal{J}_Y(X) \\
 &= \beta(\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2) \mathcal{J}_Y(X) - \alpha(\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2) \mathcal{J}_X(Y) \\
 &= (\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2)(\beta \varepsilon_Y \lambda X - \alpha \varepsilon_X \lambda Y) \\
 &= \varepsilon_{\alpha X + \beta Y} \lambda (\varepsilon_Y \beta X - \varepsilon_X \alpha Y)
 \end{aligned}$$

□

Let us investigate Conjecture 2 for low dimension  $n = 3$ . It is known that three-dimensional Einstein (consequently it holds for Osserman) curvature tensor necessarily has constant sectional curvature.

**Theorem 1** *Three-dimensional Rakić curvature tensor is of constant sectional curvature.*

**Proof.** In order to apply Rakić property we need to show that there is a pair  $(X, Y)$  of mutually orthogonal units, where  $Y$  is an eigenvector of  $\mathcal{J}_X$ . Let  $(E_1, E_2, E_3)$  be an arbitrary orthonormal basis of  $\mathcal{V}$ , such that  $\varepsilon_1 = \varepsilon_2$ . The matrix of the Jacobi operator  $\mathcal{J}_{E_3}$  is

$$\mathcal{J}_{E_3} = \begin{pmatrix} \varepsilon_1 R_{1331} & \varepsilon_1 R_{2331} & 0 \\ \varepsilon_2 R_{1332} & \varepsilon_2 R_{2332} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and therefore its reduced Jacobi operator  $\tilde{\mathcal{J}}_{E_3}$  has characteristic polynomial

$$x^2 - (\varepsilon_1 R_{1331} + \varepsilon_2 R_{2332})x + \varepsilon_1 \varepsilon_2 R_{1331} R_{2332} - \varepsilon_1 \varepsilon_2 R_{2331} R_{1332} = 0.$$

Let  $D$  be a discriminant of the previous quadratic equation, then

$$D = (\varepsilon_1 R_{1331} + \varepsilon_2 R_{2332})^2 - 4\varepsilon_1 \varepsilon_2 R_{1331} R_{2332} + 4\varepsilon_1 \varepsilon_2 (R_{1332})^2.$$

Because of  $\varepsilon_1 = \varepsilon_2$ , we have  $\varepsilon_1 \varepsilon_2 = (\varepsilon_1)^2 = 1$ , and thus

$$D = (\varepsilon_1 R_{1331} - \varepsilon_2 R_{2332})^2 + 4(R_{1332})^2 \geq 0.$$

If  $D > 0$  then our quadratic equation has two distinct real roots, which represents two distinct eigenvalues of  $\tilde{\mathcal{J}}_{E_3}$ , and hence there are two (nondegenerate) one-dimensional eigenspaces. Otherwise,  $D = 0$  gives  $\varepsilon_1 R_{1331} = \varepsilon_2 R_{2332}$  and  $R_{1332} = 0$ . Thus  $\tilde{\mathcal{J}}_{E_3}$  is diagonal with double root which is associated with a two-dimensional eigenspace.

The previous text proved that there exists an orthonormal basis  $(X, Y, Z)$ , such that  $Y$  and  $Z$  are eigenvectors of  $\mathcal{J}_X$ . Consequently, by Rakić duality,  $X$  is an eigenvector of  $\mathcal{J}_Y$ . The conditions of Lemma 1 hold, and hence  $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$  is an eigenvector of  $\mathcal{J}_{\alpha X + \beta Y}$ . Moreover, for  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha^2 \varepsilon_X + \beta^2 \varepsilon_Y \neq 0$ , both  $\alpha X + \beta Y$  and  $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$  are mutually orthogonal and nonnull. The Jacobi operator is symmetric linear operator and therefore

$$g(\mathcal{J}_{\alpha X + \beta Y}(Z), \varepsilon_Y \beta X - \varepsilon_X \alpha Y) = g(Z, \mathcal{J}_{\alpha X + \beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)) = 0.$$

Hence  $\mathcal{J}_{\alpha X + \beta Y}(Z)$  is orthogonal to both  $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$  and  $\alpha X + \beta Y$ , which gives  $\mathcal{J}_{\alpha X + \beta Y}(Z) \perp \text{Span}\{X, Y\} = \text{Span}\{Z\}^\perp$ , and consequently  $Z$  is an eigenvector of  $\mathcal{J}_{\alpha X + \beta Y}$ . According to Rakić duality,  $\alpha X + \beta Y$  is an eigenvector of  $\mathcal{J}_Z$ , which is possible only if  $\text{Span}\{X, Y\}$  is a two-dimensional eigenspace of  $\tilde{\mathcal{J}}_Z$ . Similarly one can prove that  $\text{Span}\{X, Z\}$  is an eigenspace of  $\tilde{\mathcal{J}}_Y$ . Thus arise

$$\frac{R(X, Y, Y, X)}{\varepsilon_X \varepsilon_Y} = \frac{R(Y, Z, Z, Y)}{\varepsilon_Y \varepsilon_Z} = \frac{R(X, Z, Z, X)}{\varepsilon_X \varepsilon_Z} = \kappa,$$

and  $R(Y, X, X, Z) = R(X, Y, Y, Z) = R(X, Z, Z, Y) = 0$ , which obviously completely describes  $R$ , and therefore  $R$  has constant sectional curvature  $\kappa$ .  $\square$

### 3. Rakić duality and Fiedler's tensor

In this section we try to describe Rakić duality property. Let  $(E_1, \dots, E_n)$  be an arbitrary orthonormal basis of the vector space  $\mathcal{V}$  of the signature  $(\nu, n - \nu)$ . Let us start with the left hand side of the equation (2),

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y. \quad (3)$$

The equation (3) means that  $Y$  is an eigenvector of  $\mathcal{J}_X$  for the eigenvalue  $\varepsilon_X \lambda$ . The Jacobi operator can be expressed using the curvature tensor on the following way

$$\mathcal{J}_X(Y) = \mathcal{R}(Y, X)X = \sum_l \varepsilon_l R(Y, X, X, E_l)E_l.$$

If we set nonnull  $X = \sum_i \alpha_i E_i$  and  $Y = \sum_i \beta_i E_i$ , the equation (3) become

$$\sum_{i,j,k,l} \varepsilon_l \beta_i \alpha_j \alpha_k R_{ijkl} E_l = \varepsilon_X \lambda \sum_l \beta_l E_l,$$

and finally

$$(\forall l) \sum_{i,j,k} \varepsilon_l \beta_i \alpha_j \alpha_k R_{ijkl} = \varepsilon_X \lambda \beta_l.$$

According to work of Fiedler [8], and later development by Gilkey [8, 9], for every algebraic curvature tensor  $R$ , there exist finite numbers of skew-symmetric tensors  $\Omega$  of order 2 (i.e. the coordinates of  $\Omega$  satisfy  $\Omega_{ij} = -\Omega_{ji}$ ), such that  $R$  has a representation

$$R_{ijkl} = \sum_{\Omega} \varepsilon_{\Omega} \frac{1}{3} (2\Omega_{ij}\Omega_{kl} + \Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}),$$

with  $\varepsilon_{\Omega} \in \{-1, 1\}$ . Using this fact the equation (3) is equivalent to

$$(\forall l) \sum_{\Omega} \sum_{i,j,k} \frac{1}{3} \varepsilon_{\Omega} \varepsilon_l \beta_i \alpha_j \alpha_k (2\Omega_{ij}\Omega_{kl} + \Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}) = \varepsilon_X \lambda \beta_l.$$

We can simplify the previous formula using the symmetry by  $j$  and  $k$ .

$$\begin{aligned} \sum_{i,j,k} \beta_i \alpha_j \alpha_k \Omega_{ik} \Omega_{jl} &= \sum_{i,k,j} \beta_i \alpha_k \alpha_j \Omega_{ij} \Omega_{kl} = \sum_{i,j,k} \beta_i \alpha_j \alpha_k \Omega_{ij} \Omega_{kl} \\ \sum_{i,j,k} \beta_i \alpha_j \alpha_k \Omega_{il} \Omega_{jk} &= \sum_{i,k,j} \beta_i \alpha_k \alpha_j \Omega_{il} \Omega_{kj} = - \sum_{i,j,k} \beta_i \alpha_j \alpha_k \Omega_{il} \Omega_{jk} = 0 \end{aligned}$$

Therefore the equation (3) is equivalent to

$$(\forall l) \sum_{\Omega} \sum_{i,j,k} \varepsilon_{\Omega} \varepsilon_l \beta_i \alpha_j \alpha_k \Omega_{ij} \Omega_{kl} = \varepsilon_X \lambda \beta_l.$$

Let us split sums on the left hand side

$$(\forall l) \sum_{\Omega} \varepsilon_{\Omega} \varepsilon_l \sum_{i,j} \alpha_i \beta_j \Omega_{ij} \sum_k \alpha_k \Omega_{kl} = -\varepsilon_X \lambda \beta_l.$$

If we introduce the short notation

$$\Theta_{PQ}^\Omega = \sum_{i,j} \mu_i \nu_j \Omega_{ij},$$

for  $P = \sum_i \mu_i E_i$  and  $Q = \sum_j \nu_j E_j$ , the equation (3) become equivalent to

$$(\forall l) \sum_{\Omega} \varepsilon_{\Omega} \Theta_{XY}^\Omega \Theta_{XE_l}^\Omega = -\varepsilon_l \varepsilon_X \lambda \beta_l. \quad (4)$$

Using (3)  $\Leftrightarrow$  (4) and  $\Theta_{YX}^\Omega = -\Theta_{XY}^\Omega$ , we get the equivalent form of Rakić duality condition (2)

$$(\forall l) \sum_{\Omega} \varepsilon_{\Omega} \Theta_{XY}^\Omega \Theta_{XE_l}^\Omega = -\varepsilon_l \varepsilon_X \lambda \beta_l \Rightarrow (\forall l) \sum_{\Omega} \varepsilon_{\Omega} \Theta_{XY}^\Omega \Theta_{YE_l}^\Omega = \varepsilon_l \varepsilon_Y \lambda \alpha_l. \quad (5)$$

Let us sum all  $n$  equations ( $1 \leq l \leq n$ ) from (4) multiplied by  $\beta_l$

$$\sum_l \beta_l \sum_{\Omega} \varepsilon_{\Omega} \Theta_{XY}^\Omega \Theta_{XE_l}^\Omega = - \sum_l \beta_l \varepsilon_l \varepsilon_X \lambda \beta_l,$$

After the substitutions  $\sum_l \beta_l \Theta_{XE_l}^\Omega = \Theta_{XY}^\Omega$  and  $\sum_l \varepsilon_l \beta_l^2 = \varepsilon_Y$  we get important equation.

$$\sum_{\Omega} \varepsilon_{\Omega} (\Theta_{XY}^\Omega)^2 = -\varepsilon_X \varepsilon_Y \lambda. \quad (6)$$

Let us stop here to notice the following interesting statements in the case when Fiedler's terms have a constant signs, i.e.  $\varepsilon_{\Omega} = \text{const}$ .

**Theorem 2** *If  $R$  is a curvature tensor with  $\varepsilon_{\Omega} = \text{const}$ , then it satisfies Rakić duality for the value 0.*

**Proof.** If the sign  $\varepsilon_{\Omega}$  is constant for all skew-symmetric tensors  $\Omega$  in the sense of Fiedler, then the equation (6) gives

$$0 \leq \sum_{\Omega} (\Theta_{XY}^\Omega)^2 = -\varepsilon_{\Omega} \varepsilon_X \varepsilon_Y \lambda.$$

The value  $\lambda = 0$  gives  $\sum_{\Omega} (\Theta_{XY}^\Omega)^2 = 0$  and therefore  $\Theta_{XY}^\Omega = 0$  for all  $\Omega$ . The formula (5) obviously holds and  $R$  satisfies Rakić duality for the value 0.  $\square$

**Theorem 3** *A diagonalizable Osserman curvature tensor  $R$  with  $\varepsilon_{\Omega} = \text{const}$  is Rakić.*

**Proof.** Like the previous proof, the equation (6) with  $\varepsilon_\Omega = \text{const}$  gives  $0 \leq -\varepsilon_\Omega \varepsilon_X \varepsilon_Y \lambda$ . Since by Theorem 2,  $R$  satisfies Rakić duality for the value 0, we set  $\lambda \neq 0$ , and therefore  $\varepsilon_\Omega \varepsilon_X \varepsilon_Y \lambda < 0$ . It implies the constant sign of  $\varepsilon_Y$ , which proves that an eigenspace of  $\mathcal{J}_X$  for an eigenvalue  $\varepsilon_X \lambda$  has the same type of vectors. Especially, there are no nonzero null vectors in the eigenspace of  $\mathcal{J}_X$  for an eigenvalue  $\varepsilon_X \lambda$ . According to our previous work [1, 4], diagonalizable Osserman curvature tensor, such that  $\mathcal{J}_X$  has no null eigenvector for an eigenvalue  $\varepsilon_X \lambda$ , satisfies the duality principle for the value  $\lambda$ , which completes the proof.  $\square$

The diagonalizability from Theorem 3 is a natural condition. Let us remark that, according to Gilkey and Ivanova [11], Jordan Osserman curvature tensor of a non-balanced signature ( $n \neq 2\nu$ ) is necessarily diagonalizable.

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